3. Prove Beta - Binomial conjugation

Assume prior $= p^{a-1}(1-p)^{b-1}\dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$

posterior $= P(\theta\,|\,\text{event}) = \dfrac{\text{likelihood} \times \text{prior}}{\text{marginal}}$

$$= \frac{\binom{N}{m} p^m (1-p)^{N-m}\, p^{a-1}(1-p)^{b-1}\dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}{\int_0^1 \binom{N}{m}\theta^m(1-\theta)^{N-m}\theta^{a-1}(1-\theta)^{b-1}\dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\,d\theta} = \frac{p^{m+a-1}(1-p)^{N-m+b-1}}{\int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1}\,d\theta}$$

$$\because\ 1 = \int_0^1 B(\theta, m+a, N-m+b)\,d\theta = \int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1}\frac{\Gamma(a+N+b)}{\Gamma(m+a)\Gamma(N-m+b)}\,d\theta$$

$$= \frac{\Gamma(a+N+b)}{\Gamma(m+a)\Gamma(N-m+b)}\int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1}\,d\theta$$

$$\therefore\ \int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1}\,d\theta = \frac{\Gamma(m+a)\Gamma(N-m+b)}{\Gamma(a+N+b)}$$

$$\Rightarrow P(\theta\,|\,\text{event}) = \frac{p^{m+a-1}(1-p)^{N-m+b-1}}{\int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1}\,d\theta} = \frac{p^{m+a-1}(1-p)^{N-m+b-1}}{\dfrac{\Gamma(m+a)\Gamma(N-m+b)}{\Gamma(a+N+b)}}$$

$$= \frac{\Gamma(a+N+b)}{\Gamma(m+a)\Gamma(N-m+b)}\,p^{m+a-1}(1-p)^{N-m+b-1} = B(p, a+m, b+N-m)$$