Introductory Functional Analysis
Chapter 1 Exercises

Top Maths

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1.1. Prove[1] the reverse triangle inequality: For vectors x and y in any normed linear space,

$$\|x - y\| \geq |\|x\| - \|y\||.$$

Proof. □

[1]There is a typo in the book. The reverse triangle inequality is as I stated here.

1.2. Show that $C[0,1]$ is a Banach space in the supremum norm. Hint: if $\{f_n\}$ is a Cauchy sequence in $C[0,1]$, then for each fixed $x \in [0,1]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$, which is complete.

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $C[0,1]$. That means, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$\|f_n - f_m\| = \sup_{x \in [0,1]} |f_n(x) - f_m(x)| < \varepsilon.$$

We also get that, for $m, n \geq N$, for all $a \in [0,1]$,

$$|f_n(a) - f_m(a)| \leq \sup_{x \in [0,1]} |f_n(x) - f_m(x)| < \varepsilon. \tag{1}$$

In particular, since $\{f_n\}$ is a Cauchy sequence in $C[0,1]$, $\{f_n(a)\}$ is a Cauchy sequence in $\mathbb{C}$ for all $a \in [0,1]$. Since $\mathbb{C}$ is complete, $\{f_n(a)\}$ converges (because it is Cauchy) for all $a \in [0,1]$. For each $a \in [0,1]$, define

$$f(a) := \lim_{n \to \infty} f_n(a).$$

Considering (1) again , we have by the continuity of $|\cdot|$,

$$\lim_{m \to \infty} |f_n(a) - f_m(a)| = |f_n(a) - \lim_{m \to \infty} f_m(a)| = |f_n(a) - f(a)| < \varepsilon.$$

Since this ε is independent of our choice of $a \in [0,1]$, it follows that

$$\sup |f_n(a) - f(a)| = \|f_n - f\| < \varepsilon$$

giving that $f_n \to f$ in $C[0,1]$. To establish the continuity of f, pick $a \in [0,1]$ and let $\varepsilon > 0$ arbitrary. Since each f_n is continuous, there exists $\delta > 0$ such that

$$|x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\varepsilon}{3}.$$

Also, pick n large enough so that

$$\sup_{p \in [0,1]} |f_n(p) - f(p)| < \frac{\varepsilon}{3}.$$

Then,

$$\begin{aligned}
|f(x) - f(a)| &= |f(x) - f(a) + f_n(x) - f_n(x) + f_n(a) - f_n(a)| \\
&= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\
&\leq |f(x) - f_n(x)| + \underbrace{|f_n(x) - f_n(a)|}_{<\varepsilon/3} + |f_n(a) - f(a)| \\
&\leq \|f - f_n\| + \frac{\varepsilon}{3} + \|f_n - f\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

$\square$

1.3. Let $C^1[0,1]$ be the space of continuous, complex-valued functions on $[0,1]$ with continuous first derivative. Show that with the supremum norm $\|\cdot\|_\infty$, $C^1[0,1]$ is not a Banach space, but that in the norm defined by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ it does become a Banach space.

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq C^1[0,1]$ be defined by

$$f_n(x) = \sqrt{x + \frac{1}{n}} \qquad (x \in [0,1]).$$

Observe that $\sqrt{x}$ is uniformly continuous[2] on $[0,1]$, and so for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any two points $p_1, p_2 \in [0,1]$,

$$|p_1 - p_2| < \delta \implies |f(p_1) - f(p_2)| < \varepsilon.$$

Take $p_1 = x + 1/n$ and $p_2 = x$, then the uniform continuity condition yields

$$\frac{1}{n} < \delta \implies \left|\sqrt{x + 1/n} - \sqrt{x}\right| < \varepsilon.$$

There exists $N \in \mathbb{N}$ large enough so that $1/n < \delta$, and so there exists $n \in \mathbb{N}$ large enough so that

$$\left|\sqrt{x + 1/n} - \sqrt{x}\right| < \varepsilon.$$

Since this ε comes from the uniform continuity condition, it is independent of the choice of x, as is N, and therefore there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\sup_{x\in[0,1]} \left|\sqrt{x + 1/n} - \sqrt{x}\right| < \varepsilon.$$

While $\sqrt{x + 1/n}$ converges to $\sqrt{x}$ in the supremum norm, we find that $\sqrt{x} \notin C^1[0,1]$ because the derivative of $\sqrt{x}$:

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

is not differentiable at 0, and hence $(C^1[0,1], \|\cdot\|_\infty)$ is not a Banach space.

We'll consider the new norm $\|\cdot\|$ as defined above. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C^1[0,1], \|\cdot\|)$. And so for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$,

$$\|f_n - f_m\| = \|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty < \varepsilon.$$

It follows immediately that $\{f_n\}_{n\in\mathbb{N}}$ and $\{f_n'\}_{n\in\mathbb{N}}$ are a Cauchy sequences in $C[0,1]$, and from our previous work we know $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} f_n' = g$, where

$$f(x) := \lim_{n\to\infty} f_n(x), \qquad g(x) := \lim_{n\to\infty} f_n'(x)$$

for all $x \in [0,1]$. In fact, since this convergence is in the sup-norm, it follows[3] that $\{f_n\}_{n\in\mathbb{N}}$ uniformly converges to f and $\{f_n'\}_{n\in\mathbb{N}}$ uniformly converges to g. But, we now have the conditions acquired[4] to say that

$$g(x) = f'(x) = \lim_{n\to\infty} f'(x).$$

Therefore,

$$\lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} \|f_n - f\|_\infty + \lim_{n\to\infty} \|f_n' - f'\|_\infty = 0.$$

Hence, $C^1[0,1]$ is complete under this new norm. □

[2]Continuous functions on compact sets, such as $[0,1]$ in question here, are always uniformly continuous

[3]See *Principles of Mathematical Analysis* (PMA) by Walter Rudin, 3rd Edition, Theorem 7.9 on p. 148

[4]See PMA, Theorem 7.17, p. 152

1.4. Show that the space ℓ^1 of Example 1.5 is complete.

Proof. Writing down the definition of ℓ^1 for convenience, we have:

$$\ell^1 = \left\{ \{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C} : \|\{a_n\}\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty \right\}.$$

Let $\{\{a_m^n\}_{m\in\mathbb{N}}\}_{n\in\mathbb{N}}$ be a notations nightmare, but also let it be a Cauchy sequence in ℓ^1, and so for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $k, \ell \geq N$,

$$\left\|\{a_n^\ell\} - \{a_n^k\}\right\| = \left\|\{a_n^\ell - a_n^k\}\right\| = \sum_{n=1}^{\infty} |a_n^\ell - a_n^k| < \varepsilon.$$

For a fixed n^*, we have

$$|a_{n^*}^\ell - a_{n^*}^k| < \sum_{n=1}^{\infty} |a_n^\ell - a_n^k| < \varepsilon$$

and so, for each fixed $n^* \in \mathbb{N}$,

$$\{a_{n^*}^\ell\}_{\ell\in\mathbb{N}}$$

is a Cauchy sequence in $\mathbb{C}$. For all $n \in \mathbb{N}$, let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence defined by

$$a_n = \lim_{\ell\to\infty} a_n^\ell$$

for all $n \in \mathbb{N}$. First, we check that $\sum |a_n| < \infty$. We have,

$$\sum_{n=1}^{\infty} |a_n| = \lim_{\ell\to\infty} \sum_{n=1}^{\infty} |a_n^\ell|$$

Is this limit finite? In terms of the reverse triangle inequality for norms, we have,

$$\left| \sum_{n=1}^{\infty} |a_n^\ell| - \sum_{n=1}^{\infty} |a_n^k| \right| \leq \sum_{n=1}^{\infty} |a_n^\ell - a_n^k| < \infty$$

by our Cauchy condition we obtained earlier. This result gives that the sequence of sums

$$\left\{ \sum_{n=1}^{\infty} |a_n^\ell| \right\}_{\ell\in\mathbb{N}}$$

is a Cauchy sequence of complex numbers, and therefore converges. Hence, $\{a_n\}_{n\in\mathbb{N}} \in \ell^1$. Now, to prove $\lim_{\ell\to\infty}\{a_n^\ell\}_{n\in\mathbb{N}} = \{a_n\}_{n\in\mathbb{N}}$, we have

$$\begin{aligned}
\lim_{\ell\to\infty} \left\|\{a_n^\ell\} - \{a_n\}\right\|_1 &= \sum_{n=1}^{\infty} |a_n^\ell - a_n| \\
&= \sum_{n=1}^{\infty} |\lim_{\ell\to\infty} a_n^\ell - a_n| \\
&= 0.
\end{aligned}$$

We have shown ℓ^1 is complete, as desired. $\square$

1.5. Show that a metric space is complete if every Cauchy sequence has a convergent subsequence.

Proof. Let (X, d) be a metric space having the property that every Cauchy sequence in X has a convergent subsequence. Given any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, let $\{x_{n_j}\}_{j\in\mathbb{N}}$ be a convergent subsequence to $x \in X$. Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, there exists $N_1 \in \mathbb{N}$ such that for $j, m \geq N_1$,

$$d(x_m, x_{n_j}) < \frac{\varepsilon}{2}.$$

Moreover, by the fact $\lim_{j\to\infty} x_{n_j} = x$, there exists $N_2 \in \mathbb{N}$ such that for $j \geq N_2$,

$$|x_{n_j} - x| < \frac{\varepsilon}{2}.$$

Take $N = \max\{N_1, N_2\}$, and therefore for $n, j \geq N$,

$$d(x_n, x) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\lim_{n\to\infty} x_n = x$ and we conclude that (X, d) is complete. □

1.6. Assume that you now Minkowski's inequality

$$\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}$$

for $\mathbb{C}^n$ in the Euclidean norm.

(a) Show that for $\{a_n\}$ and $\{b_n\}$ in a weighted sequence space ℓ^2_β,

$$\left(\sum_{j=0}^{\infty} |a_j + b_j|^2 \beta(j)^2\right)^{1/2} \leq \left(\sum_{j=0}^{\infty} |a_j|^2 \beta(j)^2\right)^{1/2} + \left(\sum_{j=0}^{\infty} |b_j|^2 \beta(j)^2\right)^{1/2}$$

(b) Verify directly that ℓ^2_β is complete.

Proof.

(a) For convenience, we recall the definition of ℓ^2_β. For $\{\beta(n)\}_{n=0}^\infty$ a sequence of positive numbers with $\beta(0) = 1$, $\lim_{n\to\infty} \beta(n)^{1/n} \geq 1$, we have $\{a_n\}_{n=0}^\infty \in \ell^2_\beta$ if and only if

$$\sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 < \infty.$$

Suppose that $\{a_n\}, \{b_n\} \in \ell^2_\beta$ and define $\{a'_n\}$, $\{\beta'_n\}$ by setting $a'_j = a_j\beta(j)$, $b'_j = b_j\beta(j)$. For all $m \in \mathbb{N}$, we have by the given finite Minkowski inequality over $\mathbb{C}^m$:

$$\left(\sum_{j=1}^{m} |a'_j + b'_j|^2\right)^{1/2} \leq \left(\sum_{j=1}^{m} |a'_j|^2\right)^{1/2} + \left(\sum_{j=1}^{m} |b'_j|^2\right)^{1/2}.$$

Replacing a'_j, b'_j in terms of a_j and b_j, we obtain the inequality

$$\left(\sum_{j=1}^{m} |a_j + b_j|^2 \beta(j)^2\right)^{1/2} \leq \left(\sum_{j=1}^{m} |a_j|^2 \beta(j)^2\right)^{1/2} + \left(\sum_{j=1}^{m} |b_j|^2 \beta(j)^2\right)^{1/2}$$

for all $m \in \mathbb{N}$. Given that $\{a_n\}$, $\{b_n\}$ are in ℓ^2_β, the right hand side converges as $m \to \infty$, and therefore does the left hand side. Thus, the inequality is preserved taking limits as $m \to \infty$, and we obtain the desired weighted Minkowski inequality.

(b) It follows from (a) that the weighted norm is indeed a norm over ℓ^2_β. Now, let $\{\{a^n_m\}_{m\in\mathbb{N}}\}_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ^2_β, and so for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $k, l \geq N$,

$$\left\|\{a^l_n\} - \{a^k_n\}\right\| = \left\|\{a^l_n - a^k_n\}\right\| = \left(\sum_{n=0}^{\infty} |a^l_n - a^k_n|^2 \beta(n)^2\right)^{1/2} < \varepsilon.$$

For a fixed n^*, we have

$$|a^l_{n^*} - a^k_{n^*}|\beta(n^*) \leq \left(\sum_{n=0}^{\infty} |a^l_n - a^k_n|^2 \beta(n)^2\right)^{1/2} < \infty$$

and it follows that (taking $\varepsilon' = \varepsilon/\beta(n^*)$), for each $n^* \in \mathbb{N}$,

$$\{a^l_{n^*}\}_{l\in\mathbb{N}}$$

is a Cauchy sequence in $\mathbb{C}$. For all $n \in \mathbb{N}$, let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence defined by

$$a_n = \lim_{l\to\infty} a^l_n$$

for all $n \in \mathbb{N}$. First, we check that $\sum |a_n|^2\beta(n)^2 < \infty$. We have

$$\sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 = \lim_{l\to\infty} \sum_{n=1}^{\infty} |a^l_n|^2 \beta(n)^2.$$

By the reverse triangle inequality for norms, we have

$$\left|\sum_{n=0}^{\infty} |a^l_n|^2\beta(n)^2 - \sum_{n=1}^{\infty} |a^k_n|^2\beta(n)^2\right| \leq \sum_{n=0}^{\infty} |a^l_n - a^k_n|^2 \beta(j)^2 < \infty$$

by our Cauchy condition that we obtained earlier. This result gives that the sequence of sums

$$\left\{\sum_{n=0}^{\infty} |a^l_n|^2 \beta(n)\right\}_{l\in\mathbb{N}}$$

is a Cauchy sequence of complex number, and therefore converges. Hence, $\{a_n\}_{n\in\mathbb{N}} \in \ell^2_\beta$. Now, to prove $\lim_{l\to\infty}\{a^l_n\}_{n\in\mathbb{N}} = \{a_n\}_{n\in\mathbb{N}}$, see that

$$\begin{aligned}\lim_{l\to\infty} \left\|\{a^l_n\} - \{a_n\}\right\| &= \lim_{l\to\infty} \left(\sum_{n=0}^{\infty} |a^l_n - a_n|^2 \beta(n)^2\right)^{1/2} \\ &= \left(\sum_{n=0}^{\infty} |\lim_{l\to\infty} a^l_n - a_n|^2 \beta(n)^2\right)^{1/2}\end{aligned}$$

$\square$