

Introductory Functional Analysis

Chapter 1 Exercises

Top Maths

Last Updated: Friday 12th April, 2024

1.1. Prove¹ the reverse triangle inequality: For vectors x and y in any normed linear space,

$$\|x - y\| \geq ||\|x\| - \|y\||.$$

Proof. We have,

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\| \implies \|x\| - \|y\| \leq \|x - y\|$$

Note that this argument is reflexive in x and y , we similarly obtain

$$\|y\| - \|x\| \leq \|x - y\|.$$

Putting both together, we get

$$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|.$$

That is,

$$||\|x\| - \|y\|| \leq \|x - y\|$$

as desired. □

¹There is a typo in the book. The reverse triangle inequality is as I stated here.

1.2. Show that $C[0, 1]$ is a Banach space in the supremum norm. Hint: if $\{f_n\}$ is a Cauchy sequence in $C[0, 1]$, then for each fixed $x \in [0, 1]$, $\{f_n(x)\}$ is a Cauchy sequence in \mathbb{C} , which is complete.

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $C[0, 1]$. That means, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$\|f_n - f_m\| = \sup_{x \in [0, 1]} |f_n(x) - f_m(x)| < \varepsilon.$$

We also get that, for $m, n \geq N$, for all $a \in [0, 1]$,

$$|f_n(a) - f_m(a)| \leq \sup_{x \in [0, 1]} |f_n(x) - f_m(x)| < \varepsilon. \quad (1)$$

In particular, since $\{f_n\}$ is a Cauchy sequence in $C[0, 1]$, $\{f_n(a)\}$ is a Cauchy sequence in \mathbb{C} for all $a \in [0, 1]$. Since \mathbb{C} is complete, $\{f_n(a)\}$ converges (because it is Cauchy) for all $a \in [0, 1]$. For each $a \in [0, 1]$, define

$$f(a) := \lim_{n \rightarrow \infty} f_n(a).$$

Considering (1) again, we have by the continuity of $|\cdot|$,

$$\lim_{m \rightarrow \infty} |f_n(a) - f_m(a)| = |f_n(a) - \lim_{m \rightarrow \infty} f_m(a)| = |f_n(a) - f(a)| < \varepsilon.$$

Since this ε is independent of our choice of $a \in [0, 1]$, it follows that

$$\sup |f_n(a) - f(a)| = \|f_n - f\| < \varepsilon$$

giving that $f_n \rightarrow f$ in $C[0, 1]$. To establish the continuity of f , pick $a \in [0, 1]$ and let $\varepsilon > 0$ arbitrary. Since each f_n is continuous, there exists $\delta > 0$ such that

$$|x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\varepsilon}{3}.$$

Also, pick n large enough so that

$$\sup_{p \in [0, 1]} |f_n(p) - f(p)| < \frac{\varepsilon}{3}.$$

Then,

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f(a) + f_n(x) - f_n(x) + f_n(a) - f_n(a)| \\ &= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\ &\leq |f(x) - f_n(x)| + \underbrace{|f_n(x) - f_n(a)|}_{< \varepsilon/3} + |f_n(a) - f(a)| \\ &\leq \|f - f_n\| + \frac{\varepsilon}{3} + \|f_n - f\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

1.3. Let $C^1[0, 1]$ be the space of continuous, complex-valued functions on $[0, 1]$ with continuous first derivative. Show that with the supremum norm $\|\cdot\|_\infty$, $C^1[0, 1]$ is not a Banach space, but that in the norm defined by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ it does become a Banach space.

Proof. Let $\{f_n\}_{n \in \mathbb{N}} \subseteq C^1[0, 1]$ be defined by

$$f_n(x) = \sqrt{x + \frac{1}{n}} \quad (x \in [0, 1]).$$

Observe that \sqrt{x} is uniformly continuous² on $[0, 1]$, and so for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any two points $p_1, p_2 \in [0, 1]$,

$$|p_1 - p_2| < \delta \implies |f(p_1) - f(p_2)| < \varepsilon.$$

Take $p_1 = x + 1/n$ and $p_2 = x$, then the uniform continuity condition yields

$$\frac{1}{n} < \delta \implies \left| \sqrt{x + 1/n} - \sqrt{x} \right| < \varepsilon.$$

There exists $N \in \mathbb{N}$ large enough so that $1/n < \delta$, and so there exists $n \in \mathbb{N}$ large enough so that

$$\left| \sqrt{x + 1/n} - \sqrt{x} \right| < \varepsilon.$$

Since this ε comes from the uniform continuity condition, it is independent of the choice of x , as is N , and therefore there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\sup_{x \in [0, 1]} \left| \sqrt{x + 1/n} - \sqrt{x} \right| < \varepsilon.$$

While $\sqrt{x + 1/n}$ converges to \sqrt{x} in the supremum norm, we find that $\sqrt{x} \notin C^1[0, 1]$ because the derivative of \sqrt{x} :

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

is not differentiable at 0, and hence $(C^1[0, 1], \|\cdot\|_\infty)$ is not a Banach space.

We'll consider the new norm $\|\cdot\|$ as defined above. Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(C^1[0, 1], \|\cdot\|)$. And so for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$,

$$\|f_n - f_m\| = \|f_n - f_m\|_\infty + \|f'_n - f'_m\|_\infty < \varepsilon.$$

²Continuous functions on compact sets, such as $[0, 1]$ in question here, are always uniformly continuous

It follows immediately that $\{f_n\}_{n \in \mathbb{N}}$ and $\{f'_n\}_{n \in \mathbb{N}}$ are Cauchy sequences in $C[0, 1]$, and from our previous work we know $\lim_{n \rightarrow \infty} f_n = f$ and $\lim_{n \rightarrow \infty} f'_n = g$, where

$$f(x) := \lim_{n \rightarrow \infty} f_n(x), \quad g(x) := \lim_{n \rightarrow \infty} f'_n(x)$$

for all $x \in [0, 1]$. In fact, since this convergence is in the sup-norm, it follows³ that $\{f_n\}_{n \in \mathbb{N}}$ uniformly converges to f and $\{f'_n\}_{n \in \mathbb{N}}$ uniformly converges to g . But, we now have the conditions acquired⁴ to say that

$$g(x) = f'(x) = \lim_{n \rightarrow \infty} f'_n(x).$$

Therefore,

$$\lim_{n \rightarrow \infty} \|f_n - f\| = \lim_{n \rightarrow \infty} \|f_n - f\|_{\infty} + \lim_{n \rightarrow \infty} \|f'_n - f'\|_{\infty} = 0.$$

Hence, $C^1[0, 1]$ is complete under this new norm. □

³See *Principles of Mathematical Analysis* (PMA) by Walter Rudin, 3rd Edition, Theorem 7.9 on p. 148

⁴See PMA, Theorem 7.17, p. 152

1.4. Show that the space ℓ^1 of Example 1.5 is complete.

Proof. Writing down the definition of ℓ^1 for convenience, we have:

$$\ell^1 = \left\{ \{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C} : \|\{a_n\}\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty \right\}.$$

Let $\{\{a_m^n\}_{m \in \mathbb{N}}\}_{n \in \mathbb{N}}$ be a notations nightmare, but also let it be a Cauchy sequence in ℓ^1 , and so for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $k, \ell \geq N$,

$$\|\{a_n^\ell\} - \{a_n^k\}\| = \|\{a_n^\ell - a_n^k\}\| = \sum_{n=1}^{\infty} |a_n^\ell - a_n^k| < \varepsilon.$$

For a fixed n^* , we have

$$|a_{n^*}^\ell - a_{n^*}^k| < \sum_{n=1}^{\infty} |a_n^\ell - a_n^k| < \varepsilon$$

and so, for each fixed $n^* \in \mathbb{N}$,

$$\{a_{n^*}^\ell\}_{\ell \in \mathbb{N}}$$

is a Cauchy sequence in \mathbb{C} . For all $n \in \mathbb{N}$, let $\{a_n\}_{n \in \mathbb{N}}$ be the sequence defined by

$$a_n = \lim_{\ell \rightarrow \infty} a_n^\ell$$

for all $n \in \mathbb{N}$. First, we check that $\sum |a_n| < \infty$. We have,

$$\sum_{n=1}^{\infty} |a_n| = \lim_{\ell \rightarrow \infty} \sum_{n=1}^{\infty} |a_n^\ell|$$

Is this limit finite? In terms of the reverse triangle inequality for norms, we have,

$$\left| \sum_{n=1}^{\infty} |a_n^\ell| - \sum_{n=1}^{\infty} |a_n^k| \right| \leq \sum_{n=1}^{\infty} |a_n^\ell - a_n^k| < \infty$$

by our Cauchy condition we obtained earlier. This result gives that the sequence of sums

$$\left\{ \sum_{n=1}^{\infty} |a_n^\ell| \right\}_{\ell \in \mathbb{N}}$$

is a Cauchy sequence of complex numbers, and therefore converges. Hence, $\{a_n\}_{n \in \mathbb{N}} \in \ell^1$. Now, to prove $\lim_{\ell \rightarrow \infty} \{a_n^\ell\}_{n \in \mathbb{N}} = \{a_n\}_{n \in \mathbb{N}}$, we have

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \|\{a_n^\ell\} - \{a_n\}\|_1 &= \sum_{n=1}^{\infty} |a_n^\ell - a_n| \\ &= \sum_{n=1}^{\infty} \left| \lim_{\ell \rightarrow \infty} a_n^\ell - a_n \right| \\ &= 0. \end{aligned}$$

We have shown ℓ^1 is complete, as desired. \square

1.5. Show that a metric space is complete if every Cauchy sequence has a convergent subsequence.

Proof. Let (X, d) be a metric space having the property that every Cauchy sequence in X has a convergent subsequence. Given any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X , let $\{x_{n_j}\}_{j \in \mathbb{N}}$ be a convergent subsequence to $x \in X$. Since $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy, there exists $N_1 \in \mathbb{N}$ such that for $j, m \geq N_1$,

$$d(x_m, x_{n_j}) < \frac{\varepsilon}{2}.$$

Moreover, by the fact $\lim_{j \rightarrow \infty} x_{n_j} = x$, there exists $N_2 \in \mathbb{N}$ such that for $j \geq N_2$,

$$|x_{n_j} - x| < \frac{\varepsilon}{2}.$$

Take $N = \max\{N_1, N_2\}$, and therefore for $n, j \geq N$,

$$d(x_n, x) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\lim_{n \rightarrow \infty} x_n = x$ and we conclude that (X, d) is complete. \square

1.6. Assume that you now Minkowski's inequality

$$\left(\sum_{j=1}^n |a_j + b_j|^2 \right)^{1/2} \leq \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2} + \left(\sum_{j=1}^n |b_j|^2 \right)^{1/2}$$

for \mathbb{C}^n in the Euclidean norm.

(a) Show that for $\{a_n\}$ and $\{b_n\}$ in a weighted sequence space ℓ_β^2 ,

$$\left(\sum_{j=0}^{\infty} |a_j + b_j|^2 \beta(j)^2 \right)^{1/2} \leq \left(\sum_{j=0}^{\infty} |a_j|^2 \beta(j)^2 \right)^{1/2} + \left(\sum_{j=0}^{\infty} |b_j|^2 \beta(j)^2 \right)^{1/2}$$

(b) Verify directly that ℓ_β^2 is complete.

Proof.

(a) For convenience, we recall the definition of ℓ_β^2 . For $\{\beta(n)\}_{n=0}^\infty$ a sequence of positive numbers with $\beta(0) = 1$, $\lim_{n \rightarrow \infty} \beta(n)^{1/n} \geq 1$, we have $\{a_n\}_{n=0}^\infty \in \ell_\beta^2$ if and only if

$$\sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 < \infty.$$

Suppose that $\{a_n\}, \{b_n\} \in \ell_\beta^2$ and define $\{a'_n\}, \{\beta'_n\}$ by setting $a'_j = a_j \beta(j)$, $b'_j = b_j \beta(j)$. For all $m \in \mathbb{N}$, we have by the given finite Minkowski inequality over \mathbb{C}^m :

$$\left(\sum_{j=1}^m |a'_j + b'_j|^2 \right)^{1/2} \leq \left(\sum_{j=1}^m |a'_j|^2 \right)^{1/2} + \left(\sum_{j=1}^m |b'_j|^2 \right)^{1/2}.$$

Replacing a'_j, b'_j in terms of a_j and b_j , we obtain the inequality

$$\left(\sum_{j=1}^m |a_j + b_j|^2 \beta(j)^2 \right)^{1/2} \leq \left(\sum_{j=1}^m |a_j|^2 \beta(j)^2 \right)^{1/2} + \left(\sum_{j=1}^m |b_j|^2 \beta(j)^2 \right)^{1/2}$$

for all $m \in \mathbb{N}$. Given that $\{a_n\}, \{b_n\}$ are in ℓ_β^2 , the right hand side converges as $m \rightarrow \infty$, and therefore does the left hand side. Thus, the inequality is preserved taking limits as $m \rightarrow \infty$, and we obtain the desired weighted Minkowski inequality.

(b) It follows from (a) that the weighted norm is indeed a norm over ℓ_β^2 . Now, let $\{\{a_m^n\}_{m \in \mathbb{N}}\}_{n \in \mathbb{N}}$ be a Cauchy sequence in ℓ_β^2 , and so for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $k, l \geq N$,

$$\|\{a_m^l\} - \{a_m^k\}\| = \|\{a_m^l - a_m^k\}\| = \left(\sum_{n=0}^{\infty} |a_n^l - a_n^k|^2 \beta(n)^2 \right)^{1/2} < \varepsilon.$$

For a fixed n^* , we have

$$|a_{n^*}^l - a_{n^*}^k| \beta(n^*) \leq \left(\sum_{n=0}^{\infty} |a_n^l - a_n^k|^2 \beta(n)^2 \right)^{1/2} < \infty$$

and it follows that (taking $\varepsilon' = \varepsilon / \beta(n^*)$), for each $n^* \in \mathbb{N}$,

$$\{a_{n^*}^l\}_{l \in \mathbb{N}}$$

is a Cauchy sequence in \mathbb{C} . For all $n \in \mathbb{N}$, let $\{a_n\}_{n \in \mathbb{N}}$ be the sequence defined by

$$a_n = \lim_{l \rightarrow \infty} a_n^l$$

for all $n \in \mathbb{N}$. First, we check that $\sum |a_n|^2 \beta(n)^2 < \infty$. We have

$$\sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 = \lim_{l \rightarrow \infty} \sum_{n=1}^{\infty} |a_n^l|^2 \beta(n)^2.$$

By the reverse triangle inequality for norms, we have

$$\left| \sum_{n=0}^{\infty} |a_n^l|^2 \beta(n)^2 - \sum_{n=1}^{\infty} |a_n^k|^2 \beta(n)^2 \right| \leq \sum_{n=0}^{\infty} |a_n^l - a_n^k|^2 \beta(n)^2 < \infty$$

by our Cauchy condition that we obtained earlier. This result gives that the sequence of sums

$$\left\{ \sum_{n=0}^{\infty} |a_n^l|^2 \beta(n)^2 \right\}_{l \in \mathbb{N}}$$

is a Cauchy sequence of complex number, and therefore converges. Hence, $\{a_n\}_{n \in \mathbb{N}} \in \ell_{\beta}^2$. Now, to prove $\lim_{l \rightarrow \infty} \{a_n^l\}_{n \in \mathbb{N}} = \{a_n\}_{n \in \mathbb{N}}$, see that

$$\begin{aligned} \lim_{l \rightarrow \infty} \|\{a_n^l\} - \{a_n\}\| &= \lim_{l \rightarrow \infty} \left(\sum_{n=0}^{\infty} |a_n^l - a_n|^2 \beta(n)^2 \right)^{1/2} \\ &= \left(\sum_{n=0}^{\infty} \left| \lim_{l \rightarrow \infty} a_n^l - a_n \right|^2 \beta(n)^2 \right)^{1/2} \\ &= 0. \end{aligned}$$

It follows that ℓ_{β}^2 is complete. □

1.7. Let x and y be any two vectors in an inner product space and set $\lambda = \langle y, y \rangle$. Show that

$$\lambda[\lambda \langle x, x \rangle - |\langle x, y \rangle|^2] = \langle \lambda x - \langle x, y \rangle y, \lambda x - \langle x, y \rangle y \rangle.$$

Use this to derive the Cauchy-Schwarz inequality and to determine when equality holds in the Cauchy-Schwarz inequality.

Proof. By direct computation, we have,

$$\begin{aligned}
\langle \lambda x - \langle x, y \rangle y, \lambda x - \langle x, y \rangle y \rangle &= \langle \lambda x, \lambda x - \langle x, y \rangle y \rangle - \langle \langle x, y \rangle y, \lambda x - \langle x, y \rangle y \rangle \\
&= \lambda \langle x, \lambda x - \langle x, y \rangle y \rangle - \langle x, y \rangle \langle y, \lambda x - \langle x, y \rangle y \rangle \\
&= \lambda \overline{\langle \lambda x - \langle x, y \rangle y, x \rangle} - \langle x, y \rangle \overline{\langle \lambda x - \langle x, y \rangle y, y \rangle} \\
&= \lambda (\overline{\lambda \langle x, x \rangle} - \overline{\langle x, y \rangle \langle y, x \rangle}) - \langle x, y \rangle (\overline{\lambda \langle x, y \rangle} - \overline{\langle x, y \rangle \langle y, y \rangle}) \\
&= \lambda (\overline{\lambda} \langle x, x \rangle - |\langle x, y \rangle|^2) - \langle x, y \rangle (\lambda \langle y, x \rangle - \lambda \langle y, x \rangle) \\
&= \lambda^2 \langle x, x \rangle - \lambda |\langle x, y \rangle|^2 - \lambda |\langle x, y \rangle|^2 + \lambda |\langle x, y \rangle|^2 \\
&= \lambda^2 \langle x, x \rangle - \lambda |\langle x, y \rangle|^2 \\
&= \lambda [\lambda \langle x, x \rangle - |\langle x, y \rangle|^2].
\end{aligned}$$

Now, to derive the Cauchy-Schwarz inequality: Suppose $x, y \neq 0$, otherwise the inequality holds trivially. Note that by Definition 1.13(2),

$$\begin{aligned}
\langle \lambda x - \langle x, y \rangle y, \lambda x - \langle x, y \rangle y \rangle \geq 0 &\implies \lambda [\lambda \langle x, x \rangle - |\langle x, y \rangle|^2] \geq 0 \\
&\implies \lambda \langle x, x \rangle - |\langle x, y \rangle|^2 \geq 0 \\
&\implies \lambda \langle x, x \rangle \geq |\langle x, y \rangle|^2 \\
&\implies \langle x, x \rangle \langle y, y \rangle \leq |\langle x, y \rangle|^2
\end{aligned}$$

giving the result. Now, if $|\langle x, y \rangle|^2 = \langle x, x \rangle \langle y, y \rangle$, it follows that by the formula we've derived

$$\langle \lambda x - \langle x, y \rangle y, \lambda x - \langle x, y \rangle y \rangle = 0.$$

By Definition 1.13(2), this implies that $\lambda x - \langle x, y \rangle y = 0$, or rather that

$$x = \frac{\langle x, y \rangle}{\langle y, y \rangle} y$$

so that x is a scalar multiple of y . So, for equality to hold, x must be a scalar multiple of y . To show that this is sufficient, let $x = cy$ for some $c \in \mathbb{C}$. Then,

$$\langle y, y \rangle \langle x, x \rangle - |\langle x, y \rangle|^2 = |c|^2 \langle x, x \rangle^2 - |c|^2 \langle x, x \rangle^2 = 0.$$

□

Problem 1.8.

- (a) Show that for a normed linear space X , the map $x \mapsto \|x\|$ of X into $[0, \infty)$ is continuous. Is it uniformly continuous?
- (b) Show that the mappings $X \times X \rightarrow X$ given by $(x, y) \mapsto x + y$, and $\mathbb{C} \times X \rightarrow X$ given by $(\alpha, x) \mapsto \alpha x$ are continuous. The topologies on $X \times X$ and $\mathbb{C} \times X$ are the product topologies.
- (c) Suppose that X is an inner product space. Show that the maps $x \mapsto \langle x, y \rangle$ and $x \mapsto \langle y, x \rangle$ are continuous on X for each fixed y in X . Are they uniformly continuous?

Proof.

- (a)
- (b)
- (c)

□