Introductory Functional Analysis
Chapter 1 Exercises

Top Maths

Last Updated: Tuesday 5th March, 2024

1.1. Prove the reverse triangle inequality: For vectors x and y in any normed linear space,

$$\|x+y\| \geq |\|x\| - \|y\||.$$

Proof. Without loss of generality, suppose $\|x\| \geq \|y\|$; it suffices to show that

$$\|y\| - \|x\| \leq \|x+y\| \leq \|x\| - \|y\|,$$

because for $\alpha, \beta \in \mathbb{R}$, $|\alpha| \leq |\beta|$ if and only if

$$-|\beta| \leq \alpha \leq |\beta|.$$

Assuming that $\|x\| \geq \|y\|$ is like assuming $\beta \geq 0$ in the above, so that we can remove the absolute value signs. Now,

$$\|y\| - \|x\| = \|x+y-x\| - \|x\| \leq \|x+y\| + \|x\| - \|x\| = \|x+y\|$$

giving the left-side inequality. For the right-side inequality, we have

$$\|x\| - \|y\| - \|x+y\| \geq \|x+y-x\| - \|y\| = \|y\| - \|y\| = 0$$

so we get that $\|x\| - \|y\| \geq \|x+y\|$ and the result follows. □

1.2. Show that $C[0,1]$ is a Banach space in the supremum norm. Hint: if $\{f_n\}$ is a Cauchy sequence in $C[0,1]$, then for each fixed $x \in [0,1]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$, which is complete.

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $C[0,1]$. That means, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$\|f_n - f_m\| = \max_{x \in [0,1]} |f_n(x) - f_m(x)| < \varepsilon.$$

We also get that, for $m, n \geq N$, for all $a \in [0,1]$,

$$|f_n(a) - f_m(a)| \leq \max_{x \in [0,1]} |f_n(x) - f_m(x)| < \varepsilon.$$

In particular, since $\{f_n\}$ is a Cauchy sequence in $C[0,1]$, $\{f_n(a)\}$ is a Cauchy sequence in $\mathbb{C}$ for all $a \in [0,1]$. Since $\mathbb{C}$ is complete, $\{f_n(a)\}$ converges (because it is Cauchy) for all $a \in [0,1]$. For each $a \in [0,1]$, define

$$f(a) := \lim_{n \to \infty} f_n(a).$$

□

1.3. Let $C^1[0,1]$ be the space of continuous, complex-valued functions on $[0,1]$ with continuous first derivative. Show that the supremum norm $\|\cdot\|_\infty$, $C^1[0,1]$ is not a Banach space, but that in the norm defined by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ it does become a Banach space.

1.4. Show that the space ℓ^1 of Example 1.5 is complete.