

Introductory Functional Analysis

Chapter 1 Exercises

Top Maths

Last Updated: Thursday 7th March, 2024

1.1. Prove the reverse triangle inequality: For vectors x and y in any normed linear space,

$$\|x + y\| \geq \left| \|x\| - \|y\| \right|.$$

Proof. Without loss of generality, suppose $\|x\| \geq \|y\|$; it suffices to show that

$$\|y\| - \|x\| \leq \|x + y\| \leq \|x\| - \|y\|,$$

because for $\alpha, \beta \in \mathbb{R}$, $|\alpha| \leq |\beta|$ if and only if

$$-|\beta| \leq \alpha \leq |\beta|.$$

Assuming that $\|x\| \geq \|y\|$ is like assuming $\beta \geq 0$ in the above, so that we can remove the absolute value signs. Now,

$$\|y\| - \|x\| = \|x + y - x\| - \|x\| \leq \|x + y\| + \|x\| - \|x\| = \|x + y\|$$

giving the left-side inequality. For the right-side inequality, we have

$$\|x\| - \|y\| - \|x + y\| \geq \|x + y - x\| - \|y\| = \|y\| - \|y\| = 0$$

so we get that $\|x\| - \|y\| \geq \|x + y\|$ and the result follows. □

1.2. Show that $C[0, 1]$ is a Banach space in the supremum norm. Hint: if $\{f_n\}$ is a Cauchy sequence in $C[0, 1]$, then for each fixed $x \in [0, 1]$, $\{f_n(x)\}$ is a Cauchy sequence in \mathbb{C} , which is complete.

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $C[0, 1]$. That means, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$\|f_n - f_m\| = \sup_{x \in [0, 1]} |f_n(x) - f_m(x)| < \varepsilon.$$

We also get that, for $m, n \geq N$, for all $a \in [0, 1]$,

$$|f_n(a) - f_m(a)| \leq \sup_{x \in [0, 1]} |f_n(x) - f_m(x)| < \varepsilon. \quad (1)$$

In particular, since $\{f_n\}$ is a Cauchy sequence in $C[0, 1]$, $\{f_n(a)\}$ is a Cauchy sequence in \mathbb{C} for all $a \in [0, 1]$. Since \mathbb{C} is complete, $\{f_n(a)\}$ converges (because it is Cauchy) for all $a \in [0, 1]$. For each $a \in [0, 1]$, define

$$f(a) := \lim_{n \rightarrow \infty} f_n(a).$$

Considering (1) again, we have by the continuity of $|\cdot|$,

$$\lim_{m \rightarrow \infty} |f_n(a) - f_m(a)| = |f_n(a) - \lim_{m \rightarrow \infty} f_m(a)| = |f_n(a) - f(a)| < \varepsilon.$$

Since this ε is independent of our choice of $a \in [0, 1]$, it follows that

$$\sup |f_n(a) - f(a)| = \|f_n - f\| < \varepsilon$$

giving that $f_n \rightarrow f$ in $C[0, 1]$. To establish the continuity of f , pick $a \in [0, 1]$ and let $\varepsilon > 0$ arbitrary. Since each f_n is continuous, there exists $\delta > 0$ such that

$$|x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\varepsilon}{3}.$$

Also, pick n large enough so that

$$\sup_{p \in [0, 1]} |f_n(p) - f(p)| < \frac{\varepsilon}{3}.$$

Then,

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f(a) + f_n(x) - f_n(x) + f_n(a) - f_n(a)| \\ &= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\ &\leq |f(x) - f_n(x)| + \underbrace{|f_n(x) - f_n(a)|}_{< \varepsilon/3} + |f_n(a) - f(a)| \\ &\leq \|f - f_n\| + \frac{\varepsilon}{3} + \|f_n - f\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

1.3. Let $C^1[0, 1]$ be the space of continuous, complex-valued functions on $[0, 1]$ with continuous first derivative. Show that with the supremum norm $\|\cdot\|_\infty$, $C^1[0, 1]$ is not a Banach space, but that in the norm defined by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ it does become a Banach space.

Proof. Let $\{f_n\}_{n \in \mathbb{N}} \subseteq C^1[0, 1]$ be defined by

$$f_n(x) = \sqrt{x + \frac{1}{n}} \quad (x \in [0, 1]).$$

Observe that \sqrt{x} is uniformly continuous¹ on $[0, 1]$, and so for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any two points $p_1, p_2 \in [0, 1]$,

$$|p_1 - p_2| < \delta \implies |f(p_1) - f(p_2)| < \varepsilon.$$

Take $p_1 = x + 1/n$ and $p_2 = x$, then the uniform continuity condition yields

$$\frac{1}{n} < \delta \implies \left| \sqrt{x + 1/n} - \sqrt{x} \right| < \varepsilon.$$

There exists $N \in \mathbb{N}$ large enough so that $1/n < \delta$, and so there exists $n \in \mathbb{N}$ large enough so that

$$\left| \sqrt{x + 1/n} - \sqrt{x} \right| < \varepsilon.$$

Since this ε comes from the uniform continuity condition, it is independent of the choice of x , as is N , and therefore there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\sup_{x \in [0, 1]} \left| \sqrt{x + 1/n} - \sqrt{x} \right| < \varepsilon.$$

While $\sqrt{x + 1/n}$ converges to \sqrt{x} in the supremum norm, we find that $\sqrt{x} \notin C^1[0, 1]$ because the derivative of \sqrt{x} :

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

is not differentiable at 0, and hence $(C^1[0, 1], \|\cdot\|_\infty)$ is not a Banach space.

We'll consider the new norm $\|\cdot\|$ as defined above. Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(C^1[0, 1], \|\cdot\|)$. And so for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$,

$$\|f_n - f_m\| = \|f_n - f_m\|_\infty + \|f'_n - f'_m\|_\infty < \varepsilon.$$

It follows immediately that $\{f_n\}_{n \in \mathbb{N}}$ and $\{f'_n\}_{n \in \mathbb{N}}$ are Cauchy sequences in $C[0, 1]$, and from our previous work we know $\lim_{n \rightarrow \infty} f_n = f$ and $\lim_{n \rightarrow \infty} f'_n = g$, where

$$f(x) := \lim_{n \rightarrow \infty} f_n(x), \quad g(x) := \lim_{n \rightarrow \infty} f'_n(x)$$

for all $x \in [0, 1]$. In fact, since this convergence is in the sup-norm, it follows² that $\{f_n\}_{n \in \mathbb{N}}$ uniformly converges to f and $\{f'_n\}_{n \in \mathbb{N}}$ uniformly converges to g . But, we now have the conditions acquired³ to say that

$$g(x) = f'(x) = \lim_{n \rightarrow \infty} f'_n(x).$$

Therefore,

$$\lim_{n \rightarrow \infty} \|f_n - f\| = \lim_{n \rightarrow \infty} \|f_n - f\|_\infty + \lim_{n \rightarrow \infty} \|f'_n - f'\|_\infty = 0.$$

Hence, $C^1[0, 1]$ is complete under this new norm. □

¹Continuous functions on compact sets, such as $[0, 1]$ in question here, are always uniformly continuous

²See *Principles of Mathematical Analysis* (PMA) by Walter Rudin, 3rd Edition, Theorem 7.9 on p. 148

³See PMA, Theorem 7.17, p. 152

1.4. Show that the space ℓ^1 of Example 1.5 is complete.