Introductory Functional Analysis
Chapter 1 Exercises

Top Maths

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1.1. Prove the reverse triangle inequality: For vectors x and y in any normed linear space,

$$\|x+y\| \geq |\|x\| - \|y\||.$$

Proof. Without loss of generality, suppose $\|x\| \geq \|y\|$; it suffices to show that

$$\|y\| - \|x\| \leq \|x+y\| \leq \|x\| - \|y\|,$$

because for $\alpha, \beta \in \mathbb{R}$, $|\alpha| \leq |\beta|$ if and only if

$$-|\beta| \leq \alpha \leq |\beta|.$$

Assuming that $\|x\| \geq \|y\|$ is like assuming $\beta \geq 0$ in the above, so that we can remove the absolute value signs. Now,

$$\|y\| - \|x\| = \|x+y-x\| - \|x\| \leq \|x+y\| + \|x\| - \|x\| = \|x+y\|$$

giving the left-side inequality. For the right-side inequality, we have

$$\|x\| - \|y\| - \|x+y\| \geq \|x+y-x\| - \|y\| = \|y\| - \|y\| = 0$$

so we get that $\|x\| - \|y\| \geq \|x+y\|$ and the result follows. $\square$

1.2. Show that $C[0,1]$ is a Banach space in the supremum norm. Hint: if $\{f_n\}$ is a Cauchy sequence in $C[0,1]$, then for each fixed $x \in [0,1]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$, which is complete.

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $C[0,1]$. That means, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$\|f_n - f_m\| = \sup_{x\in[0,1]} |f_n(x) - f_m(x)| < \varepsilon.$$

We also get that, for $m, n \geq N$, for all $a \in [0,1]$,

$$|f_n(a) - f_m(a)| \leq \sup_{x\in[0,1]} |f_n(x) - f_m(x)| < \varepsilon. \tag{1}$$

In particular, since $\{f_n\}$ is a Cauchy sequence in $C[0,1]$, $\{f_n(a)\}$ is a Cauchy sequence in $\mathbb{C}$ for all $a \in [0,1]$. Since $\mathbb{C}$ is complete, $\{f_n(a)\}$ converges (because it is Cauchy) for all $a \in [0,1]$. For each $a \in [0,1]$, define

$$f(a) := \lim_{n\to\infty} f_n(a).$$

Considering (1) again , we have by the continuity of $|\cdot|$,

$$\lim_{m\to\infty} |f_n(a) - f_m(a)| = |f_n(a) - \lim_{m\to\infty} f_m(a)| = |f_n(a) - f(a)| < \varepsilon.$$

Since this ε is independent of our choice of $a \in [0,1]$, it follows that

$$\sup |f_n(a) - f(a)| = \|f_n - f\| < \varepsilon$$

giving that $f_n \to f$ in $C[0,1]$. To establish the continuity of f, pick $a \in [0,1]$ and let $\varepsilon > 0$ arbitrary. Since each f_n is continuous, there exists $\delta > 0$ such that

$$|x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\varepsilon}{3}.$$

Also, pick n large enough so that

$$\sup_{p\in[0,1]} |f_n(p) - f(p)| < \frac{\varepsilon}{3}.$$

Then,

$$\begin{aligned}
|f(x) - f(a)| &= |f(x) - f(a) + f_n(x) - f_n(x) + f_n(a) - f_n(a)| \\
&= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\
&\leq |f(x) - f_n(x)| + \underbrace{|f_n(x) - f_n(a)|}_{<\varepsilon/3} + |f_n(a) - f(a)| \\
&\leq \|f - f_n\| + \frac{\varepsilon}{3} + \|f_n - f\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

$\square$

1.3. Let $C^1[0,1]$ be the space of continuous, complex-valued functions on $[0,1]$ with continuous first derivative. Show that with the supremum norm $\|\cdot\|_\infty$, $C^1[0,1]$ is not a Banach space, but that in the norm defined by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ it does become a Banach space.

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq C^1[0,1]$ be defined by

$$f_n(x) = \sqrt{x + \frac{1}{n}} \qquad (x \in [0,1]).$$

Observe that $\sqrt{x}$ is uniformly continuous[1] on $[0,1]$, and so for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any two points $p_1, p_2 \in [0,1]$,

$$|p_1 - p_2| < \delta \implies |f(p_1) - f(p_2)| < \varepsilon.$$

Take $p_1 = x + 1/n$ and $p_2 = x$, then the uniform continuity condition yields

$$\frac{1}{n} < \delta \implies \left|\sqrt{x + 1/n} - \sqrt{x}\right| < \varepsilon.$$

There exists $N \in \mathbb{N}$ large enough so that $1/n < \delta$, and so there exists $n \in \mathbb{N}$ large enough so that

$$\left|\sqrt{x + 1/n} - \sqrt{x}\right| < \varepsilon.$$

Since this ε comes from the uniform continuity condition, it is independent of the choice of x, as is N, and therefore there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\sup_{x\in[0,1]} \left|\sqrt{x + 1/n} - \sqrt{x}\right| < \varepsilon.$$

While $\sqrt{x + 1/n}$ converges to $\sqrt{x}$ in the supremum norm, we find that $\sqrt{x} \notin C^1[0,1]$ because the derivative of $\sqrt{x}$:

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

is not differentiable at 0, and hence $(C^1[0,1], \|\cdot\|_\infty)$ is not a Banach space.

We'll consider the new norm $\|\cdot\|$ as defined above. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C^1[0,1], \|\cdot\|)$. And so for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$,

$$\|f_n - f_m\| = \|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty < \varepsilon.$$

It follows immediately that $\{f_n\}_{n\in\mathbb{N}}$ and $\{f_n'\}_{n\in\mathbb{N}}$ are a Cauchy sequences in $C[0,1]$, and from our previous work we know $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} f_n' = g$, where

$$f(x) := \lim_{n\to\infty} f_n(x), \qquad g(x) := \lim_{n\to\infty} f_n'(x)$$

for all $x \in [0,1]$. In fact, since this convergence is in the sup-norm, it follows[2] that $\{f_n\}_{n\in\mathbb{N}}$ uniformly converges to f and $\{f_n'\}_{n\in\mathbb{N}}$ uniformly converges to g. But, we now have the conditions acquired[3] to say that

$$g(x) = f'(x) = \lim_{n\to\infty} f'(x).$$

Therefore,

$$\lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} \|f_n - f\|_\infty + \lim_{n\to\infty} \|f_n' - f'\|_\infty = 0.$$

Hence, $C^1[0,1]$ is complete under this new norm. $\square$

[1] Continuous functions on compact sets, such as $[0,1]$ in question here, are always uniformly continuous

[2] See *Principles of Mathematical Analysis* (PMA) by Walter Rudin, 3rd Edition, Theorem 7.9 on p. 148

[3] See PMA, Theorem 7.17, p. 152

1.4. Show that the space ℓ^1 of Example 1.5 is complete.

Proof. Writing down the definition of ℓ^1 for convenience, we have:

$$\ell^1 = \left\{ \{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C} : \|\{a_n\}\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty \right\}.$$

□