Probability and Measure
Section 1. Borel's Normal Number Theorem

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The Unit Interval

Definition 1 (Preliminary Notation)**.**

- $\Omega = (0, 1]$ denotes the unit interval.
- ω denotes a generic point of Ω.
- The ***length*** of an interval $I = (a, b]$ is given by

$$|I| = |(a, b]| = b - a.$$

- If

$$A = \bigcup_{i=1}^{n} I_i = \bigcup_{i=1}^{n} (a_i, b_i]$$

  where the intervals I_i are disjoint, then assign to A the ***probability***

$$P(A) = \sum_{i=1}^{n} |I_i| = \sum_{i=1}^{n} (b_i - a_i)$$

  which is defined only if A is a finite disjoint union of subintervals of $(0, 1]$. P is disjointly additive over such sets. If A and B are disjoint and of this form, then

$$P(A \cup B) = P(A) + P(B).$$

Definition 2 (Dyadic Expansion)**.** To each $\omega \in \Omega$, associate its *nonterminating* dyadic expansion (binary representation)

$$\omega = \sum_{n=1}^{\infty} \frac{d_n(\omega)}{2^n} = .d_1(\omega)d_2(\omega)\ldots,$$

where each $d_n(\omega)$ is either 0 or 1. Thus, $(d_1(\omega), d_2(\omega), \ldots)$ is a sequence of binary digits in the expansion of ω.

Observation 1 (Probability of n-consecutive desired coin flips)**.** Let $u_1, u_2, \ldots, u_n$ be a sequence of 0's and 1's. Then,

$$P[\omega : d_i(\omega) = u_i, i = 1, 2, \ldots, n] = \frac{1}{2^n}.$$

Definition 3 (Dyadic Intervals)**.** A dyadic interval are those whose endpoints are adjacent dyadic rationals

$$\frac{k}{2^n}, \qquad \frac{(k+1)}{2^n}$$

where n denotes the ***rank*** or ***order*** of the interval.

Observation 2 (Probability of k heads in n coin tosses)**.** Let n be a positive integer, and $0 \le k \le n$. Then,

$$P\left[\omega : \sum_{i=1}^{n} d_i(\omega) = k\right] = \binom{n}{k}\frac{1}{2^n}.$$

The Weak Law of Large Numbers

Theorem 1 (Weak Law of Large Numbers)**.** For any $\varepsilon > 0$,

$$\boxed{\lim_{n\to\infty} P\left[\omega : \left|\frac{1}{2}\sum_{i=1}^{n} d_i(\omega) - \frac{1}{2}\right| \ge \varepsilon\right]} = 0.$$

That is, if n is large, then there is a small probability that the fraction or relative frequency of heads in n tosses will deviate much from $\frac{1}{2}$.

Lemma 1. If f is a nonnegative step function, then $[\omega : f(\omega) \ge \alpha]$ is for all $\alpha > 0$ a finite union of intervals and

$$P[\omega : f(\omega) \ge \alpha] \le \frac{1}{\alpha}\int_0^1 f(\omega)\, d\omega.$$

The Strong Law of Large Numbers

Definition 4 (Normal Numbers)**.** A ***normal number*** is an element of the set

$$N = \left[\omega : \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} d_i(\omega) = \frac{1}{2}\right]$$

Definition 5 (Negligible Sets)**.** A subset A of Ω is ***negligible*** if for all $\varepsilon > 0$ there exists a finite or countable collection $I_1, I_2, \ldots$ of intervals (that may overlap) such that

$$A \subset \bigcup_k I_k \qquad \text{and} \qquad \sum_k |I_k| < \varepsilon.$$

It follows the finite or countable union of negligible sets is negligible. A finite or countable set is always negligible.

Theorem 2 (Borel's Normal Number Theorem)**.** The set of normal numbers has negligible complement.

Strong Law Versus Weak

Length

The Measure Theory of Diophantine Approximation