Probability and Measure
Section 1 Problems

Top Maths, mjachi, p1186, cedardrawers

Last Updated: Tuesday 5^{th} March, 2024

1.1.

(**a**) Show that a *discrete* probability space cannot contain an infinite sequence $A_1, A_2, \ldots$ of independent events each of probability $\frac{1}{2}$. Since A_n could be identified with heads on the nth toss of a coin, the existence of such a sequence would make this section superfluous.

(**b**) Suppose that $0 \leq p_n \leq 1$ and put $\alpha_n = \min\{p_n, 1 - p_n\}$. Show that, if $\sum_n \alpha_n$ diverges, then no discrete probability space can contain independent events $A_1, A_2, \ldots$ such that A_n has probability p_n.

Proof. Let Ω be an at most countable sample space, so that probabilities are assigned to points by the probability mass function $P : \Omega \to [0, 1]$ where P satisfies

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$

Here, for $A \subseteq \Omega$,

$$P(A) = \sum_{\omega \in A} P(\omega).$$

(**a**) Let $A_1, A_2, \cdots \subseteq \Omega$ be independent events of probability $\frac{1}{2}$. We can partition Ω into four sets:

$$\Omega = (A_1 \cap A_2) \cup (A_1 \cap A_2^c) \cup (A_1^c \cap A_2) \cup (A_1^c \cap A_2^c)\,.$$

Consequently, each $\omega \in \Omega$ has probability bounded as follows:

$$P(\omega) \leq \frac{1}{4}.$$

To see why, suppose for contradiction that $P(\omega) > \frac{1}{4}$, and without loss of generality let $\omega \in A_1$. Then, since A_1, A_2 are independent events so that $P(A_1 \cap A_2) = P(A_1)P(A_2)$,

$$\begin{aligned} P\left[(A_1 \cap A_2) \cup (A_1 \cap A_2^c)\right] &= P(A_1 \cap A_2) + P(A_1 \cap A_2^c) - P(A_1 \cap A_2 \cap A_1 \cap A_2^c) \\ &= P(A_1)P(A_2) + P(A_1)P(A_2^c) \end{aligned}$$

(**b**)

$\square$