Easier problems of December

Solutions are requested by the next training weekend on 9.-11. January. Solutions may be brought to the training weekend, or sent by post to Joni Teräväinen, Kalannintie 5, 00430 Helsinki, or emailed to joni.teravainen@helsinki.fi. One can also ask about the problems via email. The problems are not in the order of difficulty.

1. Let a and b be real numbers for which $a + b = 2$ and $ab = -1$. Determine $a^{10} + b^{10}$.

2. A rectangular grid consists of $a \times b$ unit squares, where a and b are positive integers. The number of boundary tiles is exactly one third of the number of all tiles in the grid. Determine all possible values of a and b.

3. There are 2014 stones on the table. Two players remove turn by turn $1, 2, 3$ or 4 stones. The winner is the one who removes the last stone. Can the first player force a win?

4. Let AB be a segment with midpoint M, and let Q be an arbitrary point in the plane. Show that $QB^2 = 2QM^2 + \frac{1}{2}AB^2$.

5. Let x be a positive real number. Show that $x^5 + x + 1 \geq 3x^2$.

6. Determine all pairs (a, b) of natural numbers, for which $2^a + 2^b$ is a perfect square (the perfect squares are $0^2, 1^2, 2^2, ...$).

7. Let n be a positive integer. A chess tournament has $2n$ players. How many possible strating rounds are there (on the staring round, every player plays against exactly one other)?

8. Let $ABCD$ be a square and P a point inside it in such a way that $PD = 1, PA = 2, PB = 3$. Determine the angle $\angle APD$.

9. Determine all primes p such that the numbers $p, p + a, p + 2a, p + 3a, +p + 4a$ are all primes for some positive integer $a < 30$.

10. Let $A_1, A_2, ..., A_5$ be five distinct points in the plane. What is the largest posiible value of the smallest angle among $\angle A_i A_j A_k$ $(1 \leq i, j, k \leq 5, i \neq j, j \neq k, k \neq i)$?