Easier problems for September 2014

Please return solutions to the October training event in Päivölä, or mail them to the address

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by the middle of October. The answer to a problem must be justified even if the problem text only asks for the answer!

1. How many 2014-digit numbers n are there such that the sum of the digits of n is even?

2. Determine those natural numbers n for which the number $(n^2+85)^2-(18n+3)^2$ is a prime number.

3. What is the smallest number of members a society can have, if the number of girls is more than 43.5% but less than 43.6%?

4. The lengths of the sides of a triangle are a, b and c and the radius of its circumcircle is R. Prove that $a\sqrt{bc} \leqslant (b+c)R$. Can equality occur here?

5. Determine the last two digits of the number $7^{5^{5^5}}$.

6. Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{9999}{10000} < 0.01.$$

7. Which is larger, the (arithmetical) mean of two positive numbers a and b, or the (arithmetical) mean of $x = \sqrt{ab}$ and $y = \sqrt{\frac{a^2+b^2}{2}}$?

8. Let us choose a point D from the extension of the side AC of a triangle ABC, and a point E from the extension of the side BC, so that $BD = DE$. Prove that $AD = CE$.

9. Solve the equation

$$\frac{8}{x-8} + \frac{10}{x-6} + \frac{12}{x-4} + \frac{14}{x-2} = 4.$$

10. The numbers a and b are rational numbers. Let $x = 8\left(a^4 + b^4 + (a-b)^4\right)$. Prove that $\sqrt{x}$ is a rational number.

11. What is the largest possible number of three element sets, of which each two have exactly one common element, but no element belongs to all of the sets?

12. The numbers p, q and r are prime numbers and $p+q < 111$. Furthermore,

$$\frac{p+q}{r} = p - q + r.$$

Determine the largest possible value of the product pqr.

13. Points P, Q and R are chosen from the sides AD, AB and BC of a rectangle $ABCD$ so that $AP = RC$. Prove that the fractional line PQR is at least as long as a diagonal of the rectangle $ABCD$.

14. A game is played on a table which is an $n \times 2014$ grid. In the beginning a chip is on the upper left-hand corner square of the table. Players A and B move the chip alternatively some number of squares either to right along the row on which the chip lies, of down along the column the chip lies on. The game is lost by the player who can not move the chip is his turn. For which values of n can the player A guarantee a win for himself?

15. The heights of an acute triangle are AA_1, BB_1 and CC_1. The common point of the heights is H. Let M and N be the midpoints of the segments BC and AH. Prove that MN is the perpendicular bisector of B_1C_1.