The Training Letter of April 2014

The problems are not exactly in an increasing order of difficulty, but the first problems are probably easier than the later ones. Solutions may be sent by mail to

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1. Prove that

$$(1+\sin x)(1+\cos x) \leqslant \frac{3}{2}+\sqrt{2}$$

for all real numbers x.

2. There is a heap of 1001 stones on a table. For each heap of stones (with at least three stones) on the table, we may do the following: we remove one stone from it, and then we split it into two smaller heaps (which need not be equally large). Is it possible, by repeating this operation, to reach a situation where there are only heaps of three stones each on table?

3. Let p be an odd prime. Prove that the sum

$$1^{p^p}+2^{p^p}+3^{p^p}+\ldots+p^{p^p}$$

is divisible by p.

4. Let $a,b \in \,]0,\pi/2[$. Prove that

$$\frac{\sin^3 a}{\sin b}+\frac{\cos^3 a}{\cos b} \geqslant \frac{1}{\cos(a-b)}.$$

5. Prove that

$$n^n-1 \geqslant n^{(n+1)/2}(n-1)$$

for every positive integer n.

6. Let k_1, k_2, $\ldots$ be an increasing sequence of positive integers so that $k_{n+2}+k_n > 2k_{n+1}$ for every $n \in \mathbb{Z}_+$. Prove that the number $\sum_{n=1}^{\infty} 10^{-k_n}$ is irrational.

7. Let $a,b,c,d \in [0,\pi]$ be such that

$$\begin{cases} 2\cos a+6\cos b+7\cos c+9\cos d=0, \\ 2\sin a-6\sin b+7\sin c-9\sin d=0. \end{cases}$$

Prove that $3\cos(a+d)=7\cos(b+c)$.

8. Let m and n be positive integers. Prove that the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + \ldots + x_m = n$$

is $\binom{n+m-1}{m-1}$. (Here the order of the terms on the left-hand side is to be taken into account.)

9. Let $a_1 = 1$ and $a_{n+1} = 2a_n + \sqrt{3a_n^2 - 2}$ for every $n \in \mathbb{Z}_+$. Prove that each of the numbers a_1, a_2, $\ldots$ is an integer.

10. Let the lengths of the sides of a triangle be a, b ja c, and let half of its circumference be 1. Prove that

$$1 < ab + bc + ca - abc \leqslant 1 + \frac{1}{27}.$$

11. Let a, b and c be real numbers such that $abc \neq 0$, $a + b + c = 0$ and $a^3 + b^3 + c^3 = a^5 + b^5 + c^5$. Prove that

$$a^2 + b^2 + c^2 = \frac{6}{5}.$$

12. Find all real numbers x and y for which

$$\begin{cases} x^{-1} + y^{-1} = 9, \\ \left(x^{-1/3} + y^{-1/3}\right)\left(1 + x^{-1/3}\right)\left(1 + y^{-1/3}\right) = 18. \end{cases}$$