9. 10. High School Mathematics Competition 2019

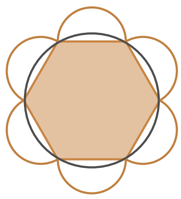
First Round, Basic Level

The problems are on two pages; the first six problems are multiple choice problems with zero to four correct answers.

1. Pihla's work week shortened p % and her hourly wage simultaneously increased p %. All in all, Pihla's weekly wage decreased 4 %. From this it follows that

a) $p < 15$	b) $p \geq 15$	c) $p = 20$	d) $p = 10$

2. A regular hexagon has been inscribed to a circle Γ, and on each side of the hexagon a semicircle has been drawn as in the figure.



The sum of the areas of the semicircles divided by the area of the circle Γ is

a) $2/3$	b) less than 0.8	c) $3/4$	d) $4/5$

3. Determine a fraction q which has the same distance to each of the periodic decimal numbers $0.0246246\ldots$ and $0.0328328\ldots$. In each case the period has three digits.

a) $q = \dfrac{612}{15000}$	b) $q = \dfrac{120}{7290}$
c) $q = \dfrac{574}{19980}$	d) Such a fraction does not exist.

4. The sides of an isosceles trapezoid have lengths $a+3$, $a-3$, $a+3$ and $a+7$. Furthermore, the diagonals of the trapezoid are perpendicular to each other. What can we say about the length a?

a) $a < 8$	b) $a = 10$	c) $a = 7$	d) a is an integer

5. We call a segment connecting two vertices of a polygon a *diagonal*, if it is not a side of the polygon. The angles of a polygon are not allowed to be straight angles. We call a polygon *convex*, if it contains all its diagonals. Which of the following statements concerning polygons are true?

a) There exists a pentagon with two parallel diagonals.
b) The diagonals of a regular polygon always intersect each other.
c) If two diagonals of a convex n-gon are parallel, then $n \geq 6$.
d) A polygon can have two diagonals which are disjoint parts of some line.

6. What can we say about the integer 7^{7^7}, when it is written out in the usual way in the decimal system?

a) It has fewer than a million digits.
b) It ends with the digit 3.
c) The sum of its digits is not divisible by three.
d) It is not a prime.

7. An arithmetic sequence a_1, a_2, a_3, $\ldots$ has the properties $\dfrac{d}{a_1} = \dfrac{a_1 + d}{d}$ and $a_{2019} = 2020 + 2018\sqrt{5}$, where $d = a_2 - a_1$. Furthermore, a_1 and d have the same sign. Determine a_1 and d.

8. Maija plays the following solitaire on an infinite grid of squares: Let k be a positive integer. Maija has a stone which lies on the square $(0, 0)$ at the beginning. In one move, the stone can be moved $k-1$ squares horizontally, k squares vertically, or $k+1$ squares diagonally. More precisely, the stone can be moved from a square (x, y) to any of the squares

- $(x - (k-1), y)$ and $(x + (k-1), y)$,
- $(x, y - k)$ and $(x, y + k)$,
- as well as $(x-(k+1), y-(k+1))$, $(x-(k+1), y+(k+1))$, $(x+(k+1), y-(k+1))$ and $(x + (k+1), y + (k+1))$.

Maija chooses randomly a goal square (a, b), where a and b are integers. She wins, if she can find a series of moves that take the stone from the square $(0, 0)$ to the square (a, b). Does Maija win regardless of the integers a and b, provided that she chooses her moves with care, when a) $k = 6$, b) $k = 2019$?

9. 10. **Basic Level Multiple Choice Answer Sheet** 2019

Basic Level Multiple Choice Answer Sheet

The first six problems are multiple choice problems. Their answers should be written in the table below. Each multiple choice problem has 0 to 4 correct answers. Put a "+" to the appropriate square, if the answer is right and a "−" if the answer is wrong. All correct marks give one point and incorrect or unintelligible marks give zero points. The answers to problems 7 and 8 can be written on a separate paper. For each of these problems, a maximum of 6 points is given.

The time allowed is 120 minutes. ***The use of calculators and tables are not allowed.*** *Please write your name and school with block letters on every paper you return.*

Name: ______________________________

School: ______________________________

	a	b	c	d
1.				
2.				
3.				
4.				
5.				
6.				

9.10. High School Mathematics Competition 2019
First Round, Intermediate Level

The problems are on two pages; the first three problems are multiple choice problems with zero to four correct answers.

1. The diameter of the base of a right circular cone is 2 and the distance of its vertex to the edge of the base is 2. Another right cone has a base which is a square with side length 2 and the distance of its vertex to a vertex of the base is 2. What can you say about the volumes of the cones?

a) The volumes are integers.
b) The volumes cannot be computed with the given data.
c) The volumes are equal.
d) The circular cone has larger volume.

2. Which of the following claims hold for the polynomial $P(x) = x^4 + x^2 + 1$ with integer coefficients?

a) It is irreducible, i.e. it cannot be written as a product of polynomials with integer coefficients of lower degree.
b) It has no real roots.
c) Its graph is symmetrical with respect to the y-axis.
d) The equation $P(x) = 7$ has a rational solution.

3. A function $f\colon\,]0,\infty[\longrightarrow]1,\infty[$ has the property that

$$f(x) = e^{f(x)-x-1}$$

for all $x \in]0,\infty[$. Which of the following statements are certainly true?

a) If $x, y \in]0,\infty[$ and $x \neq y$, then $f(x) \neq f(y)$.
b) If $x \in]0,\infty[$, then $f(x) < x$.
c) There exists $x \in]0,\infty[$, for which $f(x) = x + 1$.
d) If $x \in]0,\infty[$, then $f(x) > x$.

4. An arithmetic sequence a_1, a_2, a_3, $\ldots$ has the properties $\dfrac{d}{a_1} = \dfrac{a_1 + d}{d}$ and $a_{2019} = 2020 + 2018\sqrt{5}$, where $d = a_2 - a_1$. Furthermore, a_1 and d have the same sign. Determine a_1 and d.

5. Maija plays the following solitaire on an infinite grid of squares: Let k be a positive integer. Maija has a stone which lies on the square $(0,0)$ at the beginning. In one move, the stone can be moved $k-1$ squares horizontally, k squares vertically, or $k+1$ squares diagonally. More precisely, the stone can be moved from a square (x,y) to any of the squares

- $(x-(k-1),y)$ and $(x+(k-1),y)$,
- $(x,y-k)$ and $(x,y+k)$,
- as well as $(x-(k+1),y-(k+1))$, $(x-(k+1),y+(k+1))$, $(x+(k+1),y-(k+1))$ and $(x+(k+1),y+(k+1))$.

Maija chooses randomly a goal square (a,b), where a and b are integers. She wins, if she can find a series of moves that take the stone from the square $(0,0)$ to the square (a,b). For which values of k does Maija win regardless of the integers a and b, provided that she chooses her moves with care?

6. Solve the Diophantine equation

$$x^4y^2+2x=2019$$

that is, find all pairs of integers (x,y) solving the above equation.

9. 10.

Intermediate Level Multiple Choice Answer Sheet

2019

The first three problems are multiple choice problems. Their answers should be written in the table below. Each multiple choice problem has 0 to 4 correct answers. Put a "+" to the appropriate square, if the answer is right and a "−" if the answer is wrong. All correct marks give one point and incorrect or unintelligible marks give zero points. The answers to problems 4 to 6 can be written on a separate paper. For each of these problems, a maximum of 6 points is given.

The time allowed is 120 minutes. **Calculators and tables are not allowed.** *Please write your name and school with block letters on every paper you return.*

Name: ______________________________

School: ______________________________

	a	b	c	d
1.				
2.				
3.				

9. 10.

High School Mathematics Competition
First Round, Open Level

2019

1. Let $ABCDE$ be a regular pentagon, and suppose that the area of the star $ACEBD$ is one. Let the segments AC and BE intersect at P, and let the segments BD and CE intersect at Q. Determine the area of the quadrilateral $APQD$.

2. Maija plays the following solitaire on an infinite grid of squares: Let k be a positive integer. Maija has a stone which lies on the square $(0, 0)$ at the beginning. In one move, the stone can be moved $k - 1$ squares horizontally, k squares vertically, or $k + 1$ squares diagonally. More precisely, the stone can be moved from a square (x, y) to any of the squares

- $(x - (k - 1), y)$ and $(x + (k - 1), y)$,
- $(x, y - k)$ and $(x, y + k)$,
- as well as $(x - (k + 1), y - (k + 1))$, $(x - (k + 1), y + (k + 1))$, $(x + (k + 1), y - (k + 1))$ and $(x + (k + 1), y + (k + 1))$.

Maija chooses randomly a goal square (a, b), where a and b are integers. She wins, if she can find a series of moves that take the stone from the square $(0, 0)$ to the square (a, b). For which values of k does Maija win regardless of the integers a and b, provided that she chooses her moves with care?

3. Solve the Diophantine equation

$$x^4y^2 + 2x = 2019,$$

that is, find all pairs of integers (x, y) solving the above equation.

4. Let us consider the sequence F_1, F_2, … of Fibonacci numbers which is defined by setting $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for each positive integer n. Find the smallest positive integer k for which the interval $]F_n, F_{n+1}[$ contains a cube number for every integer $n \geqslant k$, or prove that such a number k does not exist. Positive cube numbers are $1^3 = 1$, $2^3 = 8$, $3^3 = 27$ and so on.

Time allowed: **120 minutes**.
Only writing and drawing equipments are allowed.
No calculators or tables!
Write your solutions of different problems on different sheets.
Mark every sheet with your name and school.