The problem set of the summer

Solutions can be sent by the beginning of September either by mail to

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or by email to esavesalainen@gmail.com, where also questions can be sent.

We wish you a nice summer and pleasant moments with the problems!

Easier problems

1. Let the solutions to the equation $x^2 + (p^2 + 1)x + p = 2$ be x_1 and x_2, and let us assume that they are nonzero and distinct. Determine all those values of the parameter p, for which

$$\frac{2x_1 - 1}{x_2} + \frac{2x_2 - 1}{x_1} = x_1 x_2 + \frac{55}{x_1 x_2}.$$

2. Find all positive integers x and y for which

$$(x^2 + y)(y^2 + x)$$

is the fifth power of some prime number.

3. Find all real numbers a and b for which the pair of equations

$$\begin{cases} x + a = y + b \\ x^2 - a = 2y \end{cases}$$

has a unique solution (x_0, y_0), and this solution satisfies the condition $x_0^{10} + y_0^{10} = 1025$.

4. In a triangle ABC, in which $AB > BC$, a point K lies on the side AB so that $AK = BC + BK$. A line ℓ, which passes through the point K is perpendicular to AB. Prove that the line ℓ, the bisector of the side AC and the bisector of the angle $\angle ABC$ pass through a common point.

5. Let a, b, c and d be real numbers. Prove that the smallest of the numbers

$$a - b^2, \quad b - c^2, \quad c - d^2 \quad \text{ja} \quad d - a^2$$

is at most $1/4$.

6. Let a and b be real numbers. If $a + b = 4$ and $a^2 + b^2 = 14$, then what is $a^3 + b^3$?

7. Let $ABCDE$ be a regular pentagon, and let the lines AB and DE intersect at F. Determine the angles of the triangle $\triangle BEF$.

8. The circumscribed circle of a triangle is reflected along a side of the triangle. Prove that the new circle passes through the orthocenter of the triangle.

9. Let $n \geqslant 2$ be an integer. Compute

$$S_n = \sum_{k=1}^{n-1} \sin kx \cos(n-k)x.$$

10. Let α and β be real numbers from the interval $]0, \pi/2[$, and let us assume that

$$\cos^2(\alpha - \beta) = \sin 2\alpha \sin 2\beta.$$

Prove that $\alpha + \beta = \pi/2$.

11. Let the side lengths of a triangle be a, b and c. Determine, when also a^2, b^2 and c^2 are the side lengths of a triangle.

Harder problems

12. Let us define a number a_n for each positive integer n as follows: $a_n = 0$, if the number n has an even number of prime factors greater than 2007, and $a_n = 1$, if the number n has an odd number of prime factors greater than 2007. Is the number $0{,}a_1a_2a_3\ldots$ rational?

13. Find all positive integers n for which, whenever $a, b, c \geqslant 0$ and $a+b+c = 3$, also $abc(a^n + b^n + c^n) \leqslant 3$.

14. Let $a_1 > \frac{1}{12}$ and $a_{n+1} = \sqrt{(n+2)a_n + 1}$, when $n \geqslant 1$. Prove that

1. $a_n > n - \frac{2}{n}$,
2. the sequence $b_n = 2^n \left(\frac{a_n}{n} - 1\right)$ converges $(n = 1, 2, \ldots)$.

15. Solve the system of equations in the set of integers:

$$\begin{cases} 3a^4 + 2b^3 = c^2, \\ 3a^6 + b^5 = d^2. \end{cases}$$

16. Let a and b be distinct positive real numbers. Find all pairs (x, y) of positive real numbers solving the pair of equations

$$\begin{cases} x^4 - y^4 = ax - by, \\ x^2 - y^2 = \sqrt[3]{a^2 - b^2}. \end{cases}$$