Challenging problems of December

Solutions are requested by the next training weekend on 9.-11. January. Solutions may be brought to the training weekend, or sent by post to Joni Teräväinen, Kalannintie 5, 00430 Helsinki, or emailed to joni.teravainen@helsinki.fi. One can also ask about the problems via email. The problems are not in the order of difficulty.

1. Determine all triples (x, y, z) of integers satisfying $x^2 + y^2 = z^2$ and $x + y = z + 2$.

2. Let a, b and c be real numbers. Show that

$$(a^2b + b^2c + c^2a)^2 \leq (a^2 + b^2 + c^2)(a^4 + b^4 + c^4).$$

3. Let $n \geq 2$ be a positive integer. Anna and Berta play the following game on an $n \times n$ chess board. Anna starts and chooses a row or column of the board, which she paints. Next Berta chooses some row or column and paints it (some of the squares in that row or column may already have been painted, as long as some of them are not). Then it is Anna's turn again, and the game continues like this until all the squares have been painted. The winner is the one who paints the final square. Which player has a winning startegy?

4. In a triangle ABC, the lengths of the sides BC, AC and AB are a, b and c, and their opposite angles are α, β and γ degrees. Show that

$$60^\circ \leq \frac{\alpha a + \beta b + \gamma c}{a + b + c} < 90^\circ.$$

5. Show that there exist infinitely many positive integers n for which $2^n + n$ is not a prime.

6. Define a sequence $a_1, a_2, ...$ of real numbers by setting $a_1 = 1$ and

$$a_{n+1} = a_n + \frac{1}{a_n}$$

for every integer $n \geq 1$. Is the sequence $a_1, a_2, ...$ bounded? Show that $a_{100} < 15$.

7. Show that there exist only finitely many positive integers n such that n is divisble by every integer on the interval $[1, \sqrt[10]{n}]$.

8. For a circle with center O, its chord CD and diameter AB are perpendicular. The chord AE bisects the radius OC. Show that the chord DE bisects the chord BC.

9. Let $x \neq 1$ be a positive real number and n a positive integer. Show that

$$\frac{x^n - 1}{x - 1} \geq nx^{\frac{n-1}{2}}.$$

10. Let N and n be positive integers with $2n+1 > N$. In the school of Math Town, there are n students, and in the school of Physics Town, there are $n+1$ students. A competition team of N members is randomly selected among them. What is the probability that more students are chosen to the team from the school of Math Town than from the school of Physics Town?