PROBLEMS OF FEBRUARY 2014

EASIER PROBLEMS AND MORE CHALLENGING PROBLEMS

This time both the easier and the more challenging problems are given in one problem set. Naturally, it is perfectly ok to focus only on the easier or only on the more challenging problems. It is, however, essential to observe that the activity of returning solutions has substantial weight when invitations for the training week of May (where the IMO team is selected) are sent: If no solutions or even attempts at solutions are not forthcoming, it is exceedingly unlikely to be invited to the training week.

Solutions are to be sent by the middle of February to the address Anne-Maria Ernvall-Hytönen, Matematiikan ja tilastotieteen laitos, PL 68, 00014 Helsingin yliopisto. Clarifications regarding the problems can be requested by calling 041-5228141.

The easier problems

(1) Find all those values of the parameter b for which at least one of the functions $f_1(x) = x^2 + 2011x + b$ and $f_2(x) = x^2 - 2011x + b$ is positive for all real x.

(2) Prove that there are infinitely many squares which can be written in the form $2^n + 2^m$, where n and m are distinct positive integers.

(3) The numbers $1, 2, 3, 4, 5, 6$ are written at the vertices of a prism whose bottom has the shape of a triangle, and then, for each edge, the sum of the numbers at the vertices is written on the edge. Is it possible to write the numbers at the vertices so that all the sums of the edges are pairwise distinct?

(4) Let x_1, x_2, x_3 be three pairwise distinct real numbers. Let us also assume that x_2 and x_3 are the zeros of the polynomial $f_1(x) = x^2 + p_1x + q_1$, that x_3 and x_1 are the zeros of the polynomial $f_2(x) = x^2 + p_2x + q_2$, and that x_1 and x_2 are the zeros of the polynomial $f_3(x) = x^2 + p_3x + q_3$. Does the polynomial
$$f(x) = f_1(x) + f_2(x) + f_3(x)$$
always have real zeros?

(5) Let us modify the previous problem a little bit: Let x_1, x_2, x_3 be three pairwise distinct real numbers. Let us also assume that x_2 and x_3 are the zeros of the polynomial $f_1(x) = a_1x^2 + p_1x + q_1$, that x_3 and x_1 are the zeros of the polynomial $f_2(x) = a_2x^2 + p_2x + q_2$, and that x_1 and x_2 are the zeros of the polynomial $f_3(x) = a_3x^2 + p_3x + q_3$. Does the polynomial
$$f(x) = f_1(x) + f_2(x) + f_3(x)$$
always have real zeros?

(6) Find all pairs of primes (a, b), for which $a^b b^a + 1$ is a prime.

(7) How many such divisors does the number n^2, where $n = 11^{2011} \cdot 2011^{11}$, have which are smaller than n, and which do not divide the number n?

(8) Nonnegative numbers a, b, c satisfy the inequality
$$a^2b + b^2c + c^2a \leq a + b + c.$$

Prove that

$$ab + bc + ca \leq a + b + c.$$

(9) Find the smallest positive integer n, for which the number $n^3 + n^2 + 330n + 330$ is divisible by 2011.

(10) A circle is drawn around the triangle ABC, and the tangents at the points A and B intersect at the point T. A line parallel to AC is drawn through the point T and it intersects the line BC at the point D. Prove that $AD = CD$.

(11) Let us call a positive integer a twin, if it has two positive divisors whose difference is two. Find out whether there are more twins or non-twins among the first 20112012 positive integers.

The harder problems

(1) A trapezoid $ABCD$ is drawn on a sheet of paper. Its parallel sides are $BC = a$ and $AD = 2a$. Using only a ruler (one which allows one to draw straight lines but has no markings that could be used to make measurements) draw a triangle, whose area is the same as the area of the trapezoid.

(2) Let a, b and c be nonnegative real numbers for which $a + b + c \leq 2$. Prove that

$$ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \leq 2.$$

(3) Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the conditions

(a) for all real numbers x and y,

$$f(2x) = f(x + y)f(y - x) + f(x - y)f(-x - y)$$

(b) $f(x) \geq 0$ for all x.

(4) Solve the Diophantine equation

$$y^k = x^2 + x,$$

where $k > 1$ is a positive integer.

(5) Find all primes p, q and r, for which

$$p(p + 1) + q(q + 1) = r(r + 1).$$

(6) Three circles are tangent to each other from the outside. Their diameters A_1A_2, B_1B_2 and C_1C_2 are parallel. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 all intersect each other at some point.

(7) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of n. Let us write $T(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. Let a, b, c, d be pairwise distinct nonnegative integers. Prove that if $ac + bd$ divides $ab + cd$, then $T(ab + cd) \geq 3$.

(8) Solve the equation

$$\cos \pi x = \left\lfloor \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor - \frac{1}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ is the largest integer which is not larger than x.

(9) In a triangle ABC the point M is the midpoint of BC, and a point N has been chosen from the side AB so that $NB = 2AN$. If $\angle CAB = \angle CMN$, then what is $\frac{|AC|}{|BC|}$?

(10) Solve the Diophantine equation

$$(x + y)^3 = (x - y - 6)^2$$

in the set of positive integers.