IMO Training
Homework, January 2018

Please hand in your solutions by Friday, February 23rd, in person at Päivölä, by email to `npalojar@abo.fi` or by postal mail to

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We have often been flexible about return dates, but this time the date is firm because of the EGMO team selection deadline. Owing to equality considerations, responses will not be accepted after February 23rd even from those who are not eligible to participate in EGMO.

Introductory problems

1. Outside the triangle ABC is drawn a square, whose one side is AB. Another square is drawn with side BC. Show that the centers of the squares and the center of side CA form an isosceles right triangle.

2. Let H be the intersection of the altitudes of triangle ABC, A' the center of side BC, X the center of the altitude from B, Y the center of the altitude from C, and D the foot of the altitude from A. Show that the points X, Y, D, H ja A' lie on a circle.

3. In triangle ABC, let D be the foot of the altitude from A and E the foot of the altitude from B. Let O be the circumcenter of ABC. Show that $OC \perp DE$.

4. A rectangular floor is to be tiled with tiles of shapes 2×2 and 1×4. Tiles of both shapes have been ordered in such quantities that the tiling is possible. One of the tiles has shattered, but there is an extra tile of the other shape. Show that the tiling is impossible with these tiles.

5. Show that among six people there must be either three people who know each other or three people among whom no two know each other.

6. Let $1, 4, \dots$ and $9, 16, \dots$ be two arithmetic sequences. Let set S be the union of the 2018 first elements in each sequence. How many elements are there in set S?

(In an arithmetic sequence, the difference between consecutive numbers is constant. In other words, an arithmetic sequence is of the form $a, a+d, a+2d, a+3d, \dots$.)

7. Given a strictly increasing sequence of six positive integers where each number starting from the second is a multiple of the preceding number, and where the sum of all numbers is 79, what is the largest number in the sequence?

8. A box of chocolates has 36 slots, and there are 10 kinds of chocolate. How many different boxes is it possible to make, when it is required that each box has at least one of each kind of chocolate? The ordering of the chocolates within a box does not matter, we are just interested in how many of each kind of chocolate are in each box.

Hint: to solve this problem, it is useful to find out about binomial coefficients from e.g. Wikipedia, if you are not yet familiar with them. The answer to "how many ways can you choose 10 chocolates from 36 different kinds" is the binomial coefficient $\binom{36}{10}$.

9. Show that if $2n+1$ and $3n+1$ are squares of integers, $5n+3$ is not prime.

(An integer $p > 1$ is prime if it is only divisible by p and 1.)

10. Find the three last digits of 7^{9999}. The result is probably easy to find using modern computing software, but explain how to arrive at the result without using a computer.

Advanced problems

11. Show that if p is an odd prime,

a) $1^{p-1} + 2^{p-1} + 3^{p-1} + \cdots + (p-1)^{p-1} \equiv -1 \mod p$.

b) $1^p + 2^p + 3^p + \cdots + (p-1)^p \equiv 0 \mod p$.

12. Show that $1982 \mid 222\cdots 2$ (the digit 2 repeated 1980 times).

13. Find the greatest common divisor of the 2017 numbers

$$2017 + 1, 2017^2 + 1, 2017^3 + 1, \ldots, 2017^{2017} + 1.$$

14. For positive integers n, let $\sigma(n)$ denote the sum of the divisors of n. Show that there are infinitely many positive integers n such that n divides $2^{\sigma(n)} - 1$.

15. Let n and k be two positive integers such that $1 \leq n \leq k$. Show that if $d^k + k$ is prime for all positive factors d of n, the number $n + k$ is prime.

16. Find the integer solutions to $19x^3 - 84y^2 = 1984$.

17. Find all continuous functions $f, g, h \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x+y) = g(x) + h(y)$$

for all $x, y \in \mathbb{R}$.

18. Find all functions $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x+y) - 2f(x-y) + f(x) + f(y) = 4y + 1$$

for all $x, y \in \mathbb{R}$.

19. Find all functions $f \colon \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ such that

$$f(f(m) + f(n)) = m + n$$

for all $m, n \in \mathbb{Z}_+$.

20. Find all functions $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f((x-y)^2) = f^2(x) - 2xf(y) + y^2$$

for all $x, y \in \mathbb{R}$.

21. Define $a_0 = a_1 = 3$ and $a_{n+1} = 7a_n - a_{n-1}$ for all $n \in \mathbb{Z}_+$. Show that $a_n - 2$ is a square for all $n \in \mathbb{Z}_+$.