Math olympiad training
Homework problems, February 2018

Please submit your solutions by the next Päivölä meeting, April 6th, in person, by email to `npalojar@abo.fi` or by mail to

Neea Palojärvi
Ratapihankatu 12 A 1
20100 Turku.

Introductory problems

1. What is the largest positive integer n, such that $n^3 + 100$ is divisible by $n + 10$?

2. a) Let $n > 2$ be an integer. Show that an even number of the fractions

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$$

are irreducible.

b) Show that the fraction $\dfrac{12n+1}{30n+2}$ is irreducible when n is a positive integer.

3. Prove that

$$2 \cdot 3^n \leq 2^n + 4^n, \; n = 1, 2, \ldots$$

Moreover, the inequality is strict unless $n = 1$.

4. Suppose a, b, c are positive numbers. Prove that

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1\right)^2 \geq (2a + b + c)\left(\frac{2}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

with equality if and only if $a = b = c$.

5. A and B bake cakes on a Monday. A bakes a cake every fifth day and B bakes a cake every other day. How many days will it be before they both next bake a cake on a Monday?

6. Can it happen for a whole calendar year that no single Monday falls on the first day of a month?

7. Each term of a sequence of natural numbers is obtained from the previous term by adding to it its largest digit. What is the maximal number of successive odd terms in such a sequence?

8. The positive integer n is divisible by 24. Show that the sum of all the positive divisors of $n - 1$ is also divisible by 24.

9. During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him/her a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.

10. One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed there was a positive integer such that

$$133^5 + 110^5 + 84^5 + 27^5 = n^5.$$

Find the value of n.

Advanced problems

11. Let a, b, c, d be positive integers. Prove that

$$(2a-1)(2b-1)(2c-1)(2d-1) \geq 2abcd - 1.$$

12. Let x, y, z be real numbers satisfying $x + y \geq 2z$ and $y + z \geq 2x$. Prove that

$$5(x^3 + y^3 + z^3) + 12xyz \geq 3(x^2 + y^2 + z^2)(x + y + z)$$

with equality if and only if $x + y = 2z$ or $y + z = 2x$.

13. Show that if x and y are positive real numbers, then

$$(x + y)^5 \geq 12xy(x^3 + y^3)$$

and that the constant 12 is best possible (in other words, if it is replaced by larger constant, then there are positive real numbers x and y which do not satisfy the inequality).

14. In Sixton there're n residents and m clubs. We know that in any club the number of its members isn't divisible by 6. On the other hand, for any two clubs, the number of their common members is divisible by 6. Prove that $m \leq 2n$.

15. $A_1, \ldots, A_m$ are proper subsets of $\{1, 2, \ldots, n\}$, and for any two distinct i and j from $\{1, 2, \ldots, n\}$, there is exactly one A_k that contains them both. Prove that $m \geq n$.

16. Determine all pairs (x, y) of integers such that $x^2 = y(2x - y) + 1$.

17. Determine all pairs (x, y) of positive integers such that $x^y = y^x$.

18. Determine all pairs (p, q) of primes such that $p \mid 2q + 1$ and $q \mid 2p + 1$.

19. Let x, y and z be integers such that $x^2 + y^2 + z^2 = 2xyz$. Show that $x = y = z = 0$.

20. Call a triangle *perfect* if its sides have integer lengths and the length of its perimeter equals its area. Determine all perfect triangles.