Easier Problems, February 2012

Answers by email to `laurihallila@gmail.com`, or by ordinary mail to Lauri Hallila, Kalliorinteenkuja 1, 02770 Espoo. If you have questions about the problems, use the email address above.

1. The teacher chose two postitive integers a and b such that $\frac{a}{b} \cdot \sqrt{a^2+b^2}$ is an integer.

a) Sam claims that each prime divisor of b is a divisor of a. Show that Sam's claim is correct.

b) Sam claims that $b \leq a$. Is he correct this time?

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$, for which

$$f(x + f(y)) = x + f(f(y))$$

for all real numbers x and y, and $f(2004) = 2005$.

3. We call a triangle of numbers *wonderful*, if it satisfies the following conditions:

i) All numbers are positivie integers and no two are equal.

ii) The number under two adjacent numbers is the quotient of these numbers.

Below is on wonderful triangle of side lenth 3. Find the smallest possible integer which can be the largest number in a wonderful triangle of side length 4.

$$\begin{array}{ccccc} 21 & & 84 & & 7 \\ & 4 & & 12 & \\ & & 3 & & \end{array}$$

4. The math exam of Rein contained problems in algebra, geometry and logic. When Rein saw the results, he noticed that he had given the correct answer to 50% of the algebra problems, 70% of the geometry problems and 80% of the logic problems. Altogether Rein had answerd correctly to 62% of the algebra and logic problems and 74% of the geometry and logic problems. What was the percentage of Rein's correct answers in the whole exam?

5. Write the number $\sqrt[3]{1342\sqrt{167}+2005}$ using only figures and signs of addition, subtraction, multiplication, division and square root. (You do not have to use all these signs.)

6. The real numbers x and y satisfy the conditions

$$\begin{cases} \sin x + \cos y = 1, \\ \cos x + \sin y = -1. \end{cases}$$

Show that $\cos 2x = \cos 2y$.

7. Let a, b and n be integers such that $a + b$ is divisible by n and $a^2 + b^2$ is divisible by n^2. Show that $a^m + b^m$ is divisible by n^m for every positive integer m.

8. The Mail Office of a certain country uses couriers to carry the mail. Each courier brings the mail from one town to the next one. We know that it is possible to send mail from each town to the capital P. If for two towns A and B all routes from A to P pass through B, we call B *more important* than A.

a) Prove that if for any three towns A, B and C B is more important than A and C is more important than B, then C is more important than A.

b) Prove that if for any three towns A, B and C both B and C are more important than A, then either C is more important than B or B is more important than C.

9. Does there exist an integer $n > 1$, such that $2^{2^n - 1} - 7$ is not a square of an integer?

10. Find all pairs (a, b), of real numbers such that the roots of the equations $6x^2 - 24x - 4a = 0$ and $x^3 + ax^2 + bx - 8 = 0$ are non-negative real numbers.