Challenging Problems, February 2012

Answers by email to `laurihallila@gmail.com`, or by ordinary mail to Lauri Hallila, Kalliorinteenkuja 1, 02770 Espoo. If you have questions about the problems, use the email address above.

1. Find the number of infinite arithmetic integer sequences in which the numbers 1 and 2005 are among the first ten numbers.

2. Find all triples $(a,\, b,\, c)$ of positive integers for which

$$abc + ab + c = a^3.$$

3. Compute the number

$$\left\lfloor \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots + \frac{2^{100}}{100!} \right\rfloor .$$

$\lfloor x \rfloor$ is the largest integer $\leq x$.

4. Show that

$$|n\sqrt{2005} - m| > \frac{1}{90n}$$

for all positive integers m and n.

5. Find all positive integers n such that $2^n + p$ and $2^n + q$ are prime numbers for some primes p and q satisfying $p + 2 = q$.

6. Show that

$$\frac{1}{2n} < \{n\sqrt{7}\} < 1 - \frac{1}{6n}$$

for all positive integers n ($\{x\} = x - \lfloor x \rfloor$).

7. Find all functions $f : \mathbb{N} \to \mathbb{N}$ for which

$$f(m - n + f(n)) = f(m) + f(n)$$

for all $m,\, n \in \mathbb{N}$.

8. Show that the inequality

$$\left(a^2 + b + \frac{3}{4}\right)\left(b^2 + a + \frac{3}{4}\right) \geq \left(2a + \frac{1}{2}\right)\left(2b + \frac{1}{2}\right)$$

holds for all positive real numbers a and b.

9. The numbers a, b, c and d, $0 < a, b, c, d < \pi/2$, satisfy the condition

$$\cos 2a + \cos 2b + \cos 2c + \cos 2d = 4(\sin a \sin b \sin c \sin d - \cos a \cos b \cos c \cos d).$$

Find all possible values of the sum $a + b + c + d$.

10. a) The function $f : \mathbb{N} \to \mathbb{N}$ satisfies the condition $f(n) = f(n + f(n))$ for all natural numbers n. Show that if the number of possible values of f is finite, then f is a periodic function.

b) Give an example of a non-periodic function $f : \mathbb{N} \to \mathbb{N}$, for which $f(n) = f(n + f(n))$ for all natural numbers n.