High School Mathematics Competition 2013

Problems, First Round, November 12, Basic Level

1. Which of the following pairs of numbers are equal?

a) $\sqrt{2}$ and 1,414213562373
b) $\sqrt{5-2\sqrt{6}}$ and $\sqrt{2}-\sqrt{3}$
c) $\sqrt{7}$ and 2,645751311064
d) $\sqrt{9+2\sqrt{14}}$ and $\sqrt{7}+\sqrt{2}$.

2. More air is pumped into a balloon so much that the volume of the balloon increases by 237,5 %. Then the surface area of the balloon increases by

a) 100 %
b) 129 %
c) at least 150 %
d) at most 175 %.

3. There are $n > 1$ numbers and their average is $M \neq 0$. One of the numbers, a, is removed, and the average of the remaining numbers is computed.

a) The new average is $\dfrac{M-a}{n-1}$.
b) The new average can be smaller than the original average.
c) The difference of the new average and M is $\dfrac{M-a}{n-1}$.
d) The average of the new average and M is $\dfrac{nM-a}{2(n-1)}$.

4. The expression

$$\frac{c}{a+\frac{b}{c}}+\frac{a+c}{a-\frac{b}{c}}$$

is simplified. Which of the following outcomes are correct for all values of a, b and c?

a) 0
b) $\dfrac{c(2bc+a^2c+ab)}{b^2-a^2c^2}$
c) $\dfrac{ac(2c^2+b+ac)}{a^2c^2-b^2}$
d) $\dfrac{ac}{ac-b}+\dfrac{2ac^3}{(ac+b)(ac-b)}$.

5. For a positive integer n, denote by $S(n)$ the sum of the digits of n (in base 10). Which of the following statements are true for all positive integers n?

a) $S(3n)$ is divisible by 3
b) $S(2n) \leq 2S(n)$
c) $S(2n) \geq \frac{1}{2}S(n)$.
d) $S(7n)$ is divisible by 7.

6. We know that

$$\frac{8^x}{2^{x+y}} = 64 \quad \text{and} \quad \frac{9^{x+y}}{3^{4y}} = 243.$$

Then $2xy$ is

a) a negative number
b) 5
c) 7
d) an odd integer.

7. P is a point on the hypothenuse AB of a right triangle ABC. We know that $|PB| : |PC| : |PA| = 1 : 2 : 3$. Determine the ratios of the lengths of the sides of the triangle.

8. Show that if the real numbers x, y and z satisfy $(x + y + z)^2 = 3(xy + xz + yz)$, then all the numbers are necesarily equal.

High School Mathematics Competition 2013

First Round, Basic Level

Answer Sheet

The first six problems are multiple choice problems. Their answers should be written in the table below. Each multiple choice problem has 0 to 4 correct answers. Put a "+" to the appropriate square, if the answer is right and a "−" if the answer is wrong. All correct marks give one point and incorrect or unintelligible marks give zero points. The answers to problems 7 and 8 can be written on a separate paper. For each of these problems, a maximum of 6 points is given.

The time allowed is 120 minutes. Please write your name and school to every paper you return.

Name: ______________________________

School: ______________________________

Home address: ______________________________

Email: ______________________________

	a	b	c	d
1.				
2.				
3.				
4.				
5.				
6.				

High School Mathematics Competition 2013

First Round, November 12, Intermediate Level

1. There are $n > 1$ numbers and their average is $M \neq 0$. One of the numbers, a, is removed, and the average of the remaining numbers is computed.

a) The new average is $\dfrac{M-a}{n-1}$.

b) The new average can be smaller than the original average.

c) The difference of the new average and M is $\dfrac{M-a}{n-1}$.

d) The average of the new average and M is $\dfrac{nM-a}{2(n-1)}$.

2. For a positive integer n, denote by $S(n)$ the sum of the digits of n (in base 10). Which of the following statements are true for all positive integers n? is a positive

a) $S(3n)$ is divisible by 3

b)$S(2n) \leq 2S(n)$

c) $S(2n) \geq \frac{1}{2}S(n)$.

d) $S(7n)$ is divisible by 7.

3. The equation $x^3 + ax^2 + bx + c = 0$ has three solutions which form an arithmetic sequence. (Three numbers are in an arithmetic sequence, if one of them is the average of the other two.) Then always

a) $ab = 2a^3 + c$ b) $a = 0$ c) $3a + c = 2b$ d) $b = 3ac$.

4. In the triangle ABC $|AB| < |AC|$. Let S be the circumcircle of ABC. A line through A, perbendicular to BC meets S again at the point P. The point X is on the line segment AC, and the extension of BX meets the circle S at the point Q. Prove: if $|BX| = |CX|$, then PQ is the diameter of S.

5. A board solitaire is played with one blue and three white pieces, which can be placed in the squares of a 2013×2013 board. One move consists of taking one of the pieces and moving it to the left, to the right, up or down all the way until the border of the board or another piece is encountered. Prove that the blue piece can be moved to any of the squares of the board, regardless of the initial position of the pieces.

6. Find all positive integers m and n such that n is odd and the equation

$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}$$

is fulfilled.

High School Mathematics Competition 2013

First Round, Intermediate Level

Answer Sheet

The first three problems are multiple choice problems. Their answers should be written in the table below. Each multiple choice problem has 0 to 4 correct answers. Put a "+" to the appropriate square, if the answer is right and a "−" if the answer is wrong. All correct marks give one point and incorrect or unintelligible marks give zero points. The answers to problems 4 to 6 can be written on a separate paper. For each of these problems, a maximum of 6 points is given.

The time allowed is 120 minutes. Please write your name and the name of your school on every paper you return.

Name: ______________________________

School: ______________________________

Home address: ______________________________

Email: ______________________________

	a	b	c	d
1.				
2.				
3.				

High School Mathematics Competition 2013

First Round, November 12, Open Level

1. We know that

$$\frac{8^x}{2^{x+y}} = 64 \quad \text{and} \quad \frac{9^{x+y}}{3^{4y}} = 243.$$

Determine $2xy$.

2. In average, one person in a million suffers from an uncommon disease. The disease is diagnosed with a test which gives a correct result with probability 99 %, regardless of whether the person has the disease or not. An arbitrarily chosen person is tested positive, i.e. as having the disease. What is the probability that he has the disease?

3. There are two points A and B on a sheet of paper. Their distance is more than 10 cm but less than 20 cm. You have a ruler exactly 10 cm long and a pair of compasses. How can you draw the line AB using these tools?

4. For a positive integer n, denote by $S(n)$ the sum of the digits of n (in base 10). Which rational numbers q can be written as

$$q = \frac{S(2n)}{S(n)}$$

for some positive integer n?

Time allowed: 120 minutes. Please write (legibly!) your name, school, home address and email address on at least one of the papers you return, and write your name on every paper you return.