More challenging problems for September 2014

Please return solutions to the October training event in Päivölä, or mail them to the address

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by the middle of October. The answer to a problem must be justified even if the problem text only asks for the answer!

1. Do there exist real numbers a, b, c and d for which

$$a+b+c=d \quad \text{and} \quad \frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}=\frac{1}{ad}+\frac{1}{bd}+\frac{1}{cd}?$$

2. Two real numbers a and b are such that $a^{2014}+b^{2014}=a^{2016}+b^{2016}$. Prove that $a^2+b^2 \leqslant 2$.

3. The heights of an acute triangle ABC are AA_1, BB_1 and CC_1. The point K is the perpendicular projection of the point A to the line A_1B_1, and the point L is the perpendicular projection of the point B to the line B_1C_1. Prove that $A_1K = B_1L$.

4. The diagonals of a convex 11-gon are colored with different colors. We say that two colors intersect each other if there exist diagonals colored with these two colors, which have a common interior point. What is the largest possible number of colors, if any two colors intersect each other?

5. In a triangle ABC, we have $AB + BC = 2 \cdot AC$. The bisector of the angle $\angle ABC$ intersects AC at the point L, and the points K and M are the midpoints of the sides AB and CB. Determine the angle $\angle KLM$ as a function of the angle $\angle ABC = \beta$.

6. On a blackboard, an odd number of asterisks $* * * \cdots *$ have been written. Andrei and Olga alternatively erase one asterisk and replace it by a digit. The first asterisk must be replaced by a digit which is not 0. Andrei begins. If the number, which arises when all the asterisks have been replaced by digits, turns out to be divisible by 11, then Andrei wins. Otherwise the winner is Olga. Which of the players has a winning strategy?

7. Determine those positive integers n, for which the number $(n+1)!-n$ is divisible by the number $n!+n+1$.

8. Determine all integers $n \geqslant 2014$ which have the property that a cube can be divided up into n smaller cubes.

9. Determine the largest n with the following property. There exists a set of n positive integers, in which exactly one element is divisible by n, exactly two elements are divisible by $n-1$, $\ldots$, and exactly $n-1$ elements are divisible by 2 (and all n elements are divisible by 1).

10. Determine all the functions $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ for which

$$f(n+1)f(n+2) = (f(n))^2.$$

11. Let E be a point from the side BC of a triangle ABC, and assume that E is closer to C than to B. Determine how to use ruler and compass to find points D and F from the sides AB and AC, respectively, so that the angle $\angle DEF$ is straight and DE intersects BF at its midpoint M.

12. Does there exist a geometric progression $b_0, b_1, \ldots, b_{2014}$ of positive integers so that for each i the number b_i has one digit more than the number b_{i-1} (in the decimal system!), and in which the ratio of consecutive numbers is not 10?

13. The circumcircle of a triangle ABC is Γ, and AA' is a height of the triangle. The bisector of the angle $\angle BAC$ intersects BC at the point D, and it intersects Γ at the point E. The line EA' intersects Γ also at the point F. The line FD intersects Γ also at the point G. Prove that AG is a diameter of Γ.

14. The numbers p, q and r are prime numbers. We know that $pq+1$, $pr+1$ and $qr-p$ are squares. Prove that $p+2qr+2$ is also a square.

15. Pekka must guess a positive integer n. He knows that the number n has exactly 250 positive divisors $1 = d_1 < d_2 < \ldots < d_{250} = n$. When Pekka picks an index j, $1 \leqslant j \leqslant 249$, he is told d_j. If Pekka already knows what d_j is, he will not be told what d_{250-j} is. How many guesses does Pekka need in order to determine n with certainty?