Last time:  $p: E \rightarrow B$  fibration, B CW complex,  $\sigma$  section over  $B_n$ .  $V_{nri} \in H^{nri}(B; \pi_n(F))$  varishes (=) there is a section over  $B_{nri}$ .

If  $V_{nri} = 0$ ,  $V_{nr2}$  is defined. It all obstructions vanish, there is a section of  $p: E \rightarrow B$ .

If  $\pi_n(F) = 0$   $n \le k-1$ , and  $\pi_n(F) \ne 0$ , we call  $\chi_{\mu_{+}} \in H^{h+1}(B; \pi_n(F))$  the primary obstruction.

Examples:  $P: E \rightarrow B$  a vector bundle,  $F = \mathbb{R}^d$ , so  $\pi_k(\mathbb{R}^d) = 0$ .

All obstructions to the existence of a section vanish. So every vector bundle has a section (knew that already, e.g. zero section).

More generally, if  $P: E \rightarrow B$  is a fibration with contractive  $C^{ij}$ .

More generally, if  $p: E \rightarrow B$  is a fibration with contractible fibe F, then p has a section. For instance, for a vector bundle  $V \rightarrow B$ ,  $P_{GL(A,P)}(V) \rightarrow B$  has a Ros6 to O(d) (=)  $P_{GL(A,P)}(V)/O(d) \rightarrow B$  has a Section (=) V admits a bundle metric. The fiber of  $P_{GL(A,P)}(V)/O(d) \rightarrow B$  is  $\frac{GL(A,P)}{O(d)}$  which is contractible since O(d) is a  $P_{GL(A,P)}(V)/O(d) \rightarrow B$ .

O(d) is a maximal compact subgroup of GL(d,  $\mathbb{R}$ ). So every  $V \rightarrow B$  admits a bundle metric.

· Para, m) (V) -B admin a Rosa to GL+(d, TR)

(=) Para, (V) / GL+(d, TR) -B has a section

(-) V is orientable

The fiber of Para, m) (V) / Grt(d, m) -B is \( \frac{GL(d, m)}{GL(d, m)} = \mathbb{Z}\_2.

 $T_{o}(\mathbb{Z}_{v}) = \mathbb{Z}_{v}$ ,  $T_{n}(\mathbb{Z}_{v}) = 0$  n > 0. There is one obstruction in

H'(B; To(F)) = H'(B; Zv). This is called the first Stiefel-Whitney class of

 $\forall$ , denoted  $w_1(v)$ . So V is orientable Z=V  $w_1(v)=0$ .

· Pso(d) (V) -1 B admits a ROSG to SO(d-1)

L=) Psold) (V) /sold-1) - B has a section

(=) V has a nowhere - zero section

<=> V ≥ V, ⊕ E'

The fiber  $P(SO(d))(V)/SO(d-1) \rightarrow B$  is  $\frac{SO(d)}{SO(d-1)} = S^{d-1}$ . The princy

obstruction belongs to  $H^d(B; \pi_{d-1}(S^{d-1})) = H^d(B; \mathbb{Z})$ . This is called

the Enler desi of V, dended e(V).

· Mª oriented admits can almost complex structure

(=) Pso(4) (7M) -+ M adwh 4 Rosa to U(2)

(-) /sa4) (7M)/U(2) -) M has a section.

The fiber is  $\frac{50(4)}{11(2)} = 5^2$ . So there are two obstructions to the

exitence of an ACS on  $M^4$ :  $y_3 \in H^3(M; \pi_2(S^2)) = H^3(M; \mathbb{Z})$ 

and 84 t H4 (M; 73 (52)) = H4 (M; 2). What are they?

How do we identity obstructions?

## Class; fying Spaces

Motivating Example: Mn snorth nanifold, TM >M rank n

Whitney = embedding f: M - DN, so fx: TM - TRN = RN x RN. map  $\phi: M \longrightarrow Gr_n(\mathbb{R}^N)$ We obtain a (Gauss mgs). P -> Pr2(f=(7pM)) The tautological bundle Vn To Gra (TN) Vn= 1 (W, V) & G-1 (MN) × MN | V & W}, \ \( \pi \ | W, V) = W \ \ \( \pi^{-1} (W) = W. \) \$ has the property that \$\$V\_n = TM. Every tangent bundle is a It the tactological bundle. In fact, for N large enough, every over M is the pullback of Vn by some map \$: M > PRN. Returning to principal Gr-bundles, is there on analogue? Lemma: f, fz: X -> Y, Pa -> Y principal G-bundle If f, and for are homotopic, then f, \* Por and for Por one is one sphere. Definition: A principal Go-budle Pa -> B is called a universal G-budle the map [x,B] - Pring(X) = 5 iso. classes of prin. G-bandlesy [f] - [f\*PG] is a bijection. Given a principal G-bundle Q->X, a my \$:X->B Φ\*P6 = B is called a classifing map. We call B classifying space for Theorem (Milner): Go top. group, I a universal G-burdle. The classifying space is unique up to homotopy equivalence, and

Universal G-bundle is unique up to isomorphism.

the

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We denote the hometopy type of the classifying space by BG.

Proposition: P_{G} \rightarrow B universal Z=> P_{G} is nearly contractible (i.e. T_{h}(P_{0})=0 \forall h > 0)

Long exact sequence in hometopy groups \Rightarrow T_{h}(B_{G}) = T_{h-1}(G)
```

Examples: PGL(N, M) (Vn) -9 Gry (MN) is the Stiefel neithold GL(N, M)/GL(N-n) which is (N-n-1) - connected.

It follows that  $P_{GL(n,\mathbb{R})}(V_n) \rightarrow G_{V_n}(\mathbb{R}^{\infty})$  is a universal  $GL(n,\mathbb{R})$  - bundle.

Equip  $V_n$  with a bundle natric,  $P_{O(n)}(V_n) \rightarrow G_{V_n}(\mathbb{R}^{\infty})$  is a universal O(n) - bundle.

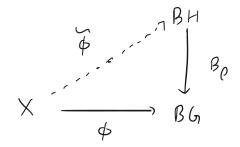
Grn (1 0 0) = B GL(4, 1 ) = B O(n).

In general, li. H -> 6 cont. group homonorphism, Bp: BH -> BG and if p is a hom. eq., so is Bp.

e, g. Z-1 M - s' is the universal Z-bundle.

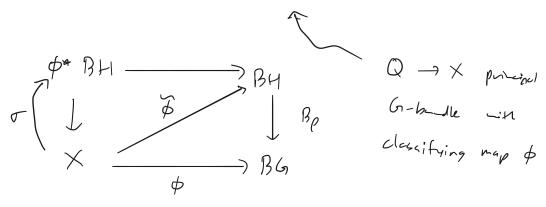
Criven a principal Go-bundle Q >X,  $\phi: X \to BG$  classifying age.

Q admits a Rosch to  $H \iff \phi: X \to BG$  lifts through  $Bp: BH \to BG$  to a wap  $\phi: X \to BH$ .



If  $\alpha \in H^{k}(Bb; \Omega)$  and  $(Bp)^{\dagger} \alpha = 0$ , then if  $\phi$  has a lift  $\overline{\phi}$ , we have  $\phi^{\dagger} \lambda = (Bp \circ \overline{\phi})^{\dagger} \lambda = \overline{\phi}^{*}(Bp)^{\dagger} \alpha = 0$ . So  $\phi^{*} \alpha = 0$  is a necessary condition for the existence of a lift  $\overline{\phi}$  and hence q  $\Omega \circ Sbr$  to H. This is related to the districtions we needled previously.

Tf H < G, Hen BH -BG is a f.be bundle with fiber G/H, and \$\$\psi^\*BH -> \times is isomorphic to Q/H -> \times.



A lift & exists (=) a section of exists.