II - INTRO TO ALGEBRAIC STRING TOPOLOGY

Let A be a dga over a commutative ring R

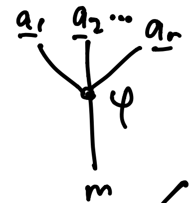
M dg module over A

DEF. <u>Hochschild cochain complex</u>

shift $\underline{A}_j := A_{j-1}$

$$CH^\bullet(A,M) := \prod_{r\geq 0} \mathrm{Hom}(\underline{A}^{\otimes r}, M)$$

$$\varphi : \underline{A}^{\otimes r} \to M$$



Differential $D = D_d + D_\bullet$

$d_{\mathrm{Hom}}(\varphi)$

$$D_d(\varphi)(\underline{a_1} \dots \underline{a_r}) := d_M \varphi(\underline{a_1} \dots \underline{a_r}) + \sum_j \pm \varphi(\underline{a_1}, \dots, \underline{da_j} \dots \underline{a_r})$$

$$D_\bullet(\varphi)(\underline{a_1}, \dots, \underline{a_{r+1}}) := a_1 \varphi(\underline{a_2} \dots \underline{a_{r+1}}) + \sum_j \pm \varphi(\underline{a_1} \dots \underline{a_j a_{j+1}} \dots \underline{a_r})$$
$$\pm \varphi(\underline{a_1} \dots \underline{a_r}) . a_{r+1}$$

$D^2 = 0$

<u>Hochschild cohomology</u>

$$HH^\bullet(A,M) := H^\bullet(CH^\bullet(A,M), D)$$

<u>normalized</u> cochains φ vanish if any input is $\underline{1} \in \underline{A}$

THM (Loday?) $\overline{CH}^\bullet(A,M) \hookrightarrow CH^\bullet(A,M)$ quasi iso

GERSTENHABER CUP PRODUCT ON $HH^\bullet(A,M)$

DEF $\varphi : \underline{A}^{\otimes r} \to A$ $\qquad \varphi \cup \psi : \underline{A}^{\otimes(r+s)} \to A$

$\psi : \underline{A}^{\otimes s} \to A$ $\qquad (\underline{a_1} \dots \underline{a_{r+s}}) \mapsto \varphi(\underline{a_1} \dots \underline{a_r}) \overset{m_A}{\cdot} \psi(\underline{a_{r+1}} \dots \underline{a_{r+s}})$

THM (Gerstenhaber 60s?) D is a derivation of $\cup$.

induces

$$HH^\bullet(A,A) \otimes H^\bullet(A,A) \xrightarrow{\ \cup\ } H^\bullet(A,A)$$

THM (Cohen-Jones 2002) M closed, oriented, simply connected. Let $A = C^\bullet(M)$ singular cochains. $\exists$ isomorphism of graded commutative algebras

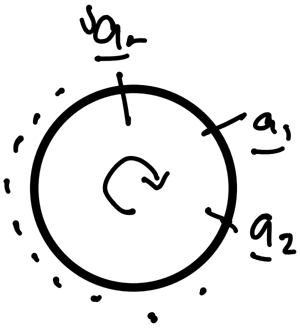
$$(H_\bullet(LM), \bullet) \longrightarrow (HH^\bullet(A,A), \cup)$$

QUESTION: What about a BV structure on $HH^\bullet(A,A)$?

CONNES B-OPERATOR on $HH^\bullet(A,A^*)$

$$\mathrm{Hom}(\underline{A}^{\otimes r}, A^*) \xrightarrow{\ B\ } \mathrm{Hom}(\underline{A}^{\otimes r-1}, A^*)$$

$$(B\varphi)(\underline{a_1}, \dots, \underline{a_{r-1}})(a_r) := \sum \pm \varphi(\underline{a_j}, \underline{a_{j+1}}, \dots, \underline{a_r}, \underline{a_1}, \dots, \underline{a_{j-1}})(1)$$



cochain map so induces

$$HH^\bullet(A,A^*) \xrightarrow{\ B\ } HH^\bullet(A,A^*), \quad B^2 = 0$$

SUMMARY:

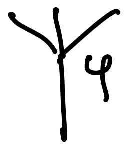
(a) $HH^\bullet(A,A) \otimes HH^\bullet(A,A) \xrightarrow{\ \cup\ } HH^\bullet(A,A)$

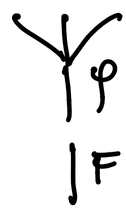
(b) $HH^\bullet(A,A^*) \xrightarrow{\ B\ } HH^\bullet(A,A^*)$

LET'S GET A BV STRUCTURE ON $HH^\bullet(A,A)$

Imagine we have a map $A \xrightarrow{F} A^*$ such that the induced map

$$HH^\bullet(A,A) \xrightarrow{\mathfrak{J}} HH^\bullet(A,A^*)$$





is an isomorphism.

Then we transfer $\mathfrak{B}$ to $HH^\bullet(A,A)$:

$$\begin{tikzcd}
HH^\bullet(A,A) \arrow[r, dashed, "\Delta^F"] \arrow[d, "\mathfrak{J}"'] & HH^\bullet(A,A) \\
HH^\bullet(A,A^*) \arrow[r] & HH^\bullet(A,A^*) \arrow[u, "\mathfrak{J}^{-1}"']
\end{tikzcd}$$

THM (Tradler, 2003?)

$$(HH^\bullet(A,A), \cup, \Delta^F) \text{ is a BV algebra}$$

REMARK If M is formal then

$$HH^\bullet(C^\bullet(M), C^\bullet(M)) \simeq HH^\bullet(H^\bullet(M), H^\bullet(M))$$

RECALL

THM (Cohen-Jones-Yan 2003, Menichi 2007)

$$H_\bullet(LS^2;\mathbb{Z}/_2)[-2] \simeq \bigoplus_{k\geq 0} \mathbb{Z}/_2[\alpha_k] \oplus \bigoplus_{k\geq 0} \mathbb{Z}/_2[\beta_k]$$

$|\alpha_k| = k$ $\quad$ $|\beta_k| = k-2$

$$\alpha_k \cdot \alpha_\ell = \alpha_{k+\ell} \qquad \Delta(\alpha_k) = 0$$

$$\beta_k \cdot \beta_\ell = 0 \qquad \Delta(\beta_k) = \underbrace{k\alpha_{k-1} + k\beta_{k+1}}$$

$$\alpha_k \cdot \beta_\ell = \beta_{k+\ell}$$

THM (Menichi)

$$HH^\bullet(H^\bullet(S^2;\mathbb{Z}/_2), H^\bullet(S^2;\mathbb{Z}/_2)) \simeq \bigoplus_{k\geq 0} \mathbb{Z}/_2[\phi_k] \oplus \bigoplus_{k\geq 0} \mathbb{Z}/_2[\psi_k]$$

$$\Delta^F(\phi_k) = 0$$

$$\Delta^F(\psi_k) = k\phi_{k-1}$$

☆ uses $H^\bullet(S^2;\mathbb{Z}/_2) \xrightarrow{F} H_\bullet(S^2;\mathbb{Z}/_2)$ ☆

usual Poincaré duality isomorphism $\cap [S^2]$

THM (Menichi)

$\exists$ isomorphisms

$$HH^\bullet(H^\bullet(S^2;\mathbb{Z}/_2), H^\bullet(S^2;\mathbb{Z}/_2)) \longrightarrow H_\bullet(LS^2;\mathbb{Z}/_2)$$

that preserve the product <u>or</u> the BV operator but no isomorphism that preserves <u>both</u> simultaneously

GOAL : $\phi_k, \psi_k, \cup, \cancel{\Delta^F}$ Δ^F $\qquad \alpha_k, \beta_k$

$$\begin{tikzcd}
HH^\cdot(H^\cdot(S^2;\mathbb{Z}/_2), H^\cdot(S^2;\mathbb{Z}/_2)) \arrow[r, "\Theta"] \arrow[d, shift right, "\mathcal{F}"'] \arrow[d, shift left, "\tilde{\mathcal{F}}"] & H_\cdot(LS^2;\mathbb{Z}/_2) \\
HH^\cdot(H^\cdot(S^2;\mathbb{Z}/_2), H_\cdot(S^2;\mathbb{Z}/_2)) &
\end{tikzcd}$$

$\bullet, \Delta$

$\theta_k, \gamma_k, \beta$

★ want a better $F: H^\cdot \xrightarrow{\sim} H_\cdot$ so Θ is isom of BV alg ★

III - SOME ALGEBRAIC MACHINERY

IDEA : Poincaré duality isomorphism

fundamendal class

$$H^\cdot(S^2;\mathbb{Z}/_2) \xrightarrow{\cap[S^2]} H_\cdot(S^2;\mathbb{Z}/_2)$$

is induced by a chain-level map

fundamental cycle

$$C^\cdot(S^2;\mathbb{Z}/_2) \xrightarrow{\cap[S^2]} C_\cdot(S^2;\mathbb{Z}/_2)$$

..... maybe it's worth seeing what's really going on in a chain model of S^2 ????

TWO ELEMENTARY OBSERVATIONS

(1) M a dg bimodule over algebra A

M^* can be given a AA bimodule structure:

$m \in M$, $n \in M^*$, $a \in A$:

R $(n.a)(m) := n(a.m)$

L $(a.n)(m) := (-1)^{|a|(|n|+|m|)}\, n(m.a)$

(2) Let $X \in A^*$, $dX = 0$

AA bimodules

Define a map $F: A \longrightarrow A^*$ by

$\alpha, \beta \in A$ $\quad F(\alpha) \in A^*$

R-mod structure on A^*

$$F(\alpha)(\beta) := (X.\alpha)(\beta) = X(\alpha \cdot \beta)$$

(think: $X \in C_\cdot(X)$ cycle $\quad F: C^\cdot(X) \xrightarrow{\cap X} C_\cdot(X)$)

$C^\cdot(X)$ bimod structures

NOTICE (i) F is a right-module map: $a \in A$

$$F(\alpha \cup a)(\beta) = (X \cap (\alpha \cup a))(\beta)$$
$$= X((\alpha \cup a) \cup \beta)$$
$$= X(\alpha \cup (a \cup \beta))$$
$$= (X \cap \alpha)(a \cup \beta)$$
$$= F(\alpha)(a \cup \beta)$$
$$= (F(\alpha) \cap \alpha)(\beta)$$

NOTICE (ii) F is not a left-module map in general:

$$F(a \cup \alpha)(\beta) = (X \cap (a \cup \alpha))(\beta)$$
$$= X((a \cup \alpha) \cup \beta)$$

$$(\alpha \cap F(\alpha))(\beta) = \pm F(\alpha)(\beta \cup a)$$
$$= \pm (X \cap \alpha)(\beta \cup a)$$
$$= \pm X(\alpha \cup (\beta \cup a))$$

UPSHOT: (with these conventions) capping with a (fundamental) cycle

$$C^{\cdot}(X) \xrightarrow{\cap X} C_{\cdot}(X)$$

is a right $C^{\cdot}(X)$-module map, but not a left-$C^{\cdot}(X)$ module map

BUT capping with a fundamental class is a map of $H^\cdot(X)$ bimodules

$$H^\cdot(X) \xrightarrow{\cap [X]} H_\cdot(X)$$

So the chain-level map should be a bimodule map up to some notion of "homotopy"

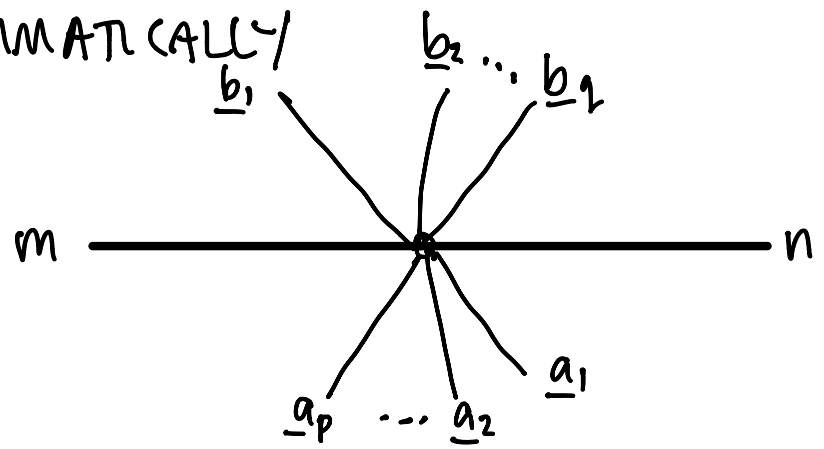
BIMODULE MAPS UP TO HIGHER HOMOTOPY AND HOMOTOPY INNER PRODUCTS

$(A, d, \cdot)$ is a dga, M dg bimod over A.

Consider a collection

$$F = \{ F_{p,q} : \underline{A}^{\otimes p} \otimes M \otimes \underline{A}^{\otimes q} \longrightarrow M^* \}$$

DIAGRAMMATICALLY



Define $DF = \{(DF)_{p,q}\}$ by

$$(DF)_{p,q} = (D_d F)_{p,q} + (D_\cdot F)_{p,q} \text{ where}$$

$$(D_d F)_{p,q}(\underline{a_1}, \dots, \underline{a_p}; m; \underline{b_1}, \dots, \underline{b_q})(n)$$
$$= \sum \pm (\text{put a } d \text{ in each spot})$$

d_{Hom}

$$(D.F)_{p,q}(\underline{a_1},\dots,\underline{a_p};m;\underline{b_1},\dots,\underline{b_q})(n)$$

$$= \pm F_{p-1,q}(\underline{a_2},\dots,\underline{a_p};m;\underline{b_1}\dots\underline{b_q})(n.a_1)$$

$$+ \sum_j \pm F_{p-1,q}(\underline{a_1},\dots\underline{a_j a_{j+1}},\dots \underline{a_p};m;\underline{b_1}\dots\underline{b_q})(n)$$

$$\pm F_{p-1,q}(\underline{a_1},\dots,\underline{a_{p-1}};a_p.m;\underline{b_1},\dots\underline{b_q})(n)$$

+ 3 more types of terms with $F_{p,q-1}$

☆ D analogous to Hochschild differential ☆

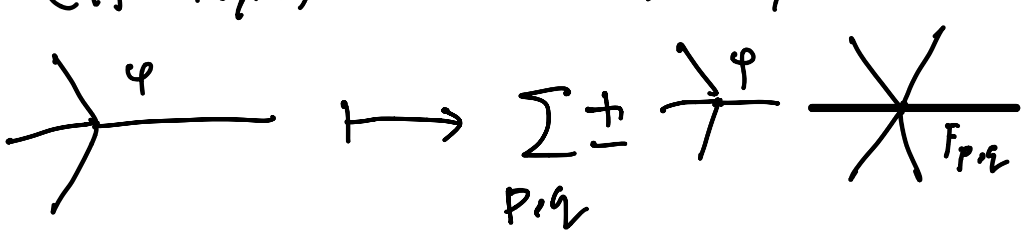
ABUSE OF NOTATION: $F: M \longrightarrow M^*$

DEF. The collection F is called a <u>homotopy inner product</u> if $DF = 0$

NOTICE: If $F_{p,q} = 0$ for $(p,q) \neq (0,0)$ then $F_{0,0}: A \longrightarrow A^*$ is a bimodule map AKA a regular inner product

DEF. F induces

$$CH^\bullet(A,A) \xrightarrow{\mathcal{F}} CH^\bullet(A,A^*)$$



$$\mathrm{Hom}(\underline{A}^{\otimes r}, A) \xrightarrow{\text{each component}} \mathrm{Hom}(\underline{A}^{\otimes r+p+q}, A^*)$$

REMARK If $DF = 0$ then $\mathcal{F}$ is a chain map, induces

$$\begin{tikzcd} HH^\bullet(A, A) \arrow[d, "\mathcal{F}"] \\ HH^\bullet(A, A^*) \end{tikzcd}$$

EXAMPLE: $A = H^\bullet(S^2; \mathbb{Z}/_2)$ $F = \{F_{0,0} : A \xrightarrow{\cap [S^2]} A^*\}$

usual poincaré duality - induces $\mathcal{F}$ from before