

Yang-Mills Instantons

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What is instanton?

- Mathematically, Instanton is a solution to the classical field equations of motion in Euclidean space.

$$\delta S_E[\psi_{\text{inst}}] = 0 \rightarrow \text{Euler-Lagrange equation} \quad (1)$$

- Instanton solutions of Euclidean EL equation are localized in (Euclidean) space and time, and have finite (Euclidean) action.
- In Minkowski QFT, it gives a non-perturbative effect: the tunneling between the classical vacua.
- Instantons are important in understanding non-perturbative effects and tunneling between vacua in quantum field theory.

Instanton effect in Path Integral

- In path integral formalism, instantons are classical solution, i.e., the saddle points of the action.
- Instantons only show up in the Euclidean path integral.
- Is it physically meaningful? Do we have to consider this effect in Minkowski space QFT?
- So the path integral is dominated by the instanton contributions in the semiclassical limit.
- The full path integral is given by the sum of all instanton contributions, with small fluctuations around them.

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \sum_{\text{instantons}} e^{-S_{\text{inst}}} \int \mathcal{D}\delta\phi e^{-S_{\text{fluct}}[\delta\phi]} \quad (2)$$

Instanton effect in Path Integral : Example

- A simple example: the one-dimensional quantum mechanics with two minima.
- The potential is given by $V(x) = \frac{1}{2}x^4 - \frac{1}{2}x^2$, and the action is $S = \int dt \left(\frac{1}{2}\dot{x}^2 - V(x) \right)$.
- The Euclidean action is $S_E = \int d\tau \left(\frac{1}{2}\dot{x}^2 + V(x) \right)$, and the Euclidean EOM has a classical solution, starting from $x = -1$ at $\tau = -\infty$ and ending at $x = 1$ at $\tau = \infty$.
- To evaluate correct path integral, We need to consider all the posible semiclassical paths, including the instanton solution. for example,
- If $T \gg t_{\text{inst}}$, all the instanton solution can be approximated as a composition of n single instanton solution, at time t_1, \dots, t_n , and the instanton action is given by $S = \sum_{i=1}^n S_{\text{inst}} = nS_{\text{inst}}$.

Instanton effect in Path Integral : Calculation

So, now we can calculate the matrix element of e^{-HT} from $x = -1$ to $x = 1$, using the path integral including instanton solution.

$$\begin{aligned}\langle 1|e^{-HT}|1\rangle &= \int \mathcal{D}x e^{-S_E} = \sum_{\text{instantons}} e^{-S_{\text{inst}}} \int \mathcal{D}\delta x e^{-S_{\text{fluct}}} \\ &= e^{-T/2} \sum_{n \text{ odd}} \int_{-T/2}^{T/2} d\tau_1 \int_{-T/2}^{\tau_1} d\tau_2 \cdots \int_{-T/2}^{\tau_{n-1}} d\tau_n e^{-nS_{\text{inst}}} \int \mathcal{D}\delta x e^{-S_{\text{fluct}}}\end{aligned}$$

Here, the fluctuation path integral can be factorized into n parts, which are the fluctuation path integrals around each instanton solution.

$$\begin{aligned}\int \mathcal{D}\delta x e^{-S_{\text{fluct}}} &= \left(\int \mathcal{D}\delta x_n e^{-S_{\text{fluct}}} \right)^n e^{-T/2} = K^n e^{-T/2} \\ \langle 1|e^{-HT}|1\rangle &= e^{-T/2} \sum_{n \text{ odd}} \int_{-T/2}^{T/2} d\tau_1 \int_{-T/2}^{\tau_1} d\tau_2 \cdots \int_{-T/2}^{\tau_{n-1}} d\tau_n (K e^{-S_{\text{inst}}})^n \\ \langle 1|e^{-HT}|1\rangle &= e^{-T/2} \sum_{n \text{ odd}} \frac{1}{n!} K^n (e^{-S_{\text{inst}}})^n T^n = e^{-T/2} \sinh(KT e^{-S_{\text{inst}}})\end{aligned}$$

Instanton effect in Path Integral : Conclusion

- The Path integral including instanton solution gives the correct matrix element of e^{-HT} from $x = -1$ to $x = 1$:
$$\langle 1 | e^{-HT} | -1 \rangle = e^{-T/2} \sinh(KTe^{-S_{\text{inst}}}).$$
- By changing the result to $T \rightarrow it$, we can get the propagator of the Minkowski QM: $\langle 1 | e^{-iHT} | -1 \rangle = e^{-it/2} \sin(Kte^{-S_{\text{inst}}})$.
- No instanton solution in Minkowski space, but the instanton effect is still important in understanding the non-perturbative effects in Minkowski QFT.
- Amplitude $\langle 1 | e^{-iHT} | -1 \rangle = e^{-it/2} \sin(Kte^{-S_{\text{inst}}/\hbar})$ is non-perturbative in \hbar , and the instanton effect is essential in understanding the non-perturbative effects in quantum field theory.

Solitons

- Solitons are the localized, finite energy solutions of the classical field equations of motion.
- Solitons ARE NOT instantons, but they are related to instantons.
- Instantons are localized in Euclidean space and time, and have finite Euclidean action.
- Solitons are localized in space, but stationary in time, and have finite energy.

Yang-Mills Instanton : Introduction

- Yang-Mills instantons are the instanton solutions of the Yang-Mills theory.
- The Yang-Mills action is given by $S = \int d^4x \left(-\frac{1}{4g^2} F_{\mu\nu}^a F^{\mu\nu a} \right)$, and the Euclidean action is $S_E = \int d^4x \left(\frac{1}{4g^2} F_{\mu\nu}^a F^{\mu\nu a} \right)$.
- The Euclidean EOM is given by $\mathcal{D}_\mu F^{\mu\nu a} = 0$, with the Bianchi identity $\sum_{\text{cyc}} \mathcal{D}_\mu \tilde{F}^{\nu\rho a} = 0$.
- The YM instanton solution is nontrivial solution of the EOM, and can be obtained by solving the self-dual equation $\tilde{F}^{\mu\nu a} = F^{\mu\nu a}$.

Yang-Mills Instanton : Self-dual equation

- Hodge dual operator : $\tilde{F}^{\mu\nu a} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}^a$.
- The self-dual equation is given by $\tilde{F}^{\mu\nu a} = F^{\mu\nu a}$.
- All the 2-forms can be decomposed into self-dual and anti-self-dual parts: $F^{\mu\nu a} = F_+^{\mu\nu a} + F_-^{\mu\nu a}$.
- If either $F_+^{\mu\nu a}$ or $F_-^{\mu\nu a}$ is zero, the solution is called self-dual or anti-self-dual, and satisfies the EOM of the YM theory, $\mathcal{D}_\mu F^{\mu\nu a} = 0$.

Yang-Mills Instanton : evaluating S_{inst}

- The instanton action is given by $S_{\text{inst}} = \int d^4x \left(\frac{1}{4g^2} F_{\mu\nu}^a F^{\mu\nu a} \right)$.
- The action can be evaluated easily when the instanton is self-dual or anti-self-dual.

$$\begin{aligned} S_{\text{YM}} &= \int d^4x \left(\frac{1}{4g^2} F_{\mu\nu}^a F^{\mu\nu a} \right) \\ &= \frac{1}{4g^2} \int d^4x \left(F_{\mu\nu}^a \pm \tilde{F}_{\mu\nu}^a \right)^2 \mp \frac{1}{2g^2} \int d^4x \tilde{F}_{\mu\nu}^a F^{\mu\nu a} \\ &\geq \pm \frac{1}{2g^2} \int d^4x \tilde{F}_{\mu\nu}^a F^{\mu\nu a} = \frac{8\pi^2}{g^2} |n| \end{aligned}$$

So, the instanton action is the bound, $8\pi^2|n|/g^2$, where n is the instanton number(the 2nd chern number). This derivation of the instanton action shows the topological nature of the instanton solution : the instanton action is quantized, and the instanton number is a topological invariant of the corresponding $SU(N)$ principal bundle.

Yang-Mills Instanton : Physical effects

- The instanton solution is a non-perturbative effect in the YM theory : $e^{-S_{\text{inst}}} = e^{-8\pi^2|n|/g^2}$ is the tunneling amplitude between the classical vacua, and non-perturbative in g .
- The instanton is a source of tunneling between classical vacua in the YM theory, and the true vacuume structure of the YM theory, and the θ term.
- The instanton solution is also important in understanding the anomalies in the YM theory.

Tunneling Effects in YM and QCD : θ -vacuum

- Classical vacume of the YM theory?
- The YM Hamiltonian is :
- So, there is only one classical vaccume, $F_\mu = 0$, and the vaccume energy is zero.
- Then, what tunneling effect appears in the YM theory?

Easy example : QM on a circle

- Consider the quantum mechanics on a circle, with the potential $V(x) = -\cos(x)$.
- There is one classical vacuum, $x = 0$, and the vacuum energy is $V(0) = -1$.
- But, in path integral formalism, the tunneling effect appears : paths with non-zero winding number.
- This contribution can be interpreted as the tunneling to itself, and the tunneling amplitude is given by $e^{-S_{\text{inst}}/\hbar}$.
- The tunneling amplitude is non-perturbative in \hbar , and this instanton effect gives non-perturbative corrections to the vacuum energy.

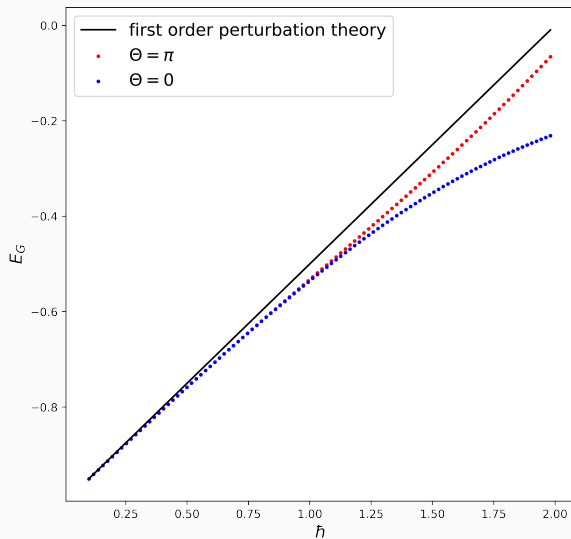
The theta-term

- One can consider A "trivial term" applied to above QM on a circle :
$$S = \int dt \left(\frac{1}{2} \dot{x}^2 + \cos(x) + \frac{\theta}{2\pi} \dot{x} \right).$$
- Classically, this term is a total derivative, and has no effect on the classical dynamics : the Hamiltonian is the same.
- Quantum mechanically : There is no effect in feynmann diagrams, no effect in the perturbative expansion in \hbar .
- This theta term affects the tunneling probability amplitude between the vacua, and the non-perturbative correction changes dramatically with the theta term.
- the tunneling amplitude gets a phase factor :
$$e^{-S_{\text{inst}}/\hbar} \rightarrow e^{-S_{\text{inst}}/\hbar + i\theta n}.$$

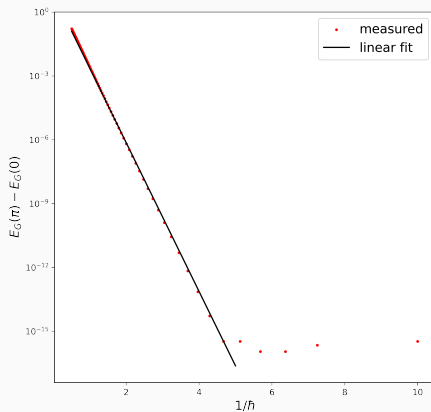
The theta-term : Continued

- The path integral calculation, including the theta term, can be done same as before, using the dilute gas approximation.
- The correction term to the vaccume energy is given by $\Delta E \propto -\cos(\theta)e^{-S_{\text{inst}}/\hbar}$.
- In the wavefunction picture, this theta term affects the ground state energy by changing the boundary condition of the wavefunction : $\psi(x + 2\pi) = e^{i\theta}\psi(x)$.
- We numerically calculated the vaccume energy of the QM on a circle, and the result is consistent with the path integral calculation.

The theta-term : Numerical Calculation



The theta-term : Numerical Calculation



$$\Delta E \propto -\cos(\theta)e^{-S_{\text{inst}}/\hbar}, \quad S_{\text{inst}} = \int_0^{2\pi} dx (2 - 2\cos(\theta))^{1/2} = 8$$

The theta-term : Conclusion

- The instanton effect and the theta term in the QM on a circle changes the vacuum energy, and the vacuum structure of the theory.
- The energy correction is non-perturbative in \hbar , asymptotically :
$$\Delta E \propto -e^{-S_{\text{inst}}/\hbar} \cos(\theta).$$
- The theta term changes the vacuum energy by changing the phase of the instanton Amplitude
- In wavefunction picture, the theta term changes the boundary condition of the wavefunction.

The QCD theta term

- The theta term in QCD is given by $S_\theta = \frac{\theta}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a$.
- The theta term is a total derivative, and has no effect on the classical dynamics, or perturbative QM of the QCD.
- But, the theta term affects the non-perturbative effects of the QCD, and the vacuum structure of the QCD.
- The theta term affects the vacuum energy of the QED, by changing the phase of the YM instanton amplitude.

$$e^{-\frac{8\pi^2}{g^2}|n|} \rightarrow e^{-\frac{8\pi^2}{g^2}|n| + i\theta n} \quad (3)$$

Effects on Real world physics

- The theta term only affects the non-perturbative effects of the QCD.
- At high energy scales, the theta term is not important, since the QCD is asymptotically free.
- At low energy scales, QCD is confining, and the theta term is important. All the things are non-perturbative!
- The theta term changes the vacuum energy of the QCD, the electric dipole moment of the neutron,

