Yang-Mills Instantons

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What is instanton?

- Mathematicaly, Instanton is a solution to the classical field equations of motion in Euclidean space.

$$\delta S_E[\psi_{\text{inst}}] = 0 \rightarrow \text{Euler-Lagrange equation} \tag{1}$$

- Instanton solutions of Euclidean EL equation are localized in (Euclidean) space and time, and have finite (Euclidean) action.
- In Minkofski QFT, it gives a non-perturbative effect: the tunneling between the classical vacua.
- Instantons are important in understanding non-perturbative effects and tunneling between vacua in quantum field theory.

Instanton effect in Path Integral

- In path integral formalism, instantons are classical solution, i.e., the saddle points of the action.
- Instantons only show up in the Euclidean path integral.
- Is it physically meaningful? Do we have to consider this effect in Minkowski space QFT?
- So the path integral is dominated by the instanton contributions in the semiclassical limit.
- The full path integral is given by the sum of all instanton contributions, with small fluctuations around them.

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \sum_{\text{instantons}} e^{-S_{\text{inst}}} \int \mathcal{D}\delta\phi e^{-S_{\text{fluct}}[\delta\phi]} \tag{2}$$

Instanton effect in Path Integral : Example

- A simple example: the one-dimensional quantum mechanics with two minima.
- The potential is given by $V(x) = \frac{1}{2}x^4 - \frac{1}{2}x^2$, and the action is $S = \int dt \left(\frac{1}{2}\dot{x}^2 - V(x)\right)$.
- The Euclidean action is $S_E = \int d\tau \left(\frac{1}{2}\dot{x}^2 + V(x)\right)$, and the Euclidean EOM has a classical solution, starting from $x = -1$ at $\tau = -\infty$ and ending at $x = 1$ at $\tau = \infty$.
- To evaluate correct path integral, We need to consider all the posible semiclassical paths, including the instanton solution. for example,
- If $T \gg t_{\text{inst}}$,all the instanton solution can be approximated as a composition of n single instanton solution, at time $t_1, \ldots t_n$, and the instanton action is given by $S = \sum_{i=1}^{n} S_{\text{inst}} = nS_{\text{inst}}$.

Instanton effect in Path Integral : Calculation

So, now we can calculate the matrix element of e^{-HT} from $x = -1$ to $x = 1$, using the path integral including instanton solution.

$$\langle 1|e^{-HT}|1\rangle = \int \mathcal{D}x e^{-S_E} = \sum_{\text{instantons}} e^{-S_{\text{inst}}} \int \mathcal{D}\delta x e^{-S_{\text{fluct}}}$$

$$= e^{-T/2} \sum_{n \text{ odd}} \int_{-T/2}^{T/2} d\tau_1 \int_{-T/2}^{\tau_1} d\tau_2 \cdots \int_{-T/2}^{\tau_{n-1}} d\tau_n e^{-nS_{\text{inst}}} \int \mathcal{D}\delta x e^{-S_{\text{fluct}}}$$

Here, the fluctuation path integral can be factorized into n parts, which are the fluctuation path integrals around each instanton solution.

$$\int \mathcal{D}\delta x e^{-S_{\text{fluct}}} = \left(\int \mathcal{D}\delta x_n e^{-S_{\text{fluct}}} \right)^n e^{-T/2} = K^n e^{-T/2}$$

$$\langle 1|e^{-HT}|1\rangle = e^{-T/2} \sum_{n \text{ odd}} \int_{-T/2}^{T/2} d\tau_1 \int_{-T/2}^{\tau_1} d\tau_2 \cdots \int_{-T/2}^{\tau_{n-1}} d\tau_n (K e^{-S_{\text{inst}}})^n$$

$$\langle 1|e^{-HT}|1\rangle = e^{-T/2} \sum_{n \text{ odd}} \frac{1}{n!} K^n (e^{-S_{\text{inst}}})^n T^n = e^{-T/2} \sinh\left(K T e^{-S_{\text{inst}}}\right)$$

Instanton effect in Path Integral : Conclusion

- The Path integral including instanton solution gives the correct matrix element of e^{-HT} from $x = -1$ to $x = 1$: $\langle 1|e^{-HT}|-1\rangle = e^{-T/2}\sinh\big(KTe^{-S_{\text{inst}}}\big)$.
- By changing the result to $T \to it$, we can get the propagator of the Minkowski QM: $\langle 1|e^{-iHT}|-1\rangle = e^{-it/2}\sin\big(Kte^{-S_{\text{inst}}}\big)$.
- No instanton solution in Minkowski space, but the instanton effect is still important in understanding the non-perturbative effects in Minkowski QFT.
- Amplitube $\langle 1|e^{-iHT}|-1\rangle = e^{-it/2}\sin\big(Kte^{-S_{\text{inst}}/\hbar}\big)$ is non-perturbative in $\hbar$, and the instanton effect is essential in understanding the non-perturbative effects in quantum field theory.

Solitons

- Solitons are the localized, finite energy solutions of the classical field equations of motion.
- Solitons ARE NOT instantons, but they are related to instantons.
- Instantons are localized in Euclidean space and time, and have finite Euclidean action.
- Solitons are localized in space, but stationary in time, and have finite energy.

Yang-Mills Instanton : Introduction

- Yang-Mills instantons are the instanton solutions of the Yang-Mills theory.
- The Yang-Mills action is given by $S = \int d^4x \left(-\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a}\right)$, and the Euclidean action is $S_E = \int d^4x \left(\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a}\right)$.
- The Euclidean EOM is given by $\mathcal{D}_\mu F^{\mu\nu a} = 0$, with the Bianchi identity $\sum_{\mathrm{cyc}} \mathcal{D}_\mu \tilde{F}^{\nu\rho a} = 0$.
- The YM instanton solution is nontrivial solution of the EOM, and can be obtained by solving the self-dual equation $\tilde{F}^{\mu\nu a} = F^{\mu\nu a}$.

Yang-Mills Instanton : Self-dual equation

- Hodge dual operator : $\tilde{F}^{\mu\nu a} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^{a}_{\rho\sigma}$.
- The self-dual equation is given by $\tilde{F}^{\mu\nu a} = F^{\mu\nu a}$.
- All the 2-forms can be decomposed into self-dual and anti-self-dual parts: $F^{\mu\nu a} = F^{\mu\nu a}_{+} + F^{\mu\nu a}_{-}$.
- If either $F^{\mu\nu a}_{+}$ or $F^{\mu\nu a}_{-}$ is zero, the solution is called self-dual or anti-self-dual, and satisfies the EOM of the YM theory, $\mathcal{D}_{\mu}F^{\mu\nu a} = 0$.

Yang-Mills Instanton : evaluating S_{inst}

- The instanton action is given by $S_{\text{inst}} = \int d^4x \left(\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a} \right)$.
- The action can be evaluated easily when the instanton is self-dual or anti-self-dual.

$$\begin{aligned} S_{\text{YM}} &= \int d^4x \left(\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a} \right) \\ &= \frac{1}{4g^2} \int d^4x \left(F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu} \right)^2 \mp \frac{1}{2g^2} \int d^4x \tilde{F}^a_{\mu\nu} F^{\mu\nu a} \\ &\geq \pm \frac{1}{2g^2} \int d^4x \tilde{F}^a_{\mu\nu} F^{\mu\nu a} = \frac{8\pi^2}{g^2} |n| \end{aligned}$$

So, the instanton action is the bound, $8\pi^2|n|/g^2$, where n is the instanton number(the 2nd chern number). This derivation of the instanton action shows the topological nature of the instanton solution : the instanton action is quantized, and the instanton number is a topological invariant of the corresponding $SU(N)$ principal bundle.

Yang-Mills Instanton : Physical effects

- The instanton solution is a non-perturbative effect in the YM theory : $e^{-S_{\text{inst}}} = e^{-8\pi^2|n|/g^2}$ is the tunneling amplitude between the classical vacua, and non-perturbative in g.
- The instanton is a source of tunneling between classical vacua in the YM theory, and the true vaccume structure of the YM theory, and the θ term.
- THe instanton solution is also important in understanding the anomalies in the YM theory.

Tunneling Effects in YM and QCD : θ-vacuum

- Classical vaccume of the YM theory?
- The YM Hamiltonian is :
- So, there is only one classical vaccume, $F_{\mu} = 0$, and the vaccume energy is zero.
- Then, what tunneling effect appears in the YM theory?

Easy example : QM on a circle

- Consider the quantum mechanics on a circle, with the potential $V(x) = -\cos(x)$.
- There is one classical vaccume, $x = 0$, and the vaccume energy is $V(0) = -1$.
- But, in path integral formalism, the tunneling effect appears : paths with non-zero winding number.
- This contribution can be interpreted as the tunneling to itself, and the tunneling amplitude is given by $e^{-S_{\text{inst}}/\hbar}$.
- The tunneling amplitube is non-perturbative in $\hbar$, and this instanton effect gives non-perturbative corrections to the vacccume energy.

The theta-term

- One can consider A "trivial term" aplied to abouve QM on a circle : $S = \int dt \left(\frac{1}{2}\dot{x}^2 + \cos(x) + \frac{\theta}{2\pi}\dot{x}\right)$.
- Classicaly, this term is a total derivative, and has no effect on the classical dynamics : the Hamiltonian is the same.
- Quantum mechanically : There is no effect in feynmann diagrams, no effect in the perturbative expansion in $\hbar$.
- This theta term affects the tunneling probability amplitude between the vaccua, and the non-perturbative correction changes dramatically with the theta term.
- the tunneling amplitude gets a phase factor : $e^{-S_{\text{inst}}/\hbar} \to e^{-S_{\text{inst}}/\hbar + i\theta n}$.

The theta-term : Continued

- The path integral calculation, including the theta term, can be done same as before, using the dilute gas approximation.
- The correction term to the vaccume energy is given by $\Delta E \propto -e^{-S_{\text{inst}}/\hbar}\cos(\theta)$.
- In the wavefuncion picture, this theta term affects the ground state energy by changing the boundary condition of the wavefunction : $\psi(x + 2\pi) = e^{i\theta}\psi(x)$.
- We numerically calculated the vaccume energy of the QM on a circle, and the result is consistent with the path integral calculation.

The theta-term : Numerical Calculation