Polynomials in categories with pullbacks

1) The operational view / extensive view

Originally, a poly. functor $\underline{\mathrm{Set}} \to \underline{\mathrm{Set}}$ is one that's in the closure of id under $\times, +$; more generally, under Π, Σ.

More generally, a multivariate poly functor $\mathrm{Set}^I \longrightarrow \mathrm{Set}$ is one in the closure of the projection fns $\mathrm{Set}^I \to \mathrm{Set}$ under Π, Σ.

(1) Even more generally, a poly functor $\mathrm{Set}^I \longrightarrow \mathrm{Set}^J$ is a J-indexed family of poly functors $\mathrm{Set}^I \longrightarrow \mathrm{Set}$.

There are many other views on poly functors:

(2) Functors $\mathrm{Set}/_I \longrightarrow \mathrm{Set}/_J$ which are composites of:

$$\begin{tikzcd}
\mathrm{Set}/_I \arrow[rr, "\Delta f" description] & & \mathrm{Set}/_J \arrow[ll, bend right=40, "\Sigma f"'] \arrow[ll, bend left=40, "\Pi f"] \arrow[ll, phantom, shift right=3, "\perp"] \arrow[ll, phantom, shift left=3, "\perp"]
\end{tikzcd}$$

$f : J \longrightarrow I$

Δf = pullback along f

Σf = postcompose with f

Πf = "dependent product"

$$\Pi_f : \mathrm{Set}^J \longrightarrow \mathrm{Set}^I$$
$$(A_{ij} : i \in I, j \in J_i) \longmapsto \left(\prod_{j \in J_i} A_{ij} : i \in I\right)$$

where think of $f : J \longrightarrow I$ as giving family of sets $(J_i : i \in I)$ with $J_i = f^{-1}(i)$

③ Functors $\mathrm{Set}/_I \xrightarrow{F} \mathrm{Set}/_J$ of form $\mathrm{Set}/_I \xrightarrow{\Delta t} \mathrm{Set}/_E \xrightarrow{\Pi p} \mathrm{Set}/_B \xrightarrow{\Sigma s} \mathrm{Set}/_J$

④ Functors $\mathrm{Set}/_I \xrightarrow{F} \mathrm{Set}/_J$ which preserve connected limits and are small.
(Girard's normal functors (88))

⑤ . . . . with a left multiadjoint (Diers '78)

⑥ . . . . whose slices $F/_x$ are right adjoints
(local right adjoint: Street '00)

What happens if I generalise these from Set to $\mathcal{E}$?

Typically:

④ = ⑤ = ⑥ if, say, $\mathcal{E}$ is locally presentable.

③ ≤ ② : but note here that $\Pi f : \mathcal{E}/_I \to \mathcal{E}/_J$ may not exist for all f: but we can restrict to the class of <u>exponentiable</u> f; by defn these are the f for which Πf exists.

① not very much to do with rest; except ① ≤ ② if $\mathcal{E}$ lextensive.

Now ② ⊆ ④ and typically ② ⊊ ④.

④=⑤=⑥ typically super interesting:

- $\mathcal{E}$ = presheaf caty, have pra. functors that David spoke of
- $\mathcal{E}$ = variety, ⇢ corings + plethories + friends (Tall-Wraith, Bergman-Hausknecht, Joyal, Borger, ...)
- $\mathcal{E} = Sh(X)$, X Stone space ⇢ topological dynamics + non-comm. geometry.

However, ② and ③ give a much more uniform theory, which is the theory of <u>polynomial functors</u> in $\mathcal{E}$.

Magical fact: in any catg $\mathcal{E}$ with pb's, ② and ③ coincide.
Why? Things in ② have normal forms in ③.
(Gambino, Kock 2013; Weber 2015)

<u>Idea</u>:
① If we have $\mathcal{E}/I \xrightarrow{\Sigma f} \mathcal{E}/J \xrightarrow{\Delta g} \mathcal{E}/K$,
I can form pb to the right

$$\begin{tikzcd} L \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "f"] \\ K \arrow[r, "g"'] & J \end{tikzcd}$$

and now the 2-cell

$$\begin{tikzcd} \mathcal{E}/J \arrow[r, "\Delta g"] \arrow[d, "\Delta f"'] \arrow[dr, phantom, "\cong"] & \mathcal{E}/K \arrow[d, "\Delta u"] \\ \mathcal{E}/I \arrow[r, "\Delta v"'] & \mathcal{E}/L \end{tikzcd}$$

transposes to one

$$\begin{tikzcd} \mathcal{E}/I \arrow[r, "\Delta v"] \arrow[d, "\Sigma f"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{E}/L \arrow[d, "\Sigma u"] \\ \mathcal{E}/J \arrow[r, "\Delta g"'] & \mathcal{E}/K \end{tikzcd}$$

which turns out to be <u>invertible</u> (Beck-Chevalley).
So now

$$\begin{tikzcd} & \mathcal{E}/J \arrow[dr, "\Delta g"] \arrow[dd, phantom, "\cong"] & \\ \mathcal{E}/I \arrow[ur, "\Sigma f"] \arrow[dr, "\Delta v"'] & & \mathcal{E}/K \\ & \mathcal{E}/L \arrow[ur, "\Sigma u"'] & \end{tikzcd}$$

② Similarly, if we have $\mathcal{E}/I \xrightarrow{\Pi f} \mathcal{E}/J \xrightarrow{\Delta g} \mathcal{E}/K$
can form pullback and the canonical 2-cell

$$\begin{tikzcd} L \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "f"] \\ K \arrow[r, "g"'] & J \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E}/I \arrow[r, "\Delta v"] \arrow[d, "\Pi f"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{E}/L \arrow[d, "\Pi u"] \\ \mathcal{E}/J \arrow[r, "\Delta g"'] & \mathcal{E}/K \end{tikzcd}$$

is again invertible (Beck-Chevalley)
so can rewrite.

③ If we have $\mathcal{E}/I \xrightarrow{\Sigma f} \mathcal{E}/J \xrightarrow{\Pi g} \mathcal{E}/K$, can form

$$\begin{tikzcd}
L \arrow[r, "u"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[dd, "\Pi_g(f) = v"] \\
I \arrow[d, "f"'] \arrow[r, phantom, "(*)"] & \\
J \arrow[r, "g"'] & K
\end{tikzcd}$$

counit of $\Delta_g \dashv \Pi_g : \mathcal{E}/J \to \mathcal{E}/K$ at f.

By messing around with adjoints, using BC isom. for Π's in this pb square, get a canonical 2-cell:

$$\begin{tikzcd}
\mathcal{E}/I \arrow[r, "\Sigma f"] \arrow[d, "\Delta\varepsilon"'] & \mathcal{E}/J \arrow[dd, "\Pi g"] \\
\mathcal{E}/L \arrow[d, "\Pi u"'] \arrow[r, phantom, "\Rightarrow"] & \\
\mathcal{E}/M \arrow[r, "\Sigma v"'] & \mathcal{E}/K
\end{tikzcd}$$

which turns out to be invertible.

In Set, this invertibility expresses the isom.

$$\prod_{a \in A} \sum_{b \in B_a} C_{ab} = \sum_{f \in \prod_{a \in A} B_a} \prod_{a \in A} C_{a, f(a)}$$

called type-theoretic axiom of choice, or complete distributivity.

Using the invertible (*), we can rewrite $\longrightarrow \longrightarrow$ as

$\longrightarrow \longrightarrow \longrightarrow$

Using these three rewrites, we can turn any ② into a ③.

REMARK So typically, have

② = ③ $\subsetneq$ ④ = ⑤ $\neq$ ⑥

However, if we interpret ④ = ⑤ $\neq$ ⑥ less naively in $\mathcal{E}$, we get another equivalent formulation of ② = ③. Namely, if we look at <u>indexed</u> functors between <u>indexed</u> slice cats of $\mathcal{E}$, which in a suitable indexed sense are local right adjoint, then we get polynomial functors. (Kock and Kock 2013).