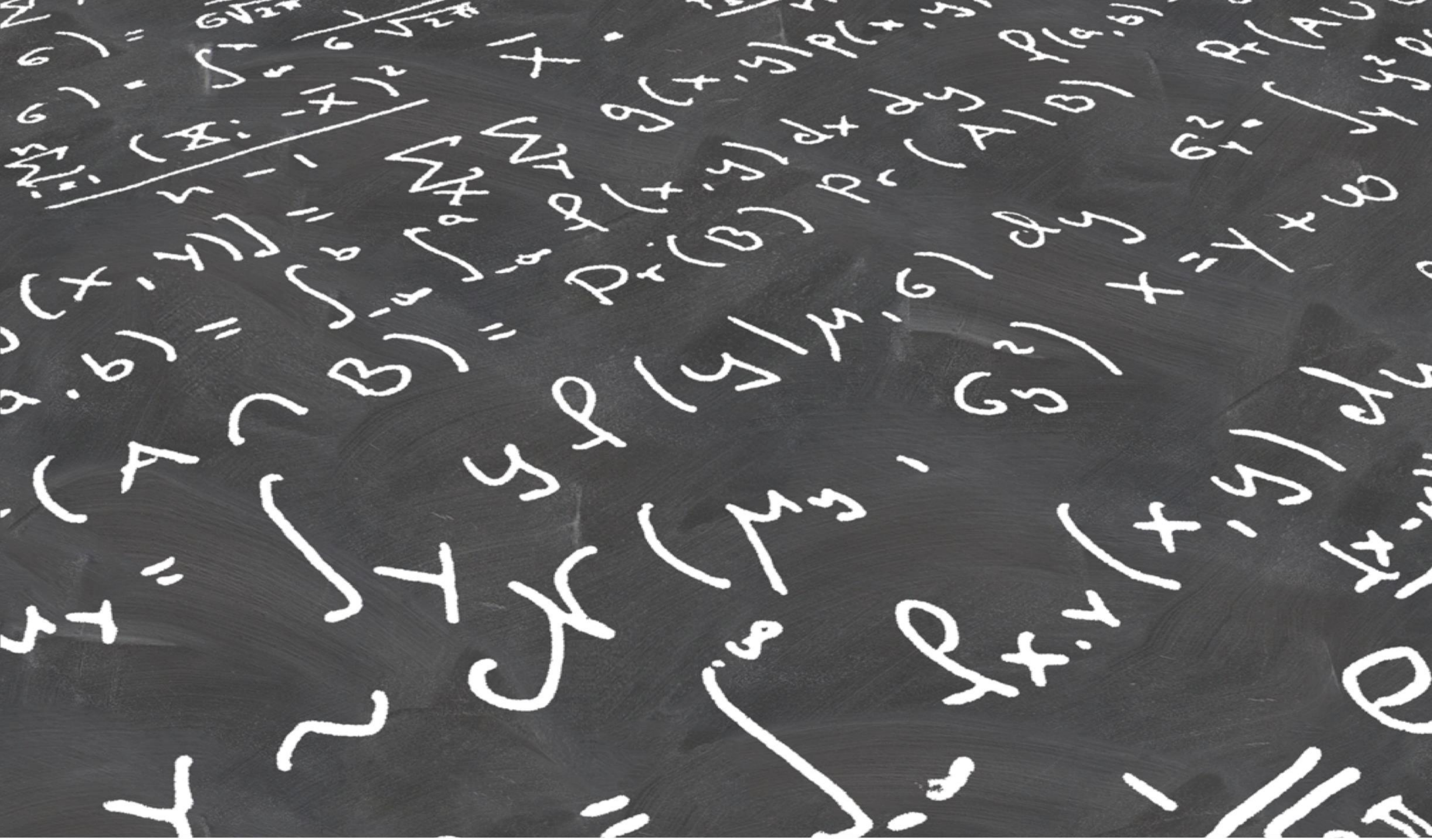


3. Discrete Random Variables

Gunvor Elisabeth Kirkelund
Lars Mandrup

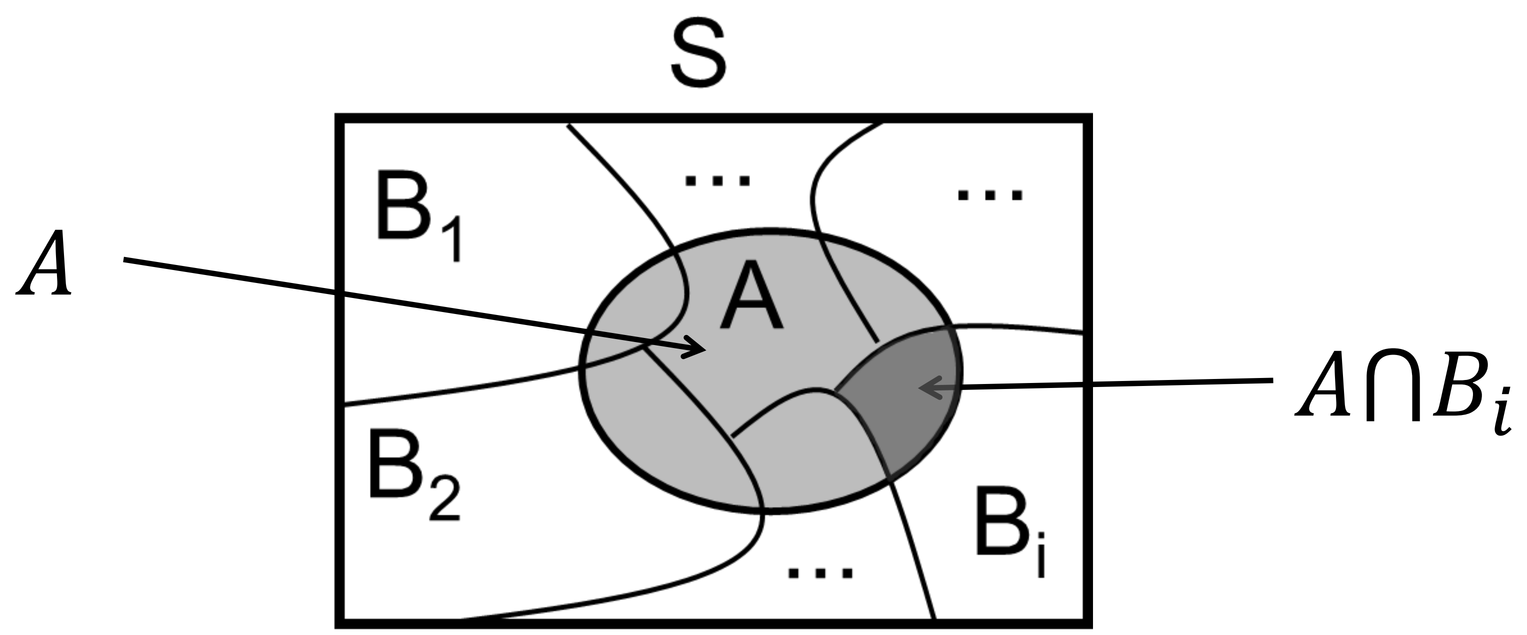
Agenda for Today

- Repetition from last time
- Definition of a Stochastic Random Variable
- Discrete Stochastic Variables

Total Probability

We sometime call it the marginal

- Pr(A) of an event is the total probability of that event.



$$Pr(A) = Pr(A \cap B_1) + Pr(A \cap B_2) + \cdots + Pr(A \cap B_i) + \cdots$$

$$= Pr(A|B_1) \cdot Pr(B_1) + Pr(A|B_2) \cdot Pr(B_2) + \cdots$$

where the B_i's are mutually exclusive ($B_i \cap B_j = \varnothing$ for $i \neq j$)
and $S = B_1 \cup B_2 \cup \ldots \cup B_i \cup \ldots$

Bayesian Terms

- **Prior**: What are the overall probability of an event E?

$$Pr(E)$$

- **Likelihood**: What are the probability of a test T given event E?

$$Pr(T|E) = \frac{Pr(T \cap E)}{Pr(E)} = \frac{Pr(E|T) \cdot Pr(T)}{Pr(E)}$$

- **Total Probability**: What is the total probability of the test?

$$Pr(T) = Pr(T|E) \cdot Pr(E) + Pr(T|\bar{E}) \cdot Pr(\bar{E})$$

- **Posterior**: What are the probability the event given the test T?

$$Pr(E|T) = \frac{Pr(T \cap E)}{Pr(T)} = \frac{Pr(T|E) \cdot Pr(E)}{Pr(T)}$$

Combinatorics

- The number of possible outcomes of k trials, sampled from a set of n objects.

Types of Experiments:

- With or without replacement
- Ordered or unordered

		Replacement	
		With	Without
Sam-pling	Ordered	n^k	$P_k^n = \dfrac{n!}{(n-k)!}$
	Unordered	$\binom{n+k-1}{k} = \dfrac{(n+k-1)!}{k!\,(n-1)!}$	$\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

The Binomial Distribution

- We have n repeated trials.
- Each trial has two possible outcomes
  - Success — probability p
  - Failure — probability q=1-p

Bernoulli trial

- What is the probability of having k successes out of n trials?
- We write this question as:

$$\boxed{Pr_n(k) = \frac{n!}{k!\,(n-k)!} p^k q^{n-k} = \binom{n}{k} p^k q^{n-k}}$$

- Faculty: $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$
  $0! = 1$

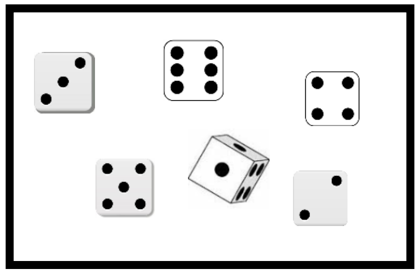
Also called a random experiment

Stochastic Experiment

- An experiment in which you can not predict the outcome

Examples:

S



- Rolling a dice
- Sample space for the experiment is: {1, 2, 3, 4, 5, 6}

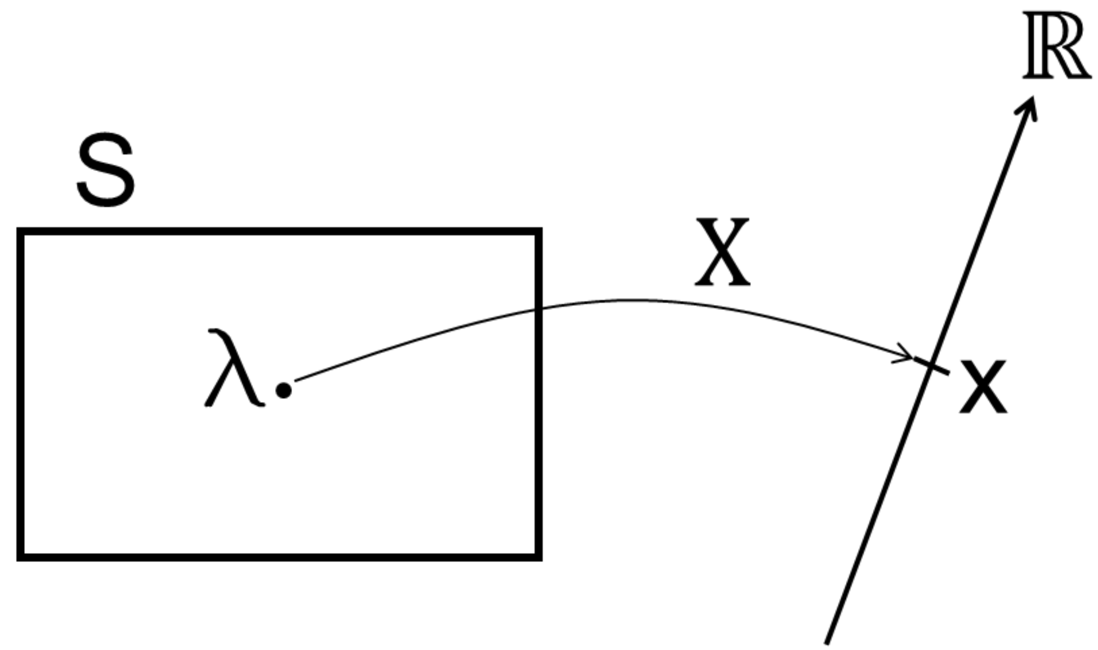
- Flip a coin
- Sample space for the experiment is: {head, tail}

S

Also just called a random variables

Stochastic Random Variables

- A random variable tells something important about a stochastic experiment.
- Can be discrete or continous



Examples:

- The numbers on a dice (discrete):
  - Sample space for variable X is : $\{1, 2, 3, 4, 5, 6\}$
  - Sample space for variable Y "Even (1)/Uneven (-1)": $\{1, -1\}$
- The hight of students at IHA (continous):
  - Sample space for variable H is all real numbers: [100;250] cm.

Probability Mass Function (PMF)

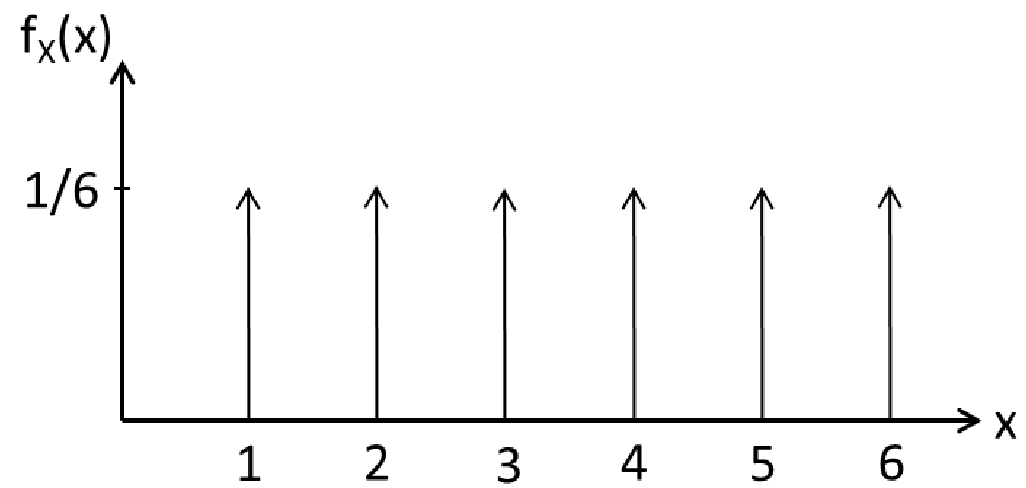
- Sample space for X.
- X is a discreet stochastic variable.

$$f_X(x) = \begin{cases} Pr(X = x_i) & for\ X = x_i \\ 0 & otherwise \end{cases}$$

$$0 \leq f_X(x) \leq 1$$

- We have that: $\sum_{i=1}^{n} f_X(x_i) = \sum_{i=1}^{n} Pr(X = x_i) = 1$

Example: Laplace Dice
(perfect dice)



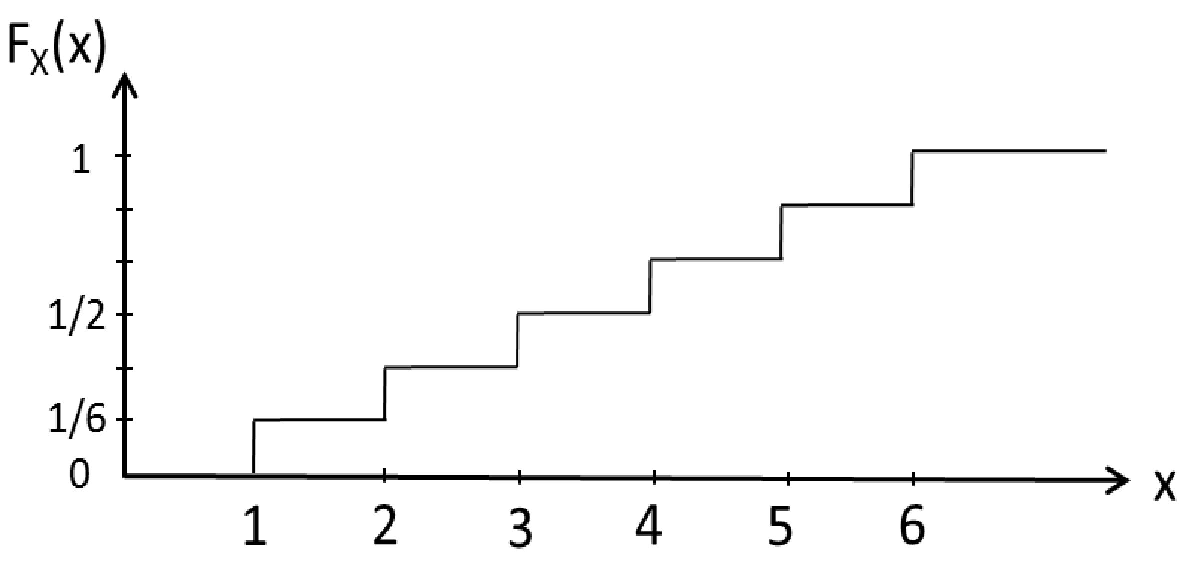
Cumulative Distribution Function (CDF)

- Sample space for X.
- X is a discreet stochastic variable.
- $F_X(x)$ is a non-decreasing step-function.

$$\boxed{F_X(x) = Pr(X \leq x)} \qquad 0 \leq F_X(x) \leq 1$$

- We have that: $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$

Example: Laplace Dice
(perfect dice)



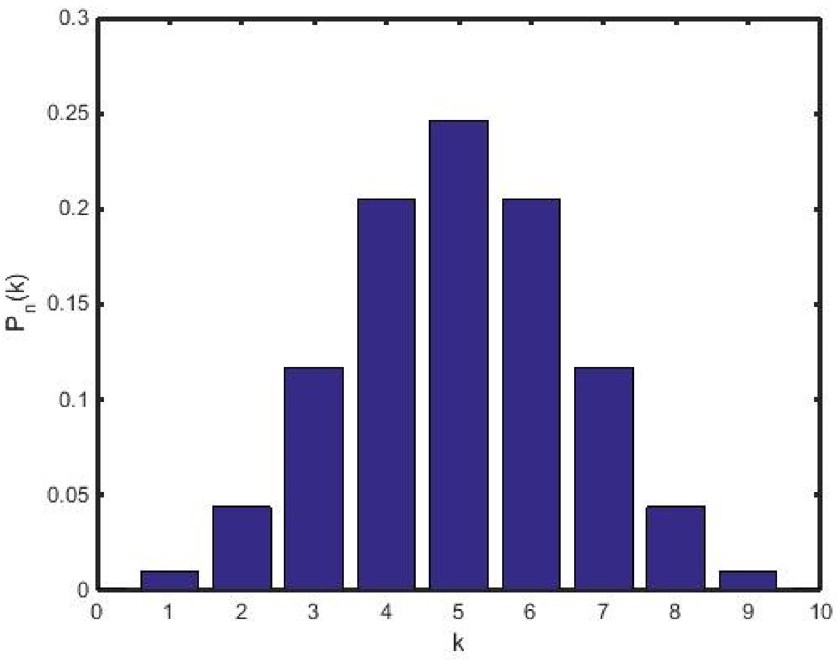
The Binomial Mass Function

- We have n repeated trials.
- Each trial has two possible outcomes
  - Success — probability p
  - Failure — probability 1-p

Also called a Bernoulli trial

- We write the mass function as:

$$\boxed{f(k|n,p) = \frac{n!}{k!\,(n-k)!}p^k(1-p)^{n-k}}$$



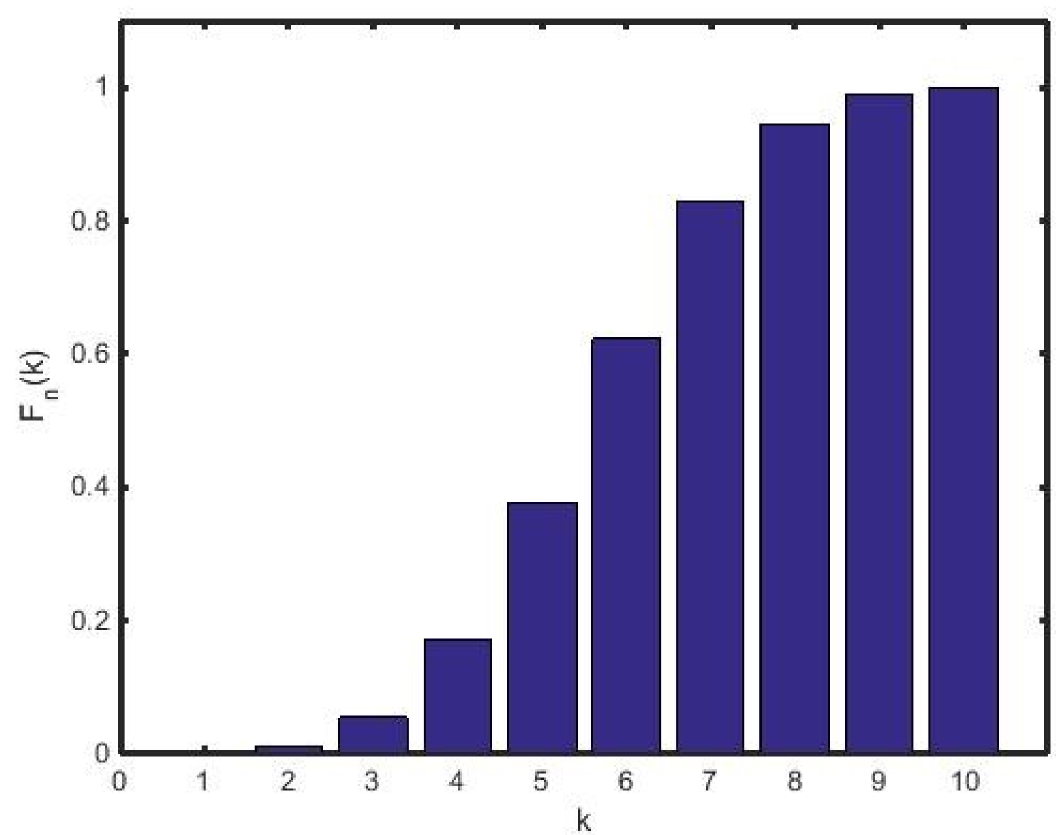
The Binomial Distribution

- The probability mass function is given as:

$$f(k|n,p) = \frac{n!}{k!\,(n-k)!}p^k(1-p)^{n-k} = \binom{n}{k}p^k(1-p)^{n-k}$$

- We write the distribution as the sum:

$$F(k|n,p) = \sum_{i=0}^{k} f(i|n,p)$$



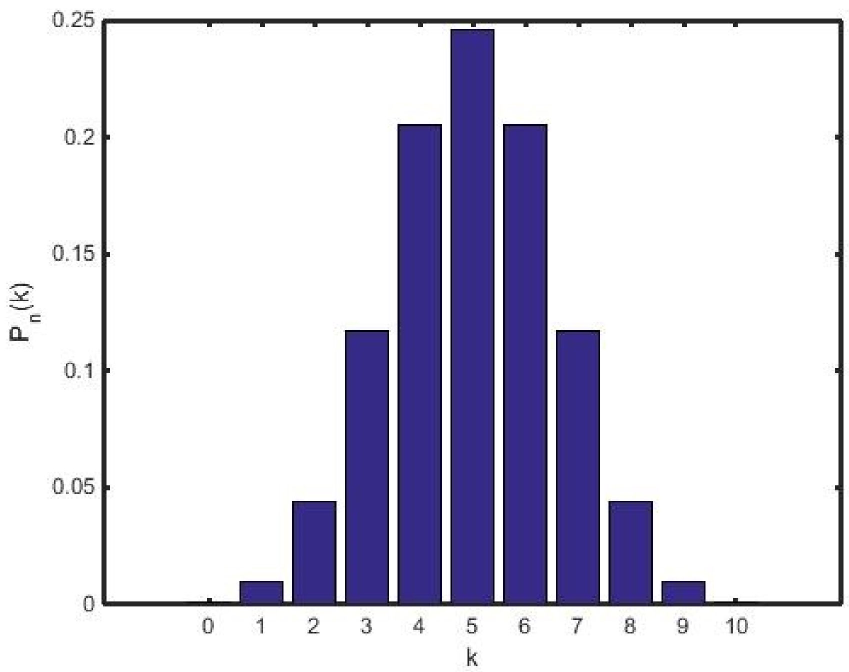
Expectation of a Discrete Random Variable

Example: If I want ten children, how many girls can I expect to get?

Answer: I assume a Binomial distribution with p=0.5:

$$f(k|10{,}0.5) = \binom{10}{k} \cdot 0.5^k \cdot 0.5^{10-k} = \binom{10}{k} \cdot 0.5^{10}$$

$$where \binom{10}{k} = \frac{10!}{k!\,(10-k)!}$$



$$E[k] = 0 \cdot f(0|10{,}0.5) + 1 \cdot f(1|10{,}0.5) + \cdots + 10 \cdot f(10|10{,}0.5)$$
$$= \left(0 + 1 \cdot \binom{10}{1} + 2 \cdot \binom{10}{2} \ldots + 10 \cdot \binom{10}{10}\right) \cdot 0.5^{10}$$
$$= (0 + 1 \cdot 10 + 2 \cdot 45 + \cdots + 10 \cdot 1) \cdot 0.5^{10} = 10 \cdot 0.5 = 5$$

Expectation of a Discrete Random Variable

- We define the <u>mean</u> or the <u>expectation</u> of a discreet random variable as:

n is the number of outcomes

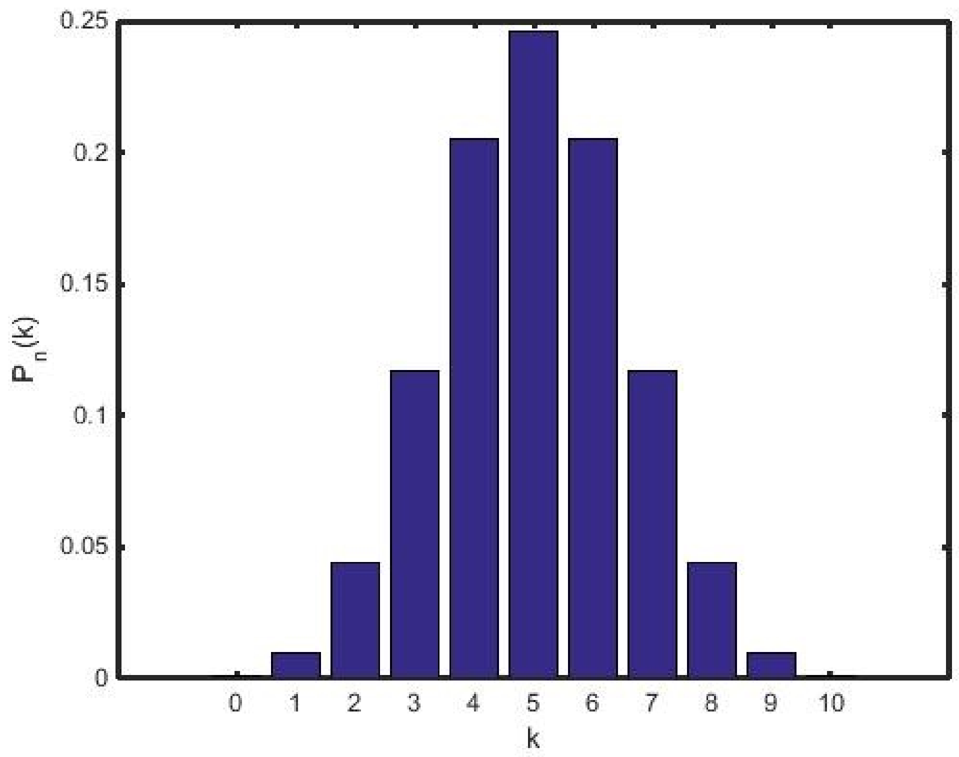
$$\boxed{\bar{X} = E[X] = \sum_{i=1}^{n} x_i \, f_X(x_i)}$$

x_i is its outcome

The Binomial Distribution (cont'd)

- For the Binomial distribution, we have:

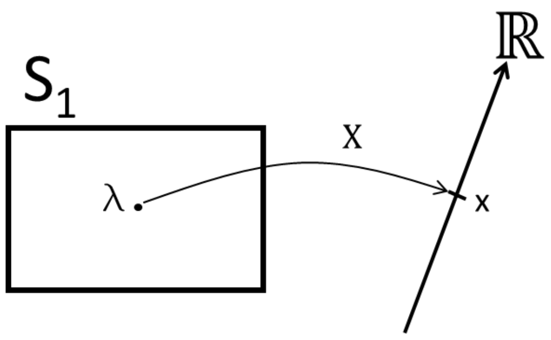
$$\boxed{\begin{aligned} E[k] &= n \cdot p \\ Var(X) &= n \cdot p \cdot (1-p) \end{aligned}}$$

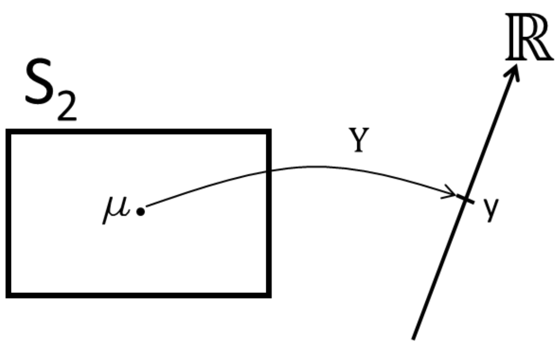


- Where the variance is defined as:

$$Var(X) = \sigma^2 = E[X^2] - E[X]^2$$

Two Simultaneous Discreet Random Variables

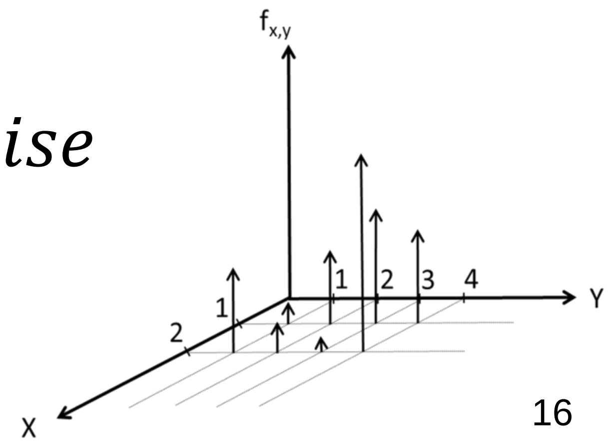




- Two (or more) discreet random variables X and Y
- We can discribe the two probabilities as a simultaneous pmf:

Joint (Simultaneous) pmfs:

$$f_{X,Y}(x,y) = \begin{cases} Pr\left((X = x_i)\cap(Y = y_j)\right) & for\ \ X = x_i \wedge Y = y_j \\ 0 & otherwise \end{cases}$$

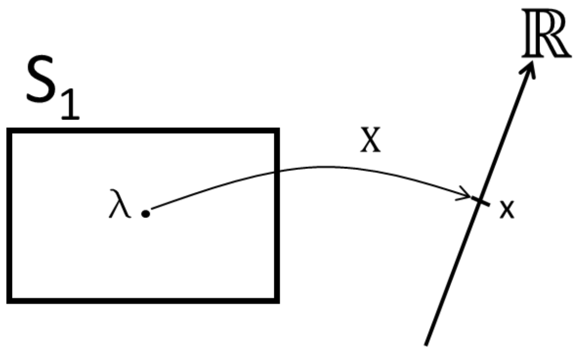


Fx.: X = The number of bicycles in front of IHA
Y = The number of people inside IHA

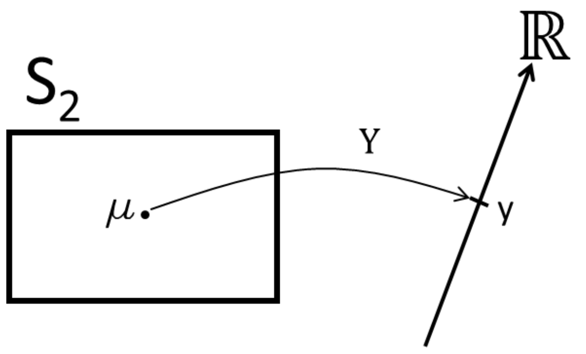
Two Simultaneous Discrete Random Variables

Marginal pmfs:

$$f_X(x) = \sum_y f_{X,Y}(x,y) \qquad f_Y(y) = \sum_x f_{X,Y}(x,y)$$



Conditional pmfs / Bayes Rule:



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \Pr(X = x|\mathrm{Y} = \mathrm{y})$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \Pr(Y = y|\mathrm{X} = \mathrm{x})$$

Orca Example

- Let us assume that the discreet simultaneous mass function (pmf) for observing a orca at a specific ocean and its gender is

$f_{X,Y}(x,y)$ $f_X(x)$

Gender (X)\ Location (Y)	Atlantic (1)	Antartica (2)	Pacific (3)	Seaworld (4)	Total
female (1)	2/60	7/60	11/60	9/60	29/60
male (2)	8/60	3/60	1/60	19/60	31/60
Total	10/60	10(60	12/60	28/60	1

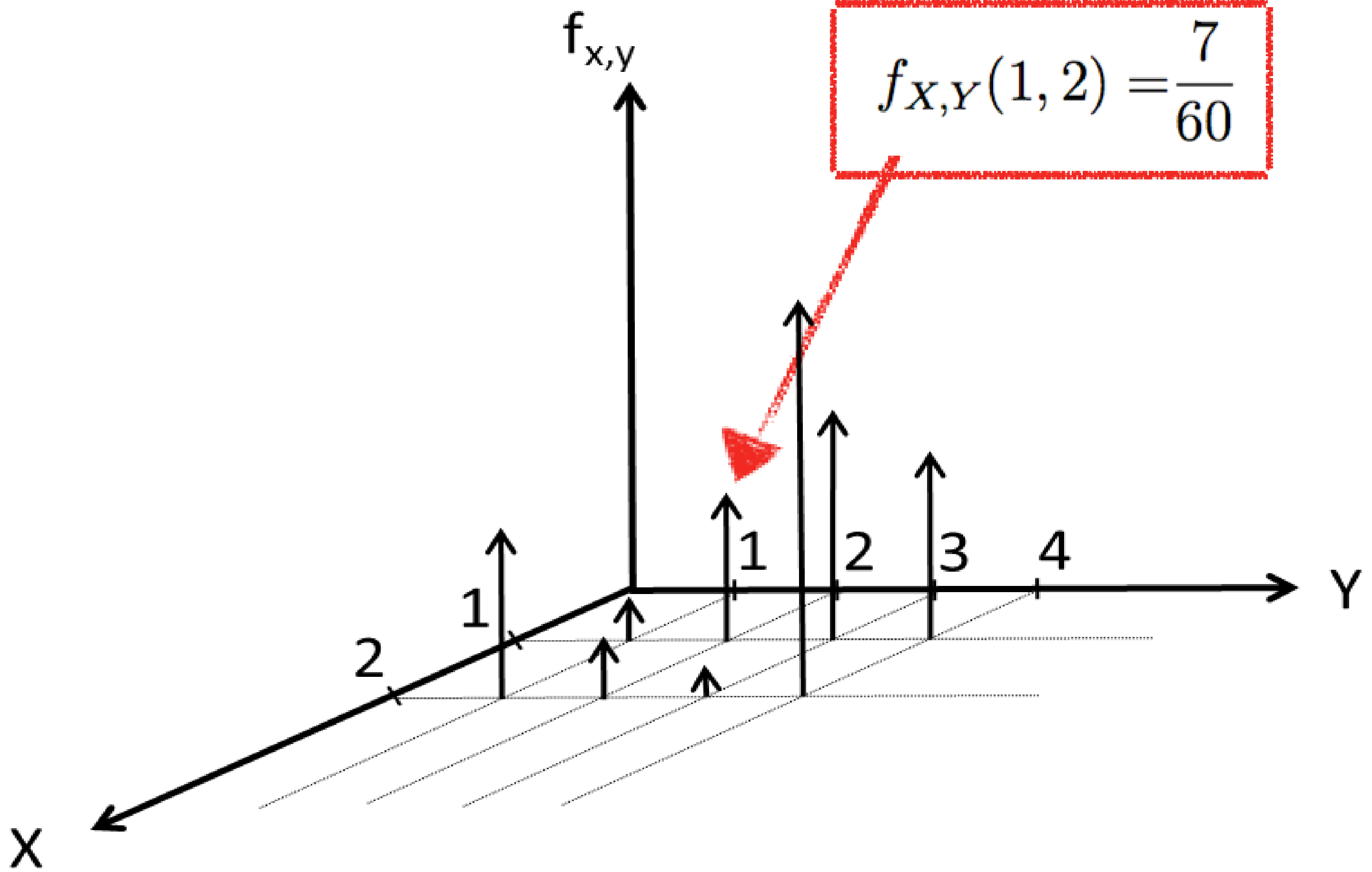
$f_Y(y)$

Fx.: $Pr(Male|Atlantica) = f_{X|Y}(2|1) = \frac{f_{X,Y}(2,1)}{f_Y(1)} = \frac{8/60}{10/60} = \frac{8}{10} = 0{,}8$

Orca Example - Joint pmf

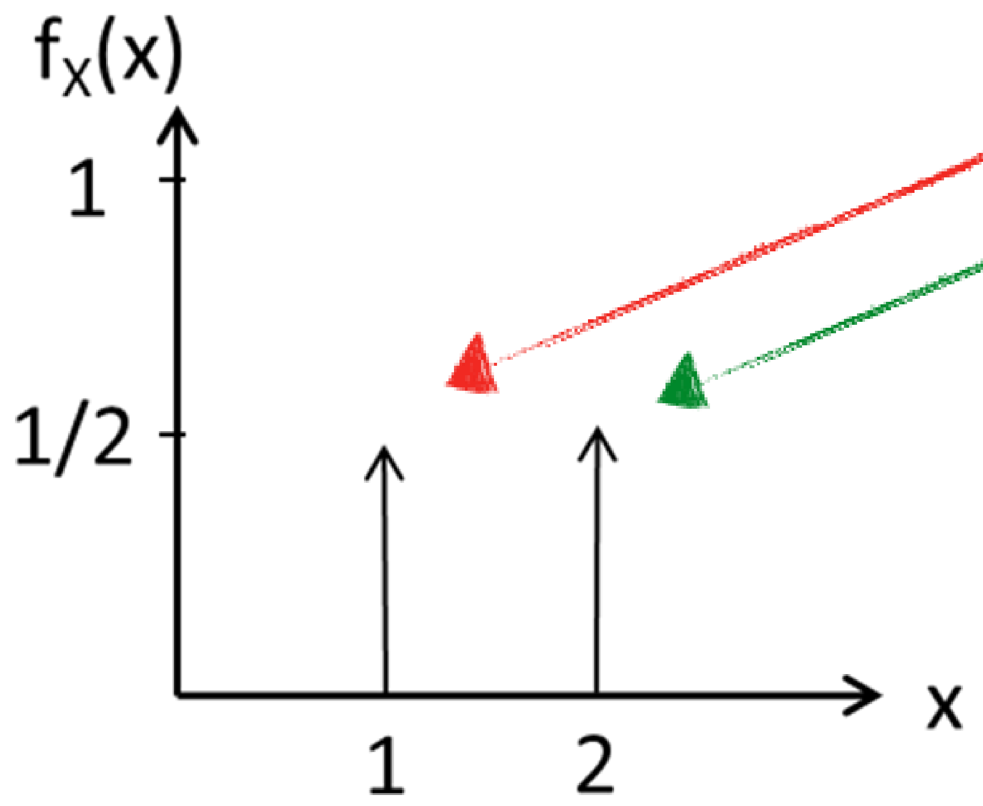
Gender (X)\ Location (Y)	Atlantic (1)	Antartica (2)	Pacific (3)	Seaworld (4)	Total
female (1)	2/60	$\boxed{7/60}$	11/60	9/60	29/60
male (2)	8/60	3/60	1/60	19/60	31/60
Total	10/60	10(60	12/60	28/60	1

$$\boxed{f_{X,Y}(1,2) = \frac{7}{60}}$$



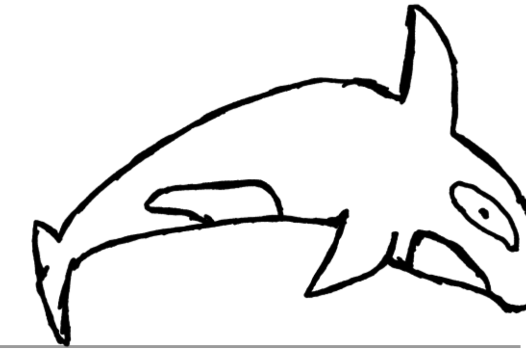
Orca Example – Marginal pmf

Gender (X)\ location (Y)	Atlantic (1)	Antartica (2)	Pacific (3)	Seaworld (4)	Total
female (1)	2/60	7/60	11/60	9/60	29/60
male (2)	8/60	3/60	1/60	19/60	31/60
Total	10/60	10(60	12/60	28/60	1

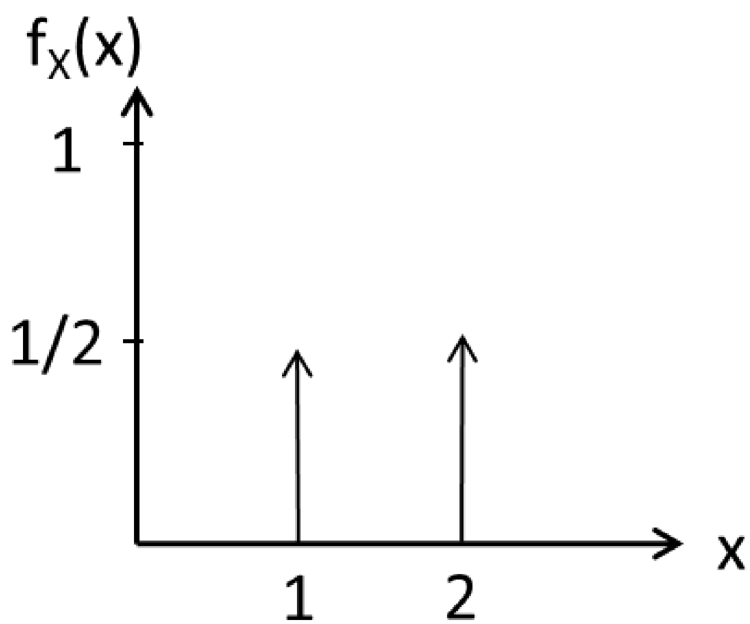


$$f_X(1) = f_{X,Y}(1,1) + f_{X,Y}(1,2) + f_{X,Y}(1,3) + f_{X,Y}(1,4)$$
$$= \frac{2}{60} + \frac{7}{60} + \frac{11}{60} + \frac{9}{60} = \frac{29}{60}$$
$$f_X(2) = f_{X,Y}(2,1) + f_{X,Y}(2,2) + f_{X,Y}(2,3) + f_{X,Y}(2,4)$$
$$= \frac{8}{60} + \frac{3}{60} + \frac{1}{60} + \frac{19}{60} = \frac{31}{60}$$

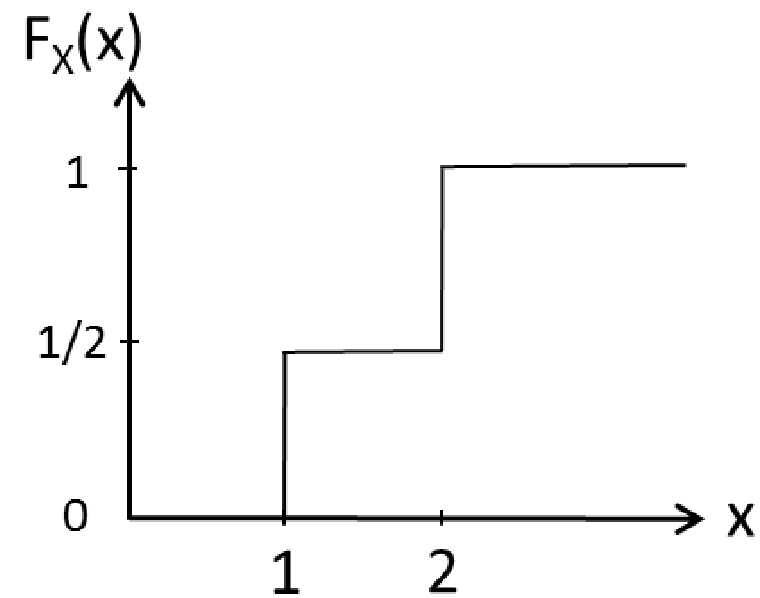
Orca Example – Quick Rewrite to cdf



- We can rewrite the pmf to the cdf



Marginal pmf



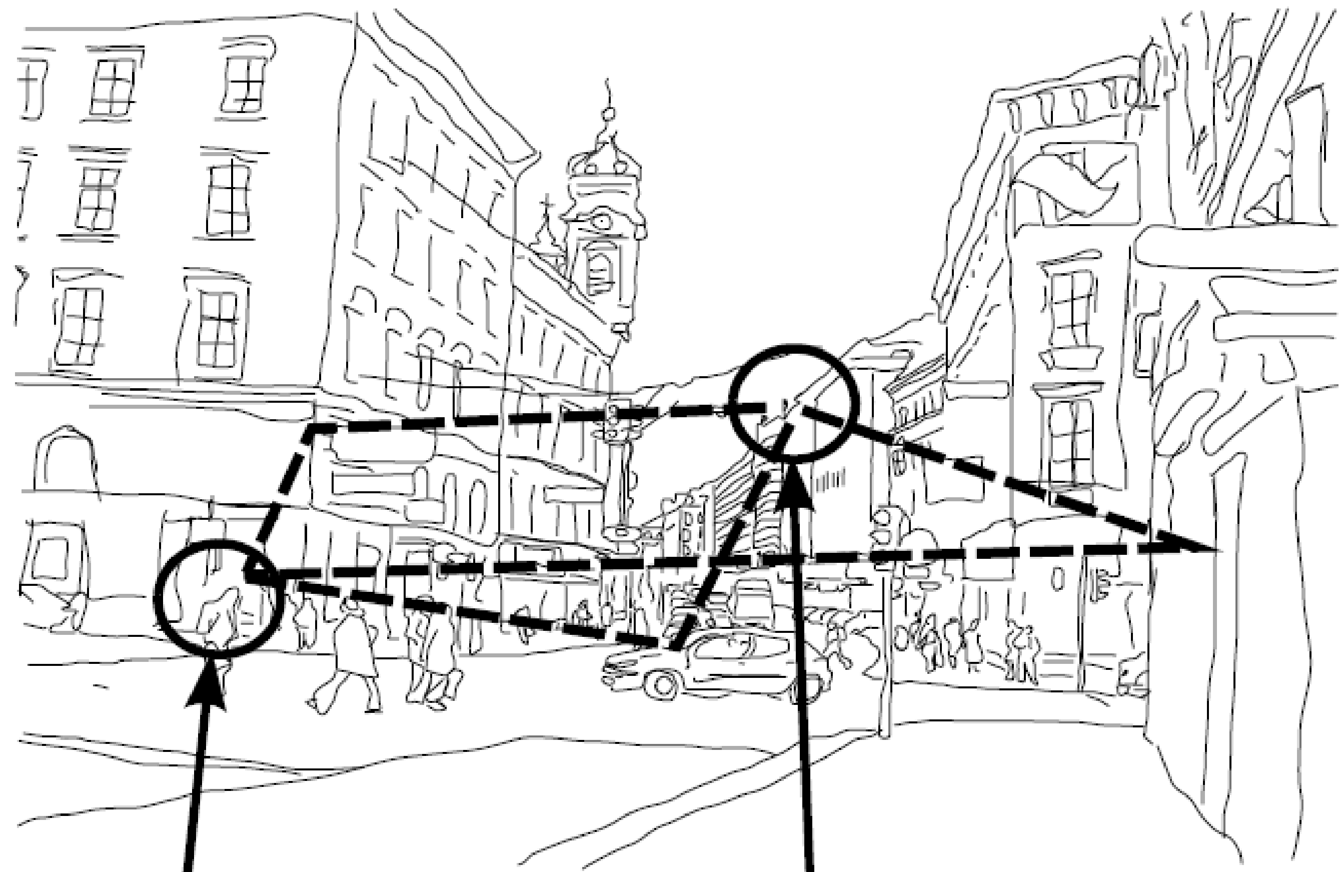
Marginal cdf

$$f_X(1) = \frac{29}{60}$$
$$f_X(2) = \frac{31}{60}$$

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{29}{60} & \text{for } 1 \leq x < 2 \\ 1 & \text{for } 2 \leq x \end{cases}$$

Example - Wireless Channel

- A signal in a wireless channel travels with equal probability of three different path from transmitter to receiver



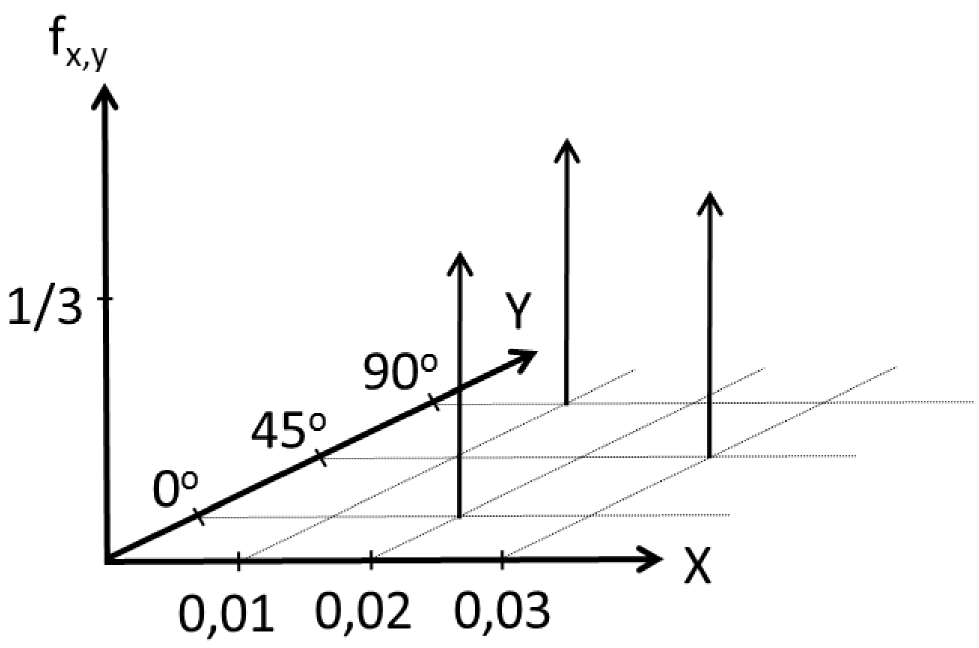
Amplitude\ Phase	0^o	45^o	90^o	**Total**
0.01	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0.02	$\frac{1}{3}$	0	0	$\frac{1}{3}$
0.03	0	$\frac{1}{3}$	0	$\frac{1}{3}$
Total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	**1**

Example - Wireless Channel: Assignment

X Y

Amplitude\ Phase	0^o	45^o	90^o	Total
0.01	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0.02	$\frac{1}{3}$	0	0	$\frac{1}{3}$
0.03	0	$\frac{1}{3}$	0	$\frac{1}{3}$
Total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	**1**

- Plot the pmf for the wireless channel.
- What is the Expected Amplitude and Phase?



$$E[X] = (0{,}01 + 0{,}02 + 0{,}03) \cdot \frac{1}{3} = 0{,}02$$

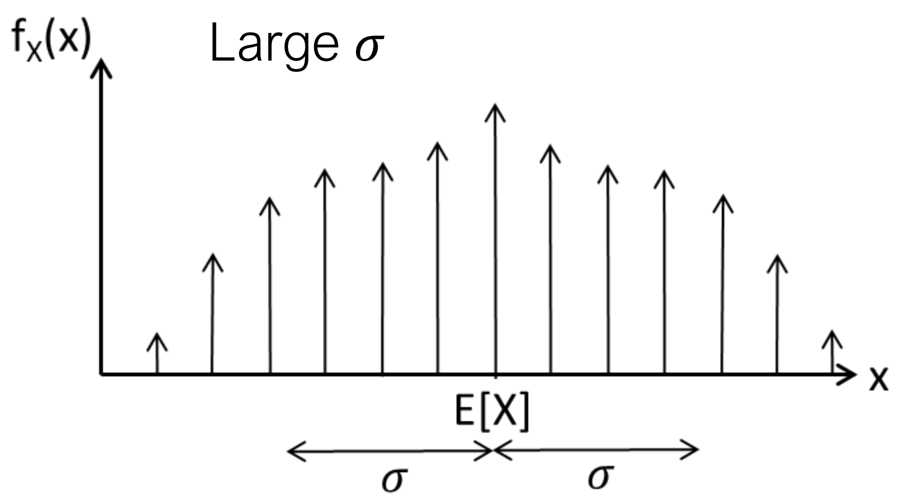
$$E[Y] = (0^o + 45^o + 90^o) \cdot \frac{1}{3} = 45^o$$

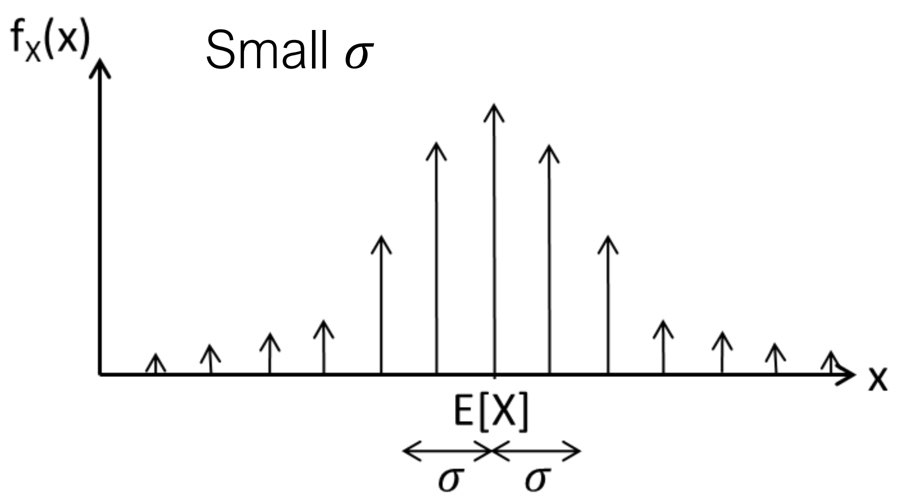
Variance and standard deviation

Variance and standard deviation tells of the spreading of the data

- The variance is an indicator on how much the values of a random variable X are spread around (deviates from) the expectation value.
- The standard deviation σ is the square root of the variance.

$$\boxed{Var(X) = {\sigma_X}^2 = E[X^2] - E[X]^2}$$





Correlation Coefficient

Correlation tells of the coupling between variables

- The correlation coefficient, is an indicator on how much two random variables X and Y are correlated.

$$\boxed{\rho = E\left[\frac{X-\bar{X}}{\sigma_X}\cdot\frac{Y-\bar{Y}}{\sigma_Y}\right] = \frac{E[XY]-E[X]E[Y]}{\sigma_X\sigma_Y}}$$

- We have that: $-1 \leq \rho \leq 1$

Independence

- We have independence between X and Y if and only if:

$$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

Example of independent random variables:

- A persons height and the current exact distance from the earth to the moon.

Example of dependent random variables:

- The time of day and the amount of bicycles parked the at the engineering college.
- The energy of a mobile signal and the length in meters to a basestation.

Independence

Independence: $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

- Bayes Rule: $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

  gives that if X and Y are independent, then:

$$\boxed{f_{X|Y}(x|y) = f_X(x)}$$

- Also:

  $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \Rightarrow E[XY] = E[X]E[Y] \Rightarrow \rho = 0$

  but the opposite is not allways true!

Dependant Variables – Simple Example

- Given a random variable X
- We define a new random variable Y=X

$$f_{X,Y}(1,1) = \frac{1}{2}$$

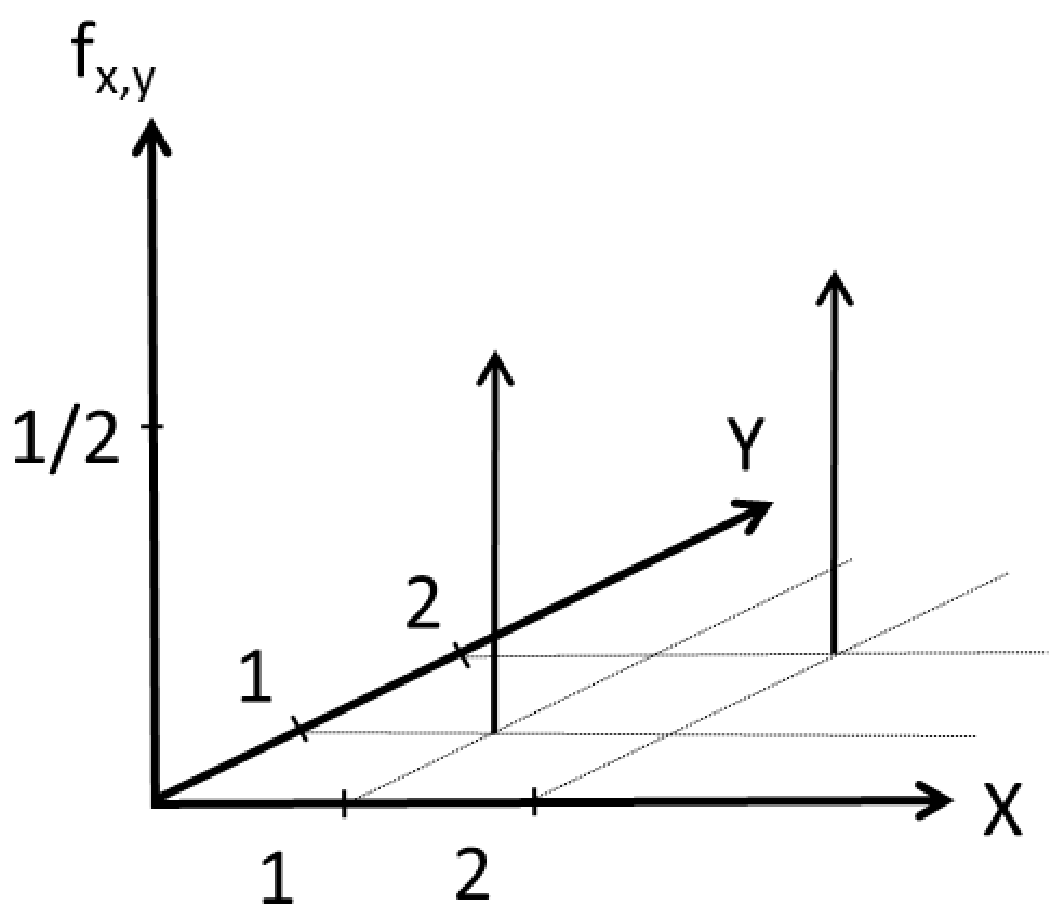
$$f_{X,Y}(2,2) = \frac{1}{2}$$

$$f_{X,Y}(1,2) = 0$$

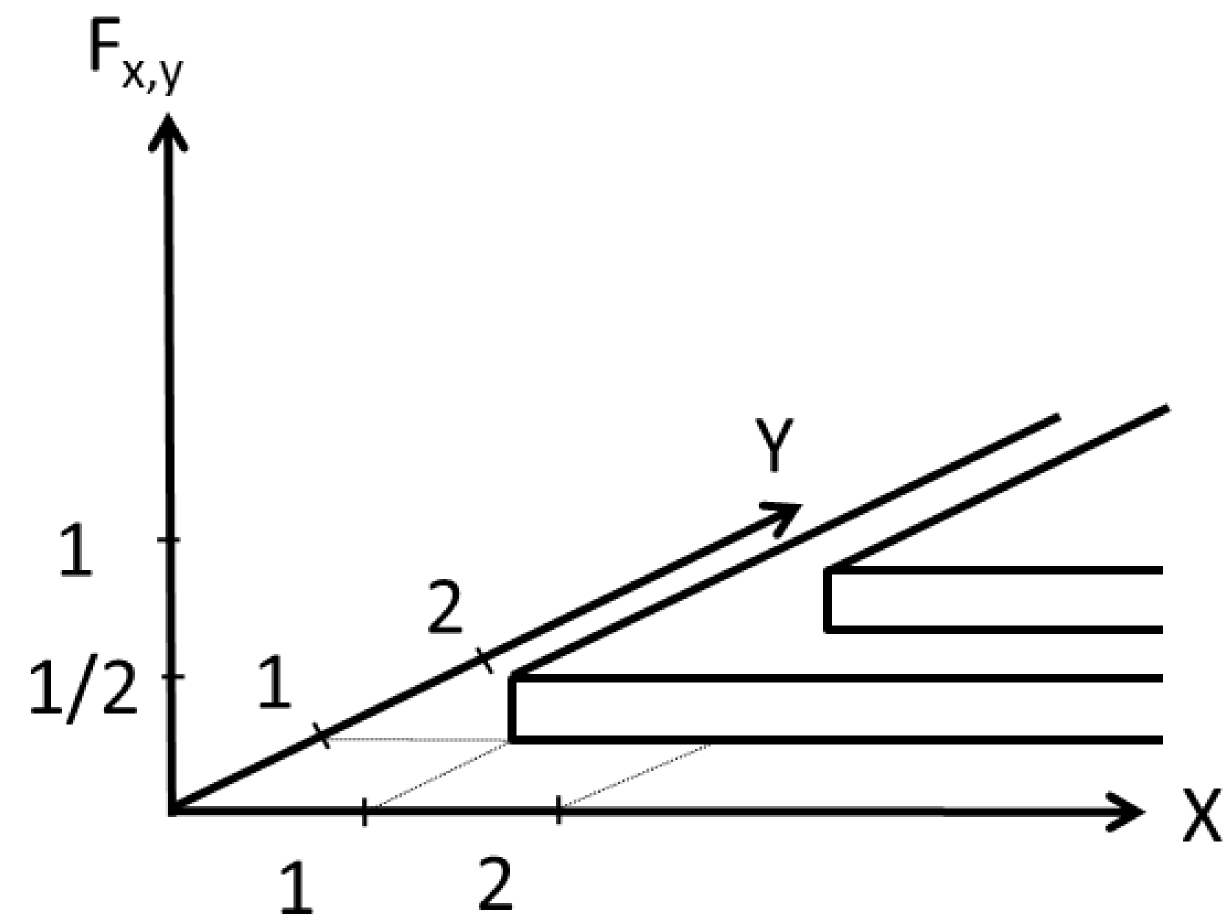
$$f_{X,Y}(2,1) = 0$$

Simple Example - Simultaneous pmf

Plots of the pmf and the cdf:



pmf



cdf

Simple Example – Marginal pmf

$$f_Y(y) = \sum_x f_{X,Y}(x,y)$$

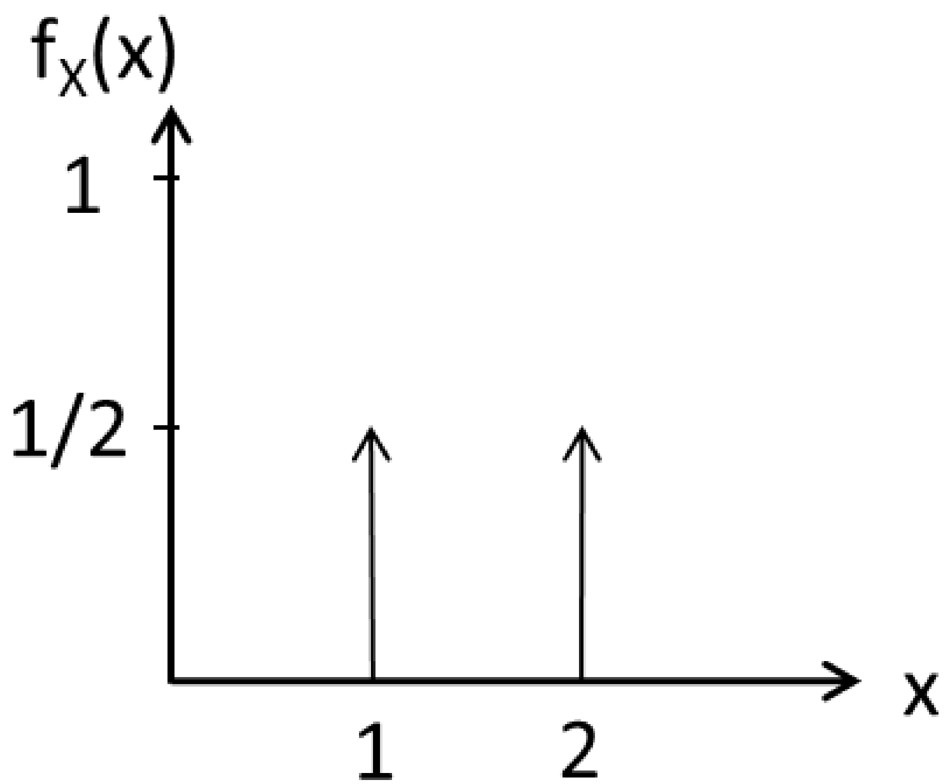
$$f_Y(1) = f_{X,Y}(1,1) + f_{X,Y}(2,1) = \frac{1}{2}$$

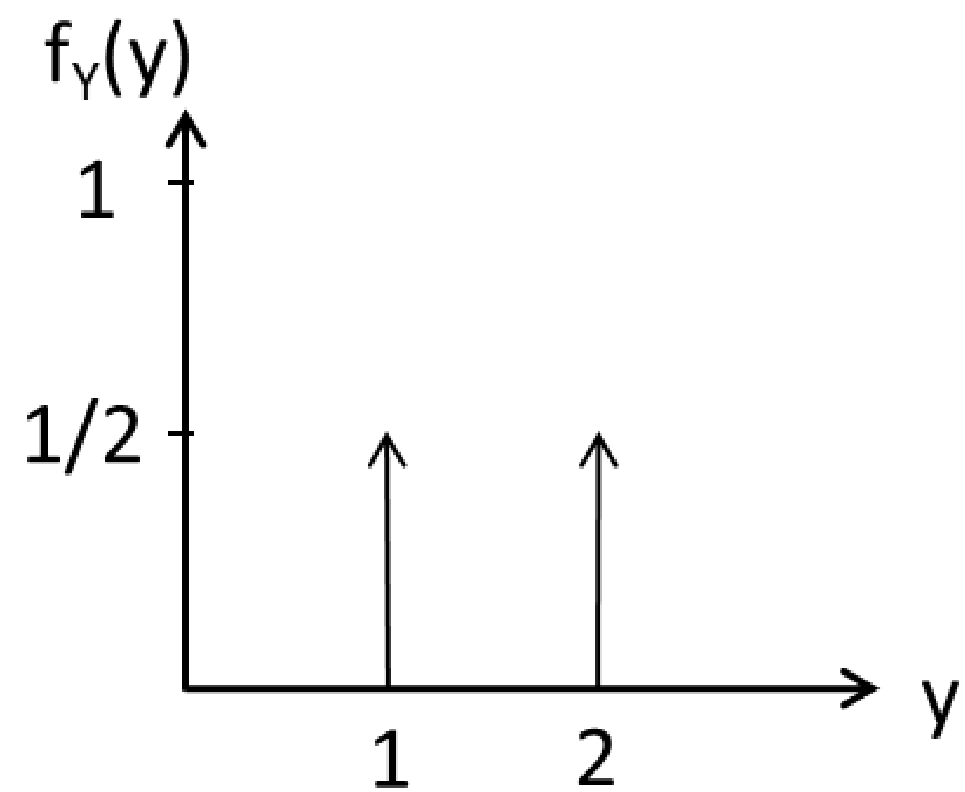
$$f_Y(2) = f_{X,Y}(1,2) + f_{X,Y}(2,2) = \frac{1}{2}$$

$$f_X(x) = \sum_y f_{X,Y}(x,y)$$

$$f_X(1) = f_{X,Y}(1,1) + f_{X,Y}(1,2) = \frac{1}{2}$$

$$f_X(2) = f_{X,Y}(2,1) + f_{X,Y}(2,2) = \frac{1}{2}$$





Dependant Variables – Simple Example

- Are X and Y independent?

$$f_{X,Y}(1,1) = \frac{1}{2} \neq \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = f_X(1) \cdot f_Y(1)$$

$$f_{X,Y}(1,2) = 0 \neq \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = f_X(1) \cdot f_Y(2)$$

...

- No, X and Y are not independent!

Words and Concepts to Know

Stochastic

Cumulative Distribution Function

Marginal

Probability Mass Function

Correlation coefficient

Simultanious pmf

cdf

Joint pmf

pmf

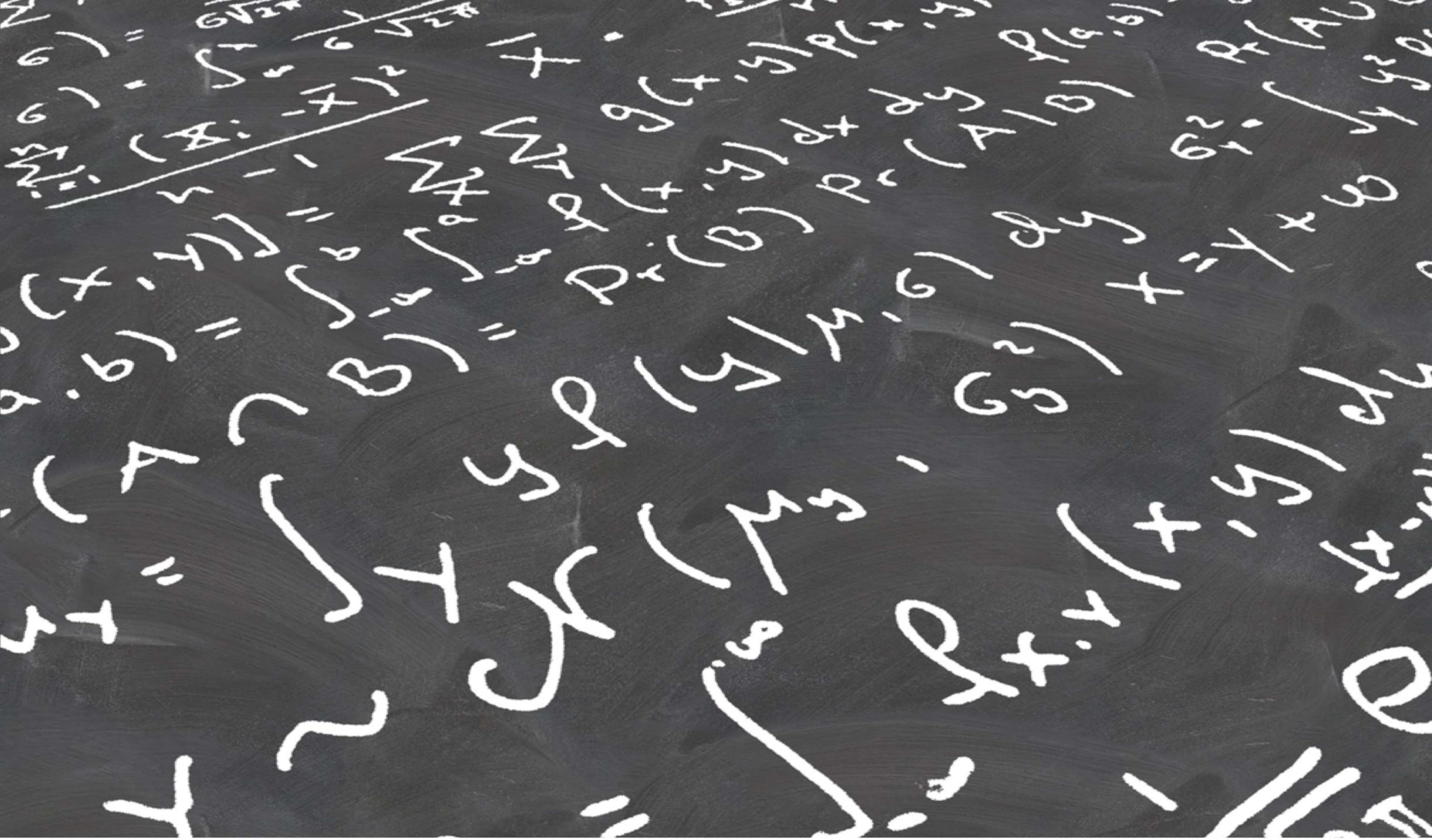
Standard deviation

Binomial Mass Function

Mean

Variance

Expectation



4. Continuous Random Variables

Gunvor Elisabeth Kirkelund
Lars Mandrup

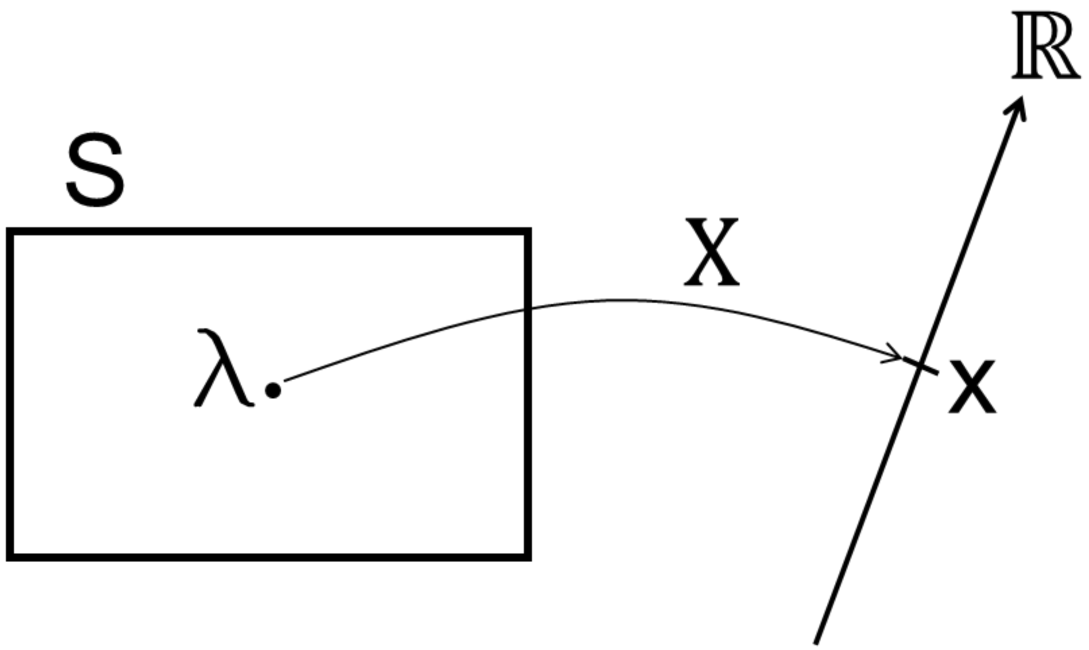
Agenda for Today

- Repetition from last time
  - Discrete Random Variables
- Continuous Random Variables

Also just called a random variables

Stochastic Random Variables

- A random variable tells something important about a stochastic experiment.
- Can be discrete or continous



Examples:

- The numbers on a dice (discrete):
  - Sample space for variable X is : $\{1, 2, 3, 4, 5, 6\}$
  - Sample space for variable Y “Even (1)/Uneven (-1)”: $\{1, -1\}$
- The hight of students at IHA (continous):
  - Sample space for variable H is all real numbers: [100;250] cm.

Probability Mass Function (PMF)

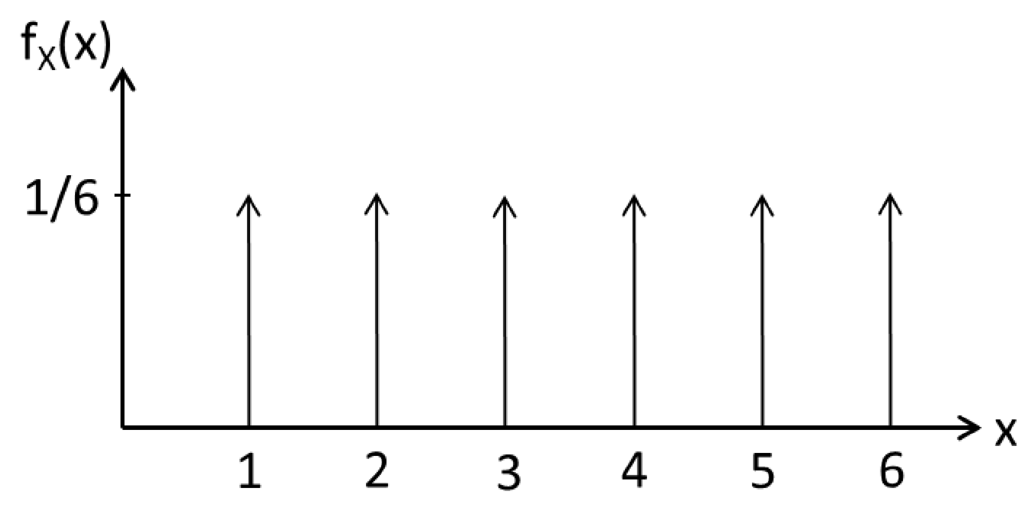
- Sample space for X.
- X is a discreet stochastic variable.

$$\boxed{f_X(x) = \begin{cases} Pr(X = x_i) & for\ X = x_i \\ 0 & otherwise \end{cases}}$$

$$0 \le f_X(x) \le 1$$

- We have that: $\sum_{i=1}^{n} f_X(x_i) = \sum_{i=1}^{n} Pr(X = x_i) = 1$

Example: Laplace Dice
(perfect dice)



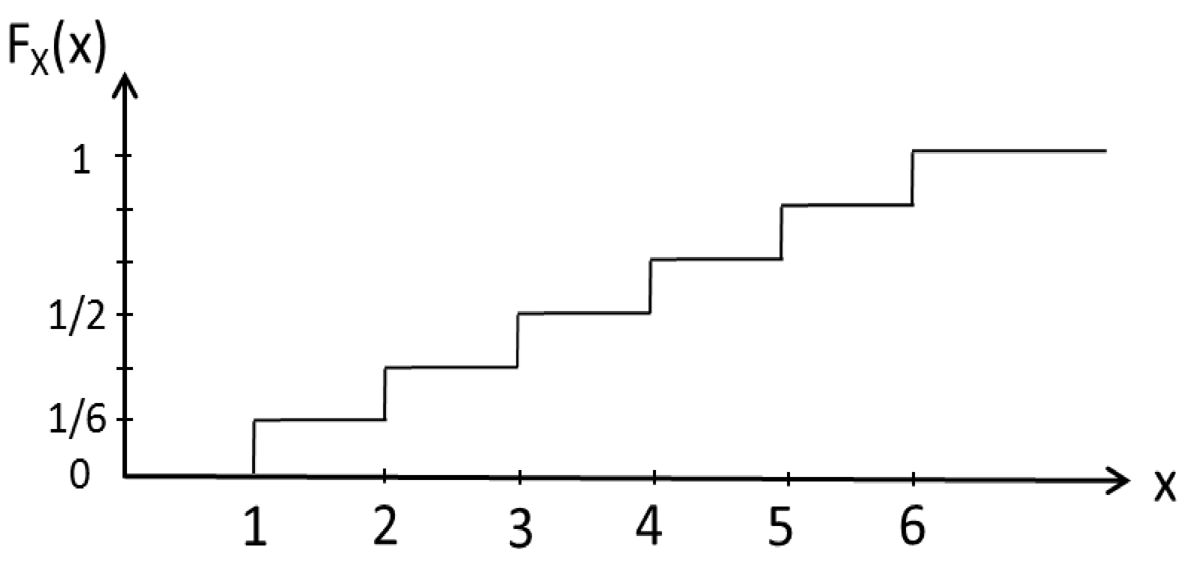
Cumulative Distribution Function (CDF)

- Sample space for X.
- X is a discreet stochastic variable.
- $F_X(x)$ is a non-decreasing step-function.

$$\boxed{F_X(x) = Pr(X \leq x)} \qquad 0 \leq F_X(x) \leq 1$$

- We have that: $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$

Example: Laplace Dice (perfect dice)



Mean, Variance and Standard deviation

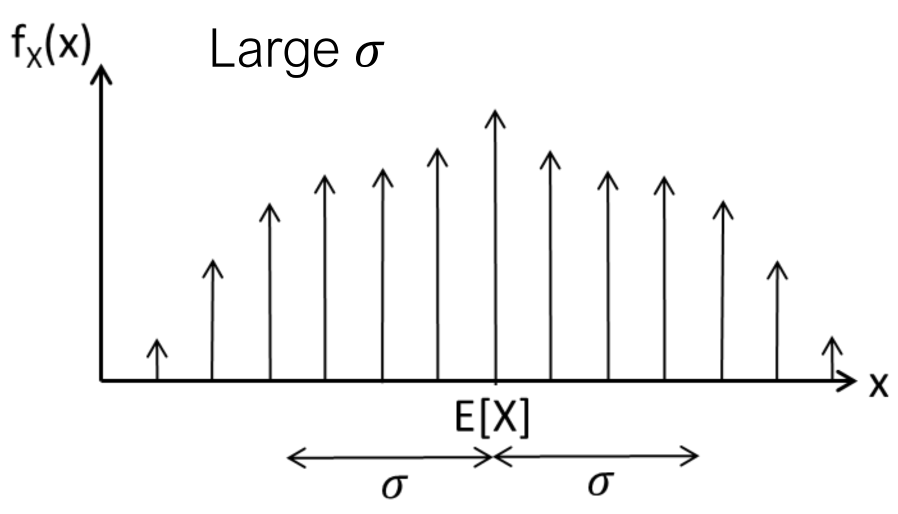
- The <u>mean</u> or the <u>expectation</u> of a discreet random variable X

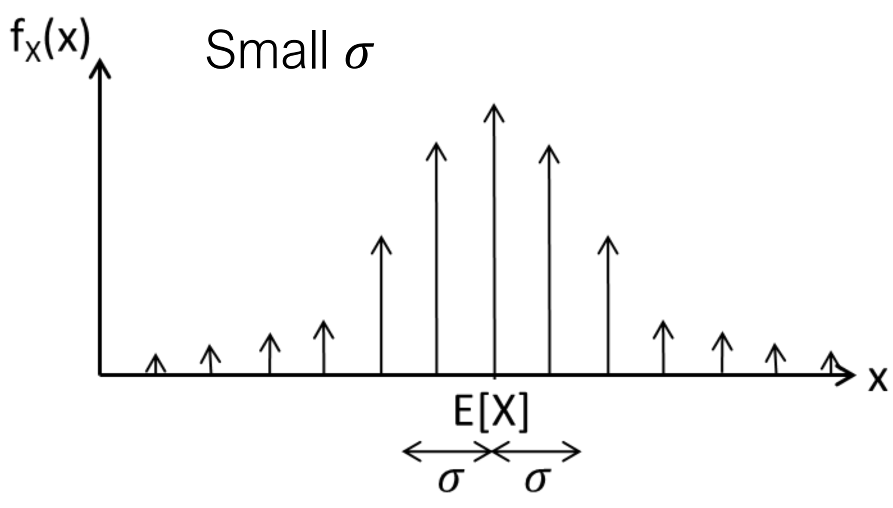
$$\boxed{\bar{X} = E[X] = \sum_{i=1}^{n} x_i\, f_X(x_i)}$$

Variance and standard deviation tells of the spreading of the data

- The <u>variance</u> σ^2 or the <u>standard deviation</u> σ of a random variable X

$$\boxed{Var(X) = {\sigma_X}^2 = E[X^2] - E[X]^2}$$

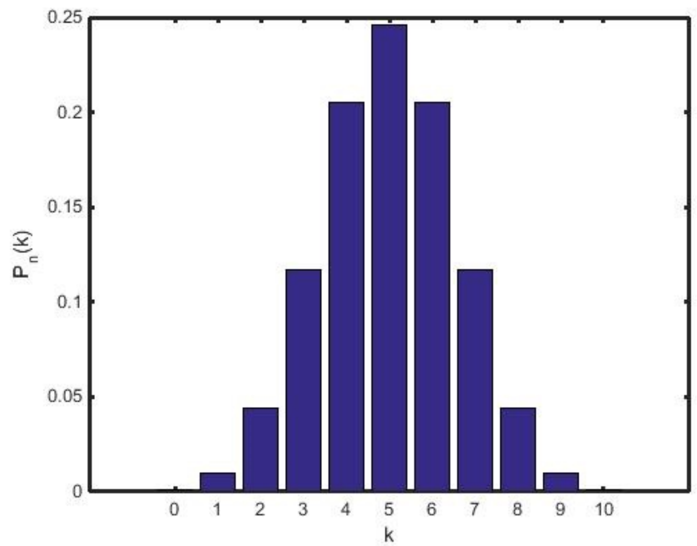




The Binomial Distribution

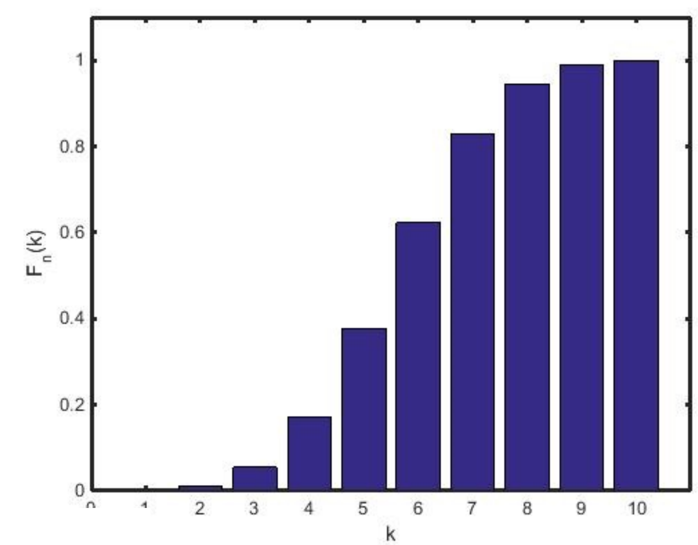
- n repeated trials – each with two possible outcomes
  - Success — probability p
  - Failure — probability 1-p

Also called a Bernoulli trial



- Probability mass function (pmf):

$$f(k|n,p) = \frac{n!}{k!\,(n-k)!}p^k(1-p)^{n-k}$$



- Cumulative distribution function (cdf):

$$F(k|n,p) = \sum_{i=0}^{k} f(i|n,p)$$

- Mean and variance:

$$E[k] = n \cdot p$$
$$Var(X) = n \cdot p \cdot (1-p)$$

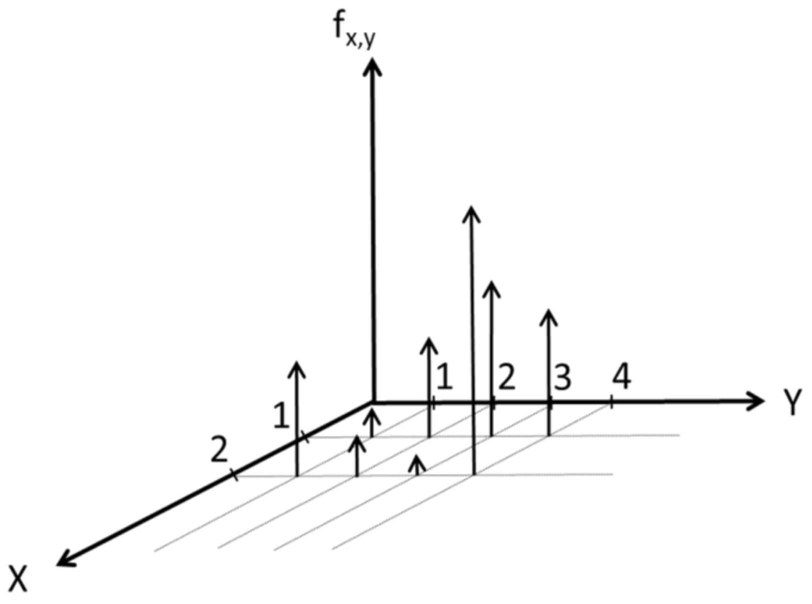
Two Simultaneous Discreet Random Variables

Joint (Simultaneous) pmfs:

$$f_{X,Y}(x,y) = \begin{cases} Pr\Big((X = x_i\,)\cap\big(Y = y_j\big)\Big) & for\ \ X = x_i\ \wedge\ Y = y_j \\ \\ 0 & otherwise \end{cases}$$

Marginal pmfs:

$$f_X(x) = \sum_y f_{X,Y}(x,y) \qquad f_Y(y) = \sum_x f_{X,Y}(x,y)$$



Conditional pmfs / Bayes Rule:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = Pr(X = x|Y = y)$$

Correlation Coefficient

Correlation tells of the coupling between variables

- The correlation coefficient, is an indicator on how much two random variables X and Y are correlated.

$$\boxed{\rho = E\left[\frac{X-\bar{X}}{\sigma_X}\cdot\frac{Y-\bar{Y}}{\sigma_Y}\right] = \frac{E[XY]-E[X]E[Y]}{\sigma_X\sigma_Y}}$$

- We have that: $-1 \leq \rho \leq 1$

Independence

Independence: $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

- Bayes Rule: $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

  gives that if X and Y are independent, then:

$$\boxed{f_{X|Y}(x|y) = f_X(x)}$$

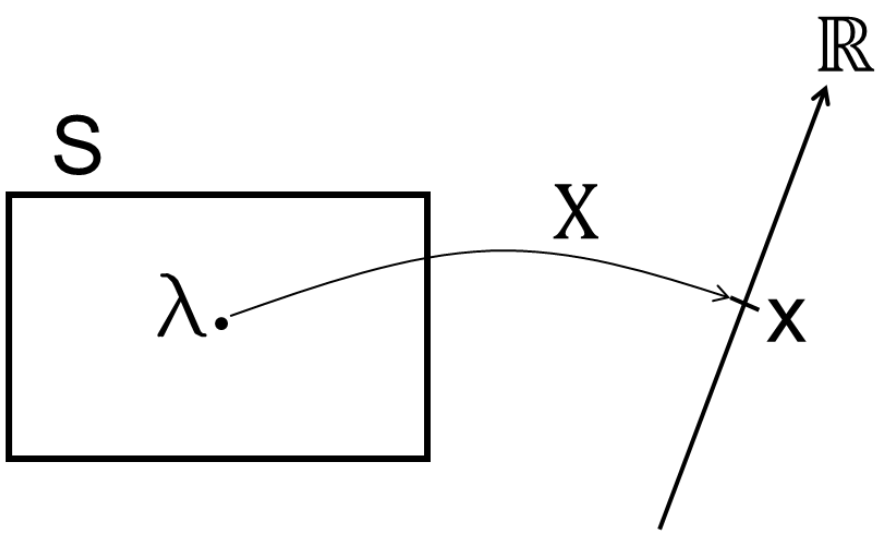
- Also:

  $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \Rightarrow E[XY] = E[X]E[Y] \Rightarrow \ \rho = 0$

  but the opposite is not allways true!

Continuous Random Variables

- We define a stochastic variable X
- X is continuous on $\mathbb{R}$

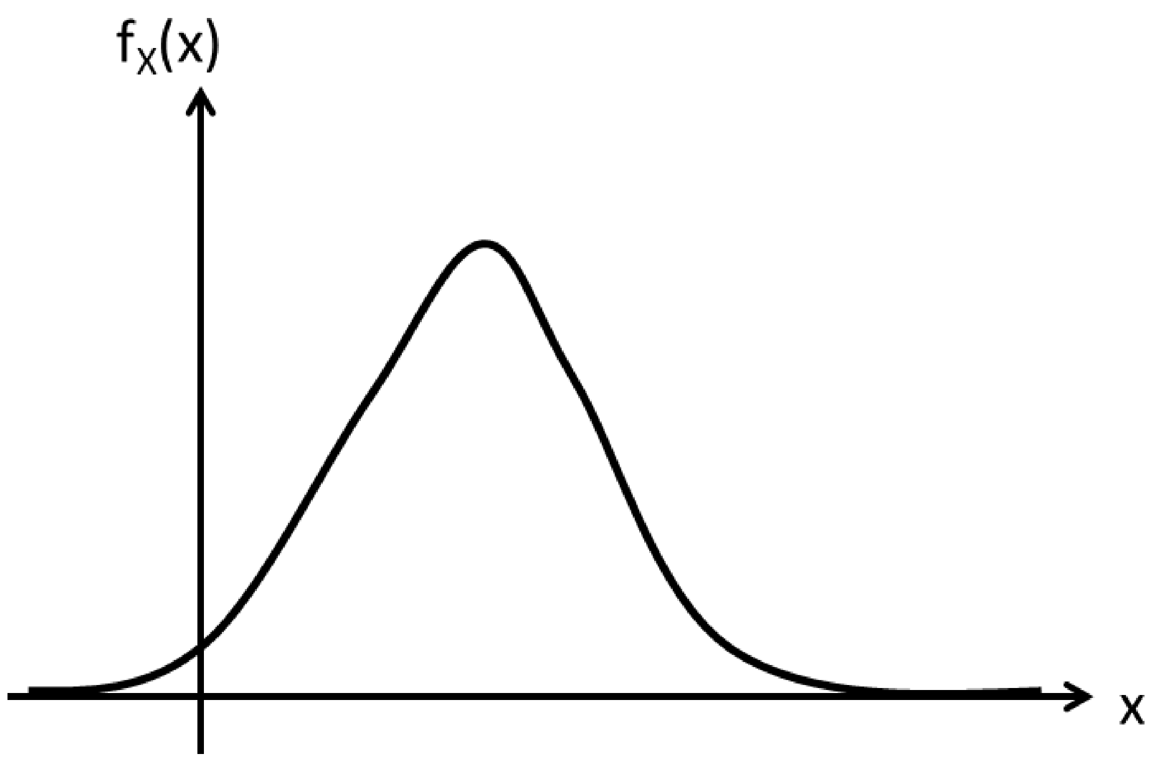


- Fx. The exact value R of a resistor

- X is defined by a density function $f_X(x)$

- The probability of one instance of the variable is always 0:

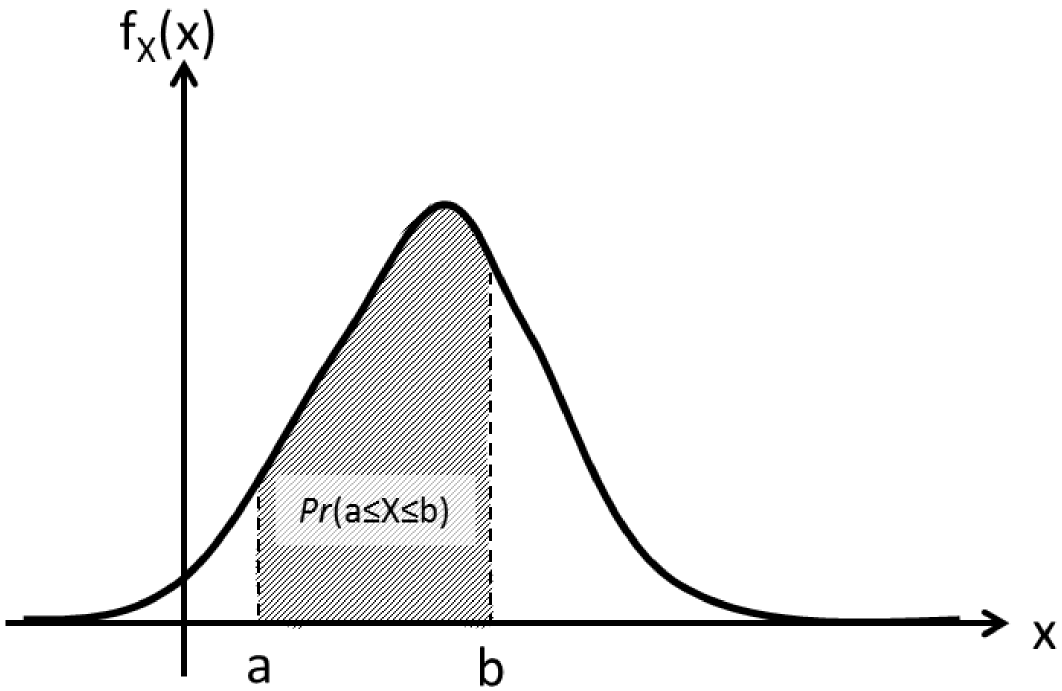
$$Pr(X = x) = 0$$



Continous Random Variables — PDF

- We define a probability density function (pdf): $f_X(x)$

$$\boxed{Pr(a \leq X \leq b) = \int_a^b f_X(x)\,dx}$$



Properties: $f_X(x) \geq 0$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

Total probability is 1.

Notice: $f_X(x) > 1$ is possible

$$Pr(X = x) = 0$$

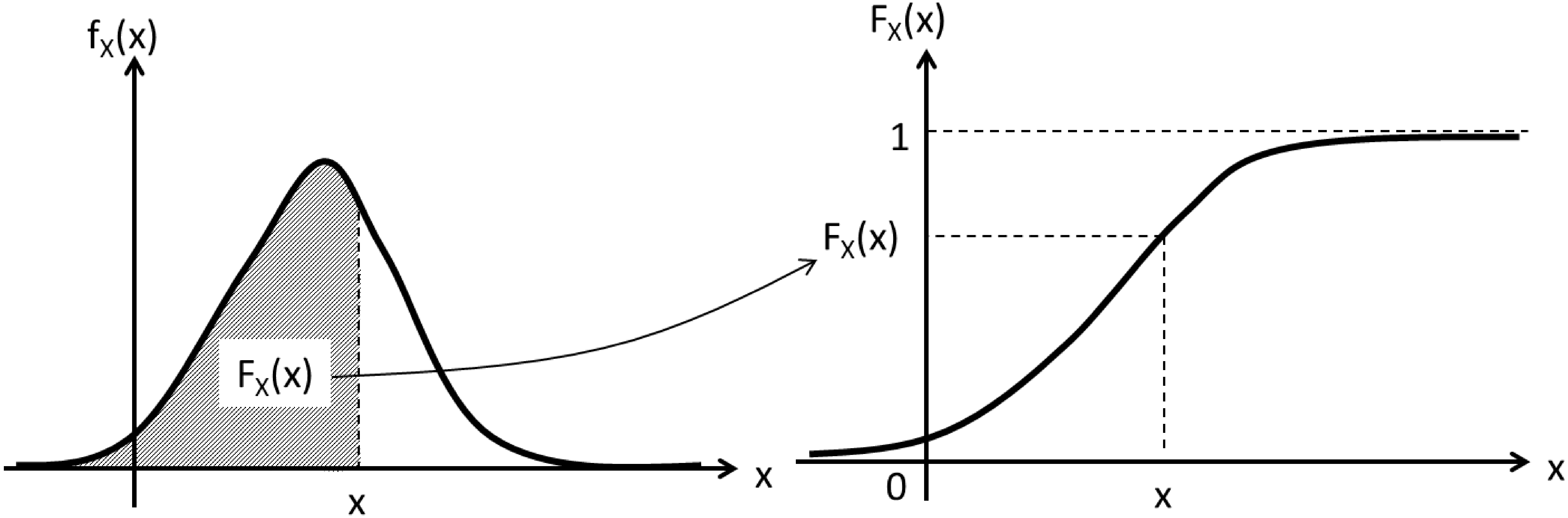
$$Pr(a < X < b) = Pr(a \leq X < b) = Pr(a < X \leq b) = Pr(a \leq X \leq b)$$

Cumulative Distribution Function (CDF)

- We define a cumulative distribution function (cdf): $F_X(x)$

Accumulates the probabilities from minus infinite to x.

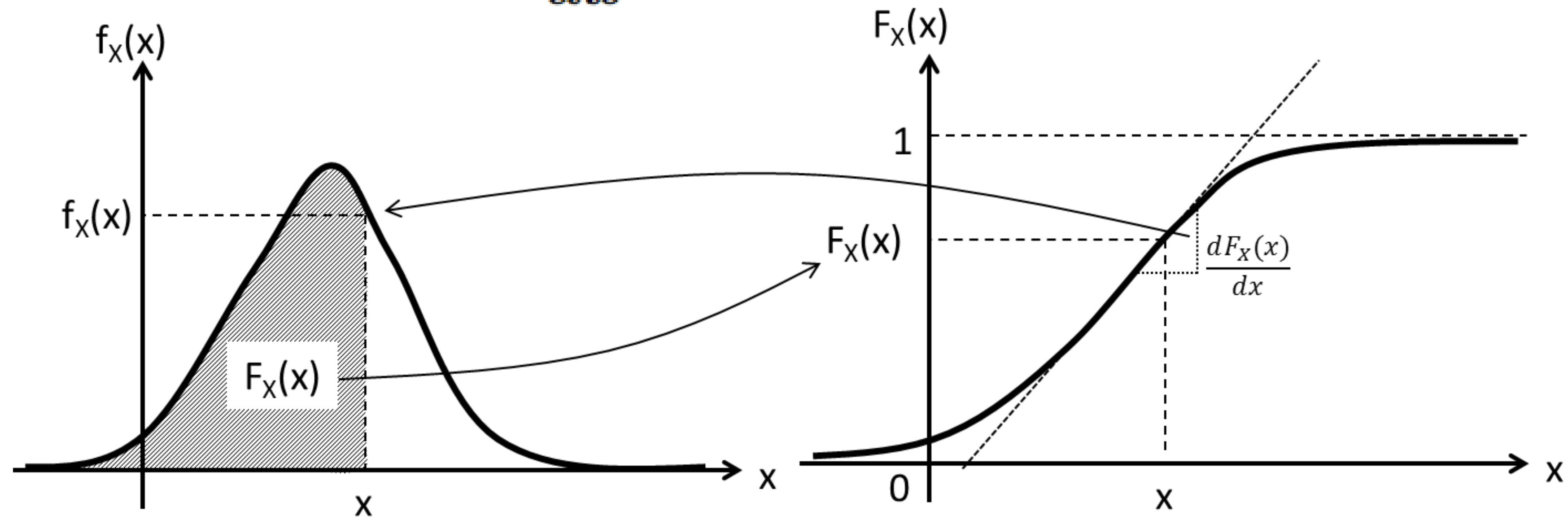
$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(u)\, du = Pr(X \leq x)}$$



The cdf and pdf contains the same information.

Cumulative Distribution Function (CDF)

- From pdf to cdf: $$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du = Pr(X \le x)$$
- From cdf to pdf: $$f_X(x) = \frac{dF_X(x)}{dx}$$



Properties:

- $0 \le F_X(x) \le 1$
- $F_X(x)$ is always non−decreasing and continuous
- $Pr(a \le X \le b) = \int_a^b f_X(x)dx = F_X(b) - F_X(a)$
- $Pr(X > x) = 1 - Pr(X \le x) = 1 - F_X(x)$

Definition of Expectation

- We define the expectation of $g(X)$ with respect to a pdf $f_X(x)$ as the integral:

$$\boxed{E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx}$$

Example:

- DC voltage with a noise-signal.

Mean Value

- The mean value is the expectation of X:

$$\boxed{E[X] = \bar{X} = \mu_X = \int_{-\infty}^{\infty} x \cdot f_X(x)dx}$$

Example:

- The value of 5% 1kΩ resistors.

Expectation

- Linear function: $g(X) = aX + b$

$$E[aX + b\ ] = \int_{-\infty}^{\infty} (ax + b) \cdot f_X\,(x)dx = a \cdot E[X] + b$$

- Square function: $g(X) = X^2$

$$E[g(X)] = E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X\,(x)dx$$

$$\neq \left(\int_{-\infty}^{\infty} x \cdot f_X\,(x)dx\right)^2 = E[X]^2$$

Definition of Variance

- We define the variance of $g(X)$ with respect to a pdf $f_X(x)$ as the integral:

$$\boxed{\begin{aligned} Var\big(g(\mathrm{X})\big) &= \int_{-\infty}^{\infty} \big(g(x) - \overline{g(x)}\big)^2 \cdot f_X(x)dx \\ &= E[g(X)^2] - E[g(X)]^2 \end{aligned}}$$

- The variance of a continuous random variable X:

$$Var(\mathrm{X}) = \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot f_X(x)dx = E[X^2] - E[X]^2$$

Variance

- Linear function: $g(X) = aX + b$

$$Var[aX + b\ ] = E[(aX + b)^2] - E[aX + b]^2$$

$$= \int_{-\infty}^{\infty} (ax + b)^2 \cdot f_X\,(x)dx - (a \cdot E[X] + b\ )^2$$

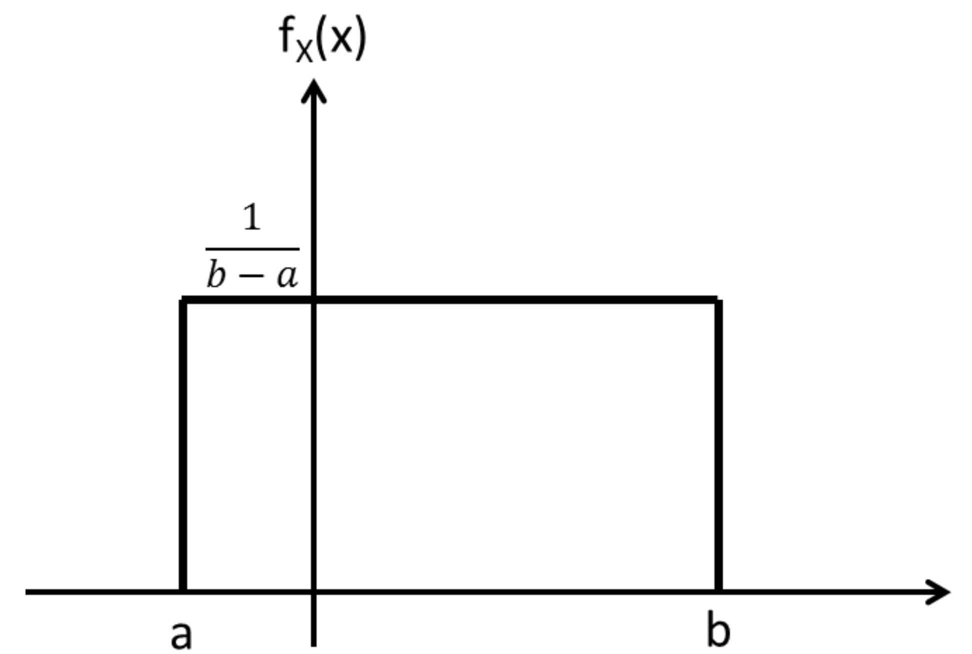
$$= (a^2E[X^2] + b^2 + 2abE[X]) - (a^2E[X]^2 + b^2 + 2abE[X])$$

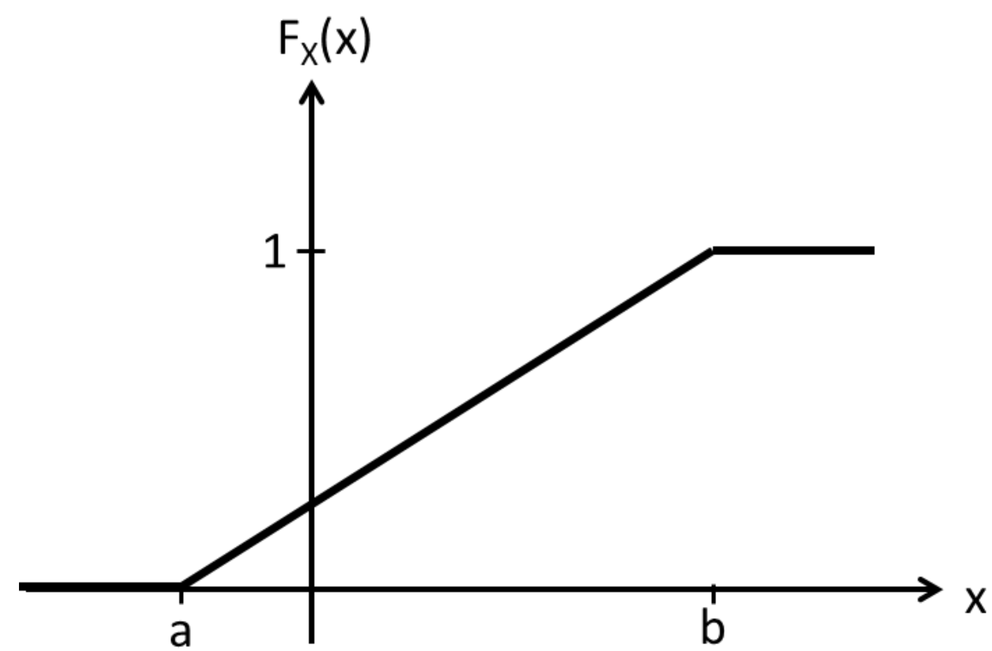
$$= a^2(E[X^2] - E[X]^2)$$

$$= a^2 \cdot Var(X)$$

Uniform Distribution

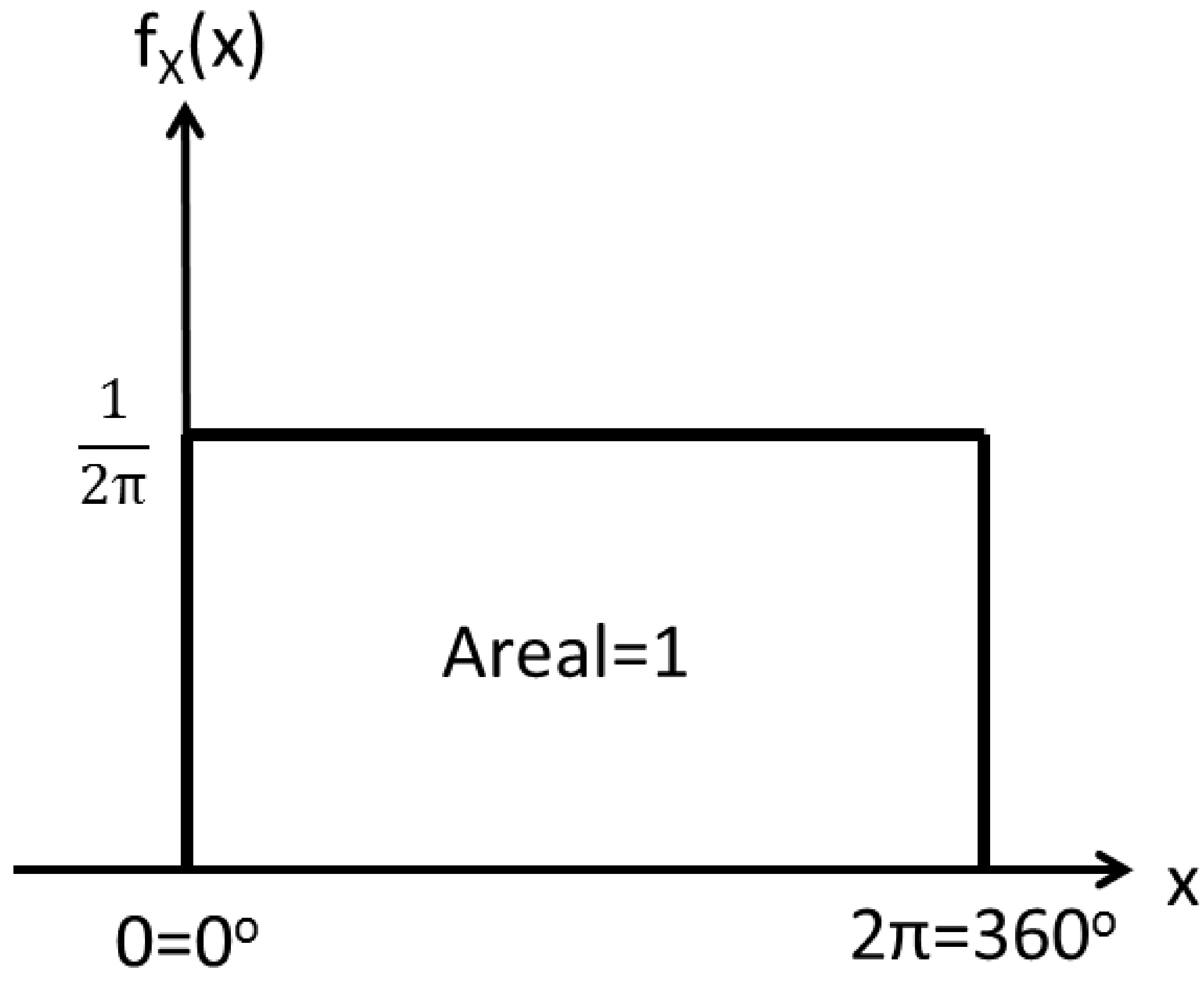
- $\mathcal{U}$(a,b)
- Mean value: $\mu = \frac{a+b}{2}$
- Variance: $\sigma^2 = \frac{1}{12}(b-a)^2$
- pdf: $f_X(x) = \begin{cases} \frac{1}{b-a} & for\ a \le x \le b \\ 0 & otherwise \end{cases}$
- cdf: $F_X(x) = \begin{cases} 0 & for\ x \le a \\ \frac{x-a}{b-a} & for\ a \le x \le b \\ 1 & for\ x \ge b \end{cases}$



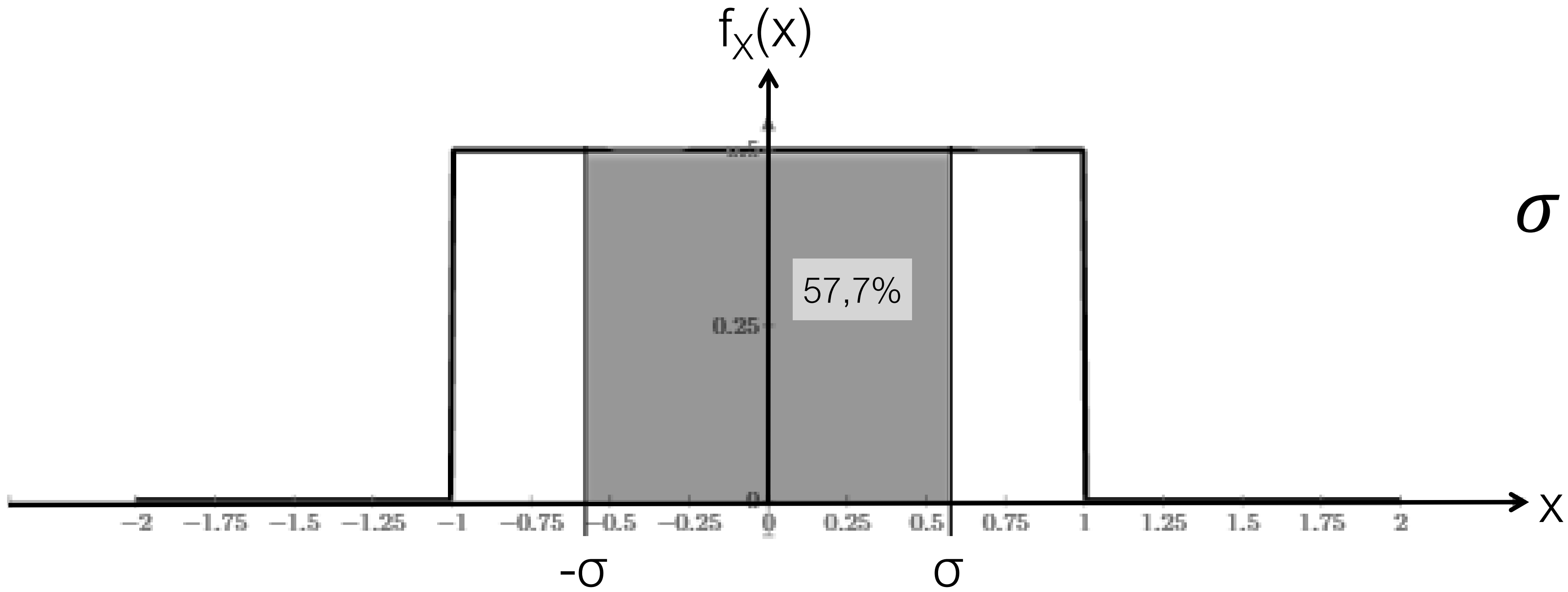


Uniform Distribution — Example

- A phase noise is uniformly distributed.



Uniform Distribution: Standard deviation



$$\sigma = \frac{b-a}{\sqrt{12}}$$

$$\Pr(|X-\mu| \leq \sigma) = 57{,}7\%$$

$$\Pr(|X-\mu| \leq 2\sigma) = 100\%$$

Gaussian Distribution = Normal Distribution

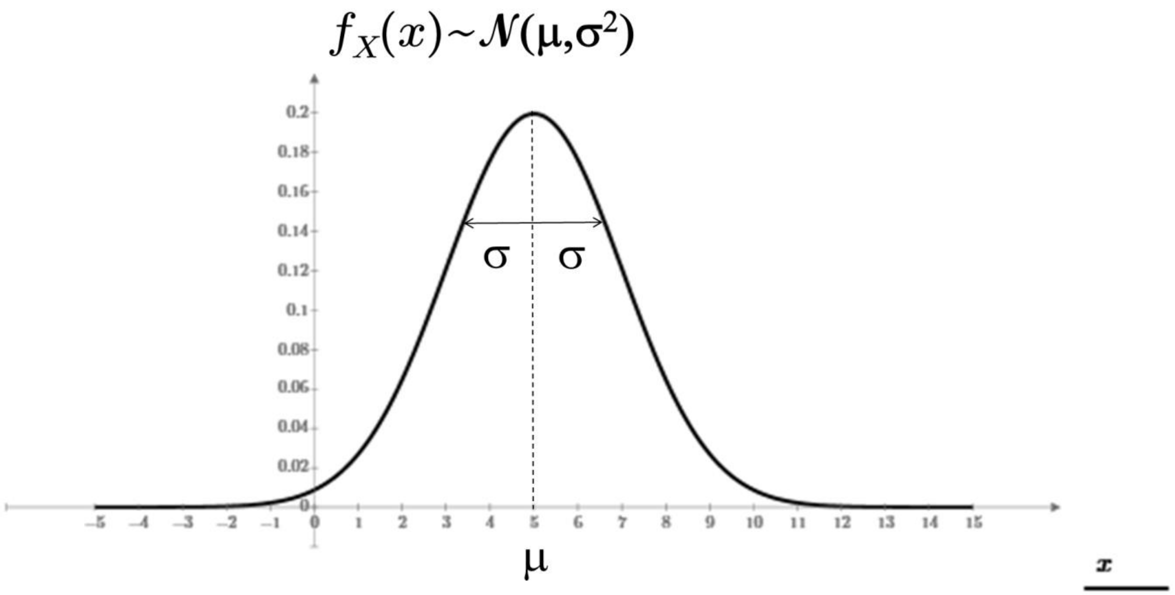
- $\mathcal{N}(\mu,\sigma^2)$
- Mean value: μ
- Variance: σ^2

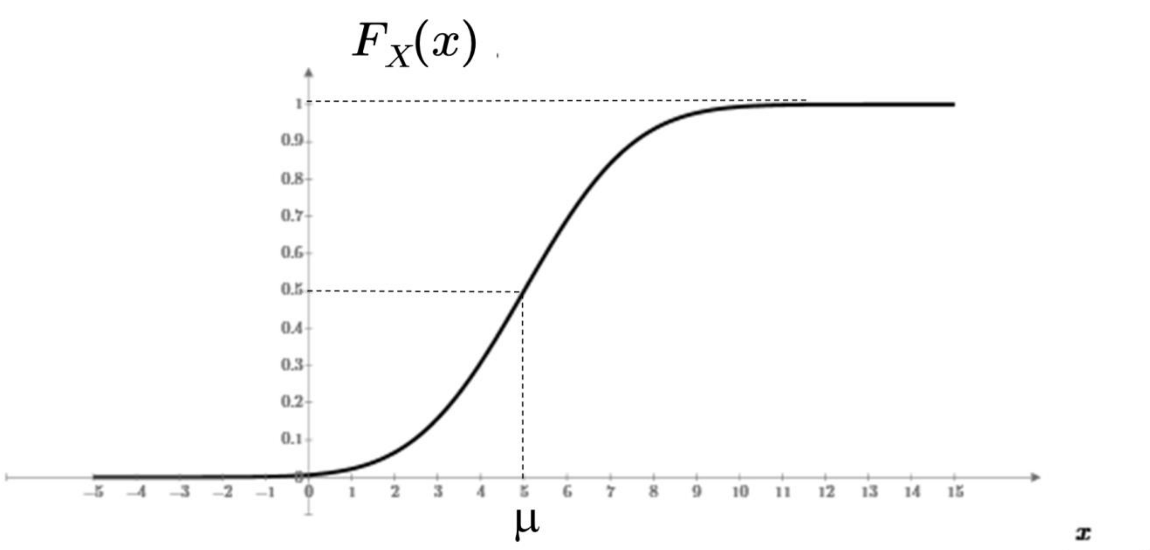
- pdf: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

- cdf: $F_X(x) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$

No closed expression for the cdf

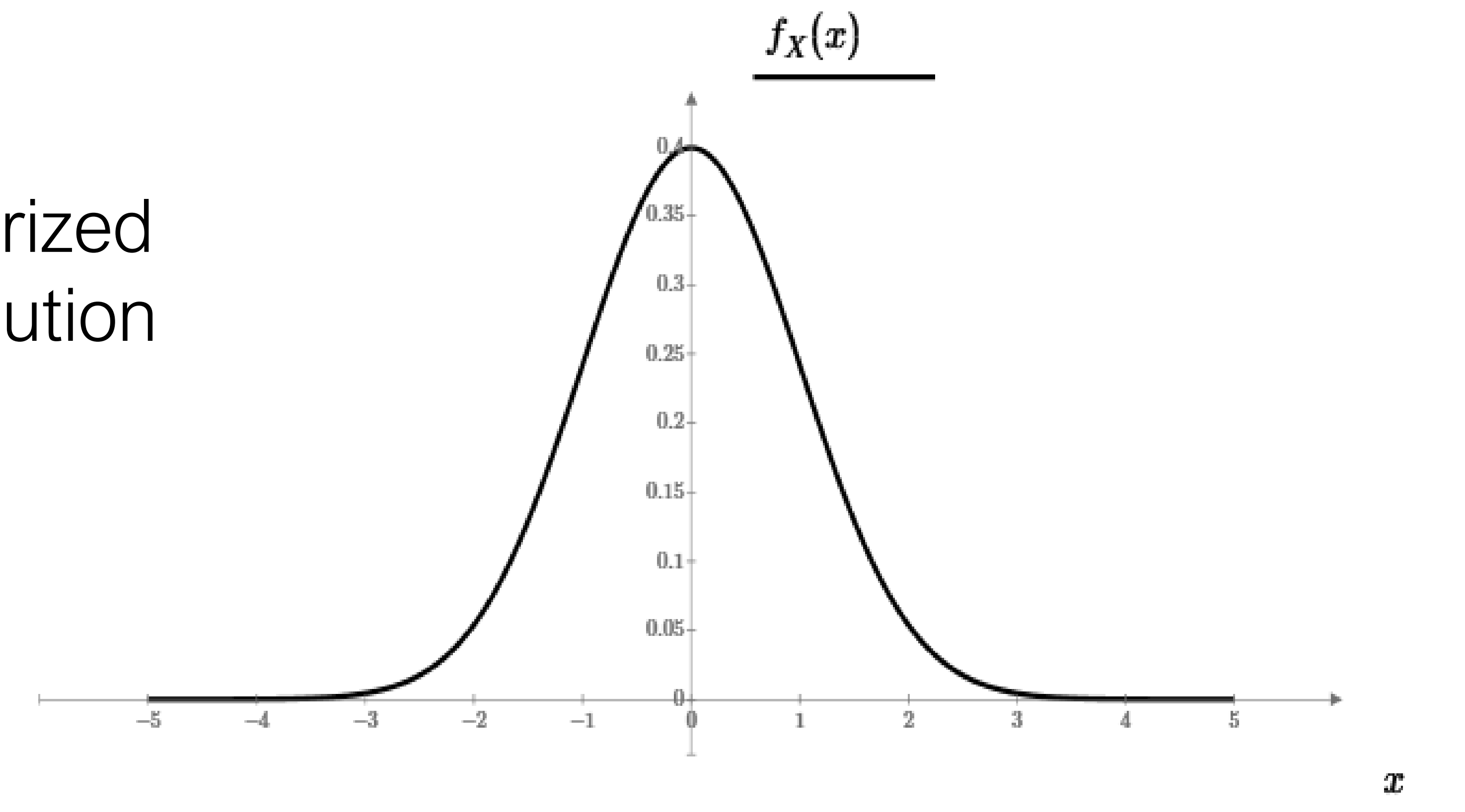
erf= error-function: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$





Gaussian Distribution = Normal Distribution

$\mathcal{N}(0,1)$
→ the standarized normal distribution

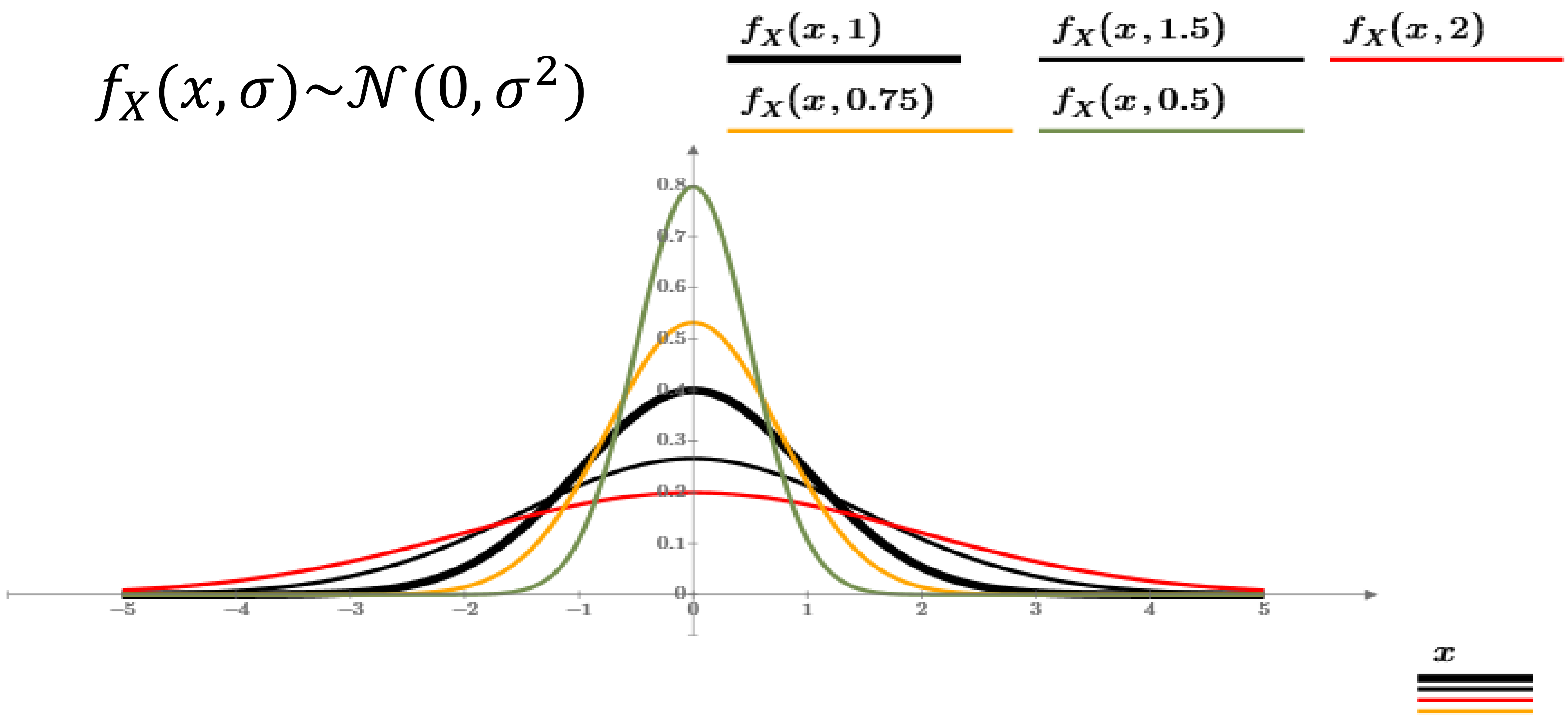


- A lot of things in nature are Gaussian distributed
  - Fx. Examination marks
- Central Limit Theorem → Gaussian distrubution

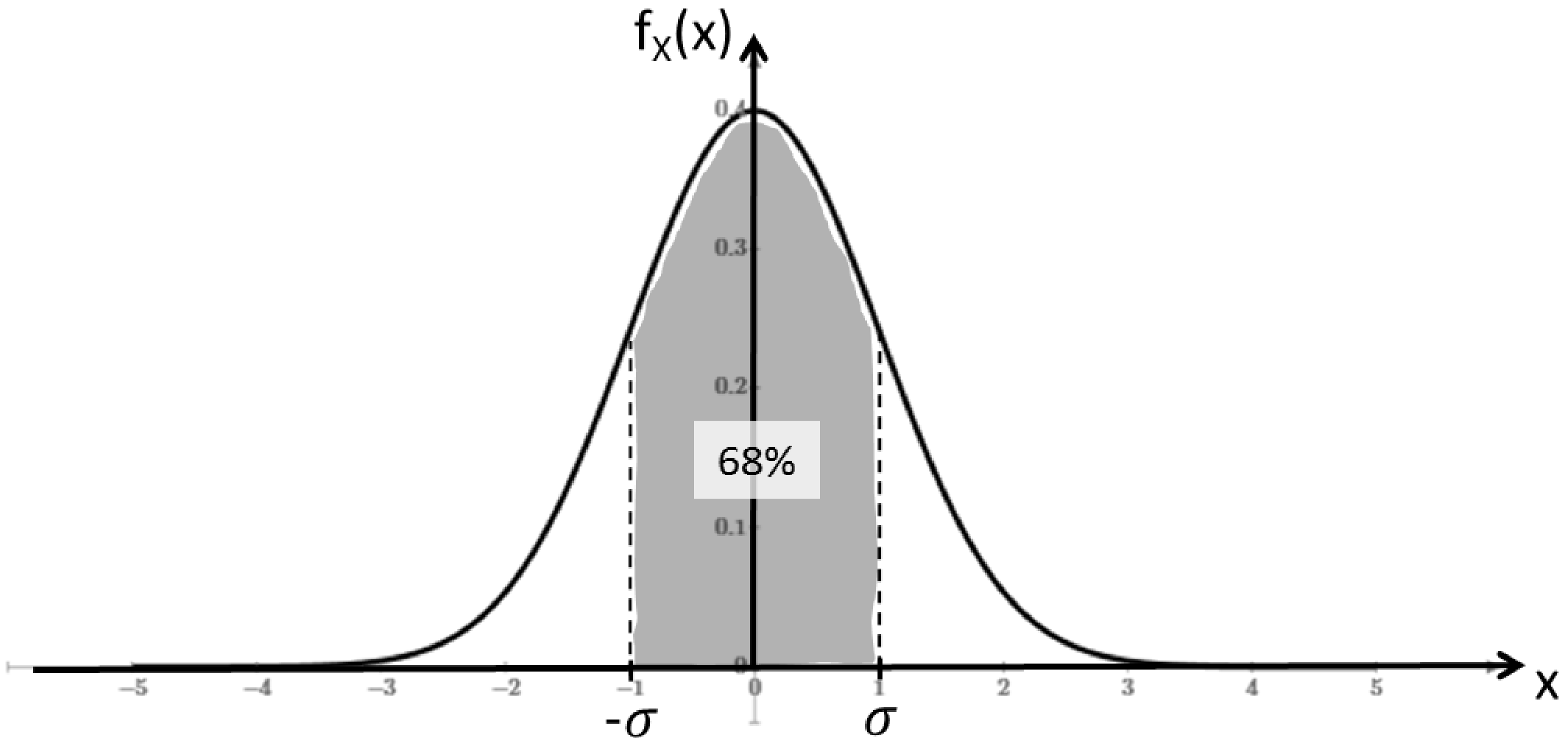
Gaussian Distribution = Normal Distribution

- Maximum probability density at the mean value μ
- The standard deviation (variance) σ determines the form (width and hight)

$$f_X(x,\sigma) \sim \mathcal{N}(0,\sigma^2)$$



Normal Distribution: Standard Deviation



$$\Pr(|X-\mu| \leq \sigma) = 68{,}3\%$$

$$\Pr(|X-\mu| \leq 2\sigma) = 95{,}4\%$$

$$\Pr(|X-\mu| \leq 3\sigma) = 99{,}7\%$$

Gaussian Distribution = Normal Distribution

- Beregninger med normalfordelinger: Tabelopslag og Matlab:
- $X \sim \mathcal{N}(\mu, \sigma^2) \;\rightarrow\; Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$ (Standard Normal Distribution)
- $F_X(x) = Pr(X \le x) = Pr\left(Z \le \frac{x-\mu}{\sigma}\right) = F_Z(z)$ hvor $z = \frac{x-\mu}{\sigma}$

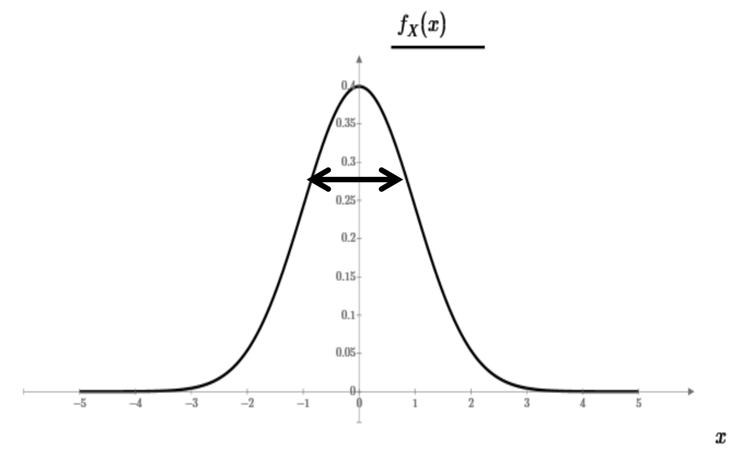
$$= \begin{cases} \Phi(z) & Tabel\ 1\ (\text{"}Statistik\ og\ Sandsynlighedsregning\text{"}) \\ 1 - \mathrm{Q}(z) & App.D\ (\text{"}Random\ Signals\text{"}) \end{cases}$$

- $\Phi(z) = Pr(Z \le z)$
- $\Phi(-z) = 1 - \Phi(z)$
- $\mathrm{Q}(z) = Pr(Z \ge z) = 1 - Pr(Z \le z) = 1 - \Phi(z)$
- $\mathrm{Q}(-z) = 1 - \mathrm{Q}(z)$

- Matlab:
  - $\Pr(X \le x) = F_X(x) = normcdf(x, \mu, \sigma)$
  - $\Pr(Z \le z) = F_Z(z) = normcdf(z, 0, 1) = normcdf(z)$

Summary of Expectations

- Mean value: $E[X] = \bar{X} = \mu_X = \int_{-\infty}^{\infty} x \cdot f_X(x)dx \quad (\sum_{i=1}^{n} x_i\, f_X(x_i))$
- Mean square: $E[X^2] = \overline{X^2} = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)dx \quad (\sum_{i=1}^{n} x_i^2\, f_X(x_i))$
- Variance: $Var(\mathrm{X}) = {\sigma_X}^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot f_X(x)dx = E[X^2] - E[X]^2$



- Standard deviation: $\sigma_X = \sqrt{Var(\mathrm{X})}$
- A function: $E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx \quad (\sum_{i=1}^{n} g(x_i)\, f_X(x_i))$

$$Var\big(g(\mathrm{X})\big) = \int_{-\infty}^{\infty} \left(g(x) - \overline{g(x)}\right)^2 \cdot f_X(x)dx = E[g(X)^2] - E[g(X)]^2$$

- Linear function: $E[aX + b\,] = a \cdot E[X] + b$

$$Var[aX + b\,] = a^2(E[X^2] - E[X]^2) = a^2 \cdot Var(X)$$

Two Stochastic Variables X,Y

- The simultaneous (joint) density function
- The marginal probability density function
- Bayes rule

- Discreet → Continous stochastic random variable

$$\sum \rightarrow \int$$

Continuous Random Variables

- We have a simultaneous (joint) pdf: $f_{X,Y}(x, y)$

- We have the probability:

$$Pr\big((a \leq X \leq b) \cap (c \leq Y \leq d)\big) = \int_c^d \int_a^b f_{X,Y}(x, y)dxdy$$

- We have for the pdf: $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy = 1$$

$$0 \leq f_{X,Y}(x, y)$$

The Marginal PDF

- For a two dimensional pdf $f_{X,Y}(x,y)$, we can find the marginals

Marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

Relationship between pdf and cdf

- For a two dimensional pdf $f_{X,Y}(x,y)$, the cdf and the pdf correspond to each other

cdf $$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(x,y)dxdy = Pr(X \leq x \wedge Y \leq y)$$

pdf $$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

The Conditional PDF

- For a two dimensional pdf $f_{X,Y}(x,y)$, we can find the conditional pdf with Bayes rule:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Independence:

- X and Y are independent if:

$$f_{X|Y}(x|y) = f_X(x) \qquad f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

Correlation

Correlation tells of the (biased) coupling between variables

- Correlation:

$$corr(X,Y) = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot y \cdot f_{X,Y}(x,y)dxdy$$

➢ If X and Y are independent: $E[XY] = E[X] \cdot E[Y]$

➢ If $X = Y$: $corr(X,X) = E[X^2]$

Covariance

Covariance is without bias from the mean

- Covariance:

$$cov(X,Y) = E[(X-\bar{X})(Y-\bar{Y})]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\bar{x})\cdot(y-\bar{y})\cdot f_{X,Y}\,(x,y)dxdy$$

$$= E[XY] - E[X]\cdot E[Y] = corr(X,Y) - E[X]\cdot E[Y]$$

➢ If X and Y are independent: $corr(X,Y) = 0$

OBS: The opposite not always true

➢ If $X = Y$: $cov(X,X) = E[X^2]-E[X]^2 = Var(X)$

Correlation Coefficient

Correlation Coefficient is the normalized Covariance

- The correlation coefficient, is an indicator on how much two random variables X and Y are correlated.

$$\rho = E\left[\frac{X-\bar{X}}{\sigma_X} \cdot \frac{Y-\bar{Y}}{\sigma_Y}\right] = \frac{E[XY]-E[X]E[Y]}{\sigma_X \cdot \sigma_Y} = \frac{cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

- We have that: $-1 \leq \rho \leq 1$
- If X and Y are independent: $\rho = 0$

Dependence

- We have independence between X and Y if and only if:

$$f_{X,Y} = f_X(x) f_Y(y)$$

Example of independent random variables:

- A persons height and the current exact distance from the earth to the moon.

Example of dependent random variables:

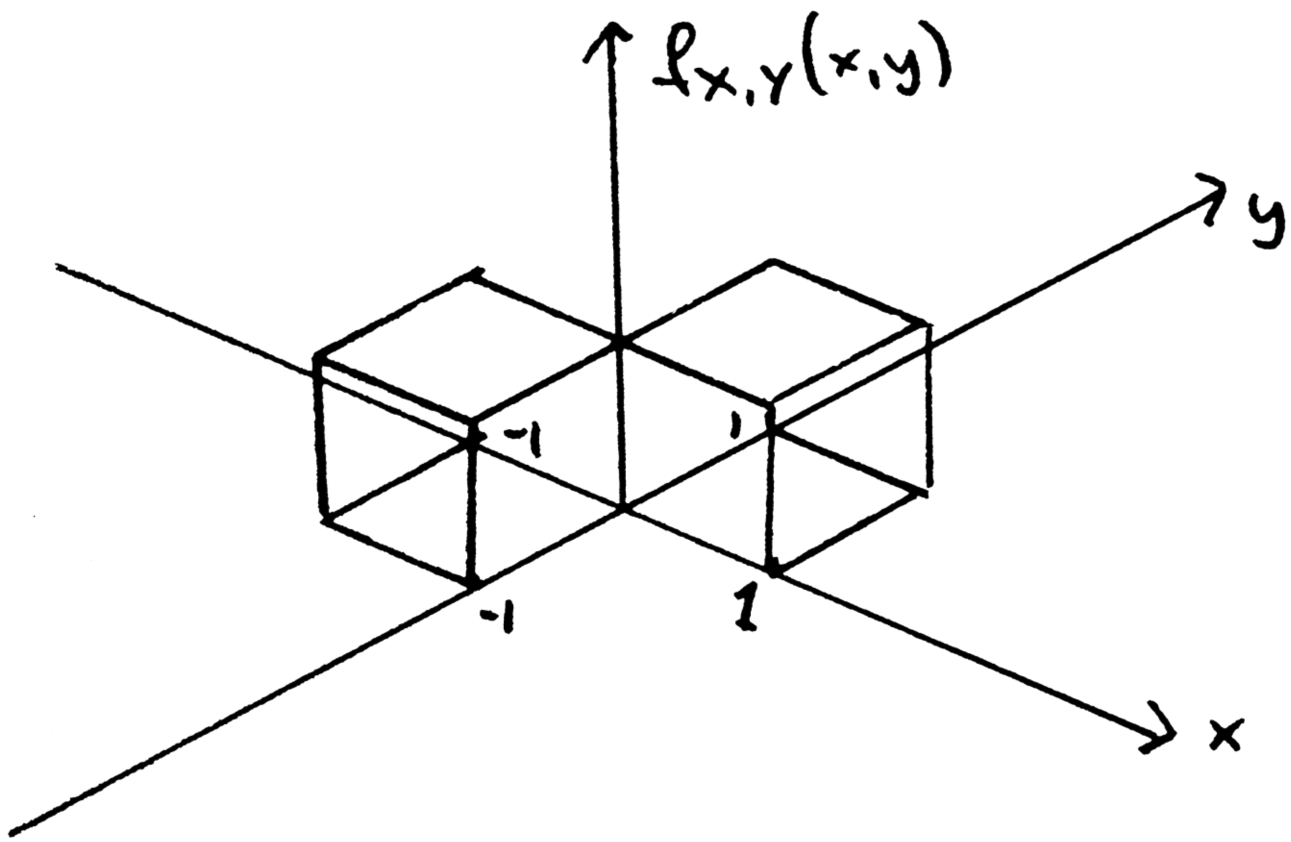
- The time of day and the amount of bicycles parked the at the engineering college.
- The energy of a mobile signal and the length in meters to a basestation.

Dependance - Example

- We want to find out whether two random variables are independent:

Simultaneous pdf for X and Y:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 0 \text{ and } -1 \leq y < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$



Dependance - Example

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 0 \text{ and } -1 \leq y < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \begin{cases} \int_{-1}^{0} \frac{1}{2}\,dy & \text{for } -1 \leq x < 0 \\ \int_{0}^{1} \frac{1}{2}\,dy & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \begin{cases} \int_{-1}^{0} \frac{1}{2}\,dx & \text{for } -1 \leq y < 0 \\ \int_{0}^{1} \frac{1}{2}\,dx & \text{for } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2} & \text{for } -1 \leq y < 0 \\ \frac{1}{2} & \text{for } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

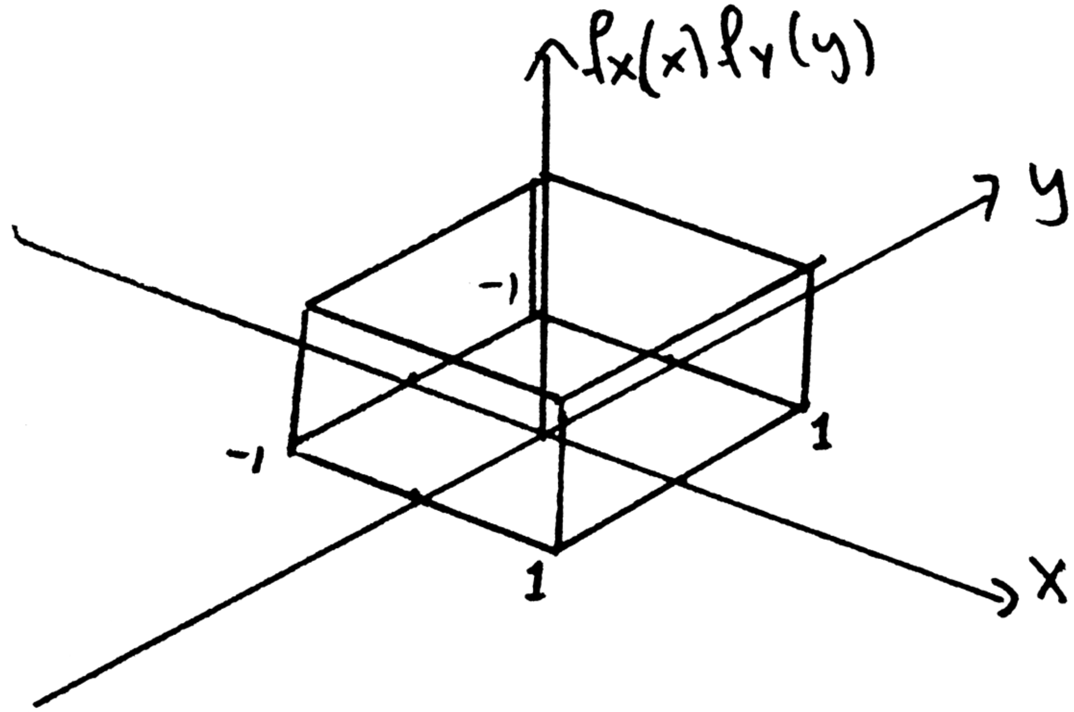
Dependance - Example

- Independence if and only if: $f_{X,Y} = f_X(x) f_Y(y)$

Multiply marginals:

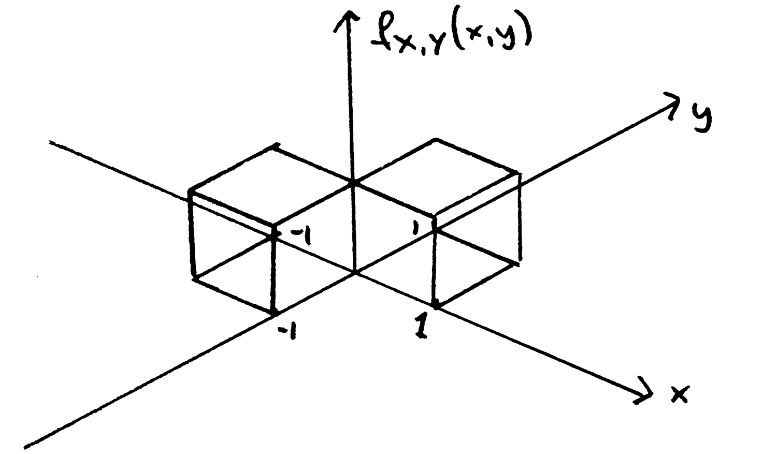
$$f_X(x) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) f_Y(y) = \begin{cases} \frac{1}{4} & \text{for } -1 \leq x < 1 \text{ and } -1 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

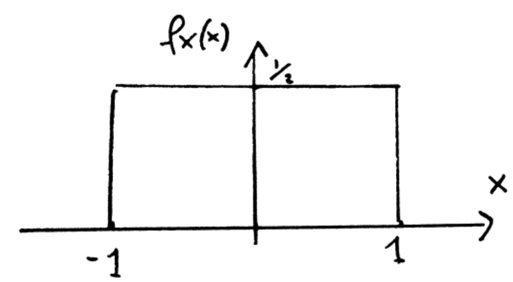


Dependance - Example

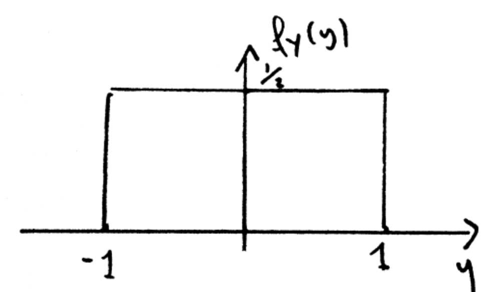
$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 0 \text{ and } -1 \leq y < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$



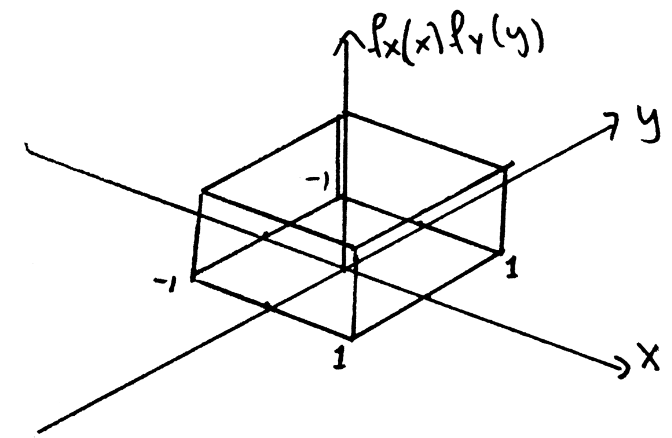
$$f_X(x) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$f_X(x)f_Y(y) = \begin{cases} \frac{1}{4} & \text{for } -1 \leq x < 1 \text{ and } -1 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$



$f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y) \;\Rightarrow\; X$ and Y er <u>ikke</u> uafhængige

Correlation calculation

Assignment:
Verify the results by doing the detailed calculations

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot y \cdot f_{X,Y}\,(x,y)dxdy = \frac{1}{4}$$

$$E[X] = E[Y] = \int_{-\infty}^{\infty} x \cdot f_X\,(x)dx = \int_{-\infty}^{\infty} y \cdot f_Y\,(y)dx = 0$$

$$\sigma_X^2 = \sigma_Y^2 = E[X^2] - E[X]^2 = E[Y^2] - E[Y]^2 = \frac{1}{3}$$

$$corr(X,Y) = E[XY] = \frac{1}{4}$$

$$cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{4} - 0 \cdot 0 = \frac{1}{4}$$

$$\rho = \frac{cov(X,Y)}{\sigma_X\ \sigma_Y} = \frac{1/4}{1/3} = \frac{3}{4} = 0{,}75$$

Very important!

i.i.d.: Independent and Identically distributed

- We define that for series of random variables that is taken from the same distribution (identically distributed), and are sampled independent of each other, that they are i.i.d.

i.i.d. = Independent and Identically distributed

- i.i.d. is a very important characteristic in stochastic variable processing and statistics

Example:

- Quantisation noise.

Words and Concepts to Know

Probability density function

i.i.d.

Correlation

Marginal probability density function

Continuous random variable

Gaussian distribution

pdf

Uniform distribution

Independent and Identically Distributed

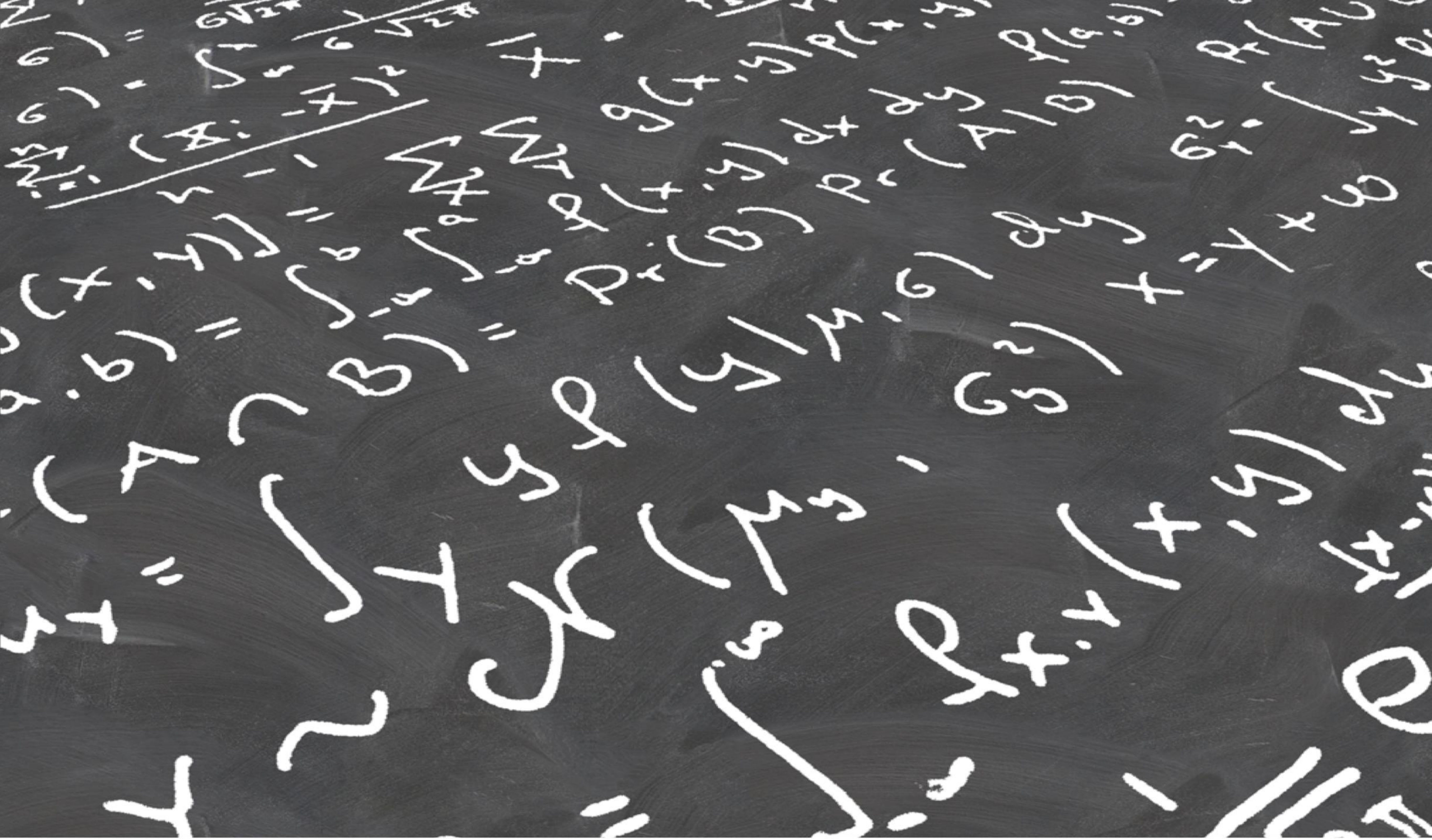
Normal distribution

Correlation coefficient

Simultaneous density function

Joint density function

Covariance



5. Transformations and Multivariate Random Variables

Gunvor Elisabeth Kirkelund
Lars Mandrup

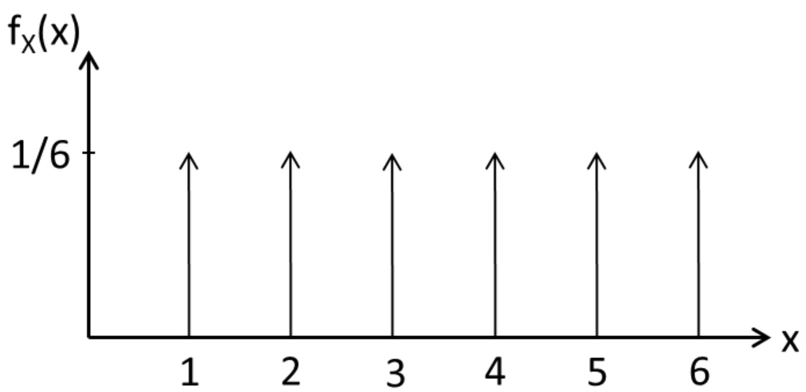
Agenda for Today

- One Random Variable - repetition
- Two Random Variables - repetition
- Sum of two random variables
- Central limit theorem

One Stochastic Variable – Discrete

- Probability mass function (pmf):

$$f_X(x) = \begin{cases} Pr(X = x_i) & for\ X = x_i \\ 0 & otherwise \end{cases}$$

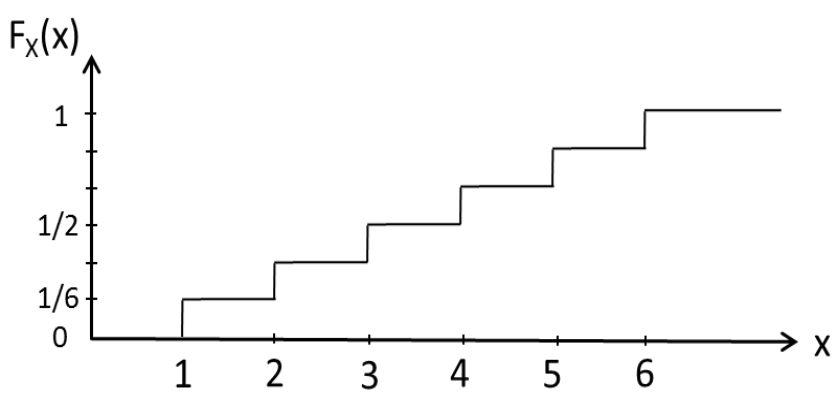


$$0 \leq f_X(x) \leq 1$$

$$\sum_{i=1}^{n} f_X(x_i) = \sum_{i=1}^{n} Pr(X = x_i) = 1$$

- Cumulative distribution function (cdf):

$$F_X(x) = P\,r(X \leq x) = \sum_{i=1}^{n_x} f_X(x_i)$$



$$0 \leq F_X(x) \leq 1$$

$$\lim_{x \to -\infty} F_X(x) = 0$$

$$\lim_{x \to \infty} F_X(x) = 1$$

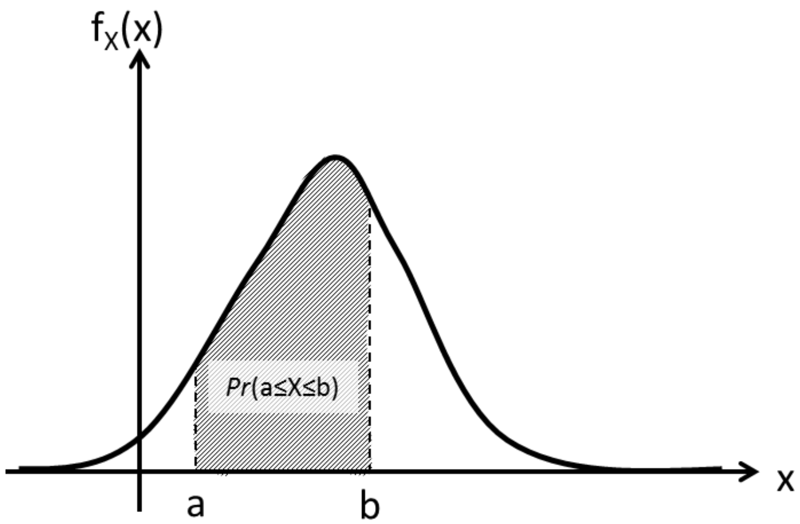
One Stochastic Variable – Continuous

- Probability density function (pdf):

$$Pr(a \le X \le b) = \int_a^b f_X(x)\,dx$$

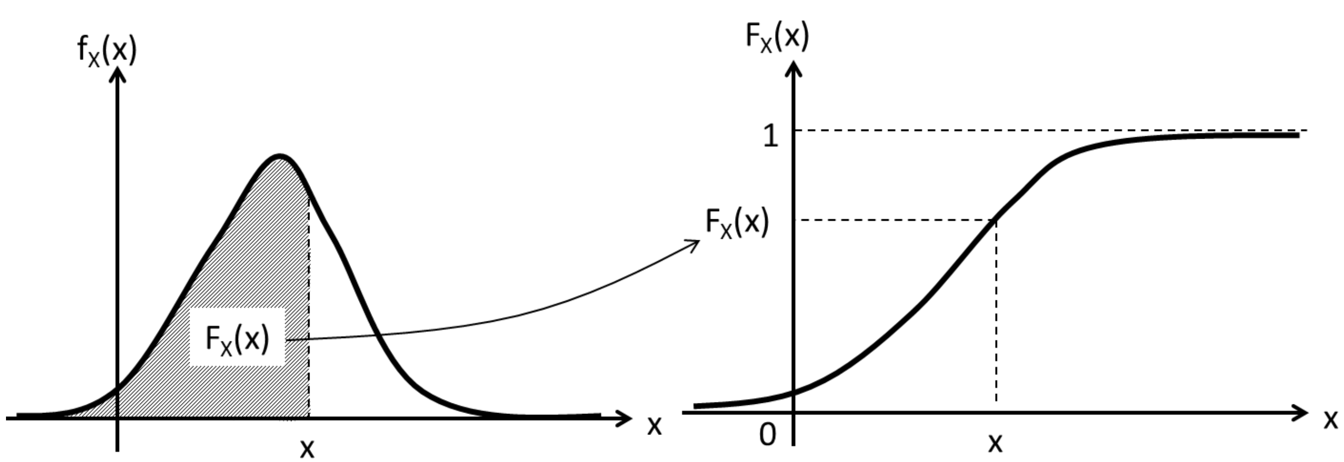
$$f_X(x) \ge 0$$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$



- Cumulative distribution function (cdf):

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = Pr(X \le x)$$



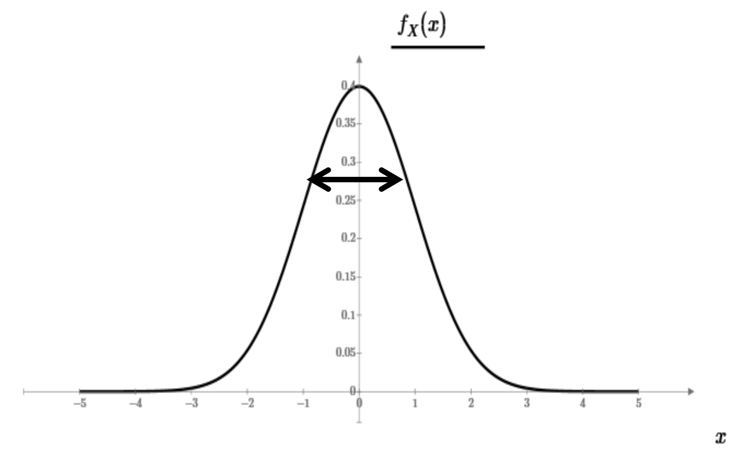
$$0 \le F_X(x) \le 1$$

$$\lim_{x \to -\infty} F_X(x) = 0$$

$$\lim_{x \to \infty} F_X(x) = 1$$

Expectations

- Mean value: $E[X] = \bar{X} = \mu_X = \int_{-\infty}^{\infty} x \cdot f_X(x)dx \quad (\sum_{i=1}^{n} x_i\, f_X(x_i))$
- Mean square: $E[X^2] = \overline{X^2} = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)dx \quad (\sum_{i=1}^{n} x_i^2\, f_X(x_i))$
- Variance: $Var(\mathrm{X}) = {\sigma_X}^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot f_X(x)dx = E[X^2] - E[X]^2$
- Standard deviation: $\sigma_X = \sqrt{Var(\mathrm{X})}$



- A function: $E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx \quad (\sum_{i=1}^{n} g(x_i)\, f_X(x_i))$

$$Var\big(g(\mathrm{X})\big) = \int_{-\infty}^{\infty} \left(g(x) - \overline{g(x)}\right)^2 \cdot f_X(x)dx = E[g(X)^2] - E[g(X)]^2$$

- Linear function: $E[aX + b\,] = a \cdot E[X] + b$

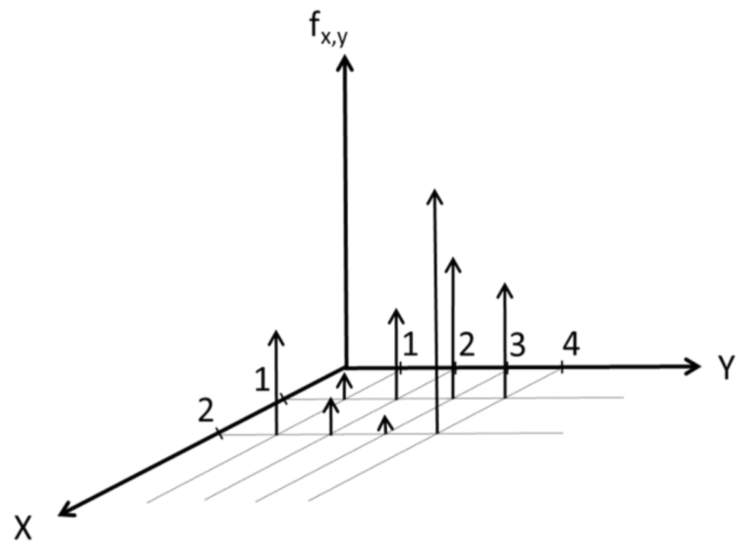
$$Var[aX + b\,] = a^2(E[X^2] - E[X]^2) = a^2 \cdot Var(X)$$

Two Stochastic Variables X, Y – Discrete

Joint (Simultaneous) pmf:

$$f_{X,Y}(x,y) = \begin{cases} Pr\left((X = x_i\ ) \cap (Y = y_j)\right) & for\ \ X = x_i\ \wedge\ Y = y_j \\ 0 & otherwise \end{cases}$$

$$0 \le f_{X,Y}(x,y) \le 1 \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} f_{X,Y}(x_i, x_j) = 1$$



Marginal pmfs:

$$f_X(x) = \sum_y f_{X,Y}(x,y) \qquad f_Y(y) = \sum_x f_{X,Y}(x,y)$$

Cumulative Distribution Function cdf:

$$F_X(x_j)\ = Pr\left(X \le x_j\right) = \sum_{i=1}^{j} f_X(x_j)$$

Two Stochastic Variables X, Y – Continuous

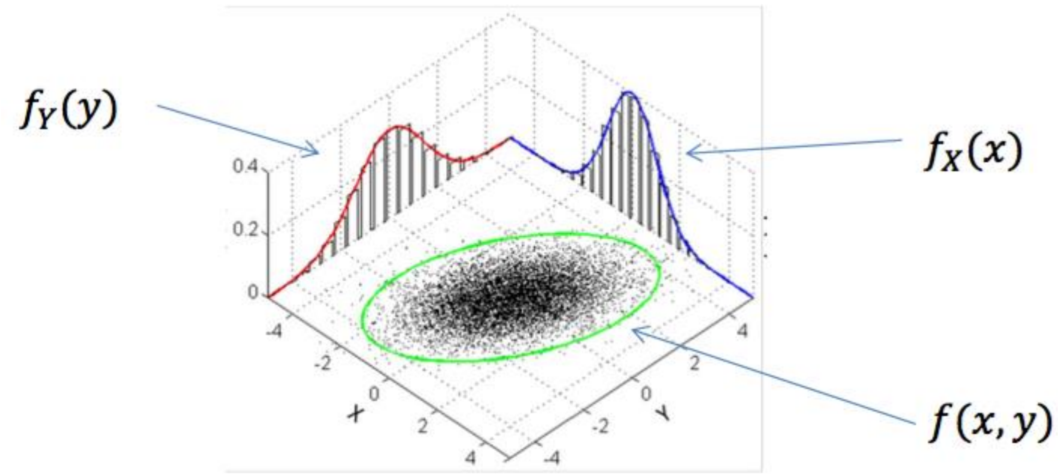
Joint (Simultaneous) pdf: $f_{X,Y}(x,y) \geq 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$$

Marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$



Cumulative Distribution Function cdf:

cdf $$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)dxdy = Pr(X \leq x \wedge Y \leq y)$$

pdf $$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Bayes Rule, Conditional PDF and Independence

Bayes rule:

- The joint/simultaneous pmf/pdf for two stochastic variables:

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x)$$

Conditional pdf:

- For a two dimensional pmf/pdf $f_{X,Y}(x,y)$, we can find the conditional pdf with Bayes rule:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Independence:

- X and Y are independent if and only if:

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad \text{or} \quad f_{X|Y}(x|y) = f_X(x) \quad \text{for all x and y}$$

Correlation and Covariance

Correlation tells of the (biased) coupling between variables

- Correlation: $corr(X,Y) = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot y \cdot f_{X,Y}(x,y)dxdy$

Covariance is without bias from the mean

- Covariance: $cov(X,Y) = E[(X-\bar{X})(Y-\bar{Y})] = E[XY] - E[X] \cdot E[Y]$

Correlation Coefficient is the normalized Covariance

- Correlation coefficient: $\rho = E\left[\frac{X-\bar{X}}{\sigma_X} \cdot \frac{Y-\bar{Y}}{\sigma_Y}\right] = \frac{E[XY] - E[X]E[Y]}{\sigma_X \cdot \sigma_Y}$

  $-1 \leq \rho \leq 1$

- If X and Y are independent:

  $E[XY] = E[X] \cdot E[Y]$ and $cov(X,Y) = \rho = 0$

The Conditional PDF and Independence

Conditional pdf:

- For a two dimensional pdf $f_{X,Y}(x, y)$, we can find the conditional pdf with Bayes rule:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Independence:

- X and Y are independent if and only if:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) \quad \text{or} \quad f_{X|Y}(x|y) = f_X(x)$$

for all x and y

Very important!

i.i.d.: Independent and Identically distributed

- We define that for series of random variables that is taken from the same distribution (identically distributed), and are sampled independent of each other, that they are i.i.d.

i.i.d. = Independent and Identically distributed

- i.i.d. is a very important characteristic in stochastic variable processing and statistics

Example:

- Quantisation noise.

Bivariate (2D) Normal Distribution

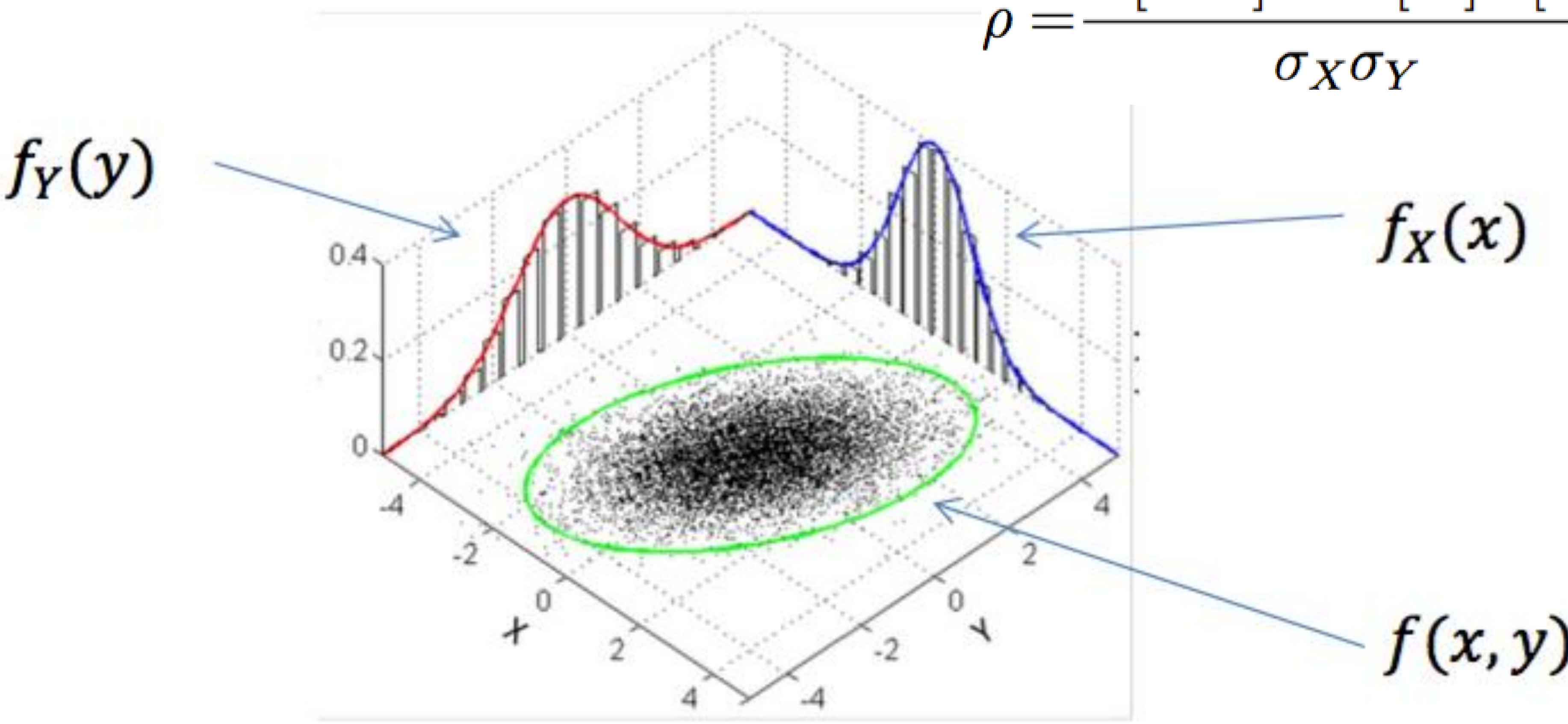
$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{z}{2(1-\rho^2)}\right)$$

Two dimensional Gaussian

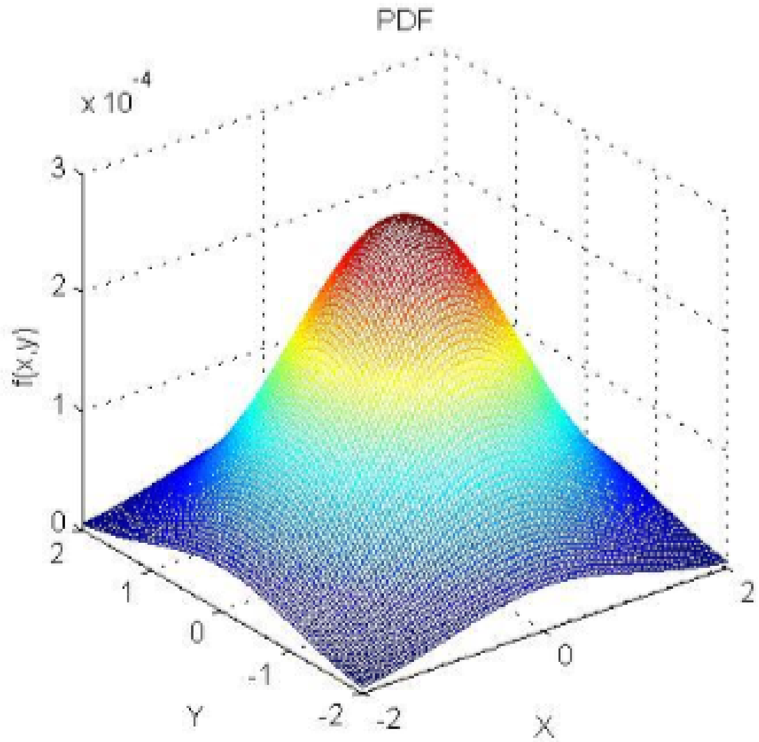
$$z = \frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}$$

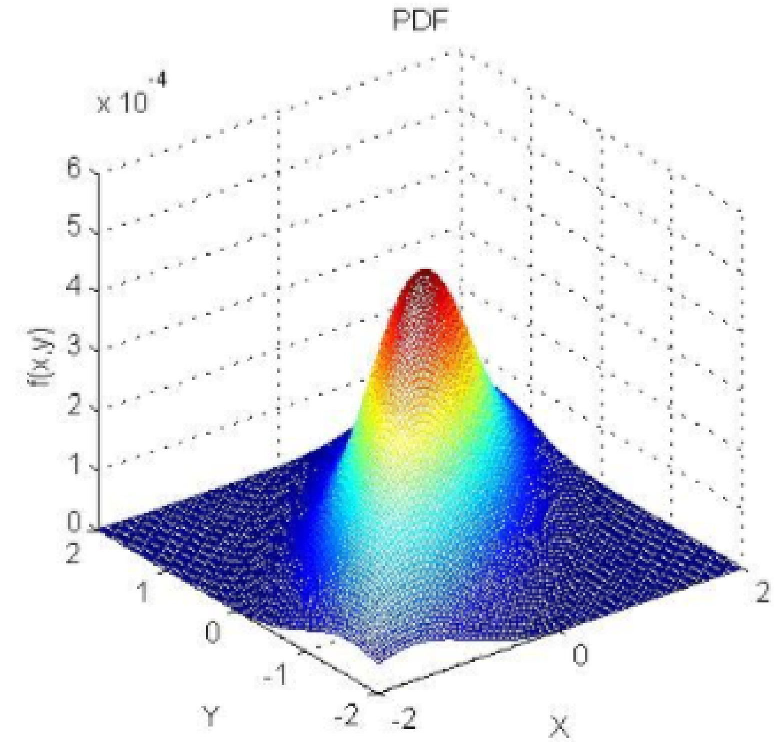
$$\rho = \frac{E[XY]-E[X]E[Y]}{\sigma_X\sigma_Y}$$

Correlation coefficient



Bivariate Normal Distribution





Symmetric PDF:
$\rho = 0$

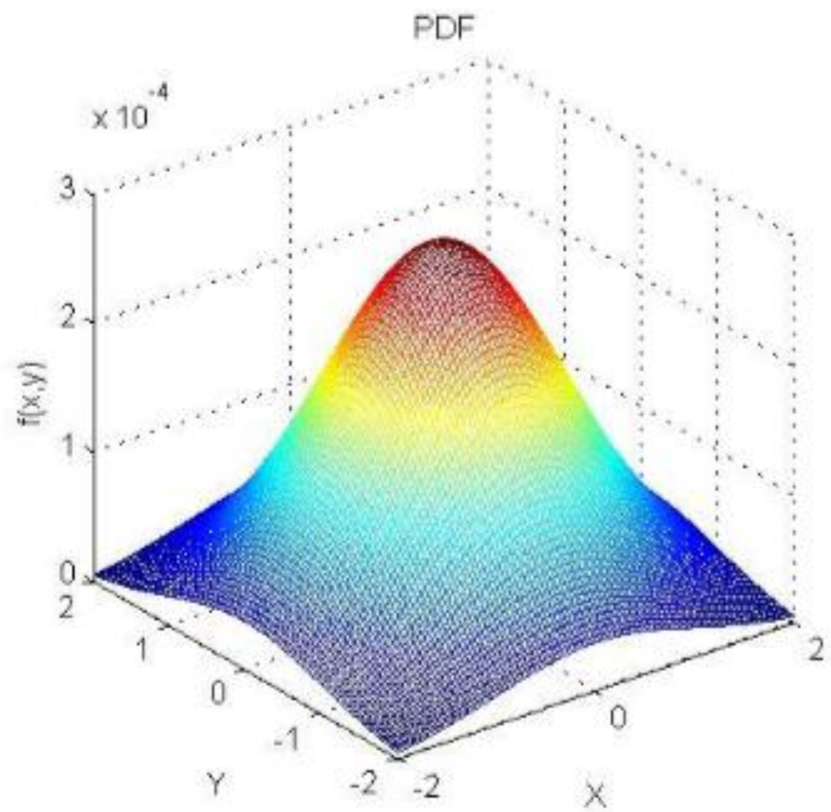
X and Y independent

Asymmetric PDF:
$\rho = 0.8$

X and Y **dependent**

Symmetric Case

Bivariate Normal Distribution



Symmetric PDF:
$\rho = 0$

X and Y independent

Because of the independence, we should have

$$f(x|y) = f_X(x)$$

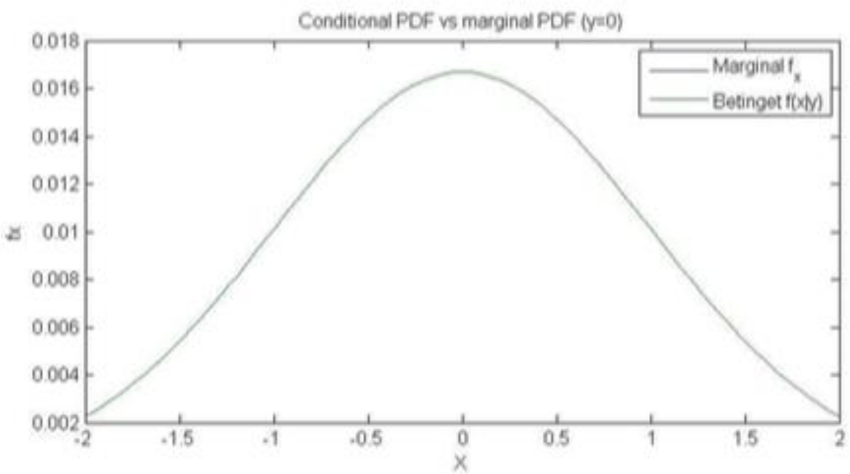
$$f(y|x) = f_Y(y)$$

$$f(x, y) = f_X(x) \cdot f_Y(y)$$

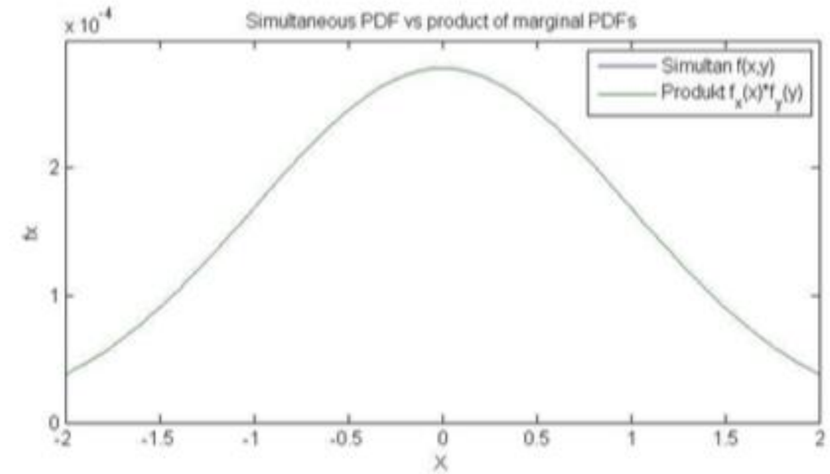
Bivariate Normal Distribution

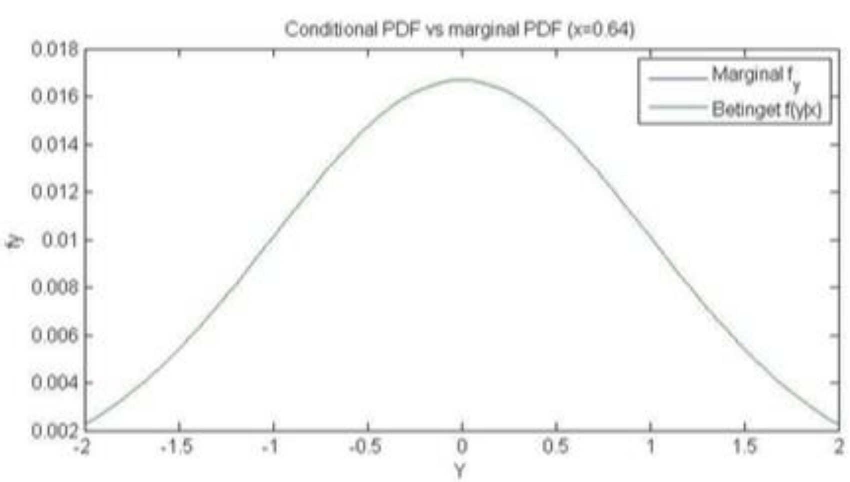
The graphs ($f_{X|Y}(x|y=0)$, $f_{X,Y}(x,y=0)$) and $f_X(x)$ has the same shape (proportional)

$$f(x|y=0) = f_X(x)$$

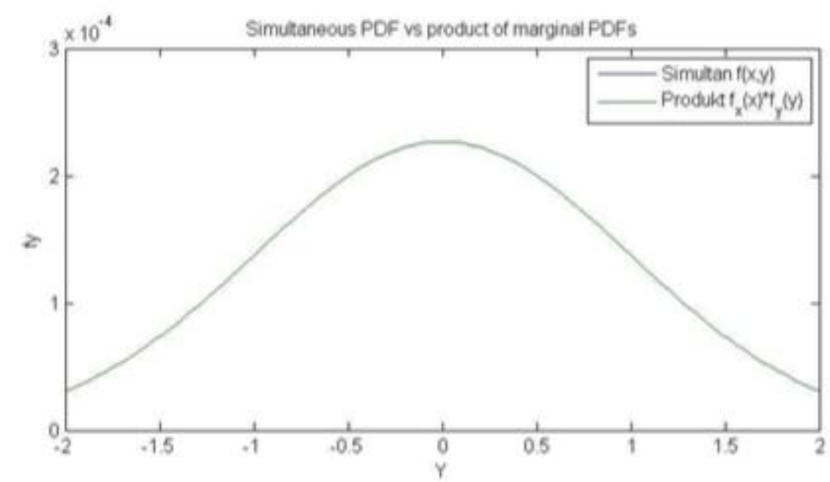


$$f(x,y=0) = f_X(x) \cdot f_Y(y=0)$$





$$f(y|x=0.64) = f_Y(y)$$

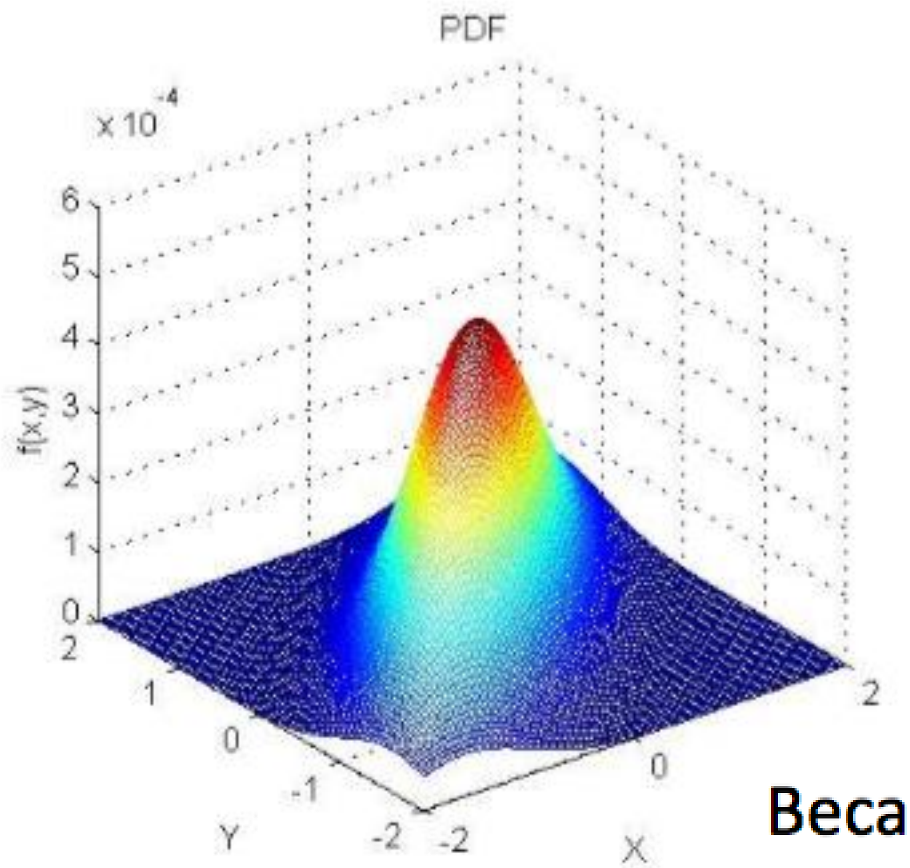


$$f(x=0.64,y) = f_X(x=0.64) \cdot f_Y(y)$$

The graphs $f_{Y|X}(y|x=0.64)$, $f_{X,Y}(x=0.64,y)$ and $f_Y(y)$ has the same shape (proportional)

Asymmetric Case

Bivariate Normal Distribution



Asymmetric PDF:

$\rho = 0.8$

X and Y **dependent**

Because of the dependence, we should have

$$f(x|y) \neq f_X(x)$$

$$f(y|x) \neq f_Y(y)$$

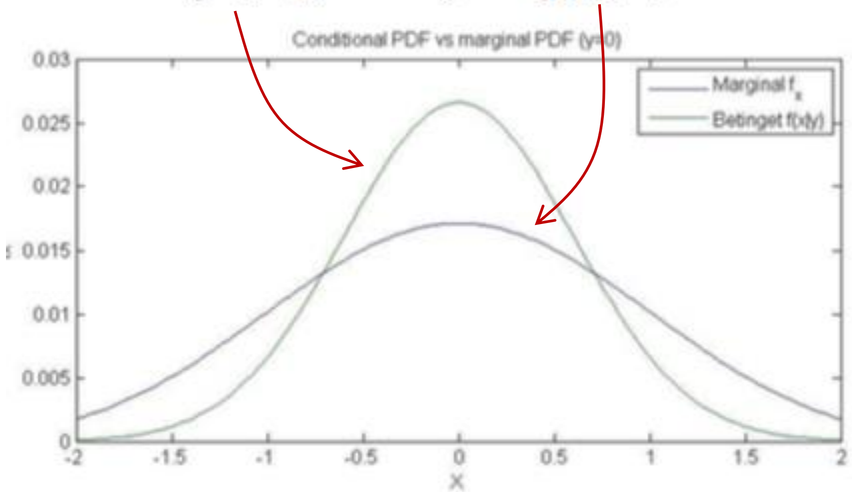
$$f(x,y) \neq f_X(x) \cdot f_Y(y)$$

Asymmetric Case

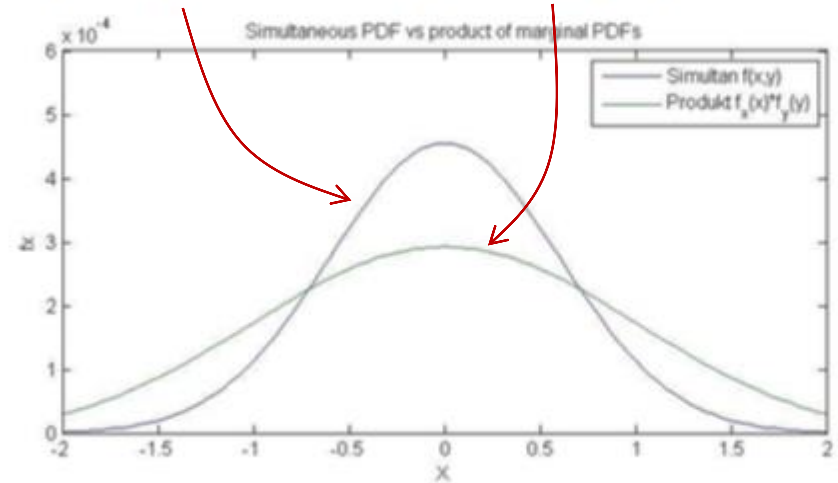
Bivariate Normal Distribution

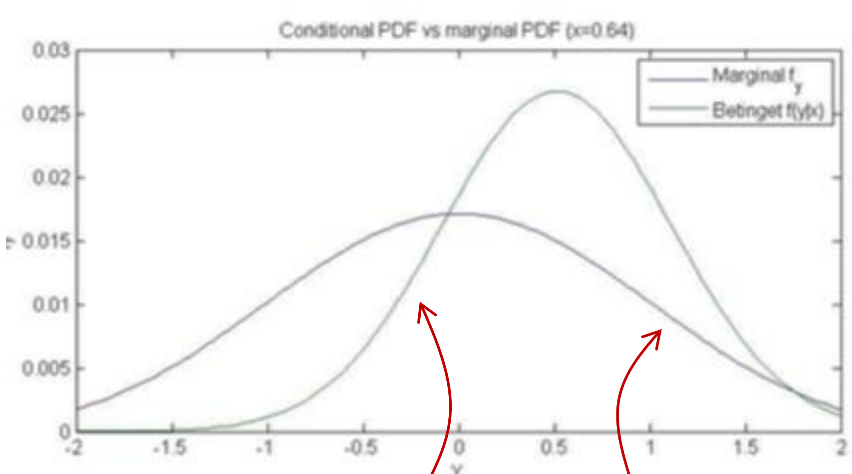
The graphs ($f_{X|Y}(x|y=0)$, $f_{X,Y}(x,y=0)$) and $f_X(x)$ do not have the same shapes.

$f(x|y=0) \neq f_X(x)$

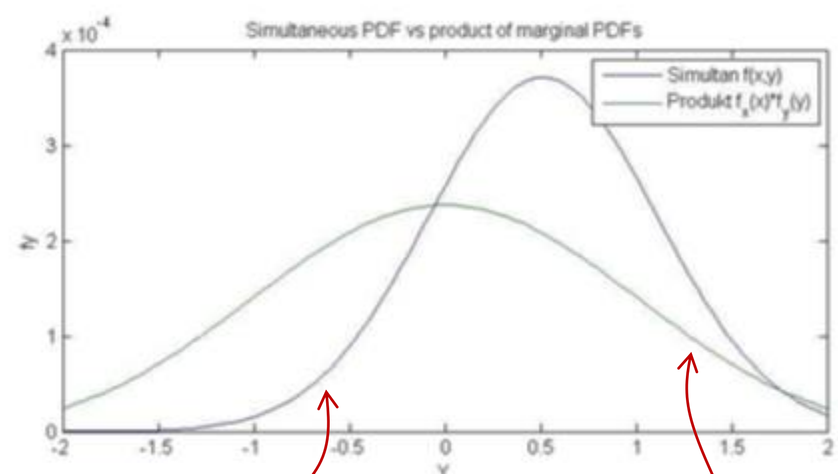


$f(x,y=0) \neq f_X(x) \cdot f_Y(y=0)$





$f(y|x=0.64) \neq f_Y(y)$

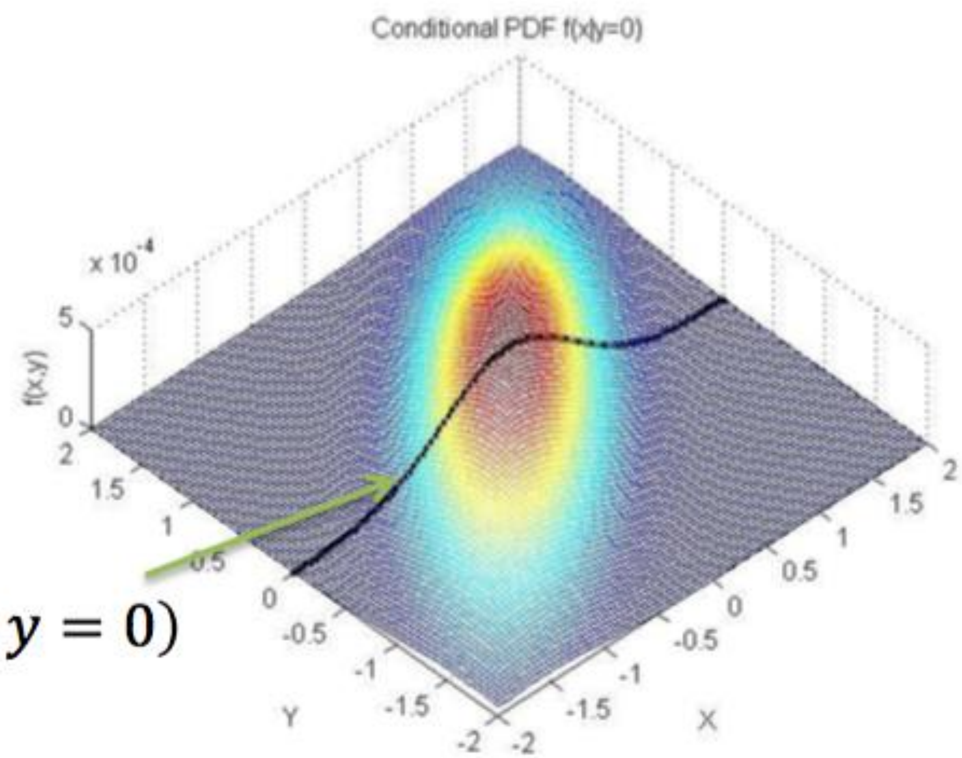


$f(x=0.64,y) \neq f_X(x=0.64) \cdot f_Y(y)$

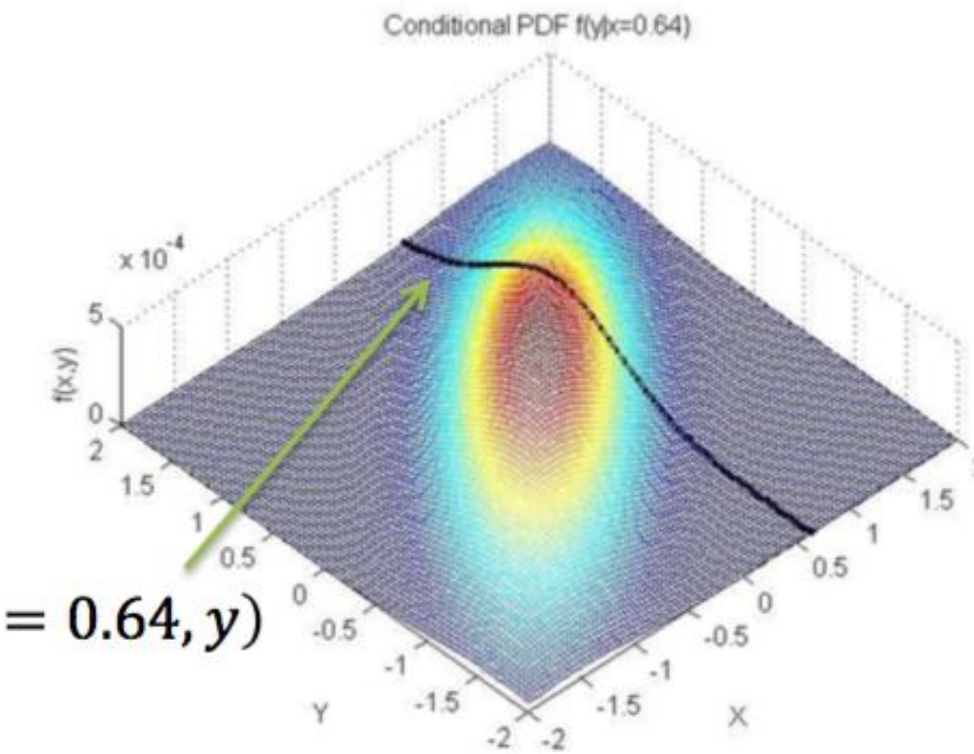
The graphs ($f_{Y|X}(y|x=0.64)$, $f_{X,Y}(x=0.64,y)$) and $f_Y(y)$ do not have the same shapes.

The Conditional pdf's

Bivariate Normal Distribution



$f(x, y = 0)$



$f(x = 0.64, y)$

Area under the curve =

$$\int_{-\infty}^{\infty} f(x, y = 0)dx = f_Y(y = 0)$$

Area under the curve =

$$\int_{-\infty}^{\infty} f(x = 0.64, y)dx = f_X(x = 0.64)$$

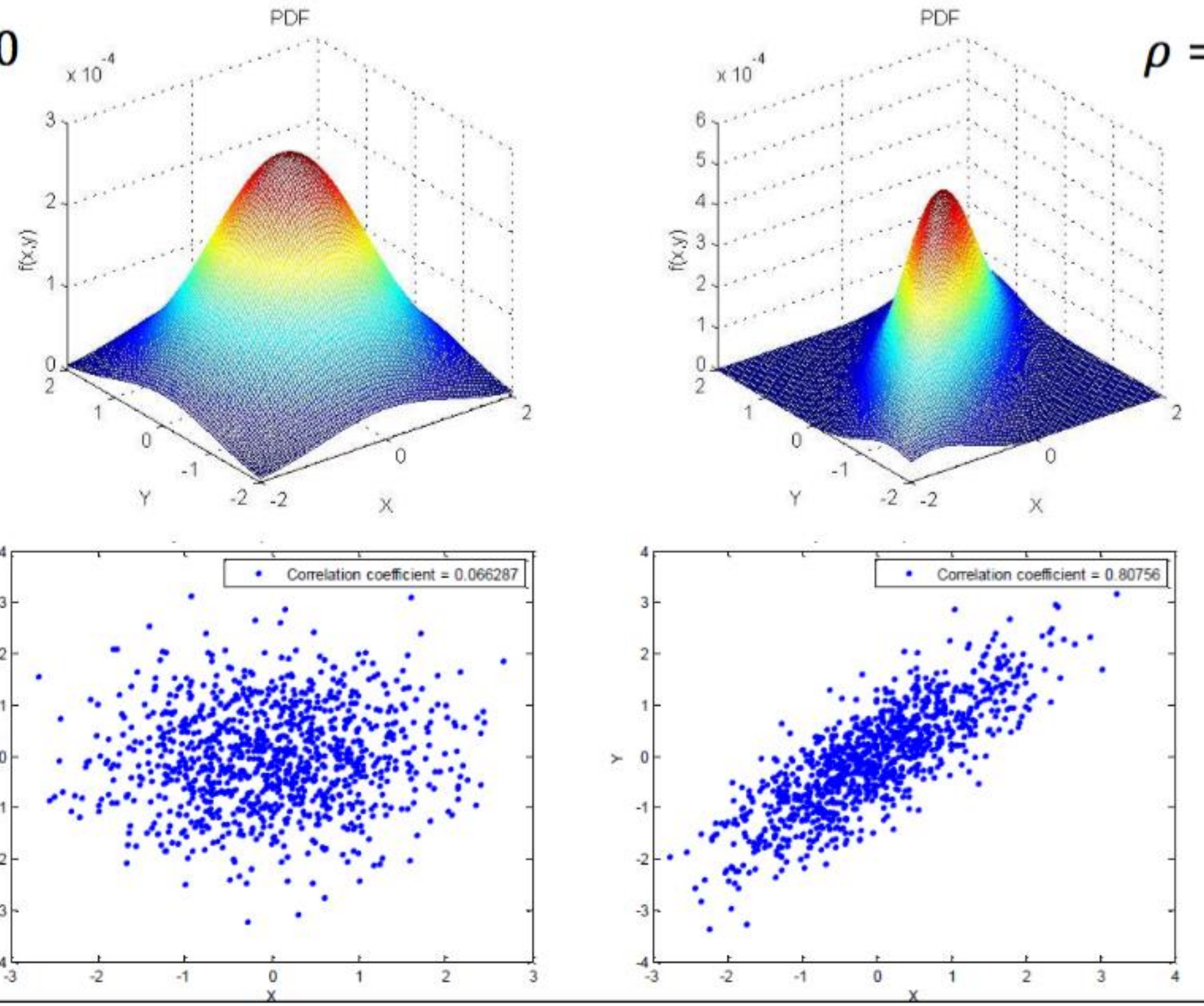
$$f(x|y = 0) = \frac{f(x, y = 0)}{f_Y(y = 0)}$$

$$f(y|x = 0.64) = \frac{f(x = 0.64, y)}{f_X(x = 0.64)}$$

Sampling

Bivariate Normal Distribution

$\rho = 0$ $\rho = 0.8$



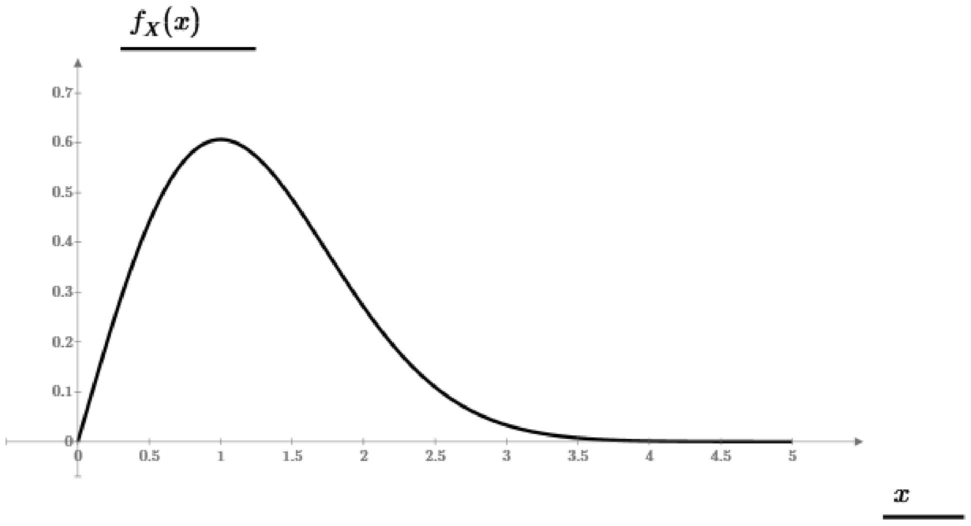
Sampling From Any Distribution

For test or simulation you need testdata (”measurements”) randomly sampled from a given distribution:

- Find the cdf of the distribution: $F_X(x)$
- Find the inverse of the cdf: $y = F_X(x) \;\Rightarrow\; x = F_X^{-1}(y)$
- Draw a ramdom sample: $y \sim \mathcal{U}[0;1]$
- Insert into the inverse cdf: $x = F_X^{-1}(y)$
- The samples $X = x$ is distributed according to: $F_X(x)$

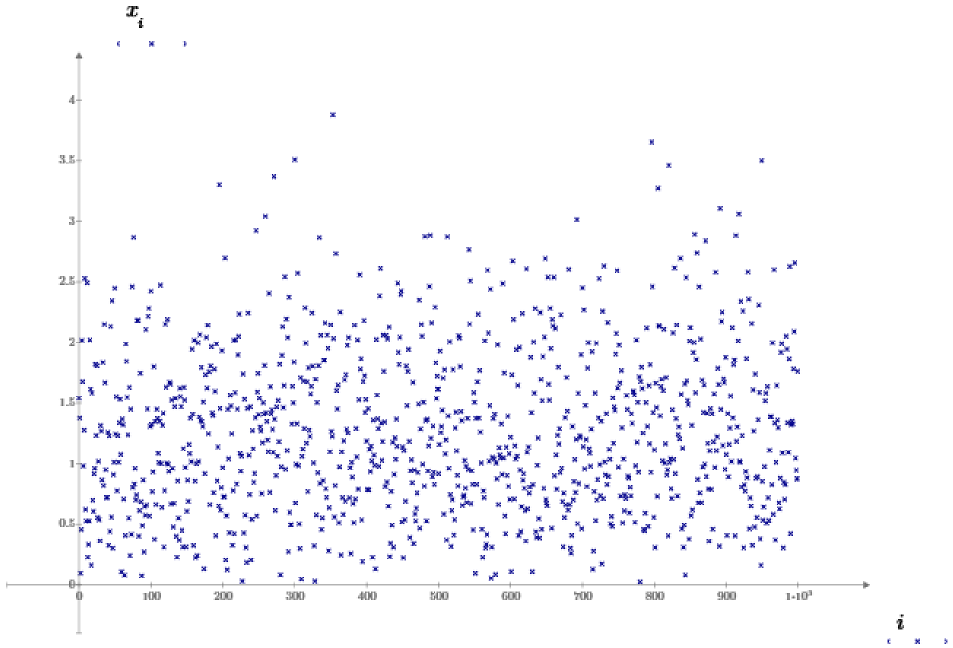
Example – Flight Simulator

- In a flight simulator, the altitude of the plane is simulated to be Rayleigh distributed.
- For a given initial height, draw a Rayleigh distributed sample.



Flight Simulator Example

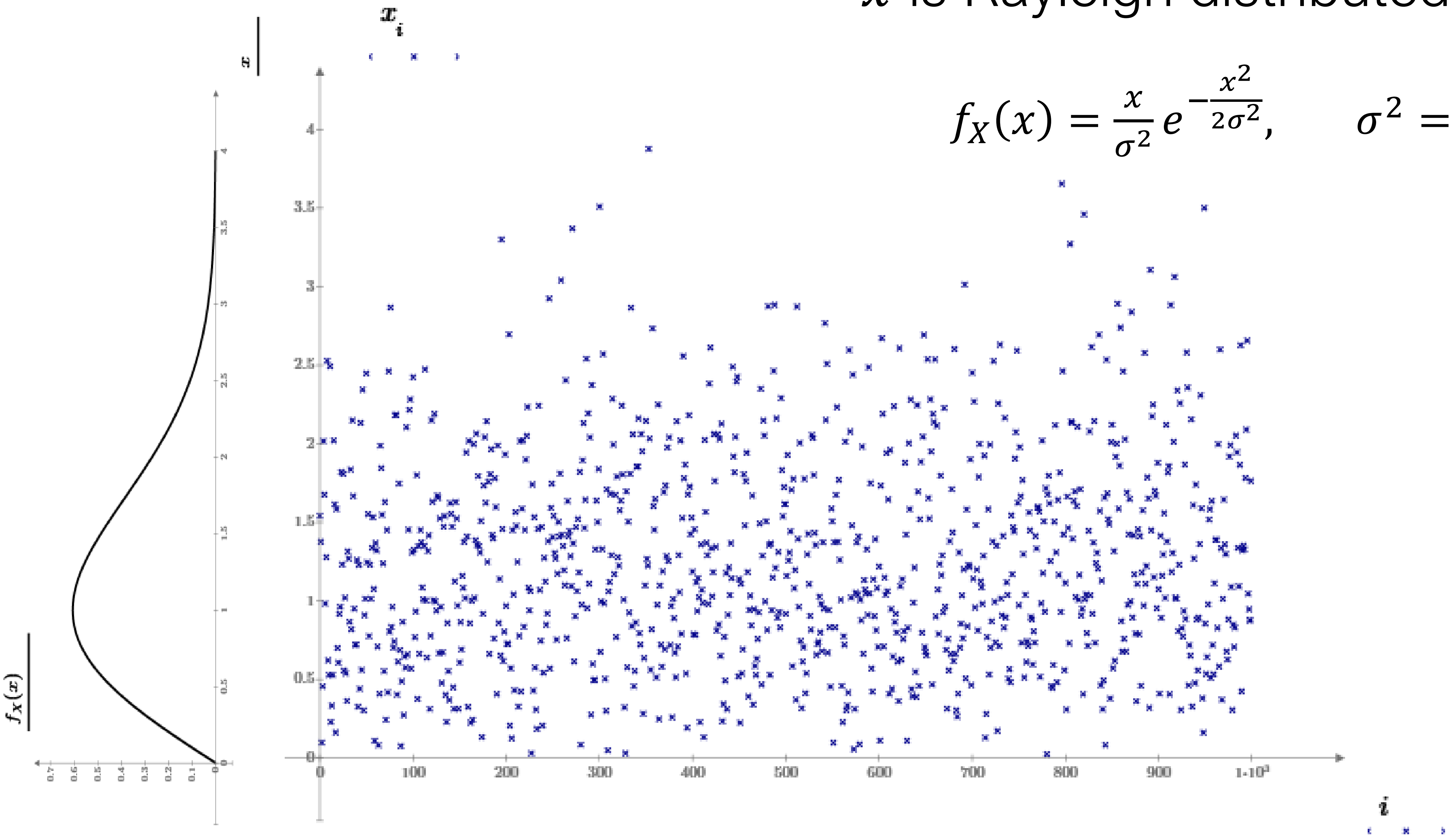
- Rayleigh pdf: $f_X(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$ for $x \geq 0$
- Rayleigh cdf: $F_X(x) = \int_0^x \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx = 1 - e^{-\frac{x^2}{2\sigma^2}}$
- Invers of cdf: $\mathrm{y} = 1 - e^{-\frac{x^2}{2\sigma^2}} \Rightarrow x = \sqrt{-2\sigma^2 \ln(1-y)}$
- Draw $\mathrm{y} \sim \mathcal{U}[0;1]$ and insert into $x = \sqrt{-2\sigma^2 \ln(1-y)}$
- x is Rayleigh distributed



Flight Simulator Example

x is Rayleigh distributed:

$$f_X(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, \qquad \sigma^2 = 1$$



Assignment

- Choose an exponential pdf: $f_X(x) = \lambda e^{-\lambda \cdot x}$
- Make a Matlab program that samples from that distribution

Transformation of Variable X to Y

- Given:
  - Pdf: $f_X(x)$
  - Function/Transformation: $\mathrm{Y} = g(X)$
  - Limits: $\mathrm{a} \leq X \leq b$

- Find new pdf: $f_Y(y)$:
  1. Inverse: $x = g^{-1}(y)$
  2. Differentiate: $\frac{dg^{-1}(y)}{dy} = \frac{dx(y)}{dy} = \frac{1}{\frac{dg(x)}{dx}}$
  3. Limits: Find $g(a) = a_Y \leq Y \leq b_Y = g(b)$ based on $\mathrm{a} \leq X \leq b$
  4. New pdf: $f_Y(y) = \sum \left|\frac{dx(y)}{dy}\right| f_X\big(g^{-1}(y)\ \big) = \sum \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}$

Example with Transformation of Random Variable

- We have a random sample x.
- The Noise is known to be Gaussian distributed.
- The signal of the noise is amplified.
- What is the pdf of the amplified noise?

- Given:
  - function: $Y = 2$ X
  - pdf: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\sim\mathcal{N}(\mu, \sigma^2)$
  - Support: $x \in \mathbf{R}$
- Steps:
  1. Inverse: $x = \frac{1}{2}y$
  2. Differentiate: $\frac{d}{dy}\frac{1}{2}y = \frac{1}{2}$
  3. Support: y $\in \mathbf{R}$
  4. New pdf: $f_Y(y) = \frac{1}{2} f_X(\frac{1}{2}y)$.
- Then: $f_Y(y) = \frac{1}{2}\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\frac{y}{2}-\mu)^2}{2\sigma^2}}$

$\sim\mathcal{N}(2\mu, 4\sigma^2)$

Distribution of the Sum of Two Random Variables

- Two random variables X and Y have density functions $f_X(x)$ and $f_Y(y)$.

- If we define a new random variable $Z = X + Y$, and Z have density function $f_Z(z)$.

Convolution of Two functions

- Then $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$

Expectation of the Sum of Two Random Variables

- For a random variables $Z = X + Y$.
- X, Y can be both dependent and independent.
- The expectation of Z is:

$$\boxed{E[Z] = E[X] + E[Y]}$$

Expectation of the Sum of Two Random Variables

- For a random variables $Z = X + Y$.
- X, Y can be both dependent and independant.

Proof:

$$
\begin{aligned}
E[X+Y] &= \int_x \int_y (x+y) f_{X,Y}(x,y)\, dx\, dy \\
&= \int_x \int_y x f_{X,Y}(x,y)\, dx\, dy + \int_x \int_y y f_{X,Y}(x,y)\, dx\, dy \\
&= \int_x x \int_y f_{X,Y}(x,y)\, dy\, dx + \int_y y \int_x f_{X,Y}(x,y)\, dx\, dy \\
&= \int_x x f_X(x)\, dx + \int_y y \; f_Y(y)\, dy \\
&= E[X] + E[Y]
\end{aligned}
$$

Variance of the Sum of Two Random Variables

- We have $Z = X + Y$.
- For independent random variables X, Y, the variance of Z is:

$$\boxed{var(Z) = var(X) + var(Y).}$$

- For correlated random variables X, Y, the variance of Z is:

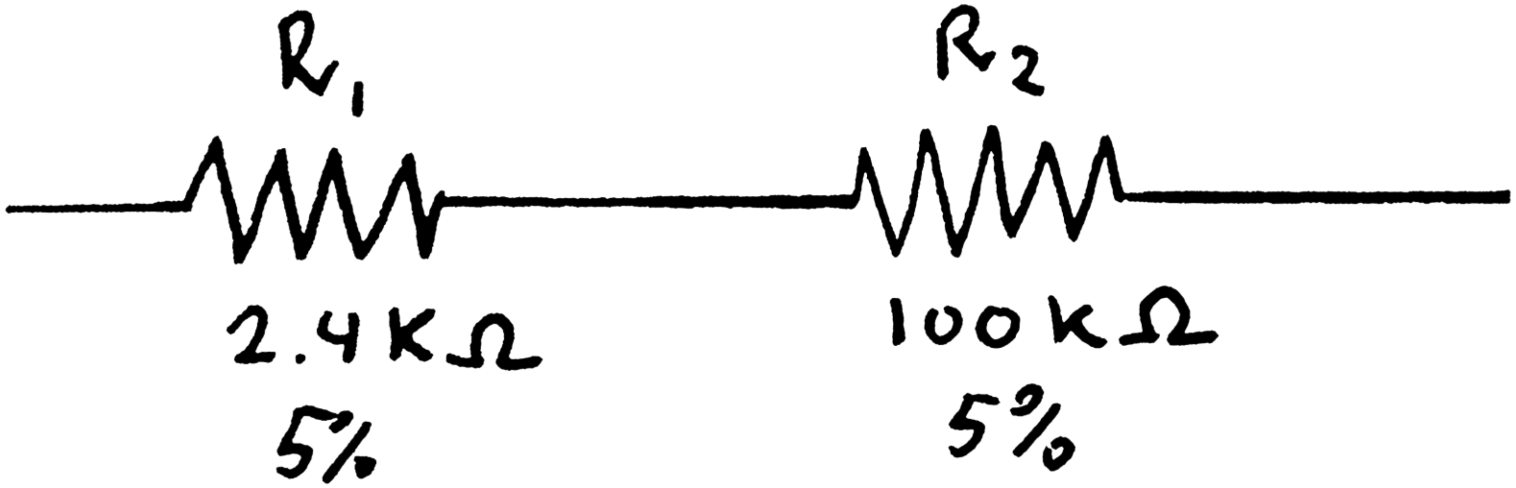
$$\boxed{var(Z) = var(X) + var(Y) + 2cov(X, Y).}$$

where: $cov(X,Y) = E[XY] - E[X]E[Y]$

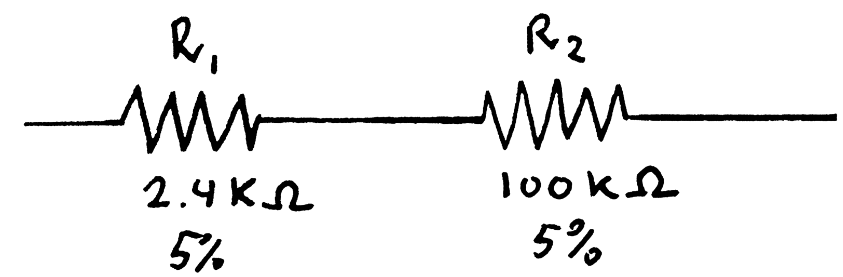
Proof: Similar to the proof of the expectation value

Precision of Resistors in Series

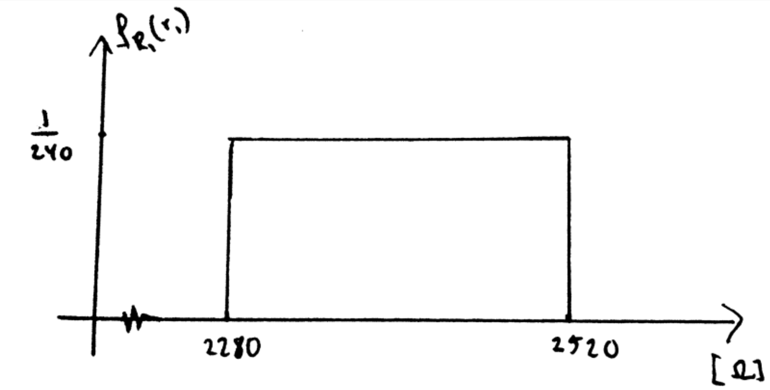
- In a analog filter a resister of size $2.5K\Omega$ is needed.
- We use two 5% resisters of $2.4K\Omega$ and 100Ω respectively.
- What is the resulting uncertainty of the resister?
- X and Y are independent random variables with pdfs: $f_X(x)$ and $f_Y(y)$
- What is the pdf of a random variable Z, where $Z = X + Y$

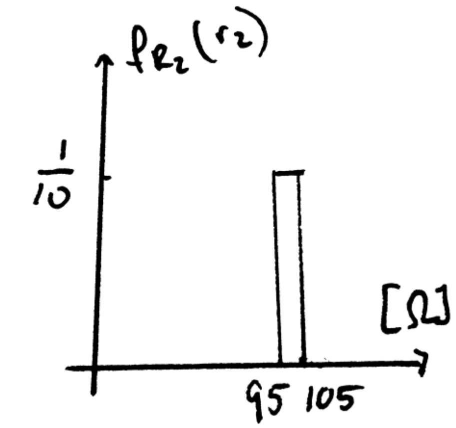


Precision of Resistors in Series

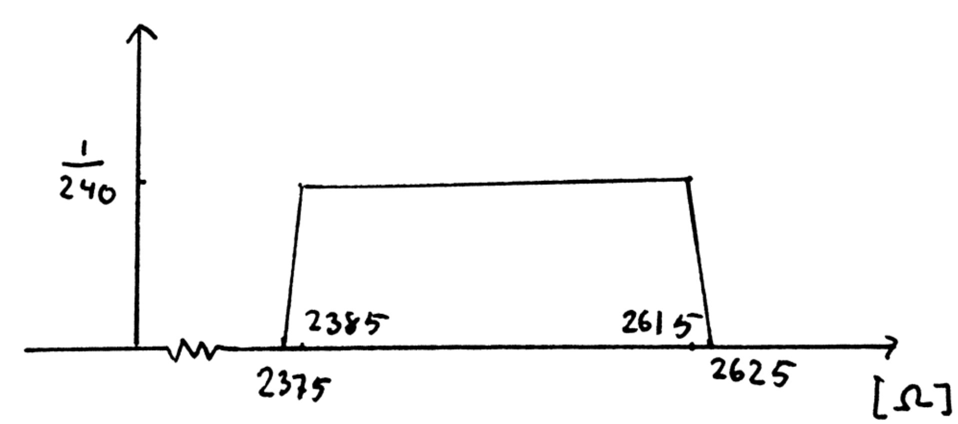


- We assume that the resistance of the resisters are uniformly distributed.
- $R_1 \sim \mathcal{U}[2280; 2520]$
- $R_2 \sim \mathcal{U}[95; 105]$
- The resisters are in series: $R_3 = R_1 + R_2$.
- We have: $f_{R_3}(r_3) = \int_{-\infty}^{\infty} f_X(\rho) f_Y(r_3 - \rho) d\rho$
- We can find that:



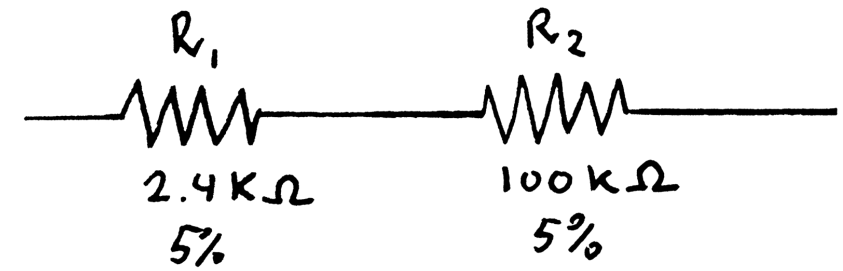


$$f_{R_3}(r_3) = \begin{cases} \frac{1}{2400} r_3 - \frac{95}{96} & \text{for } 2375 \leq x < 2385 \\ \frac{1}{240} & \text{for } 2385 \leq x < 2615 \\ -\frac{1}{2400} r_3 + \frac{35}{32} & \text{for } 2615 \leq x < 2625 \\ 0 & \text{otherwise} \end{cases}$$



R3 is still a 5% resistor – but no longer uniform distributed!

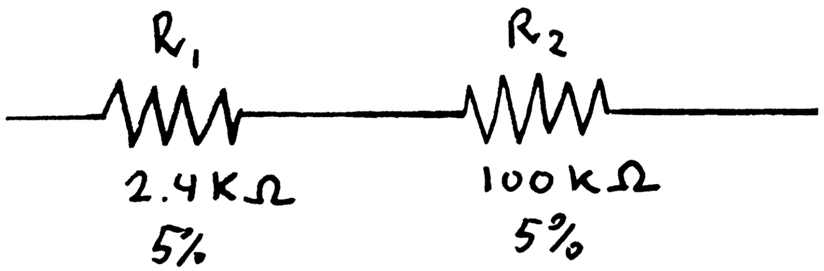
Expected Value of the Resistor



- We assume that R_1 and R_2 are independent
- For a uniform distibution: $\mathrm{E}[R_1] = \frac{1}{2}(2520 + 2280) = 2400\Omega$
- For a uniform distribution: $var(R_2) = \frac{1}{2}(105 + 95) = 100\Omega$
- For the sum $R_3 = R_1 + R_2$ we have:

$$E[R_3] = E[R_1] + E[R_2] = 2400\Omega + 100\Omega = \underline{2500\Omega}$$

Variance of the Resistor



- We assume that R_1 and R_2 are independent
- For a uniform distibution: $var(R_1) = \frac{1}{12}(2520 - 2280)^2 = 4800$
- For a uniform distribution: $var(R_2) = \frac{1}{12}(105 - 95)^2 = 8{,}333$
- For the sum $R_3 = R_1 + R_2$ we have:

$$var(R_3) = var(R_1) + var(R_2) = \underline{4808} \rightarrow \sigma_3 = \underline{69\Omega}$$

- For one uniform distributed 5%-resistor $R_0 = 2500 \sim \mathcal{U}[2375; 2625]$:

$$var(R_0) = \frac{1}{12}(2625 - 2375)^2 = \underline{5208} \rightarrow \sigma_0 = \underline{72\Omega}$$

- So: $var(R_3) = \underline{var(R_1) + var(R_2) < var(R_0)}$ $(\sigma_3 \underline{< \sigma_0})$

Two Random Variables

Two random variables: X and Y

- Simultaneous pdf: $f_{X,Y}(x, y)$
- Marginal pdf: $f_X(x)$ and $f_Y(y)$
- Conditional pdf: $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$
- Simultaneous cdf: $F_{X,Y}(x, y)$
- Correlation: $corr(X, Y) = E[XY]$
- Covariance: $cov(X, Y) = E[XY] - E[X]E[Y]$
- Correlation coefficient: $\rho = \frac{E[XY]-E[X]E[Y]}{\sigma_X \cdot \sigma_Y}$
- Sum: $\mathrm{Z} = X + Y$
- Expectation: $\mathrm{E[Z]} = E[X] + E[Y]$
- Variance: $\mathrm{Var[Z]} = Var[X] + Var[Y]$ if <u>independent</u>
  $\mathrm{Var[Z]} = Var[X] + Var[Y] + 2cov(X, Y)$ if <u>dependent</u>

Central Limit Theorem

- Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2
- Let $\bar{X}$ be the random variable (average):

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Then in the limit: $n \to \infty$ we have that: $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$

i.e. in the limit $\bar{X}$ will be normally distributed with mean = μ and variance = $\frac{\sigma^2}{n}$.

The variance is reduced with a factor 1/n

Central Limit Theorem

- Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2
- Let X be the random variable:

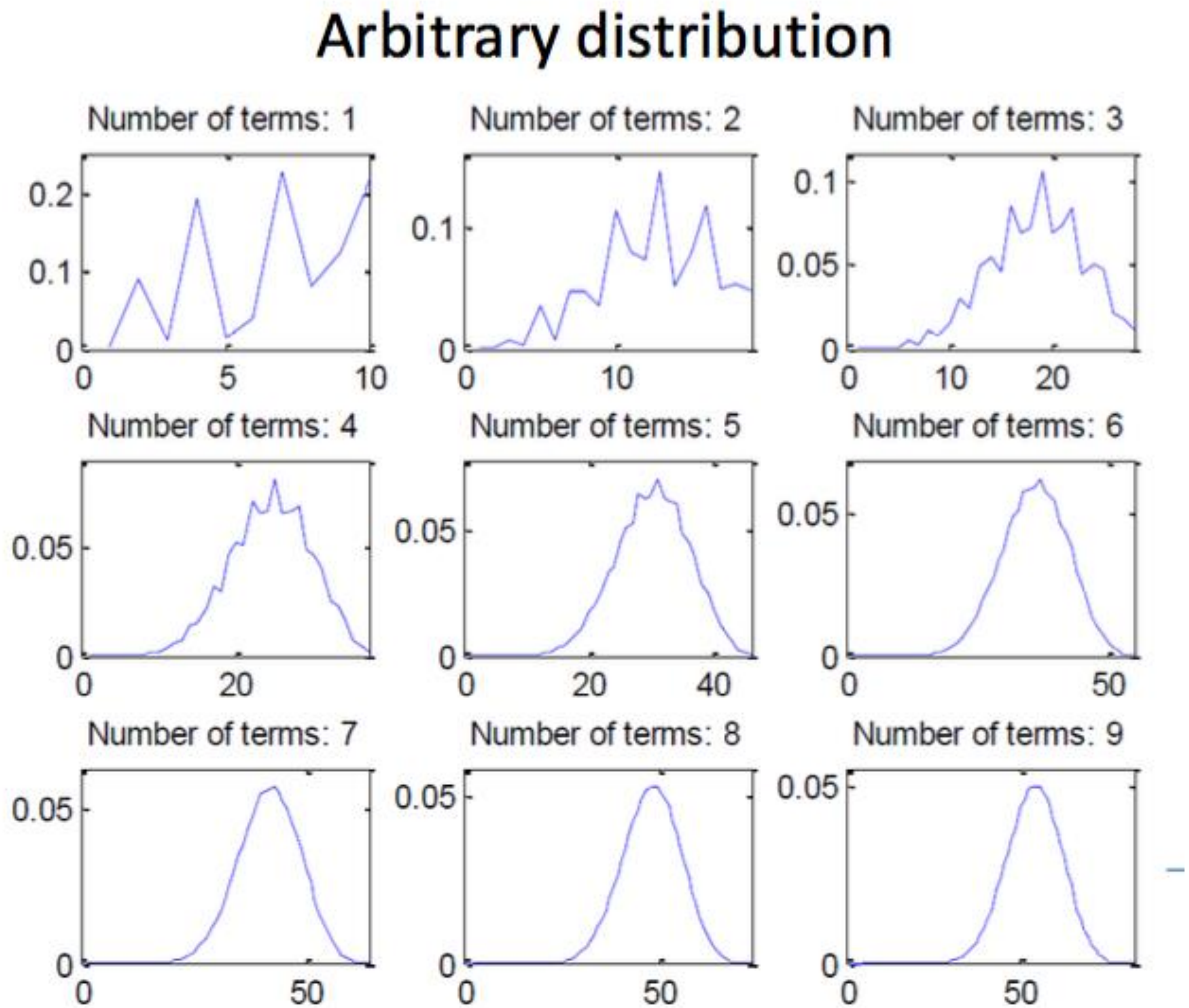
$$X = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} = \frac{\sum_{i=1}^{n} \frac{1}{n} X_i - \mu}{\sqrt{\sigma^2/_{n}}} = \frac{\bar{x} - \mu}{\sqrt{\sigma^2/_{n}}}$$

- Then in the limit: $n \rightarrow \infty$ we have that: $X \sim \mathcal{N}(0,1)$

  i.e. in the limit X will be normally distributed with mean = 0 and variance = 1 (standard normal distributed).

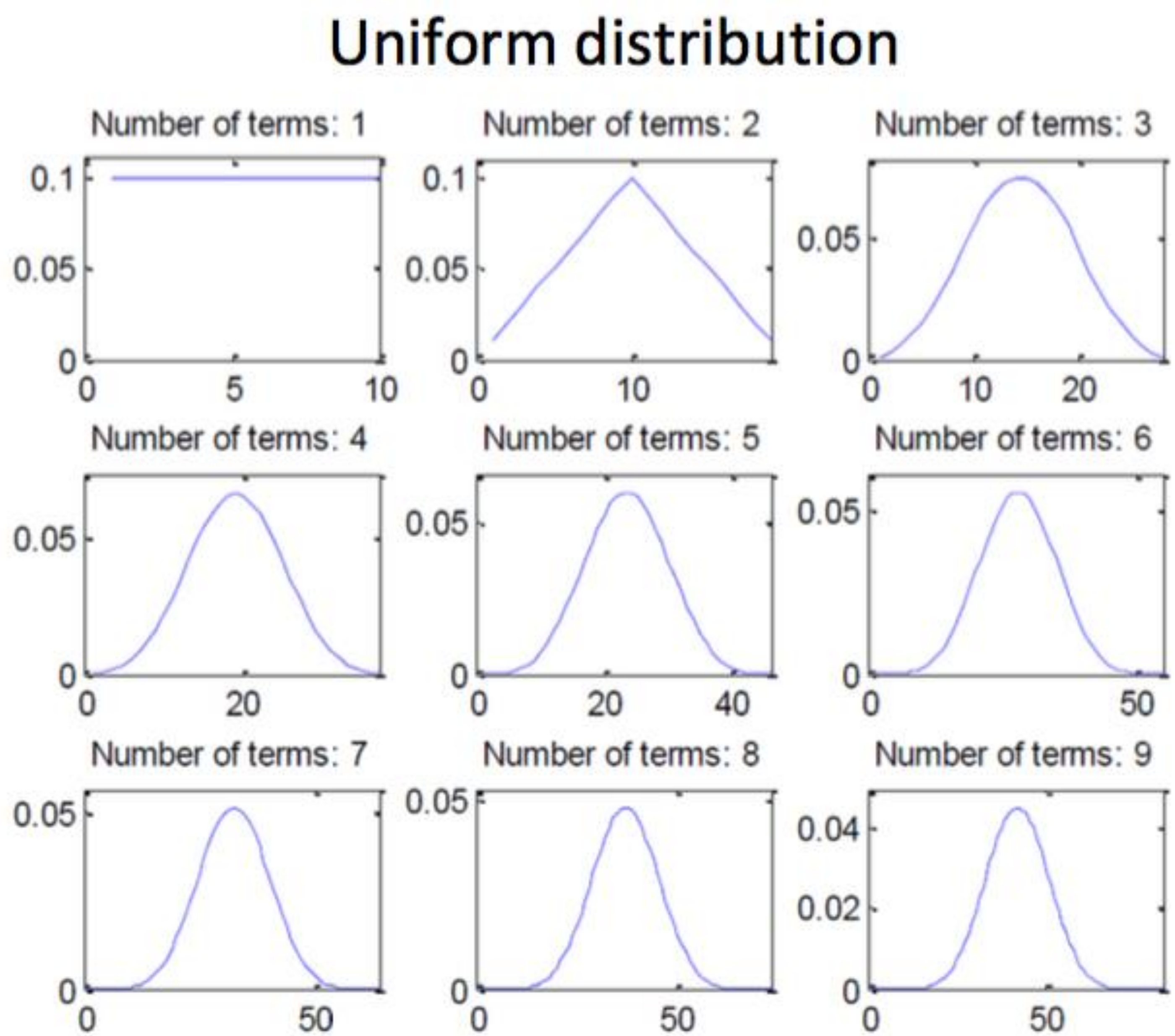
Sum of Random Variables

- The random variables are i.i.d and taken from the same distribution.



Sum of Random Variables

- The random variables are i.i.d and taken from the same uniform distribution.



Words and Concepts to Know

Central Limit Theorem

Convolution

Transformation of stochastic variables

Rayleigh Distribution

Randomly Sampled Data

Bivariate Normal Distribution