Shanmugam

problem 2.18

a) Show that $E[a + bX] = a + bE[X]$

$$E[a+bX] = \int_x (a+bX)\, f_X(x)\, dx$$

(pr. definition)

hvor $f_X(x)$ er en funktion af X og er pdf for X.

$$E[\cancel{x}a+bX] = \int_x a \cdot f_X(x)\, dx + \int_x bX \cdot f_X(x)\, dx$$

$$= a \cdot \underbrace{\int_x f_X(x)\, dx}_{=1} + b \underbrace{\int_x X \cdot f_X(x)\, dx}_{=E[X]}$$

$$= a + bE[X].$$

b) $$E[aX + bY] = \iint_{X,Y} (aX + bY)\, f_{X,Y}(x,y)\, dx\, dy$$

hvor $f_{X,Y}(x,y)$ er joint pdf for X og Y

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problem 2.18
(continued)

b) fordsad

$$E[aX + bY] = \int_{x,y} (aX + bY) f_{x,y}(x,y)\, dx\, dy$$

$$= \int_{x,y} aX \cdot f_{x,y}(x,y)\, dx\, dy \; +$$

$$\int_{x,y} bY\, f_{x,y}(x,y)\, dx\, dy$$

$$= \int_x ax \int_y f_{x,y}(x,y)\, dy\, dx$$

$$+ \int_y by \int_x f_{x,y}(x,y)\, dx\, dy$$

pr. definition:

$$\boxed{\int_y f_{x,y}(x,y)\, dy = f_x(x)}$$

$$E[aX + bY] = \int_x aX \cdot f_x(x)\, dx$$

$$+ \int_y bY\, f_y(y)\, dy$$

$$= aE[X] + bE[Y]$$

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problem 2.18 (fortsat)

c) $$\mathrm{var}[aX + bY] = \int_{x,y} (aX + bY)^2 f_{xy}(x,y)\,dx\,dy$$

$$- E[aX + bY]^2$$

$$= \int_{x,y} (a^2X^2 + b^2Y^2 + 2aXbY) f_{xy}(x,y)\,dx\,dy$$

$$- (aE[x] + b[Y]])^2$$

$$= a^2\int_x x^2 f_x(x)\,dx + b^2\int_y y^2 f_y(y)\,dy$$

$= a^2\mathrm{var}[x]$

$$+ 2ab \int_{xy} xY f_{xy}(x,y)\,dx\,dy$$

$$- a^2E[x]^2 - b^2E[y]^2 - [2a\cdot b\,E[x]\,E[Y]]$$

Covar(x, y)

$$= a^2\,\mathrm{var}[X] + b^2\,\mathrm{var}[Y] + 2ab\,\mathrm{Covar}[x,Y]$$

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problem 2.24

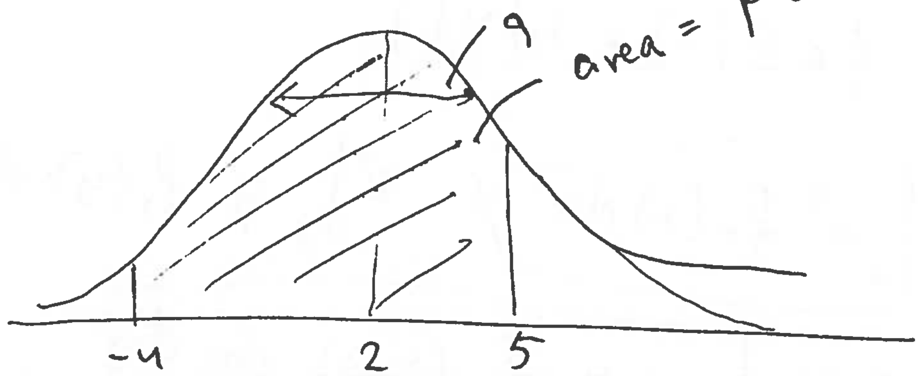
X is a Gaussian Random Variable

$\mu_x = 2 \qquad \sigma_x^2 = 9 \qquad \sigma_x = \sqrt{9} = 3$

thus: $X \sim \mathcal{N}(2, 9)$

find $P(-4 < X \le 5)$

area $= P(-4 < x \le 5)$

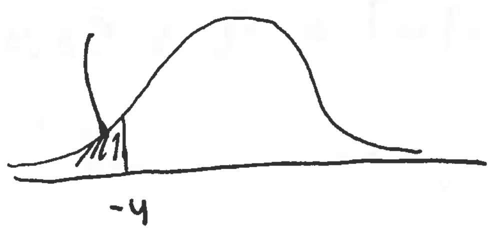


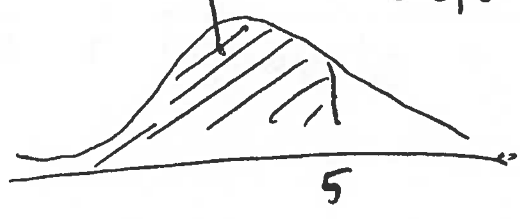
use matlabs normcdf(x, μ, σ)

p = normcdf(-4,2,3) = 0,023

p = normcdf ($\cancel{4}$ 5, 2, 3)

= 0,84





$P(-4 < X \le 5) = 0{,}84 - 0{,}0227 \cong \underline{\underline{0{,}819}}$

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problem 2.29

$f_{x,y}(x,y) = \frac{1}{2} \qquad 0 \le x \le y \qquad 0 \le y \le 2$

$0 \le x \le 2 \Rightarrow x \le y \le 2$

a)

$f_x(x) = \int_{y} f_{x,y}(x,y)\, dy$

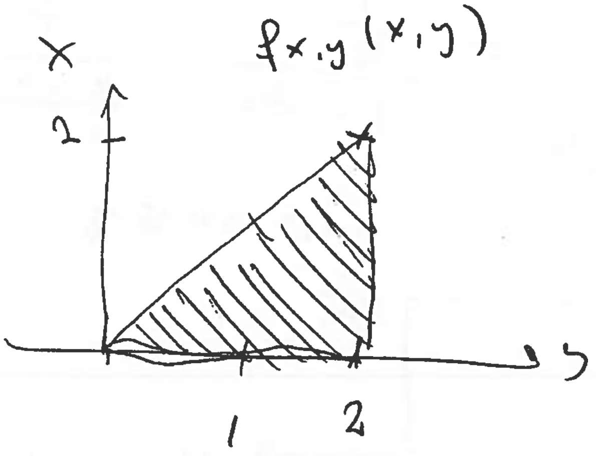
$= \int_{x}^{2} \frac{1}{2}\, dy = \left[\frac{1}{2} y \right]_{x}^{2}$

$= 1 - \frac{1}{2}x$ fra $0 \le x \le 2$

$f_y(y) = \int_x f_{x,y}(x,y)\, dx$

$= \int_0^y \frac{1}{2}\, dx = \left[\frac{1}{2} x \right]_0^y$

$= \frac{1}{2} y \qquad 0 \le y \le 2$



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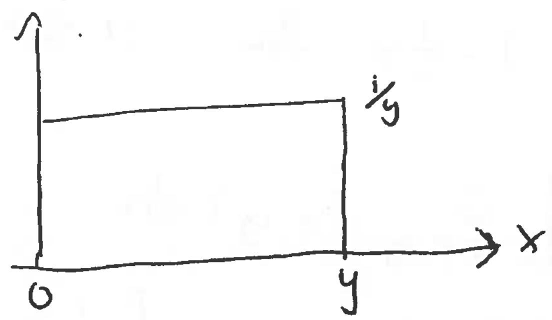
problem 2.29

b) conditional pdfs

$$f_{x|y}(x|y) = \frac{f_{x,y}(x,y)}{f_y(y)}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}\cdot y} = \frac{1}{y}$$

$f_{x|y}(x|y)$ $\quad 0 \le x \le y \qquad 0 \le y \le 2$

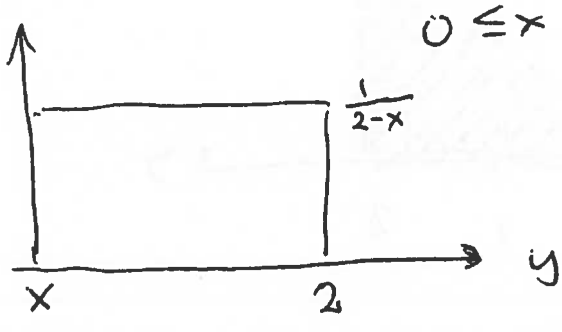


$$f_{y|x}(y|x) = \frac{f_{x,y}(x,y)}{f_x(x)}$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}x} = \frac{\frac{1}{2}}{\frac{2-x}{2}} = \frac{1}{2-x}$$

$f_{y|x}(y|x)$

$0 \le x \le 2 \qquad x \le y \le 2$



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problem 2.29

d) Independence between X and Y?

No, since

$$f_{x,y}(x,y) \neq f_x(x) \cdot f_y(y)$$

c) find ρ_{XY}

$$\rho_{XY} = \frac{E[XY] - E[X] \cdot E[Y]}{\sqrt{E[x^2] - E[x^2]} \cdot \sqrt{E[y^2] - E[y^2]}}$$

$$E[XY] = \int_{XY} XY \, f_{XY}(x,y) \, dx \, dy$$

$$= \int_0^2 y \cdot \int_0^y x \; \cancel{3} \; \frac{1}{2} \, dx \, dy$$

$$= \frac{1}{2} \int_0^2 y \left[\frac{1}{2} x^2 \right]_0^y dy$$

$$= \frac{1}{4} \int_0^2 y \cdot y^2 \, dy$$

$$= \frac{1}{4} \left[\frac{1}{4} y^4 \right]_0^2 = \frac{1}{16} \cdot 16 = 1$$

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problem 2.29 (cont'd)

c)

$$E[X \mid Y=1]$$

$$f_{X|Y=1}(x \mid y=1) = \frac{1}{1} = 1$$

$$0 \leq x \leq 1$$

$$E[X \mid Y=1] = \int_x x \cdot f_x(x)\, dx$$

$$= \int_0^1 x \cdot 1 \, dx = \left[\frac{1}{2} x^2\right]_0^1$$

$$= \frac{1}{2} \cdot 1^2 = \underline{\underline{\frac{1}{2}}}$$

$$E[X \mid Y=0,5] = \int_0^{0,5} x \cdot 2 \, dx$$

$$= 2\left[\frac{1}{2} x^2\right]_0^{0,5}$$

$$2 \cdot \frac{1}{8} = \underline{\underline{\frac{1}{4}}}$$

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problem 2.29

$$E[X] = \int_0^2 x\cdot\left(1 - \tfrac{1}{2}x\right)\,dx$$

$$= \left[\tfrac{1}{2}x^2 - \tfrac{1}{6}x^3\right]_0^2 = 2 - \frac{4}{3} = \frac{2}{3}$$

$$E[Y] = \int_0^2 y\cdot\tfrac{1}{2}y\,dy$$

$$= \left[\tfrac{1}{2}\cdot\tfrac{1}{3}y^3\right]_0^2 = \frac{8}{6} = \frac{4}{3}$$

$$E[X^2] = \int_0^2 x^2\left(1 - \tfrac{1}{2}x\right)dx$$

$$\left[\tfrac{1}{3}x^3 - \tfrac{1}{8}x^4\right]_0^2$$

$$= \frac{8}{3} - 2 = \frac{2}{3}$$

$$E[Y^2] = \int_0^2 y^2\cdot\tfrac{1}{2}y\,dy$$

$$= \left[\tfrac{1}{2}\cdot\tfrac{1}{4}y^4\right]_0^2$$

$$= 2$$

$$\rho_{xy} = \frac{1 - \frac{2}{3}\cdot\frac{4}{3}}{\sqrt{\frac{2}{3} - \frac{4}{9}}\sqrt{2 - \frac{\cancel{8}16}{9}}} = \underline{\underline{+\frac{1}{2}}}$$