Solutions

1 The test size is

$$z = \frac{\hat{\mu}_{obs} - \mu}{\sigma/\sqrt{n}} = \frac{260 - 250}{2.5/\sqrt{25}} = 20$$

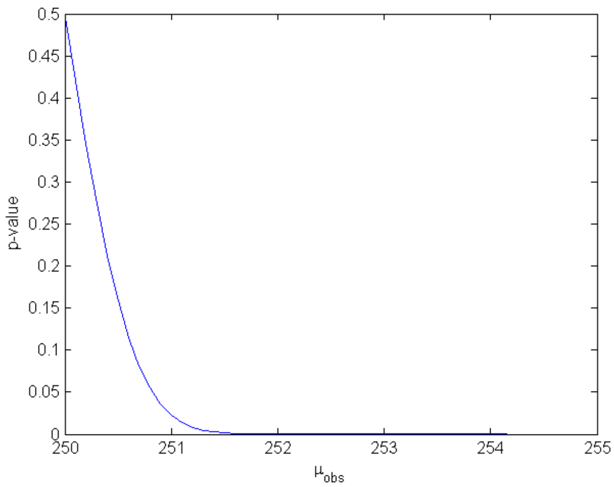
and the p-value is

$$\Pr(\hat{\mu} \geq \hat{\mu}_{obs}) = \Pr(Z \geq z) = 1 - \Pr(Z \leq z) = 1 - \Phi(z) = 1 - \Phi(20) = 1 - 1 = 0$$

Since the p-value is smaller than 0.05, we reject the hypothesis that $\hat{\mu}_{obs} = 260$ is a random sample from a normal distribution with mean 250 and variance 0.25. In other words, if the machine was adequately calibrated, it would be unlikely to observe a sample mean that is more extreme than 260 grams. Hence, we conclude that the machine is *not* adequately calibrated.

2 Your task is to find $\hat{\mu}_{obs}$, such that the p-value is 0.05. You can do this by trial and error. The Matlab code below calculates the p-value for $\hat{\mu}_{obs}$ ranging from 250 to 250 at intervals of 0.1. By plotting the p-value against $\hat{\mu}_{obs}$ I can find out where the p-value is 0.05.

```
range=250:0.1:255;
pval=1-normcdf((range-250)/(2.5/sqrt(25)));
plot(range,pval)
xlabel('\mu_{obs}')
ylabel('p-value')
```



By inspecting this plot, I find that the p-value is approximately 0.05 for $\hat{\mu}_{obs} = 250.825$ grams. Thus, the interval for which we would conclude that the machine is adequately calibrated is $\hat{\mu}_{obs} \in [250; 250.825]$.

3 We have

$$\begin{aligned}\Pr(\hat{\mu} \geq (\mu + |\mu - \hat{\mu}_{obs}|)) &= \Pr(\hat{\mu} \geq (250 + |250 - 250.2|)) = \Pr(\hat{\mu} \geq 250 + 0.2)\\ &= \Pr(\hat{\mu} \geq 250.2) = \Pr\left(Z \geq \frac{250.2 - \mu}{\sigma/\sqrt{n}}\right) = \Pr\left(Z \geq \frac{250.2 - 250}{2.5/\sqrt{25}}\right)\\ &= \Pr(Z \geq 0.4) = 1 - \Pr(Z \leq 0.4) = 1 - \Phi(0.4) = 1 - 0.6554 = 0.34\end{aligned}$$

$$\Pr\big(\hat{\mu} \le (\mu - |\mu - \hat{\mu}_{obs}|)\big) = \Pr\big(\hat{\mu} \le (250 - |250 - 250.2|)\big) = \Pr(\hat{\mu} \le 250 - 0.2)$$
$$= \Pr(\hat{\mu} \le 249.8) = \Pr\left(Z \le \frac{249.8 - \mu}{\sigma/\sqrt{n}}\right) = \Pr\left(Z \le \frac{249.8 - 250}{2.5/\sqrt{25}}\right)$$
$$= \Pr(Z \le -0.4) = \Phi(-0.4) = 0.34$$

Accordingly, the p-value is approximately $0.34 + 0.34 = 0.68$. Hence, if we ignore the sign of the deviation from the mean, which makes good sense in practice, there is a 68% chance of observing a sample mean that is more extreme than 250.2 grams. Using the same arguments is in the example, we can conclude that the machine is adequately calibrated.

4 Looking at the solution of Problem 3, you can verify yourself that if

$$z = \frac{\hat{\mu}_{obs} - \mu}{\sigma/\sqrt{n}}$$

and $\hat{\mu}_{obs} \ge \mu$, then z is positive (or equal to zero) and the p-value is

$$pval = \Pr(Z \ge z \cup Z \le -z) = \Pr(Z \ge z) + \Pr(Z \le -z) = \big(1 - \varphi(z)\big) + \varphi(-z)$$

Now, if $\hat{\mu}_{obs} < \mu$ then z is negative, and the p-value is

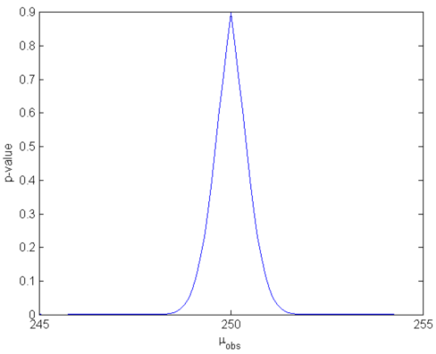
$$pval = \Pr(Z \le z \cup Z \ge -z) = \Pr(Z \le z) + \Pr(Z \ge -z) = \varphi(z) + \big(1 - \varphi(-z)\big)$$

so in general (i.e., regardless of the sign of z), we have

$$pval = \big(1 - \varphi(|z|)\big) + \varphi(-|z|)$$

Following the same procedure is in Problem 3, we can now plot the p-value vs $\hat{\mu}_{obs}$:

```
range=245:0.1:255;
z = (range-250)/(2.5/sqrt(25));
pval = 1-normcdf(abs(z)) + normpdf(abs(z));
plot(range,pval)
xlabel('\mu_{obs}')
ylabel('p-value')
```



According to my inspection of the plot, the p-value is larger than 0.05 for $\hat{\mu}_{obs} \in [248.9; 251.1]$.