#part1

**a)(Theorem 2.33)Suppose K** ⊂ **Y** ⊂ **X. Then (K is compact relative to X.)⬄(K is compact relative to Y.) Question will ask only => or <=**

**Proof :**

Suppose K is compact relative to X, i.e., every open (relative to X) cover of K contains a finite subcover (relative to X).

Let {Gα} be an open (relative to Y ) cover of K.

Then for every α, there exists an open subset, Hα, of X such that Gα = Hα ∩ Y (Gα is same to Hα intersected Y).

Since {Gα} covers K, then K ⊂ ∪α Gα = ∪α(Hα ∩ Y ) ⊂ ∪α Hα, which implies that {Hα} is an open (relative to X) cover of K. Since K is compact relative to X, there are α1,...,αn such that K ⊂ Hα1 ∪ ... ∪ Hαn. Since K ⊂ Y , K ⊂ (Hα1∪...∪Hαn)∩Y = (Hα1∩Y )∪...∪(Hαn∩Y ) = Gα1 ∪ ... ∪ Gαn.

Then {Gα1,...,Gαn} is a finite subcover (relative to Y ) of K.

Now suppose that K is compact relative to Y , i.e., every open (relative to Y ) cover of K contains a finite subcover (relative to Y ). Let {Vα} be an open (relative to X) cover of K.

For every α, let Wα = Vα ∩ Y . (Wα is same to Vα intersected Y)

Note that, for every α, Wα is open relative to Y , and hence {Wα} forms an open (relative to Y ) cover of K.

Since K is compact relative to Y , then there exist α1,...,αn such that {Wα1,...,Wαn} is a finite subcover (relative to Y ) of K.

Since Wα1 ⊂ Vαn for i = 1,...,n, then {Vα1,...Vαn forms a finite subcover (relative to X) of K.

b)(Theorem 2.34)Compact subsets of metric spaces are closed.

**Proof** :

Suppose K is a compact subset of a metric space X.

Let p ∈ Kc. For every q ∈ K, let Wq and Vq be open neighborhoods of q and p, respectively, such that Wq ∩ Vq 6= ∅ (we can do this by choosing the radius of each neighborhood to be less than d(p, q), which is nonzero since p 6= q and X is a metric space.

Note that {Wq} is an open cover of K. Since K is compact, then there exist a finite number of points q1,...,qn, such that {Wq1,...,Wqn} is a finite subcover of K.

Since K ⊂ Wq1 ∪...∪Wqn, which implies that (Wq1 ∪ ... ∪ Wqn)c ⊂ Kc, and Vαi ⊂ Wc αi for i = 1,...,n, then V = Vα1 ∩ ... ∩ Vαn ⊂ Wc α1 ∩ ... ∩ Wc αn = (Wα1 ∪ ... ∪ Wαn)c ⊂ Kc .

Since V is a neighborhood of p contained in Kc, then p is an interior point of Kc. Since p was arbitrarily chosen from Kc, then Kc is open, which implies that K is closed.

c)(Theorem 2.35)Closed subsets of compact sets are compact.

**Proof** :

Suppose F ⊂ K ⊂ X, F is closed (relative to X), and K is compact. Let {Vα} be an open cover of F.

Note that Ω = {Vα}∪{Fc} is an open cover of K.

Since K is compact, there exists a finite subcover Φ of K. Note also that since F ⊂ K, then Φ covers F. Thus, F is compact.

d)(Theorem 2.36)if{Kα} is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of {Kα} is nonempty, then Ո Kα is nonempty.

**Proof:**

Suppose that ∩α∈A Kα is empty.

That is, U α∈A Kcα = Φc = X,Since compact sets are closed, each Kcα is open.

Take any one of the Kα s;call it Kβ. Since Kβ is compact and since { Kcα } α∈A is an open cover of Kβ,

There exist α1, α2,…,αk∈A such that Kβ⊂ Kcα1U…U Kcαk(the finite subcover).

So Kcα1∩…∩ Kcαk ⊂ Kcβ, which inplies K α1∩…∩ K αk∩ Kβ = Φ.

This is a contradiction to the hypothesis that the intersection of every finite subcollection of {Kα} is nonempty.

e)(Theorem 2.37)If E is an infinite subset of a compact set K, then E has a limit point in K.

**Proof:**

Suppose, for the purposes of contradiction, that E has no limit point in K.

Then, for every q in K, there exists a neighborhood Vq such that Vq ∩ E ⊂ {q}. {Vq} is an open cover of K, but any finite subcollection of {Vq} is then a subset of a finite set, which could not contain the infinite set K (K is infinite since it includes an infinite set, E).

This contradicts the compactness of K.

f)(Theorem 2.38)If{In} is a sequence of (closed ) intervals in R, such that In ͻ In+1 (n = 1,2,3,…), then Ո1∞ In is not empty.

**Proof:**

If T In has two or more points, say x, y ∈ T In with x 6= y, then x, y ∈ In for all n,

so diam In ≥ d(x, y) > 0, contradicting the fact that diam In → 0. Therefore, we know it has at most one point; all that is left is to prove it is nonempty.

For each n, pick a point xn in In (which is nonempty).

Since In ⊃ In +1, we know {xn,xn+1,...} ⊂ In.

The fact that diam In → 0 implies that for any ² > 0 we can pick some N ∈ N such that diam In < ² for n ≥ N, which means for any n,m ≥ N we have |xn − xm| ≤ diam({xN ,xN+1,xN+2,...}) ≤ diam In < ², so {xn} is a Cauchy sequence.

Since X is complete, that means that {xn} converges, say xn → x. En is closed and it contains {xn,xn+1,...}, (i.e. the tail of the sequence {xn}) so x ∈ En for all n.

Therefore, x ∈ T In, so it is nonempty, finishing the proof

g)(Theorem 2.41) If E ᴄ Rk has one of the following three properties, then it has the other two:

(a) E is closed and bounded .

(b) E is compact.

(c) Every infinite subset of E has a limit point in E. Question will ask only 1 step among 3.

**Proof:**

If (a) holds, then E ⊂ I for some k-cell I , and (b).

Theorem 2.37 shows that (b) implies (c).

It remains to be shown that (c) implies(a).

If E is not bounded, then E contains points xn with

|xn| > n (n = 1,2,3, … )

The set S consisting of these points xn is infinite and clearly has no limit point in Rk, hence has none in E, Thus(c) implies that E is bounded.

If E is not closed, then there is a point x0∈Rk which is a limit point of E but not a point of E. For n=1,2,3,. . ., there are points xn. Then S is infinite (otherwise |xn – x0| would have a constant positive value, for infinitely many n), S has x0 as a limit point, and S has no other limit point in Rk .

For if y ∈Rk, y ≠x0, then

|xn - y| >= |x0 – y| - |xn - x0|

>= |x0 – y| - 1/n >= ½|x0 - y|

For all but finitely many n; this shows that y is not a limit point of S.

Thus S has no limit point in E; hence E must be closed if(c) holds.

h)(Theorem 2.42;Weierstrass) Every bounded infinite subset of Rk has a limit point in Rk.

**Proof :**

Let S be a bounded infinite subset of R , and let I be a finite closed interval which contains S .

Cut I in half. At least one of the resulting two closed subintervals of I contains infinitely many elements of S . Call this interval I1 .

Continue in this manner to build the closed interval In , for each natural number n , with the length of In equal to 1/2n times the length of I and In contains infinitely many elements of S .

The nested interval property of R tells us that the intersection T ∞ n=1 In is non-empty. Let p be an element of T ∞ n=1 In .

We will show that p is a limit point of S . Given ε > 0 , there exists n large enough that the length of In is less than ε .

We know that p ∈ In . It follows that In ⊆ Nε(p) . Furthermore, there is at least one element q of S with q 6= p and q ∈ Nε(p) ; since In ∩ S is infinite.

i)(Theorem 2.43) Let P be a nonempty perfect set in Rk. Then P is uncountable.

**Proof :**

 Assume P is countable, and let {x1,x2,x3,…}{x1,x2,x3,…} be an enumeration of P.

Since each singleton is closed, each Xi=X∖{xi}Xi=X∖{xi} is open for each ii.

Moreover, each of them is dense, since each point is an accumulation point of XX.

By Baire's Theorem, ⋂i∈NXi⋂i∈NXi must be dense, hence nonempty, but it is readily seen it is empty, which is absurd.

j)(Theorem 2.47) A subset E of R is connected if and only if it has the following property. If x Є E, y Є E, and x<z<y, then z Є E. Question will ask only => or <=

**Proof**:

To prove the converse suppose that E is not connected.

Then there are nonempty separated sets A and B such that A∪B=E. Pick x∈A,y∈B and assume (without loss of generality) x<y.

Define

z=sup(A∩[x,y])z=sup(A∩[x,y])

By Theorem 2.28, z∈A----; hence z∉B.

In particular, x≤z<y. If z∉A, it follows that x<z<y, and z∉E.

If z∈A, then z∉B----, hence there exists z1 such that z<z1<y and z1∉.

Then x<z1<y and z1∉.