#part4 .**Prove the statement.**

**a)(Theorem 3.11)**

**(a)In a metrix space X, every convergent sequence is a Cauchy sequence.**

**Proof:**

Let {Pn} be a convergent sequence and let p Є X be the point to which it converges.

Let Ꜫ > 0. Then there exists N Є **N** such that if n >= N,then d(pn,p)< Ꜫ /2.

Thus, if m,n >= N, then d(pm,p)< Ꜫ /2 and d(pn,p) < Ꜫ /2, so that

d(pm,pn) <= d(pm,p) + d(pn,p) < Ꜫ /2+ Ꜫ /2 = Ꜫ .

We have proved that {pn } is a Cauchy sequence.

**(b)If X is a compact metric space and if (Pn) is a Cauchy sequence in X, then {Pn}converges to some point of X.**

**Proof:**

E is finite. Let E = {q1,…, qk}.Let Fi = {n Є **N**: pn = qi} for I = 1, … k.Then **N** = Uki=1Fi.

Hence there exists I such that Fi is infinite.

Let Fi = {n1,n2,…}, where 1 <= n1 <=n2,… .

Then pnk = qi for k Є **N.**

We shall show that {pn} converges to qi.

Let Ꜫ>0. Since {pn} is a Cauchy sequence, there exists N Є **N** such that for all m,n >= N we have d(pm,p)< Ꜫ.

Since Fi is infinite, there exists nk Є Fi such that nk >= N.

Then, if n>=N, we then have

d(pn,qi) = d(pn,pnk) < Ꜫ

since pnk = qi and nk >= N.

**b) (Theorem 3.14) Suppose {Sn} is monotonic. Then,**

**({Sn}converges.)⬄({Sn} is bounded.)**

**Proof =>:**

Suppose sn → s.

There exists a positive integer N such that |s−sn| < 1 for all n>N.

Therefore, it follows that |sn| = |sn − s + s|≤|sn − s| + |s| < 1 + |s| for all n > N.

Let M = max {|s1|, |s2|, ..., |sN |, 1 + |s|}. Then |sn| < M for all n. Therefore {sn} is bounded.

**c)(Theorem 3.17)Let {Sn} be a sequence of real numbers.**

**Let E and s\* have the same meaning as in Definition.**

**Then , s\* has the following two properties:**

1. **S\* Є E**
2. **If x > s\*, there is an integer N such that Sn < x for n >= N, Moreover, s\* is the only number with the properties (a) and (b).**

**Proof:**

**d) (Theorem 3.23) (∑an converges.) =>( limn=∞an = 0)**

**Proof:**

First let’s suppose that the series starts at n=1.

If it doesn’t then we can modify things as appropriate below. Then the partial sums are,

sn−1=∑ n−1 i=1ai=a1+a2+a3+a4+⋯+an−1

sn= ∑ni=1 ai=a1+a2+a3+a4+⋯+an−1+an

Next, we can use these two partial sums to write,

an=sn−sn−1

Now because we know that ∑an is convergent we also know that the sequence {sn}∞n=1 is also convergent and that limn→∞sn=s for some finite value s.

However, since n−1→∞ as n→∞ we also have limn→∞sn−1=s.

We now have,

limn→∞an=limn→∞(sn−sn−1)=limn→∞sn−limn→∞sn−1=s−s=0

**e)(Theorem 3.26) Let 0 <= x.(0 <= x < 1) ⬄(∑∞n=0xn converges to 1/(1-x)).Also,(x > 1) ⬄(∑∞n=0xn diverges.)**

**Proof:**

**f) (Theorem 3.27) Suppose a1>=a2>=a3>=…>=0.**

**(∑∞n=1an converges.) ⬄(∑∞k=02ka2k(=a1+ 2 \*a2+4 \*a4+…) Converges.)**

**Proof:**

**g) (Theorem 3.28) (p>1)⬄( ∑1/np converges. ) and (p<=1)⬄( ∑1/np diverges.)**

**Proof:**