Optimal Control & Modelling of 2D Quadcopter

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***Abstract*—A model for a 2D quadcopter with elevation and pitch state was derived using Euler-Lagrange. To control the system to a desired elevation and pitch a LQR controller with feedforward and integral action was developed. The full state of the system was estimated using a Luenberger observer. Simulation of the closed loop nonlinear system showed that the controller indeed stabilized the system around the equilibrium point and demonstrated good tracking performance.**

I. Introduction

THE use of quadcopters and other unmaned aerial vehicles (UAVs) has increased drastically the past years. UAVs has revolutionized numerous industries, from agriculture to surveillance, due to their agility, versatility and cost-effectiveness. Quadcopters has emerged as a prominent choice due to their simple design, low cost and maneuverability. A important problem is ensuring that the quadcopter is stable in a wide range of operating conditions.

The project focuses on the modelling and control of a simplified quadcopter. The quadcopter is modelled as a 2D side-view of the drone. I.e. it has a elevation and pitch angle that should be controlled. To control the elevation and pitch two motors on either side of the main body is to be utilized.

There are many methods that can be used to control quadcopters. A common method is PID controllers in a cascaded structure [1]. This is what common open source software's like ArduPilot often use [2]. However, many other techniques has been shown to work. This includes Linear-Quadratic Regulator, Feedback Linearization (FL), Sliding Mode Control (SMC), Model Predictive Control (MPC), H_∞ and many more [3].

II. Model Derivation

The configuration of the 2D drone is represented using two variables; elevation, e, and the pitch p. The dynamics are modelled using three point masses; one for the body, m_b, and one for each of the motors, m_m. The motors are located at a distance l away from the center of the drone. In addition to the gravitational forces, two external forces, F_f and F_r, can be used to control the state of the drone. These forces represents the thrust from the propellers, and act perpendicular to the front and rear motor respectively.

The equations of motion is derived using the Euler-Lagrange method. The potential energy and kinetic energy is given in (1). Where r_i is the position of mass i and $\mathbf{F}_{g,i}$ is gravitational force acting on mass i.

$$\begin{aligned} U &= -\mathbf{r}_b^T\mathbf{F}_{g,b} - \mathbf{r}_f^T\mathbf{F}_{g,f} - \mathbf{r}_r^T\mathbf{F}_{g,r} \\ T &= \frac{1}{2}(m_b\dot{\mathbf{r}}_b^T\dot{\mathbf{r}}_b + m_m\dot{\mathbf{r}}_f^T\dot{\mathbf{r}}_f + m_m\dot{\mathbf{r}}_r^T\dot{\mathbf{r}}_r) \end{aligned} \tag{1}$$

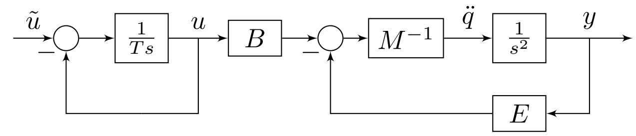

Fig. 1: Block diagram of linearized system.

The relation of the external forces from the motors and internal torques are related through $\tau_{ext} = \sum_{j=1}^{2}\mathbf{J}_{p,j}^T\mathbf{F}_j$, where $\mathbf{J}_{p,j} = \frac{\partial \mathbf{r}_j}{\partial \mathbf{q}}$ is the translational jacobian and $\mathbf{F}_j$ is the force vector from motor j. This yields the nonlinear system (2).

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \tau_{ext} \tag{2}$$

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} m_b + 2m_m & 0 \\ 0 & 2l^2 m_m \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{g} &= \begin{bmatrix} g(m_b + 2m_m) \\ 0 \end{bmatrix}, \tau_{ext} = \begin{bmatrix} cos(p)(F_f + F_r) \\ l(F_f - F_r) \end{bmatrix} \end{aligned} \tag{3}$$

Linearizing (2) around $\mathbf{x}_0 = [\mathbf{q}_0^T, \dot{\mathbf{q}}_0^T]^T = [0, \pi/4, 0, 0]^T$ with $\mathbf{u}_0$ chosen to ensure steady state yields the linearized system (4)

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{E}\mathbf{q} + \mathbf{B}\mathbf{u}) \tag{4}$$

$$\mathbf{E} = \begin{bmatrix} 0 & -g(m_b + 2m_m) \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ l & -l \end{bmatrix} \tag{5}$$

Where $\mathbf{M}$ is the same generalized mass-intertia-matrix. The values of the matrices $\mathbf{E}$ and $\mathbf{B}$ are shown in (5).

Furthermore, since the motors cannot provide the requested thrust immediately they are modelled as first-order systems with a time constant T. This yields that

$$\dot{u}_i = \frac{1}{T}(\tilde{u}_i - u_i), \forall i = 1, 2 \tag{6}$$

Combinging the two systems gives the system depicted in (Fig. 1)

III. Control Law

To control the system to a desired elevation and pitch a LQR controller with feedforward and intergral action was developed (7). Integral action was added by introducing two new states ξ and ζ such that $\dot{\xi} = e - e_r$ and $\dot{\zeta} = p - p_r$. Where e_r and p_r is the references for the elevation and pitch respectively. The full augmented state is then $\tilde{\mathbf{x}} = [\xi, \zeta, e, p, \dot{e}, \dot{p}, u_f, u_r]^T$. The feedback gain $\mathbf{K}$ was solved for given the cost matrices $\mathbf{Q} = diag\{[10^2, 10^2, 10^3, 10^1, 10^3, 10^2, 1, 1]\}$ and $\mathbf{R} =$

Symbol[Unit]	value
m_b[Kg]	2.0
m_m[Kg]	0.1
l[m]	0.2
T[s]	0.2

TABLE I: Parameter values used in simulation.

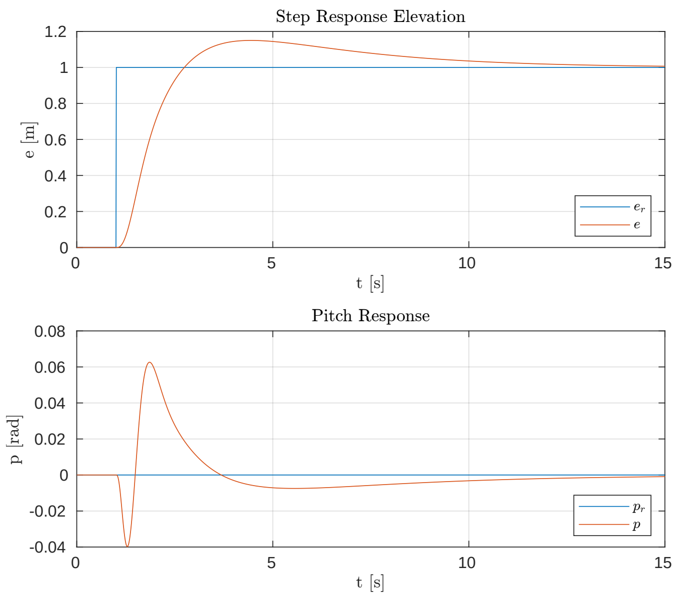

Fig. 2: Simulation of nonlinear response to step in elevation reference.

$diag\{[1, 1]\}$. The feedforward gain matrix, $\mathbf{K_f}$, was designed to ensure that $e = e_r \wedge p = p_r$ is a equilibrium of the closed loop system.

$$\tilde{\mathbf{u}} = \mathbf{K_f}\mathbf{r} - \mathbf{K}\hat{\mathbf{x}} \tag{7}$$

The LQR controller needs the entire state x to calculate the control effort. Since we are only measuring $y = q = [e, p]^T$ we need to estimate the other states $\dot{q}$ and u. This is done using a Luenberger observer with constant feedback gain $\mathbf{L}$ (8), where $\mathbf{L}$ was found using pole-placement. The poles where linearly spaced on a circle arc in the left-half plane with a radius of 15 and angle of $20°$.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\tilde{\mathbf{u}} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{8}$$

IV. Simulation Results

The nonlinear closed loop system was simulated with the parameters given in (TABLE I). A step in the elevation reference after one second was applied to the system (Fig 2). The elevation responds well with a bit of over-shoot. The pitch stays close to the zero reference.

A step with magnitude 0.2 was similarly applied to the pitch reference (Fig 3). The tracking of the pitch is very good, however we see that the elevation drops a bit before it climbs back to its reference. Without the integral action one would observe a constant offset in the elevation due to the nonlinearity of the system.

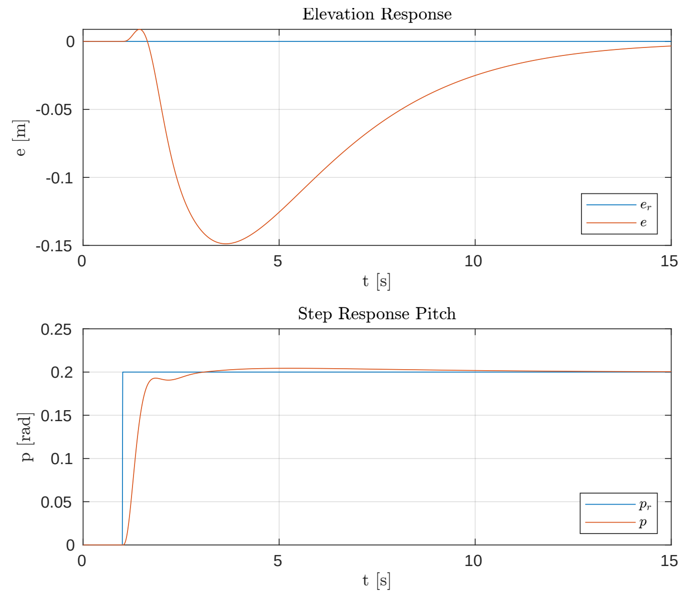

Fig. 3: Simulation of nonlinear response to step in pitch reference angle.

V. Conclusion

In this study, a nonlinear model of a 2D quadcopter was derived. The model was linearized around a given pitch angle. We proceeded to develop a Linear-Quadratic Regulator (LQR) controller augmented with integral action to stabilize the system. A feedforward term was added to make the quadcopter track a reference elevation and pitch. To estimate the full state a Luenberger observer was develop using pole placement.

The performance of the controller was evaluated through simulations of the nonlinear system operating in closed loop with the proposed controller. The results demonstrates that the controller effectively stabilizes the system around the linearization point. Due to the integral action the system does not exhibit any steady-state error. It remains to see how robust the controller is to noise, modelling errors and other disturbances.

References

[1] H. Bolandi, M. Rezaei, R. Mohsenipour, H. Nemati, and S. M. Smailzadeh, "Attitude control of a quadrotor with optimized pid controller," *Scientific Research*, vol. 4, no. 3, 2013. [Online]. Available: https://www.scirp.org/journal/paperinformation?paperid=35654

[2] Arudupilot. [Online]. Available: http://www.ardupilot.org/

[3] M. Rinaldi, S. Primatesta, and G. Guglieri, "A comparative study for control of quadrotor uavs," *Applied Sciences*, vol. 13, no. 6, 2023. [Online]. Available: https://www.mdpi.com/2076-3417/13/6/3464