Støddfrøðiligt grundarlag til tilgjørt vit - 2025

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Contents

1 Linear Algebra

1.1 Matrix

A matrix is usually written as $\mathbf{A} \in \mathbb{R}^{m \times n}$ where m means the amount of rows and n is the amount of columns. An example of a matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad a_{ij} \in \mathbb{R}$$

1.2 Matrix addition

Property	**Description**
Associative	$(A+B)+C = A+(B+C)$
Commutative	$A+B = B+A$
Identity	$A+O = O+A = A$, where O is the zero matrix
Inverse	$A+(-A) = O$, where $-A$ is the additive inverse

Example

1.3 Matrix multiplication

Property	**Description**
Associative	$(AB)C = A(BC)$
Distributive over Addition	$A(B+C) = AB+AC$ $(A+B)C = AC+BC$
Identity	$AI = IA = A$, where I is the identity matrix
Non-commutative	$AB \neq BA$ (in general)

Make sure to check the dimensions of the matrices, that they are compatible and that the resulting matrix has the correct dimensions. On the example below $A \cdot B = C$ notice the dimensions.

$$\begin{aligned} A &\in \mathbb{R}^{n \times m} \\ B &\in \mathbb{R}^{m \times p} \\ C &\in \mathbb{R}^{n \times p} \end{aligned}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 3},$$
$$B = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2},$$
$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} \in \mathbb{R}^{2\times 2},$$
$$BA = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 2 \\ -2 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

1.4 Inverse and determinant

A matrix only has a inverse if its determinant is not 0 so always calculate it before finding the inverse. If it exists the following holds:

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

The determinant is easy to find for a 2×2 matrix:

$$Det(\mathbf{A}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a12$$

After checking that the determinant is not zero proceed to find the inverse. If you're working with a 2×2 matrix the following setup gives the inverse, fig. **??**
For a 3×3 matrix Sarrus's rule can be used.
For matrices 3×3 and larger one can use Laplace expansion.

1.5 Transposed matrix

To find the transpose

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

then

$$\mathbf{A}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

1.6 Particular and general solution

1.7 Gauss elimination

When doing gaussian elimination you are allowed to do the following operations:

- Exchange of two equations (rows in the matrix representing the system of equations)
- Multiplication of an equation (row) with a constant $\lambda \in \mathbb{R}\{0\}$
- Addition of two equations (rows)

Definition 2.6 (Row-Echelon Form). A matrix is in row-echelon form if:

- All rows that contain only zeros are at the bottom of the matrix; correspondingly, all rows that contain at least one nonzero element are on top of rows that contain only zeros.
- Looking at nonzero rows only, the first nonzero number from the left (also called the pivot or the leading coefficient) is always strictly to the right of the pivot of the row above it.

Remark (Reduced Row Echelon Form). An equation system is in *reduced row-echelon form* (also: *row-reduced echelon form* or *row canonical form*) if

- It is in row-echelon form.
- Every pivot is 1.
- The pivot is the only nonzero entry in its column.

Minus-1 trick

1.8 Groups

Definition 2.7 (Group). Consider a set G and an operation $\otimes : G \times G \to G$ defined on G. Then $G := (G, \otimes)$ is called a *group* if the following hold:

1. **Closure** of G under $\otimes$: $\forall x, y \in G : x \otimes y \in G$
2. **Associativity**: $\forall x, y, z \in G : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3. **Neutral element**: $\exists e \in G \, \forall x \in G : x \otimes e = x$ and $e \otimes x = x$
4. **Inverse element**: $\forall x \in G \, \exists y \in G : x \otimes y = e$ and $y \otimes x = e$, where e is the neutral element. We often write x^{-1} to denote the inverse element of x.

Remark The inverse element is defined with respect to the operation $\otimes$ and does not necessarily mean $\frac{1}{x}$. $\diamondsuit$

1.9 Vector spaces

Definition 1.1 (Vector Space)**.** A real-valued *vector space* $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

$$+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \tag{2.62}$$

$$\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V} \tag{2.63}$$

where

1. $(\mathcal{V}, +)$ is an Abelian group.
2. Distributivity:
   (a) $\forall \lambda \in \mathbb{R},\ x, y \in \mathcal{V} : \lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$.
   (b) $\forall \lambda, \psi \in \mathbb{R},\ x \in \mathcal{V} : (\lambda + \psi) \cdot x = \lambda \cdot x + \psi \cdot x$.
3. Associativity (outer operation): $\forall \lambda, \psi \in \mathbb{R},\ x \in \mathcal{V} : \lambda \cdot (\psi \cdot x) = (\lambda\psi) \cdot x$.
4. Neutral element with respect to the outer operation: $\forall x \in \mathcal{V} :\ 1 \cdot x = x$.

1.10 Basis and rank

To determine a basis take the vectors that span the vector subspace and put them into matrix form:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -7 \\ 1 \end{bmatrix}$$

We check that the following holds:

$$\sum_{i=1}^{4} \lambda_i \mathbf{x}_i = \mathbf{0}$$

$$\begin{aligned}[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4] &= \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \\ &\rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

The resulting matrix has three columns with pivots. Those form a basis for the subspace spanned by those four vectors.
We see that there are two dependent rows, those that are all zeros and that there is one free variable $\mathbf{x}_3$ hence the rank is 3:

$$rk(A) = 3$$

The column with no pivot, $\mathbf{x}_3$,

A matrix is full rank if the rank equals the largest possible rank for a matrix of the same dimensions.

1.11 Linear mappings

A **linear mapping** (or linear transformation) $\Phi : V \to W$ between vector spaces V and W over the same field preserves vector addition and scalar multiplication. For all $\mathbf{x}, \mathbf{y} \in V$ and $\lambda, \psi \in \mathbb{R}$:

$$\Phi(\mathbf{x} + \mathbf{y}) = \Phi(\mathbf{x}) + \Phi(\mathbf{y}), \quad \Phi(\lambda\mathbf{x}) = \lambda\Phi(\mathbf{x}).$$

This can be summarized as:

$$\Phi(\lambda\mathbf{x} + \psi\mathbf{y}) = \lambda\Phi(\mathbf{x}) + \psi\Phi(\mathbf{y}).$$

1.12 Special Cases of Linear Mappings

- **Isomorphism**: A linear mapping $\Phi : V \to W$ that is both injective and surjective (bijective). Two vector spaces are isomorphic if and only if $\dim(V) = \dim(W)$.
- **Endomorphism**: A linear mapping $\Phi : V \to V$ from a vector space to itself.
- **Automorphism**: An endomorphism that is also bijective.
- **Identity Mapping**: $\mathrm{id}_V : V \to V, \mathbf{x} \mapsto \mathbf{x}$.

1.13 Injective, Surjective, and Bijective Mappings

- **Injective (One-to-One)**: $\Phi(\mathbf{x}) = \Phi(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$.
- **Surjective (Onto)**: $\Phi(V) = W$.
- **Bijective**: Both injective and surjective. A bijective mapping Φ has an inverse Φ^{-1}.

1.14 Matrix Representation of Linear Mappings

For finite-dimensional vector spaces V and W with ordered bases $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $C = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$, the **transformation matrix** $\mathbf{A}_\Phi$ of Φ is defined by:

$$\Phi(\mathbf{b}_j) = \sum_{i=1}^{m} \alpha_{ij}\mathbf{c}_i, \quad \text{where } \mathbf{A}_\Phi(i,j) = \alpha_{ij}.$$

If $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the coordinate vectors of $\mathbf{x} \in V$ and $\Phi(\mathbf{x}) \in W$ with respect to B and C, then:

$$\hat{\mathbf{y}} = \mathbf{A}_\Phi \hat{\mathbf{x}}.$$

1.15 Example: Transformation Matrix

For a linear mapping $\Phi : V \to W$ with bases $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ and $C = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4)$, if:

$$\Phi(\mathbf{b}_1) = \mathbf{c}_1 - \mathbf{c}_2 + 3\mathbf{c}_3 - \mathbf{c}_4, \quad \Phi(\mathbf{b}_2) = 2\mathbf{c}_1 + \mathbf{c}_2 + 7\mathbf{c}_3 + 2\mathbf{c}_4, \quad \Phi(\mathbf{b}_3) = 3\mathbf{c}_2 + \mathbf{c}_3 + 4\mathbf{c}_4,$$

the transformation matrix $\mathbf{A}_\Phi$ is:

$$\mathbf{A}_\Phi = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 3 \\ 3 & 7 & 1 \\ -1 & 2 & 4 \end{bmatrix}.$$

1.16 Basis Change

If the bases in V and W are changed to $\tilde{B}$ and $\tilde{C}$, the new transformation matrix $\tilde{\mathbf{A}}_\Phi$ is related to the original $\mathbf{A}_\Phi$ by:

$$\tilde{\mathbf{A}}_\Phi = \mathbf{T}^{-1}\mathbf{A}_\Phi\mathbf{S},$$

where $\mathbf{S}$ maps coordinates from $\tilde{B}$ to B, and $\mathbf{T}$ maps coordinates from $\tilde{C}$ to C.

1.17 Example: Basis Change

For a linear mapping $\Phi : \mathbb{R}^3 \to \mathbb{R}^4$ with standard bases B and C, and new bases $\tilde{B}$ and $\tilde{C}$, the transformation matrices $\mathbf{S}$ and $\mathbf{T}$ are constructed by expressing the new basis vectors in terms of the old basis vectors. The new transformation matrix is:

$$\tilde{\mathbf{A}}_\Phi = \mathbf{T}^{-1}\mathbf{A}_\Phi\mathbf{S}.$$

1.18 Image and Kernel

For a linear mapping $\Phi : V \to W$:

- **Kernel (Null Space)**: $\ker(\Phi) = \{\mathbf{v} \in V \mid \Phi(\mathbf{v}) = \mathbf{0}_W\}$. The kernel is a subspace of V.
- **Image (Range)**: $\mathrm{Im}(\Phi) = \{\Phi(\mathbf{v}) \mid \mathbf{v} \in V\}$. The image is a subspace of W.

1.19 Rank-Nullity Theorem

The **Rank-Nullity Theorem** states:

$$\dim(\ker(\Phi)) + \dim(\mathrm{Im}(\Phi)) = \dim(V).$$

This theorem is fundamental in linear algebra and has several important consequences:

- If $\dim(\mathrm{Im}(\Phi)) < \dim(V)$, then $\ker(\Phi)$ is non-trivial.
- If $\dim(V) = \dim(W)$, then Φ is injective if and only if it is surjective if and only if it is bijective.

1.20 Example: Image and Kernel

For a linear mapping $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ defined by:

$$\Phi(\mathbf{x}) = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \mathbf{x},$$

the image is the span of the columns of the transformation matrix:

$$\mathrm{Im}(\Phi) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

The kernel is found by solving $\mathbf{Ax} = \mathbf{0}$:

$$\ker(\Phi) = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

1.21 Key Theorems and Results

- Two finite-dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.
- The composition of linear mappings is linear. If $\Phi : V \to W$ and $\Psi : W \to X$ are linear, then $\Psi \circ \Phi : V \to X$ is also linear.
- If $\Phi : V \to W$ is an isomorphism, then $\Phi^{-1} : W \to V$ is also an isomorphism.

1.22 Examples of Linear Mappings

- The mapping $\Phi : \mathbb{R}^2 \to \mathbb{C}, \Phi(\mathbf{x}) = x_1 + ix_2$ is a homomorphism, justifying the representation of complex numbers as tuples in $\mathbb{R}^2$.
- Linear transformations can represent geometric operations such as rotations, stretches, and reflections in $\mathbb{R}^2$.

1.23 Example: Geometric Transformations

For a rotation by 45° in $\mathbb{R}^2$, the transformation matrix is:

$$\mathbf{A}_1 = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}.$$

For a stretch along the horizontal axis by a factor of 2, the transformation matrix is:

$$\mathbf{A}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

1.24 Affine spaces