

Smooth Operator – Modifying the Anhøj Rules to Improve Runs Analysis in Statistical Process Control

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Abstract An abstract of less than 150 words.

Introduction

Within statistical process control (SPC) runs analysis is being used to detect persistent shifts in process location over time.

Runs analysis deals with the natural limits of number of runs and run lengths in random processes. A run is a series of one or more consecutive elements of the same type, for example heads and tails, odd and even numbers, or numbers above or below a certain value. A run chart is a point-and-line chart showing data over time with the median as reference line. In a random process, the data points will be randomly distributed around the median and the number and lengths of runs will be predictable within limits. All things being equal, if the process shifts, runs tend to become longer and fewer. Consequently, runs analysis may help detect shifts in process location. Process shifts are one form of non-random variation in time series data that are of particular interest to quality control and improvement. If a process shifts, it may be the result of planned improvement or unwanted deterioration.

Several tests (or rules) for shift detection based on the principles of runs analysis exist. In previous papers we demonstrated that the currently best performing rules with respect to sensitivity and specificity to shifts in process location are two simple tests (Anhøj and Olesen, 2014; Anhøj, 2015; Anhøj and Wentzel-Larsen, 2018):

- Shift test: one or more unusually long runs of data points on the same side of the centre line.
- Crossings test: the curve crosses the centre line unusually few times.

Collectively, we refer to these tests as the Anhøj rules. Critical values for run length and number of crossings depend on the total number of data points in the chart. The number of crossings follow a binomial distribution, $b(n - 1, 0.5)$, where n is the number of data points and 0.5 the success probability. Thus the lower prediction limit for number of crossings may, for example, be set to the lower 5th percentile of the corresponding cumulative binomial distribution (Chen, 2010). However, no closed form expression exists for the distribution of longest runs. Consequently, the upper prediction limit for longest runs has traditionally been either a fixed value (usually 7 or 8) or an approximate value depending on n as with the Anhøj rules: $\log_2(n) + 3$ rounded to the nearest integer (Schilling, 2012).

Using simulations, we have previously shown that runs analysis using the Anhøj rules are comparable or even better at detecting minor to moderate *persistent* shift than the widely used Western Electric control chart rules (Anhøj and Wentzel-Larsen, 2018).

Each of the two tests has a specificity (true negative proportion) around 95%. The sensitivity (true positive proportion) of a test depends on the size of the shift (signal) relative to the random variation inherent in the process (noise). When applied together, the sensitivity increases, while the specificity decreases a bit and varies around 92.5% depending on the number of available data points in the chart (Figure 1).

Historically, runs tests have mainly been studied in isolation as individual tests. But what is really of interest – because the rules are linked, when one goes up, the other goes down – is the properties of the joint distribution of longest runs and number of crossings.

We recently released an R package, **crossrun** (Wentzel-Larsen and Anhøj, 2018), that includes functions for calculating the joint probabilities of the number of crossings (C) and longest runs (L) in random data series of different lengths (n) and with and without shifts in process location expressed in standard deviation units (sd). Figure 2 illustrates this for a run chart with $n = 11$ and $sd = 0$ (no shift). To avoid very small numbers, the probabilities are shown using the times representation, that is the probabilities times 2^{n-1} . The red box encloses the combinations of C and L that would indicate random variation according to the Anhøj rules (true negatives). The area outside the box represents combinations of C and L that would indicate non-random variation (false positives).

With the **crossrun** package at hand it is feasible to calculate exact joint probabilities of C and L for different numbers of observations and shift sizes. And consequently, it is feasible to investigate the

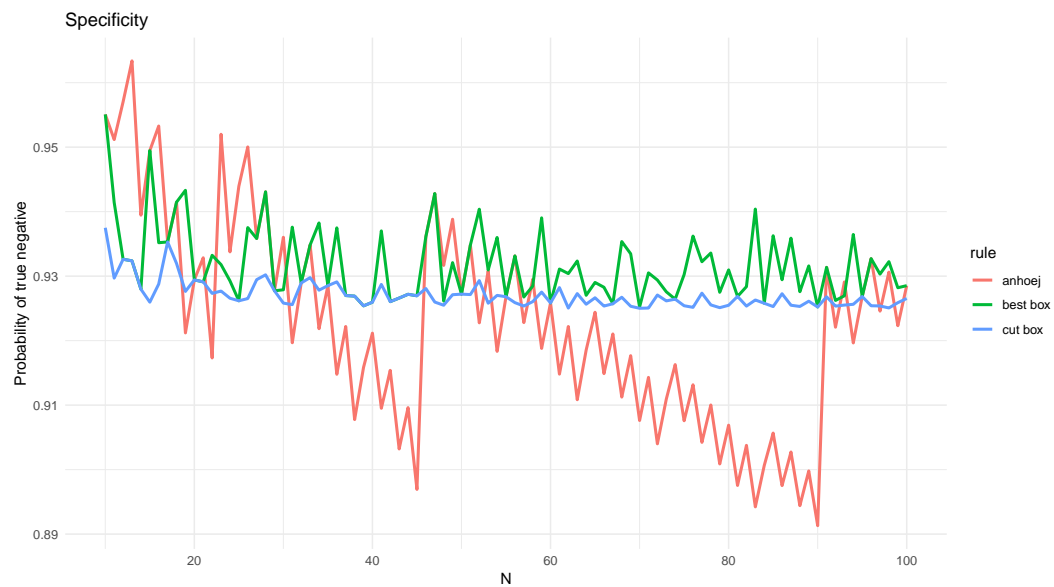


Figure 1: Specificity of the anhoej, best box, and cut box rules. N = number of data points in run chart.

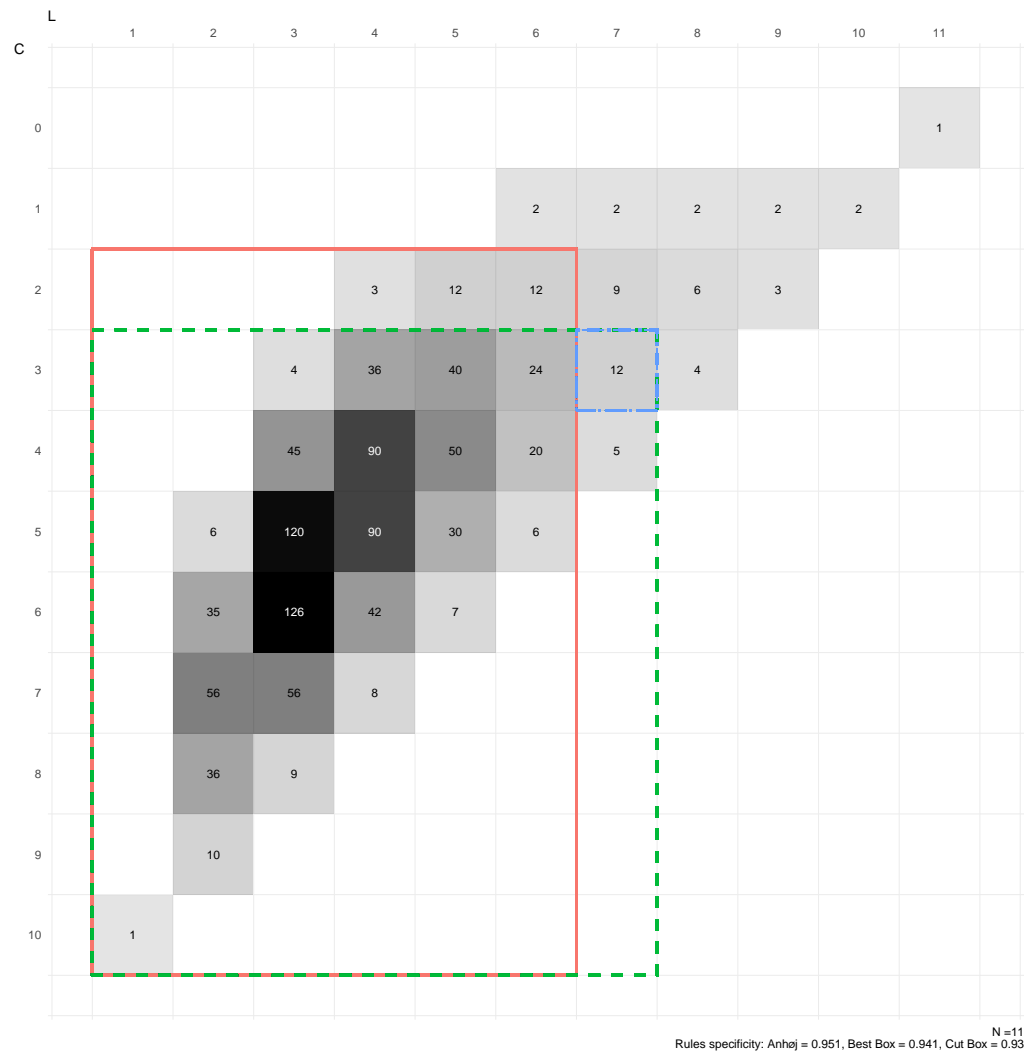


Figure 2: Borders of the anhoej, best box, and cut box rules for 11 data points. C = number of crossings, L = longest run. The numbers in the cells are times representation of of the joint probabilities of longest run and number of crossings. Anhoej = red solid, best box = green dashed, cut box = blue dot-dashed.

diagnostic properties of run charts using exact values for specificity and sensitivity rather than values based on time consuming and complicated simulation studies.

As is clearly visible in Figure 1 the specificity of the Anhøj rules (red line) jumps up and down as n changes. This is a result of the discrete nature of the two tests – especially the shift test. Although the specificity of the Anhøj rules does not decrease continuously as n increases like other rules do, we hypothesised that it would be possible to improve their diagnostic value further by smoothing the specificity using minor adjustments to C and L depending on n .

The aim of this study was to suggest a procedure for smoothing the diagnostic properties of the Anhøj runs rules.

Methods

Likelihood ratios to quantify the diagnostic value of runs rules

The value of a diagnostic test is traditionally described using terms like sensitivity and specificity. These parameters express the probability of detecting the condition being tested for when it is present and not detecting it when it is absent:

$$\text{Specificity} = P(\text{no signal} \mid \text{no shift}) = P(\text{true negative}) = 1 - \alpha$$

$$\text{Sensitivity} = P(\text{signal} \mid \text{shift}) = P(\text{true positive}) = 1 - \beta$$

However, we usually seek to answer the opposite question: what is the likelihood that a positive or negative test actually represents the condition being tested for, which in our case is a shift in the underlying process? Likelihood ratios (LR) do this:

$$LR+ = TP/FP = \text{sensitivity} / (1 - \text{specificity})$$

$$LR- = FN/TN = (1 - \text{sensitivity}) / \text{specificity}$$

A likelihood ratio greater than 1 speaks in favour of the condition, while a likelihood ratio less than 1 speaks against the condition. As a rule of thumb, a positive likelihood ratio greater than 10 is considered strong evidence that the condition is present. A negative likelihood ratio smaller than 0.1 is considered strong evidence against the condition (Deeks and Altman, 2004). Thus, likelihood ratios are useful measures of the diagnostic value of run charts (Anhøj, 2015; Anhøj and Wentzel-Larsen, 2018).

Best box and cut box adjustments to improve the Anhøj rules

We developed two functions, `bestbox()` and `cutbox()` that automatically seek to adjust the critical values for C and L in such a way that the specificity is kept as close as possible to a specified target value. In this study we used a target of 92.5%, which is close to the actual average specificity for the Anhøj rules for $n = 10$ -100.

Specifically, the `bestbox()` function finds the box with lowest specificity for the target shift, among boxes with specificity \geq target for shift = 0; and the `cutbox()` cuts corners from a best box while keeping specificity \geq target for shift = 0. Figure 2 illustrates these principles for a run chart with 11 data points.

While `bestbox()` may add or remove rows or columns from the corresponding Anhøj box, `cutbox()` cuts none or more squares in vertical or horizontal direction from the upper right corner of the corresponding best box.

Thus, for $n = 11$, the Anhøj rules would signal a shift if $C < 2$ or $L > 6$; best box would signal if $C < 3$ or $L > 7$; and cut box would signal if $C < 3$ or $L > 7$ except when $C = 3$ and $L = 7$.

Results

We calculated the limits for the Anhøj, best box, and cut box rules together with their corresponding positive test proportions and log-likelihood ratios for $n = 10$ -100 and shift = 0-3 SD (in 0.2 SD increments). The results are available in the supplementary R data set 'cr_bounds.rds'. The R code to produce the results and the figures in this article are available in the supplementary materials 'crossrunbox.R' and 'figs.R' respectively.

Figure 1 illustrates the effect of the best box and cut box procedures on the specificity of the runs analysis. As expected, the variability in specificity with varying n is markedly reduced and kept above and closer to the specified target – more with cut box than with best box.

Figure 3 shows the probabilities of getting a signal as a function of n and shift size. The upper left facet (shift = 0 SD) contains the same data as Figure 1. As expected and shown previously in our simulations studies, the power of the runs analysis increases with increasing n and shift size. The smoothing effect of best box and cut box appear to wear off as n and shift size increases.

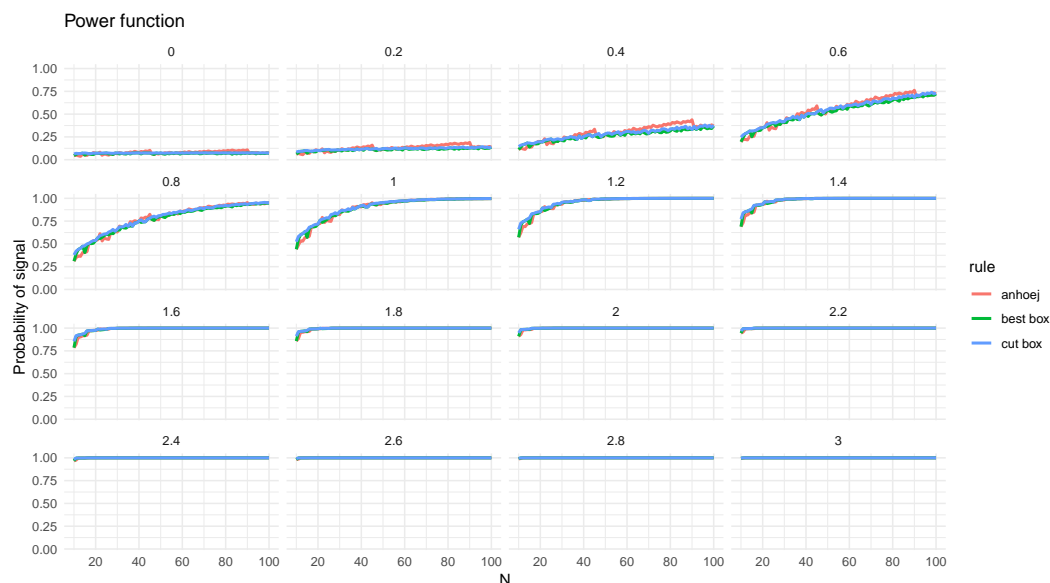


Figure 3: Power function of Anhøj, best box, and cut box rules. N = number of data points in run chart. Numbers above each facet represent the size of the shift in standard deviation units that is present in data.

Figures 4 and 5 compares the positive and negative likelihood ratios of the Anhøj rules to the box adjustments. The smoothing effect appear to be of practical value only for positive tests.

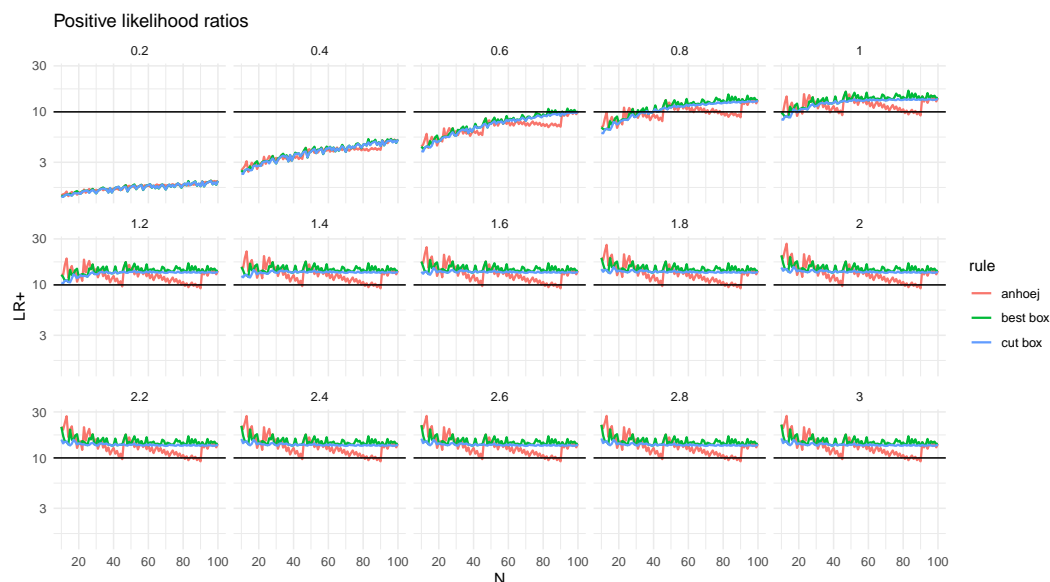


Figure 4: Positive likelihood ratio of Anhøj, best box, and cut box rules. N = number of data points in run chart. Numbers above each facet represent the size of the shift in standard deviation units that is present in data.

Discussion

In this study we demonstrated that it is feasible to reduce variation in run chart specificity from varying number of data points by using the best box and cut box adjustments of the Anhøj rules.

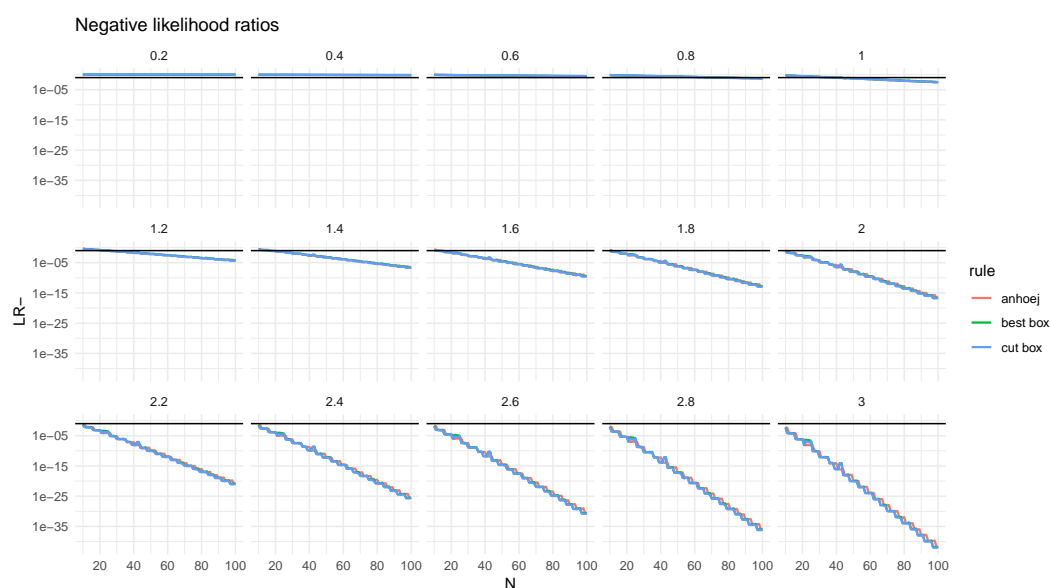


Figure 5: Negative likelihood ratio of Anhøj, best box, and cut box rules. N = number of data points in run chart. Numbers above each facet represent the size of the shift in standard deviation units that is present in data.

Confirmation that runs analysis is a valuable tool even with small n .

What is already known on this topic? (nothing)

What are the limitations of this study?

What are the strengths of this study?

What are the practical implications of our findings? (maybe none)

How can our findings be operationalised (method argument to `qic()`)

Conclusion

In this study we demonstrated that it is feasible to reduce variation in run chart specificity from varying number of data points by using the best box and cut box adjustments of the Anhøj rules.

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Appendix: Bounds table

Table 1: Signal limits for the anhöj and best box rules and borders for the cut box rules. N = number of trials. L = upper limit for longest run, C = lower limit for number of crossings, Cbord and Lbord = cut box borders to keep.

N	Anhøj		Best box		Cut box	
	L	C	L	C	Cbord	Lbord
10	2	6	2	6	3	5
11	2	6	3	7	4	6
12	3	7	3	6		
13	3	7	3	6		
14	4	7	3	6		
15	4	7	4	7	6	6
16	4	7	5	8	6	7
17	5	7	5	7		
18	5	7	5	7	6	6
19	6	7	5	7	6	5
20	6	7	6	7		
21	6	7	7	8		
22	7	7	6	7	7	6
23	7	8	6	7	7	6
24	8	8	6	7	7	6
25	8	8	6	7		
26	8	8	9	9	10	7
27	9	8	9	8	10	7
28	9	8	9	8	11	7
29	10	8	10	8		
30	10	8	11	10	12	9
31	11	8	11	9	14	8
32	11	8	11	8		
33	11	8	11	8	12	7
34	12	8	11	8	13	7
35	12	8	12	8		
36	13	8	13	9	15	8
37	13	8	14	10		
38	14	8	13	8		
39	14	8	15	11		
40	14	8	15	9		
41	15	8	15	9	17	8
42	15	8	14	8		
43	16	8	14	8		
44	16	8	17	10		
45	17	8	17	9		
46	17	9	17	9	19	8
47	17	9	17	9	20	7
48	18	9	19	12	20	11
49	18	9	19	10	21	9
50	19	9	19	9		
51	19	9	19	9	21	8
52	20	9	19	9	21	7
53	20	9	21	11	23	9
54	21	9	21	10	23	8
55	21	9	21	9		
56	21	9	21	9	23	8
57	22	9	23	12	25	11
58	22	9	23	10	24	9
59	23	9	23	10	26	8

Table 1: Signal limits for the anhøj and best box rules and borders for the *(continued)*

N	Anhøj		Best box		Cut box	
	L	C	L	C	Cbord	Lbord
60	23	9	23	9		
61	24	9	23	9	24	8
62	24	9	25	11	27	9
63	25	9	25	10	27	9
64	25	9	26	11	27	10
65	25	9	26	10	27	9
66	26	9	27	12	29	10
67	26	9	27	10		
68	27	9	27	10	29	8
69	27	9	28	11	29	8
70	28	9	29	14	30	13
71	28	9	29	11	31	9
72	29	9	29	10	30	9
73	29	9	30	11	31	10
74	29	9	30	10		
75	30	9	31	12	32	9
76	30	9	31	11	34	8
77	31	9	31	10	33	9
78	31	9	32	11	33	8
79	32	9	33	13	37	11
80	32	9	33	11	35	9
81	33	9	33	10		
82	33	9	34	11	36	10
83	34	9	33	10	36	7
84	34	9	35	11		
85	34	9	35	11	38	8
86	35	9	35	10	36	9
87	35	9	35	10	38	8
88	36	9	37	12	38	10
89	36	9	37	11	39	9
90	37	9	38	12		
91	37	10	37	10	39	9
92	38	10	39	13	41	12
93	38	10	39	11	40	10
94	39	10	39	11	42	8
95	39	10	39	10		
96	39	10	39	10	41	8
97	40	10	41	12	42	9
98	40	10	41	11	44	9
99	41	10	42	12	43	10
100	41	10	41	10	42	9

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