Similar and Diagonalizable Matrices (10/30/2020)

Similar Matrices:
Two $n \times n$ matrices A and C are similar if there is another $n \times n$ matrix B such that $A = BCB^{-1}$. This property allows us to conclude that if two matrices A and C are similar, then they have the same eigenvalues (but not necessarily the same eigenvectors). Suppose $\vec{x}$ is an eigenvector with eigenvalue λ for C, that is $C\vec{x} = \lambda\vec{x}$. It follows that $B\vec{x}$ is an eigenvector for A with eigenvalue λ. This is because $A(B\vec{x}) = (BCB^1)(B\vec{x}) = BC\vec{x} = B(\lambda\vec{x}) = \lambda(B\vec{x})$. Thus, the corresponding eigenvector is $B\vec{x}$.

Diagonalizable Matrices:
An $n \times n$ matrix is diagonalizable if it is similar to a diagonal matrix. Not all matrices are diagonalizable, but most are. Note that if we have an $n \times n$ matrix A with n independent eigenvectors $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$, with eigenvalues $\{\lambda_1, \lambda_2, \dots \lambda_n\}$, then A is diagonalizable. We actually already know that if we take the eigenvalues and eigenvectors, we can produce A! Let $A = XCX^{-1}$, where the columns of X are all of the eigenvectors, and C is a diagonal matrix with the corresponding eigenvalues at each value on the diagonal. We know by def. of eigenvectors/eigenvalues:

$$\begin{aligned} AX &= A(\vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n) \\ &= (A\vec{x}_1 \mid A\vec{x}_2 \mid \cdots \mid A\vec{x}_n) \\ &= (\vec{x}_1\lambda_1 \mid \vec{x}_2\lambda_2 \mid \cdots \mid \vec{x}_n\lambda_n) \\ &= (\vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n)C \\ &= XC \end{aligned}$$

Now that we know $AX = XC$, it is simple to conclude $A = XCX^{-1}$, as $XCX^{-1} = AXX^{-1} = A$. And we are done! Diagonalizable matrices are similar to the matrix with eigenvalues at each value on the diagonal.

Thinking About Diagonalization:
Our matrices A and C of the last section can be thought of as an <u>change of basis</u>. As we multiply by the matrix X, we cause $X\vec{e}_1 = \vec{x}_1$, $X\vec{e}_2 = \vec{x}_2$, $\dots X\vec{e}_n = \vec{x}_n$, effectively mapping the standard basis vectors to the eigenvectors. We see that $X^{-1]}$ will have the reverse effect, as $X^{-1}\vec{x}_1 = \vec{e}_1$, $X^{-1}\vec{x}_2 = \vec{e}_2, \dots X^{-1}\vec{x}_n = \vec{e}_n$. Thus, it follows that A and C are the same transformation with respect to different bases. As X is effectively shifting the space of $\mathbb{R}^n$ to a different set of standard basis vectors, C will produce the same linear transformation in this new space with these modified standard bases that A will produce in normal space with normal standard bases. Similarly, X and X^{-1} provide the linear transformations by which we switch between these standard bases.

Clarifying the Notion of Diagonalization:
Consider multiplying our eigenvectors by the different vectors we have discussed. By multiplying by A, we scale each eigenvector by its corresponding eigenvalue. As we multiply by X^{-1}, we send the eigenvectors into the standard basis vectors. Now, if we multiply these standard basis vectors by C, it scales each standard basis vector by the eigenvalues. Finally, if we multiply these scaled standard basis vectors by X, we convert the standard basis vectors into the eigenvectors. Through this process, we see that multiplication by A is the same as multiplication by X^{-1}, then C, than X, that is XCX^{-1}. It follows that by finding the diagonalization, we can think of multiplication by a matrix A as a liner transformation of a diagonal matrix, something much easier to think about conceptually!