SINGULAR VALUE DECOMPOSITION CONTINUED(11/11/2020)

Review:
Remember that we discussed singular value decomposition (SVD) in class yesterday. This process allows us to take any $m \times n$ matrix A with rank r, and show $A = U_r \Sigma_r V_r^T$, where U_r and V_r are orthogonal and Σ_r is diagonal, and $A = U\Sigma V^T$, where U and V are orthogonal and all elements on the diagonal of Σ are non-zero. Furthermore, the columns of U are a basis for the column space and the left nullspace, and the columns of V are a basis for the row space and the nullspace, while the columns of U_r are a basis for the column space while the columns of V_r are a basis for the row space.

Finding SVD:
So, this is a really nice idea! But how do we get it to actually work! Well, let's start with our $m \times n$ matrix A. Now, given matrices U, Σ, and V, let:

- $\vec{v}_i$'s are the eigenvectors or $n \times n$ matrix $A^T A$.
- $\vec{u}_i$'s are the eigenvectors or $m \times m$ matrix AA^T.
- σ's in the diagonal or Σ are the square root of the eigenvalues of $A^T A/AA^T$.

Why does this work? Well, suppose we have $A = U\Sigma V^T$. It would follow that:

$$\begin{aligned} & A = U\Sigma V^T \quad \wedge \quad A^T = V\Sigma^T U^T \\ \implies \quad & A^T A = V\Sigma^T \Sigma V^T \quad \wedge \quad AA^T = U\Sigma^T \Sigma U^T \\ \implies \quad & A^T A = V\Sigma^2 V^T \quad \wedge \quad AA^T = U\Sigma^2 U^T \end{aligned}$$

By the spectral theorem, the eigenvalues of $A^T A$ are the values of Σ^2 and the eigenvectors of $A^T A$ are the columns of V, and the eigenvalues of AA^T are the values of Σ^2 and the eigenvectors of AA^T are the columns of U. So, this system of $\vec{v}_i$, $\vec{u}_i$, and σ works!
One last property we need to prove before continuing is that $N(A) = N(A^T A)$. This is a simple set equality proof that doesn't need to be included here.

Proving A Method For SVD:
So, if we have matrix $A^T A$, it must be positive semi-definite, so by spectral theorem, $A^T A = Q\Lambda Q^T$, where Q is orthonormal and the first 1 through r (rank) values on the diagonal of Λ are the eigenvalues of $A^T A$, while the rest are zero. So, let $\vec{v}_i = Q_{*i}$, that is let $V = Q$. This will give us all the properties we want! It follows that the last $r + 1$ through n columns must be a members of $N(A^T A)$, and thus members of $N(A)$, while the first 1 through r columns of V must be orthogonal to $N(A^T A)$, and thus orthogonal to $N(A)$ and thus a basis of the row space. If we simply let σ be equal to $\sqrt{\lambda}$ for each item on the diagonal of Λ, it must follow that $A\vec{v}_i = \sigma_i \vec{u}_i$, and thus $\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$ for $1 \leq i \leq r$. Now, we can prove that $\vec{u}_i, \ldots, \vec{u}_r$ are orthonormal by simply showing $\vec{u}_i \vec{u}_j = 0$ if $i \neq j$, and $\vec{u}_i \vec{u}_j = 1$ if $i = j$. However, this will only get us U_r – it will not give us U! We still need $\vec{u}_i$ for $r + 1 \leq i \leq m$. Thus, if we want to find all the columns of U, the only way we have is to use the same spectral decomposition of AA^T. It would be useful to go through a few numerical examples to better understand exactly what is going on!
By diagonalization of a square matrix, we were able to see the "change of basis" brought on my our original matrix A. With singular value decomposition, we can as well!