SPECTRAL THEOREM (11/06/2020)

Statement of Theorem:
The statement of spectral theorem is actually quite simple. Given a symmetric $n \times n$ matrix S, the matrix is not only diagonalizable, but diagonalizable in the form $S = Q\Lambda Q^-1$, where the columns of Q are orthonormal. This is equivalent to saying the eigenvectors of S are orthonormal and S diagonalizable.

A Complex Review:
We have to go over a few different concepts before we have the tools prove spectral theorem. First, let us go over exactly what complex numbers are. There are members of the set:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i^2 = 1\}$$

Note that these complex numbers are closed under addition and multiplication. Furthermore, there is an important additional operation we can take on the imaginary numbers. The complex conjugate of $a+bi$ is $\overline{a + bi} = a - bi$. Note that a few properties of the conjugate are true. Given complex numbers x and y and real number a, we see that:

- $\overline{x} + \overline{y} = \overline{x + y}$
- $\overline{xy} = \overline{x} \cdot \overline{y}$, which implies $(\overline{x})^n = \overline{x^n}$
- $\overline{a} = a$

If we have a polynomial p with real coefficients, and a root of p is complex number x, it follows that $\overline{x}$ is a root of the polynomial as well. This means that complex roots come in conjugate pairs. Finally, the magnitude of a complex number is $|x| = \sqrt{x\overline{x}}$. This is equivalent to the length of the vector (a, b) where $x = a + bi$.

Towards a Proof:
Let's prove that if we have a real, symmetric matrix, then it has only real eigenvalues. Let S be real and symmetric, with eigenvector $\vec{x}$, eigenvalue λ. We can define $\vec{x}$ and $\overline{\vec{x}}$, which takes the complex conjugate of each and every element of $\vec{x}$. We will see that:

$$S\vec{x} = \lambda\vec{x} \implies \overline{\vec{x}}^T S\lambda = \overline{\vec{x}}^T \lambda\vec{x}$$

$$S\vec{x} = \lambda\vec{x} \implies \overline{S\vec{x}} = \overline{\lambda\vec{x}} \implies S \cdot \overline{\vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}} \implies \overline{\vec{x}}^T S = \overline{\lambda} \cdot \overline{\vec{x}}^T$$

$$\overline{\vec{x}}^T S = \overline{\lambda} \cdot \overline{\vec{x}}^T \implies \overline{\vec{x}}^T S\vec{x} = \overline{\lambda} \cdot \overline{\vec{x}}^T \vec{x} \implies \lambda \cdot \overline{\vec{x}}^T \vec{x} = \overline{\lambda} \cdot \overline{\vec{x}}^T \vec{x} \implies \lambda = \overline{\lambda}$$

As we have shown that $\lambda = \overline{\lambda}$, it follows that λ must be real!

Final Proof:
We want to show that a real, symmetric matrix S has orthogonal eigenvectors. Let S be real and symmetric with eigenvectors $\vec{x}_1$ and $\vec{x}_2$. It follows that $S\vec{x}_1 = \lambda_1\vec{x}_1$, $S\vec{x}_2 = \lambda_2\vec{x}_2$, and $\lambda_1 \neq \lambda_2$. We want to show that $\vec{x}_1 \perp \vec{x}_2$. As S is symmetric, we can conclude that $S - \lambda I$ is symmetric for $\lambda \in \{\lambda_1, \lambda_2\}$. Furthermore, as $S - \lambda_1 I$ symmetric, row space orthogonal to nullspace we know:

$$C(S - \lambda_1 I) = C(S - \lambda_1 I)^T \quad \perp \quad N(S - \lambda_1 I)$$

Since $\vec{x}_1$ is an eigenvector for S, it follows that $\vec{x}_1 \in N(S - \lambda_1 I)$. Also, we see that:

$$(S - \lambda_1 I)\vec{x}_2 = S\vec{x}_2 - \lambda_1\vec{x}_2 = (\lambda_2 - \lambda_1)\vec{x}_2$$

As $\lambda_1 \neq \lambda_2$, we know that $\lambda_2 - \lambda_1 \neq 0$. From the equation above, we can conclude $\vec{x}_2 \in C(S - \lambda_1 I)$, and by the same process $\vec{x}_1 \in C(S - \lambda_2 I)$. As we showed above that the column space is orthogonal to the nullspace, we can conclude that $\vec{x}_1 \perp \vec{x}_2$.

Some Matrix Definitions:
We say that an $n \times n$ symmetric matrix S is positive definite if:

- All n pivots are positive.
- All upper left determinants are positive.
- All n eigenvalues are positive.
- For all $\vec{x} \neq \vec{0}$, $\vec{x}^T S\vec{x} > \vec{0}$
- There is an $m \times n$ matrix A with independent columns such that $S = A^T A$.

A matrix is positive semi-definite if it meets all the requirements above, except A does not necessarily have independent columns.