Geometry of Determinants (10/14/2020)

A determinant is a measurement of *signed volume*. We will investigate this by calculating linear transformations at looking at geometric results. When thinking about a linear transformation (i.e. matrix multiplication), the basis is important for determining what is going to happen! Any linear transformation of a subspace can be generalized to a linear transformation of the bases of the subspace.

An Example:
Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This matrix via multiplication (i.e. $A\vec{x} = \vec{y}$) will map vectors from $\mathbb{R}^2$ to $\mathbb{R}^2$. We can use some simple matrices to get rows and columns!

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A(x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

The important thing to note here is that for any (x, y), multiplication to A is really a linear transformation of the columns of A. Accordingly, this is going to form a parallelogram quadrilateral of vectors with points of $(0, 0)$, $(b, d)y$, $(a, c)x$, and finally $(a, c)x + (b, d)y$.

Thinking About Area:
So, based on that values in our 2×2 matrix A, what is the area going to be? Well, he have a parallelogram with the points listed above. We can use some geometry to subscribe the parallelogram into a rectangle of sides $c + d$ and $b + a$, with the final result that the area for $\vec{x} = (1, 1)$ is $ad - bc$. This formula is important!
If we take any random area (think a blob around the origin), we can think of multiplication by A as a transformation of the grid of the cartesian plane itself. Each and every separate point in the blob is transformed just like (x, y) was above. It follows that multiplication by a 2×2 matrix A can be generalized as this transformation of the grid. Specifically, if we consider $A\vec{x} = \vec{y}$, then the ratio of the area of our original $\vec{x}$ to the area of our final $\vec{y}$ will be:

$$\frac{\text{area}(\vec{y})}{\text{area}(\vec{x})} = \frac{ad - bc}{1} = ad - bc$$

Thinking About Signed Area:
Consider if we switch the columns of A, that is $A_M = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$. Given our (x, y) to multiply, we're going to get a new but very similar parallelogram quadrilateral, with $(0, 0)$, $(b, d)x$, $(a, c)y$, and finally $(b, d)x + (a, c)y$. Furthermore, if we calculate our **determinant**, we get:

$$\det(A_M) = bd - ac = -\det(A_M)$$

This means while we get almost the same shape with A and A_M, the area of one is going to be negative and the other positive (or both zero)! This makes sense because we are talking about *signed* area, not unsigned area. The determinant isn't just giving us the area of the parallelogram, but the orientation of this area. For instance, note that $A(x, y) = A_M(y, x)$, and not only are both shapes the exact same – they have the same determinant!

Properties of Determinants:

Let's consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$. It follows that $\det(A) = (1)(8) - (4)(2) = 0$. This is important! We can conjecture that for any square matrix A, if the columns are linearly dependent, then $\det(A) = 0$.

Note these interesting determinant calculations:

- $\begin{pmatrix} qa & qc \\ b & d \end{pmatrix} = qad - qbc = q(ad - bc) = (q)\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}$
- $\begin{pmatrix} a_1 + a_2 & c_1 + c_2 \\ b & d \end{pmatrix} = (a_1 + a_2)d - (c_1 + c_2)b = \det \begin{pmatrix} a_1 & c_1 \\ b & d \end{pmatrix} + \det \begin{pmatrix} a_2 & c_2 \\ b & d \end{pmatrix}$

These rules more generally apply to any of the rows of A. Furhtermore, note that we can extend a determinant to any $n \times n$ matrix of any size (even if it quickly gets to be a headache!). See below:

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

As a final note, see that for $n \in \mathbb{Z}$, $n \geq 2$ the formula for the determinant of an $n \times n$ matrix can be represented recursively as determinants of $(n-1) \times (n-1)$ matrixes. For example, given our formula for 3×3 matrix A, we can show:

$$\det(A) = (a_{11})\det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - (a_{12})\det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + (a_{13})\det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

In the future, we will explore more about the determinant and its implications.