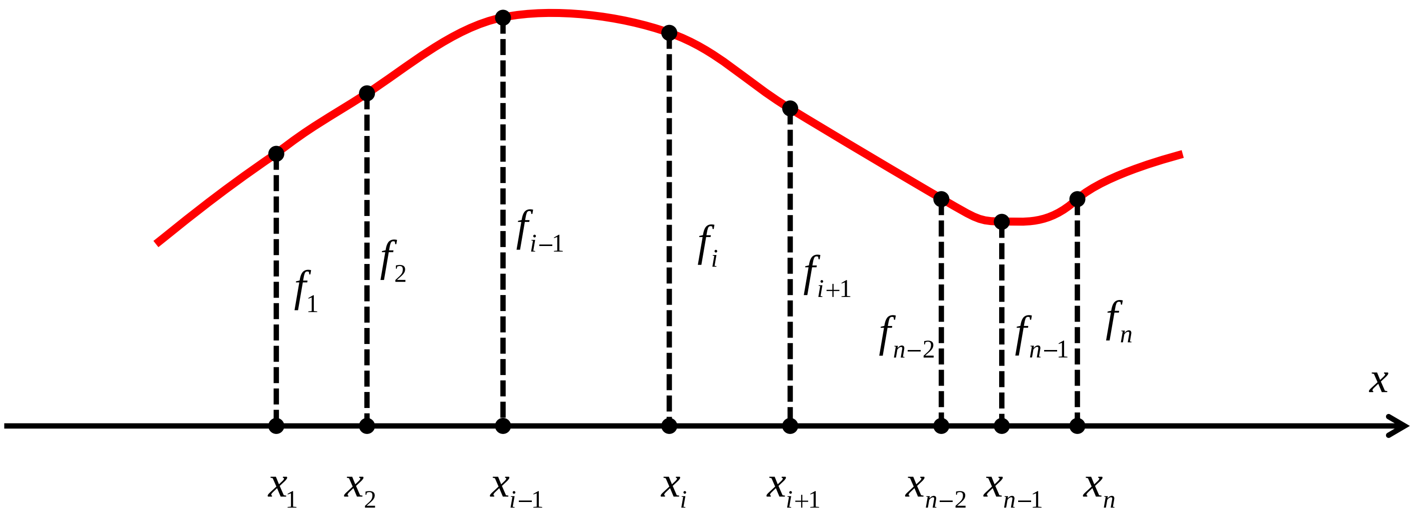
Interpolation by linear combination of known (base) functions

Problem statement



Interpolation:

Given a set of n values $f_1, \ldots, f_i, \ldots, f_{n\text{-}1}, f_n$

at n points $x = x_1, \ldots, x_i, \ldots, x_{n\text{-}1}, x_n$

produce an interpolating function $f(x)$

which reproduces (**exact interpolation**)

or is close (**smoothing interpolation**)

to the given data:

$$f(x_i) = f_i, \quad i = 0,1,\ldots,\mathrm{n}$$

Interpolation using a linear combination of known base functions

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots + a_k\varphi_k(x) + \ldots + a_m\varphi_m(x)$$

Interpolation using a linear combination of known base functions

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots + a_k\varphi_k(x) + \ldots + a_m\varphi_m(x)$$

known "base" functions

Interpolation using a linear combination of known base functions

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots + a_k\varphi_k(x) + \ldots + a_m\varphi_m(x)$$

unknown numerical coefficients

Interpolation using a linear combination of known base functions

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unknown numerical coefficients

Each set of numerical coefficients corresponds to a different interpolating function.

Finding the "best" interpolating function reduces to finding the corresponding values of the numerical coefficients

Interpolation using a linear combination of known base functions

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots + a_k\varphi_k(x) + \ldots + a_m\varphi_m(x)$$

Examples of base functions

Polynomials: $\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x^2, \quad \ldots, \quad \varphi_k(x) = x^{k-1}, \quad \ldots, \quad \varphi_m(x) = x^{m-1}$

$$f(x) = a_1 + a_2 x + a_3 x^2 + \ldots + a_k x^{k-1} + \ldots + a_m x^{m-1}$$

Interpolation using a linear combination of known base functions

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \dots + a_k\varphi_k(x) + \dots + a_m\varphi_m(x)$$

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$$f(x) = a_1 + a_2 x + a_3 x^2 + \dots + a_k x^{k-1} + \dots + a_m x^{m-1}$$

Fourier series:

$$\varphi_{1c}(x) = \cos(\omega x), \quad \varphi_{2c}(x) = \cos(2\omega x), \quad \dots, \quad \varphi_{kc}(x) = \cos(k\omega x), \quad \dots, \quad \varphi_{mc}(x) = \cos(m\omega x)$$
$$\varphi_{1s}(x) = \sin(\omega x), \quad \varphi_{2s}(x) = \sin(2\omega x), \quad \dots, \quad \varphi_{ks}(x) = \sin(k\omega x), \quad \dots, \quad \varphi_{ms}(x) = \sin(m\omega x)$$

$$f(x) = a_{1c}\cos(\omega x) + a_{1s}\sin(\omega x) + a_{2c}\cos(2\omega x) + a_{2s}\sin(2\omega x) + \dots + a_{mc}\cos(m\omega x) + a_{ms}\sin(m\omega x)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{x_n - x_1}$$

Interpolation using a linear combination of known base functions

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots + a_k\varphi_k(x) + \ldots + a_m\varphi_m(x)$$

Examples of base functions

Equidistant cubic splines:

$$\varphi_k(x) = S_k(x) = \begin{cases} c_0^k + c_1^k(x - z_{k-1}) + c_2^k(x - z_{k-1})^2 + c_3^k(x - z_{k-1})^3 & z_{k-1} \le x \le z_k \\ 0 & \text{otherwise} \end{cases}$$

Attention: **In this case the spline "knots"**

$z_0, z_1, \ldots, z_{k\text{-}1}, z_k, \ldots, z_N$

do not coincide with the data points

$x_1, x_2, \ldots, x_k, \ldots, x_n$

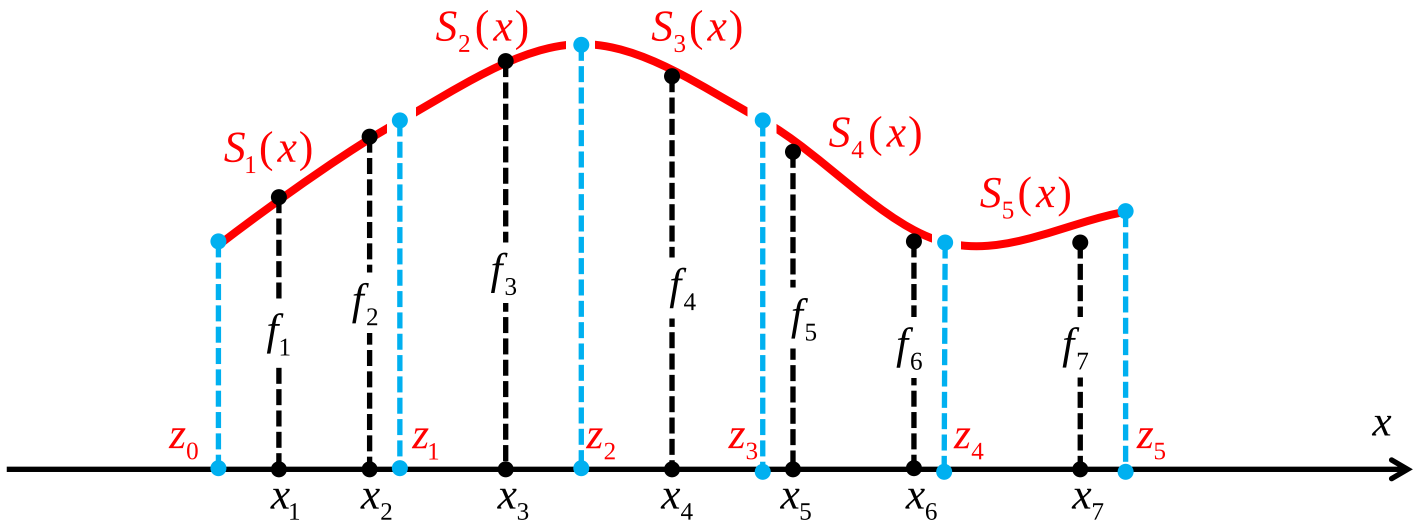
Interpolation using a linear combination of known base functions

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots + a_k\varphi_k(x) + \ldots + a_m\varphi_m(x)$$

Examples of base functions

Equidistant cubic splines:

$$\varphi_k(x) = S_k = \begin{cases} c_0^k + c_1^k(x - z_{k-1}) + c_2^k(x - z_{k-1})^2 + c_3^k(x - x_{k-1})^3 & z_{k-1} \le x \le z_k \\ 0 & \text{otherwise} \end{cases}$$



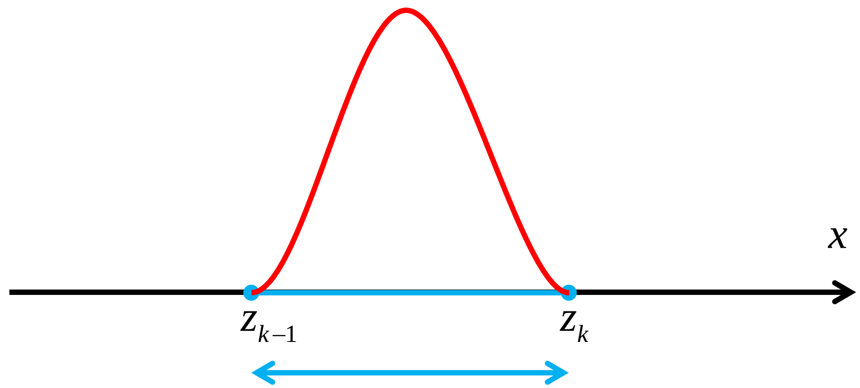
Interpolation using a linear combination of known base functions

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots + a_k\varphi_k(x) + \ldots + a_m\varphi_m(x)$$

Examples of base functions

Overlapping functions with finite support:

$$\varphi_k(x)\begin{cases} \neq 0 & z_{k-1} \le x \le z_k \\ = 0 & \text{otherwise} \end{cases}$$



Support of function:
where function is not zero

Interpolation using a linear combination of known base functions

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots + a_k\varphi_k(x) + \ldots + a_m\varphi_m(x)$$

Examples of base functions

Overlapping functions with finite support:

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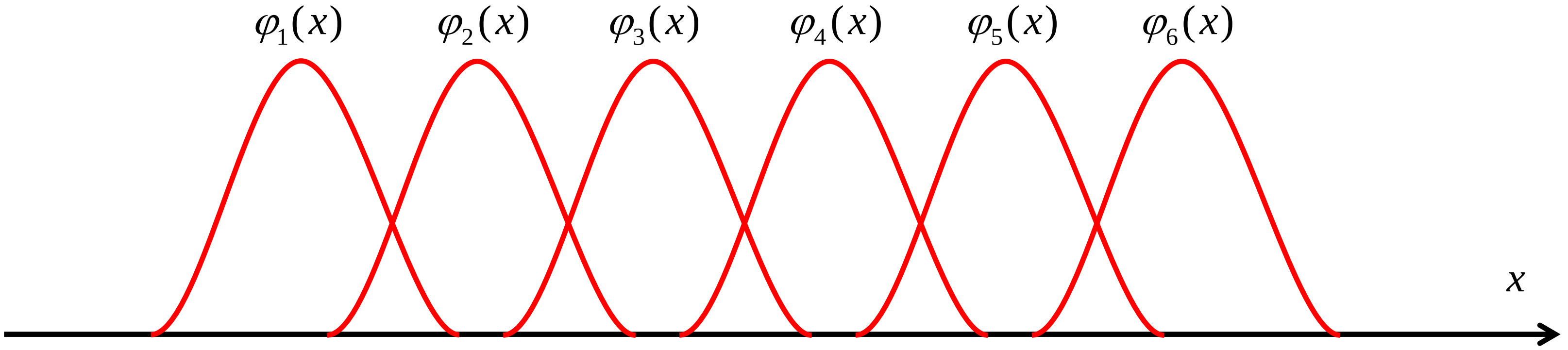

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

Satisfying the data values

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_i) \\ \vdots \\ f(x_n) \end{bmatrix} = \begin{bmatrix} a_1\varphi_1(x_1) + a_2\varphi_2(x_1) + \ldots + a_k\varphi_k(x_1) + \ldots + a_m\varphi_m(x_1) \\ a_1\varphi_1(x_2) + a_2\varphi_2(x_2) + \ldots + a_k\varphi_k(x_2) + \ldots + a_m\varphi_m(x_2) \\ \vdots \\ a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \\ \vdots \\ a_1\varphi_1(x_n) + a_2\varphi_2(x_n) + \ldots + a_k\varphi_k(x_n) + \ldots + a_m\varphi_m(x_n) \end{bmatrix}$$

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

Satisfying the data values

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_k(x_1) & \cdots & \varphi_m(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_k(x_2) & \cdots & \varphi_m(x_2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_i) & \varphi_2(x_i) & \cdots & \varphi_k(x_i) & \cdots & \varphi_m(x_i) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_k(x_n) & \cdots & \varphi_m(x_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_m \end{bmatrix} = \mathbf{Fa}$$

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

Satisfying the data values

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_k(x_1) & \cdots & \varphi_m(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_k(x_2) & \cdots & \varphi_m(x_2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_i) & \varphi_2(x_i) & \cdots & \varphi_k(x_i) & \cdots & \varphi_m(x_i) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_k(x_n) & \cdots & \varphi_m(x_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_m \end{bmatrix} = \mathbf{Fa}$$

each row corresponds to a data point

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

Satisfying the data values

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_k(x_1) & \cdots & \varphi_m(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_k(x_2) & \cdots & \varphi_m(x_2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_i) & \varphi_2(x_i) & \cdots & \varphi_k(x_i) & \cdots & \varphi_m(x_i) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_k(x_n) & \cdots & \varphi_m(x_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_m \end{bmatrix} = \mathbf{Fa}$$

each column corresponds to a base function

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

Satisfying the data values

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_k(x_1) & \cdots & \varphi_m(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_k(x_2) & \cdots & \varphi_m(x_2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_i) & \varphi_2(x_i) & \cdots & \varphi_k(x_i) & \cdots & \varphi_m(x_i) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_k(x_n) & \cdots & \varphi_m(x_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_m \end{bmatrix} = \mathbf{Fa}$$

each column corresponds to a base function

$$F_{ik} = \varphi_k(x_i)$$

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

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a system of n equations with m unknowns

$\mathbf{f}, \mathbf{F}$ **= known,** $\mathbf{a}$ **= unknown**

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

$$\underset{n\times m}{\mathbf{F}}\;\underset{m\times 1}{\mathbf{a}} = \underset{n\times 1}{\mathbf{f}}$$

a system of *n* equations with *m* unknowns

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

$$\underset{n\times m}{\mathbf{F}} \; \underset{m\times 1}{\mathbf{a}} = \underset{n\times 1}{\mathbf{f}}$$

a system of *n* equations with *m* unknowns

3 cases:

n **>** ***m*** **(more equations than unknowns)**

n **=** ***m*** **(as many equations as unknowns)**

n **<** ***m*** **(more unknowns than equations)**

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

$$\underset{n\times m}{\mathbf{F}} \; \underset{m\times 1}{\mathbf{a}} = \underset{n\times 1}{\mathbf{f}}$$

a system of *n* equations with *m* unknowns

3 cases:

***n* > *m* (more equations than unknowns)** — **no solution $\mathbf{a}$ exists**

***n* = *m* (as many equations as unknowns)**

***n* < *m* (more unknowns than equations)**

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

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a system of *n* equations with *m* unknowns

3 cases:

n > *m* (more equations than unknowns) — no solution $\mathbf{a}$ exists

n = *m* (as many equations as unknowns) — unique solution $\mathbf{a}$

n < *m* (more unknowns than equations)

Interpolation using a linear combination of known base functions

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) \qquad i = 1, 2, \ldots, n$$

$$\underset{n\times m}{\mathbf{F}}\ \underset{m\times 1}{\mathbf{a}} = \underset{n\times 1}{\mathbf{f}}$$

a system of n equations with m unknowns

3 cases:

$n > m$ (more equations than unknowns) — no solution $\mathbf{a}$ exists

$n = m$ (as many equations as unknowns) — unique solution $\mathbf{a}$

$n < m$ (more unknowns than equations) — infinitely many solutions $\mathbf{a}$

Smoothing least squares interpolation

Least squares smoothing interpolation

$$\underset{n\times m}{\mathbf{F}}\ \underset{m\times 1}{\mathbf{a}} = \underset{n\times 1}{\mathbf{f}}$$

Case 1: ***n* > *m* (more equations than unknowns). No solution $\mathbf{a}$ exists**

$$\mathbf{Fa} \neq \mathbf{f}$$

$$f(x_i) \neq a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i)$$

$$f(x_i) = a_1\varphi_1(x_i) + a_2\varphi_2(x_i) + \ldots + a_k\varphi_k(x_i) + \ldots + a_m\varphi_m(x_i) + e_i$$

e_i **= interpolation error**

Least squares smoothing interpolation

$$\underset{n\times m}{\mathbf{F}}\ \underset{m\times 1}{\mathbf{a}} = \underset{n\times 1}{\mathbf{f}}$$

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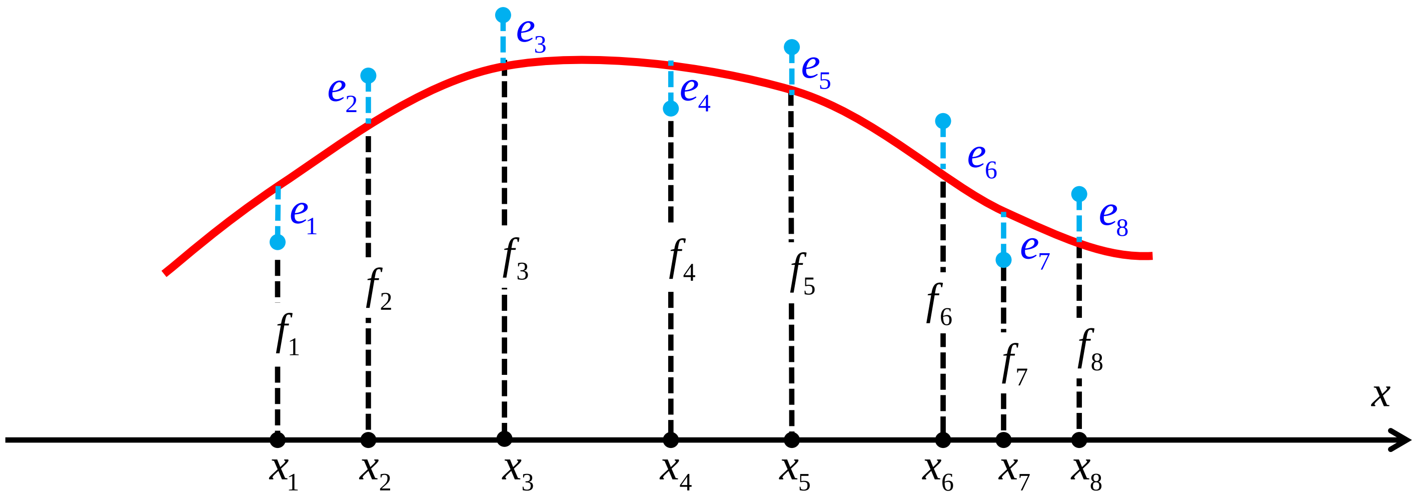


Least squares smoothing interpolation

$$\underset{n\times m}{\mathbf{F}}\ \underset{m\times 1}{\mathbf{a}} = \underset{n\times 1}{\mathbf{f}}$$

Case 1: $n > m$ (more equations than unknowns). No solution $\mathbf{a}$ exists

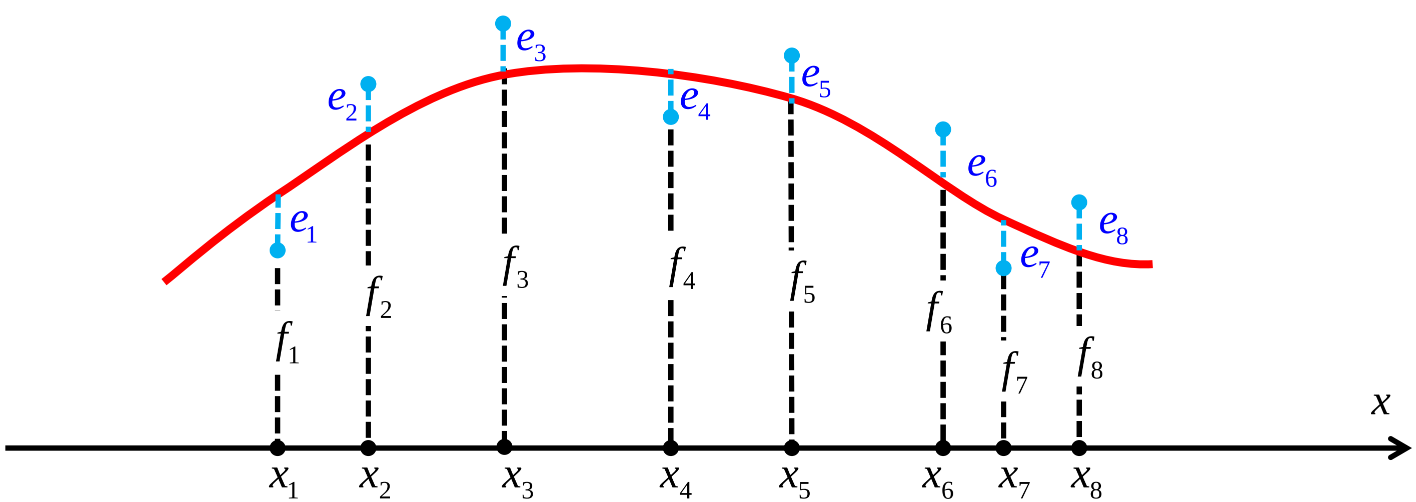
$$\mathbf{Fa} \neq \mathbf{f}$$

Smoothing interpolation:
The interpolating function does not pass through the data point values. Thus it is smoother than the data (which may oscillate due to observation errors)

$$) + \ldots + a_k \varphi_k(x_i) + \ldots + a_m \varphi_m(x_i)$$

$$) + \ldots + a_k \varphi_k(x_i) + \ldots + a_m \varphi_m(x_i) + e_i$$

= interpolation error



Least squares smoothing interpolation

Case 1: $n > m$ (more equations than unknowns). No solution $\mathbf{a}$ exists

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e}$$

Least squares solution: Find $\mathbf{a}$ which satisfies $\mathbf{e}^T \mathbf{P} \mathbf{e} = \min$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution $\mathbf{a}$ exists**

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e}$$

Least squares solution: **Find $\mathbf{a}$ which satisfies** $\mathbf{e}^T \mathbf{P} \mathbf{e} = \min$

$\mathbf{P} =$ **weight matrix (symmetric, positive definite)**

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution** $\mathbf{a}$ **exists**

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e}$$

Least squares solution: **Find** $\mathbf{a}$ **which satisfies** $\mathbf{e}^T\mathbf{P}\mathbf{e} = \min$

$\mathbf{P} =$ **weight matrix (symmetric, positive definite)**

symmetric: $\mathbf{P} = \mathbf{P}^T$

Least squares smoothing interpolation

Case 1: $n > m$ (more equations than unknowns). No solution $\mathbf{a}$ exists

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e}$$

Least squares solution: Find $\mathbf{a}$ which satisfies $\mathbf{e}^T\mathbf{P}\mathbf{e} = \min$

$\mathbf{P} =$ weight matrix (symmetric, positive definite)

symmetric: $\mathbf{P} = \mathbf{P}^T$

positive definite: $\mathbf{z}^T\mathbf{P}\mathbf{z} > 0, \quad \forall \mathbf{z}$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution** $\mathbf{a}$ **exists**

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e}$$

Least squares solution: **Find** $\mathbf{a}$ **which satisfies** $\mathbf{e}^T\mathbf{P}\mathbf{e} = \min$

Usually: diagonal weight matrix

$$\mathbf{P} = \begin{bmatrix} P_{11} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & P_{22} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{ii} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & P_{nn} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & p_n \end{bmatrix}$$

$$\mathbf{e}^T\mathbf{P}\mathbf{e} = p_1 e_1^2 + p_2 e_2^2 + p_i e_i^2 + p_n e_n^2 = \sum_{i=1}^{n} p_i e_i^2 = \min$$

Least squares smoothing interpolation

Case 1: $n > m$ (more equations than unknowns). No solution $\mathbf{a}$ exists

Derivation of the least squares solution

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e} \qquad\qquad \mathbf{e} = \mathbf{f} - \mathbf{F}\mathbf{a}$$

$$\phi(\mathbf{a}) = \mathbf{e}^T \mathbf{P}\mathbf{e} = \min$$

Minimum when: $$\frac{\partial \phi}{\partial \mathbf{a}} = \frac{\partial \phi}{\partial \mathbf{e}} \frac{\partial \mathbf{e}}{\partial \mathbf{a}} = \mathbf{0}$$

Least squares smoothing interpolation

Case 1: ***n*** **>** ***m*** **(more equations than unknowns). No solution a exists**

Derivation of the least squares solution

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e} \qquad\qquad \mathbf{e} = \mathbf{f} - \mathbf{F}\mathbf{a}$$

$$\phi(\mathbf{a}) = \mathbf{e}^T \mathbf{P}\mathbf{e} = \min$$

Minimum when:

$$\frac{\partial \phi}{\partial \mathbf{a}} = \frac{\partial \phi}{\partial \mathbf{e}} \frac{\partial \mathbf{e}}{\partial \mathbf{a}} = \mathbf{0}$$

$$\frac{\partial (\mathbf{e}^T \mathbf{P}\mathbf{e})}{\partial \mathbf{e}} = 2\mathbf{e}^T \mathbf{P} = 2(\mathbf{f} - \mathbf{F}\mathbf{a})^T \mathbf{P} \qquad\qquad \frac{\partial \mathbf{e}}{\partial \mathbf{a}} = \frac{\partial (\mathbf{f} - \mathbf{F}\mathbf{a})}{\partial \mathbf{a}} = -\mathbf{F}$$

Least squares smoothing interpolation

Case 1: $n > m$ (more equations than unknowns). No solution $\mathbf{a}$ exists

Derivation of the least squares solution

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e} \qquad \mathbf{e} = \mathbf{f} - \mathbf{F}\mathbf{a}$$

$$\phi(\mathbf{a}) = \mathbf{e}^T \mathbf{P}\mathbf{e} = \min$$

Minimum when: $$\frac{\partial \phi}{\partial \mathbf{a}} = \frac{\partial \phi}{\partial \mathbf{e}} \frac{\partial \mathbf{e}}{\partial \mathbf{a}} = \mathbf{0}$$

$$\frac{\partial(\mathbf{e}^T \mathbf{P}\mathbf{e})}{\partial \mathbf{e}} = 2\mathbf{e}^T \mathbf{P} = 2(\mathbf{f} - \mathbf{F}\mathbf{a})^T \mathbf{P} \qquad \frac{\partial \mathbf{e}}{\partial \mathbf{a}} = \frac{\partial(\mathbf{f} - \mathbf{F}\mathbf{a})}{\partial \mathbf{a}} = -\mathbf{F}$$

$$\frac{\partial \phi}{\partial \mathbf{a}} = \frac{\partial \phi}{\partial \mathbf{e}} \frac{\partial \mathbf{e}}{\partial \mathbf{a}} = 2(\mathbf{f} - \mathbf{F}\mathbf{a})^T \mathbf{P}(-\mathbf{F}) = \mathbf{0}$$

Least squares smoothing interpolation

Case 1: $n > m$ (more equations than unknowns). No solution $\mathbf{a}$ exists

Derivation of the least squares solution

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e} \qquad\qquad \mathbf{e} = \mathbf{f} - \mathbf{F}\mathbf{a}$$

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$$\frac{\partial \phi}{\partial \mathbf{a}} = \frac{\partial \phi}{\partial \mathbf{e}} \frac{\partial \mathbf{e}}{\partial \mathbf{a}} = 2(\mathbf{f} - \mathbf{F}\mathbf{a})^T \mathbf{P}(-\mathbf{F}) = \mathbf{0}$$

$$\frac{1}{2}\left(\frac{\partial \phi}{\partial \mathbf{a}}\right)^T (\hat{\mathbf{a}}) = -\mathbf{F}^T \mathbf{P}(\mathbf{f} - \mathbf{F}\hat{\mathbf{a}}) = \mathbf{0} \qquad \Rightarrow \qquad (\mathbf{F}^T \mathbf{P} \mathbf{F})\hat{\mathbf{a}} = \mathbf{F}^T \mathbf{P} \mathbf{f}$$

(normal equations)

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution** $\mathbf{a}$ **exists**

Derivation of the least squares solution

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e}$$

$$\mathbf{e}^T \mathbf{P}\mathbf{e} = \min$$

normal equations: $$(\mathbf{F}^T \mathbf{P}\mathbf{F})\hat{\mathbf{a}} = \mathbf{F}^T \mathbf{P}\mathbf{f} \qquad \mathbf{N}\hat{\mathbf{a}} = \mathbf{u}$$

$$\hat{\mathbf{a}} = (\mathbf{F}^T \mathbf{P}\mathbf{F})^{-1} \mathbf{F}^T \mathbf{P}\mathbf{f} = \mathbf{N}^{-1}\mathbf{u}$$

Least squares smoothing interpolation

Case 1: $n > m$ (more equations than unknowns). No solution $\mathbf{a}$ exists

Derivation of the least squares solution

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e}$$

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Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Derivation of the least squares solution

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e}$$

$$\mathbf{e}^T \mathbf{P} \mathbf{e} = \min$$

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$$\hat{\mathbf{a}} = (\mathbf{F}^T \mathbf{P} \mathbf{F})^{-1} \mathbf{F}^T \mathbf{P} \mathbf{f} = \mathbf{N}^{-1}\mathbf{u}$$

Interpolating function:

$$\hat{f}(x) = \hat{a}_1 \varphi_1(x) + \ldots + \hat{a}_i \varphi_i(x) + \ldots + \hat{a}_n \varphi_n(x) = \begin{bmatrix} \varphi_1(x) & \cdots & \varphi_i(x) & \cdots & \varphi_n(x) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_i \\ \vdots \\ \hat{a}_n \end{bmatrix} = \boldsymbol{\varphi}(x)^T \hat{\mathbf{a}}$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Derivation of the least squares solution

$$\mathbf{f} = \mathbf{F}\mathbf{a} + \mathbf{e}$$

$$\mathbf{e}^T \mathbf{P} \mathbf{e} = \min$$

normal equations: $$(\mathbf{F}^T \mathbf{P} \mathbf{F})\hat{\mathbf{a}} = \mathbf{F}^T \mathbf{P} \mathbf{f} \qquad \mathbf{N}\hat{\mathbf{a}} = \mathbf{u}$$

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Interpolating function:

$$\hat{f}(x) = \hat{a}_1 \varphi_1(x) + \ldots + \hat{a}_i \varphi_i(x) + \ldots + \hat{a}_n \varphi_n(x) = \begin{bmatrix} \varphi_1(x) & \cdots & \varphi_i(x) & \cdots & \varphi_n(x) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_i \\ \vdots \\ \hat{a}_n \end{bmatrix} = \boldsymbol{\varphi}(x)^T \hat{\mathbf{a}}$$

$$\hat{f}(x) = \boldsymbol{\varphi}(x)^T \hat{\mathbf{a}} = \boldsymbol{\varphi}(x)^T (\mathbf{F}^T \mathbf{P} \mathbf{F})^{-1} \mathbf{F}^T \mathbf{P} \mathbf{f} = \boldsymbol{\varphi}(x)^T \mathbf{N}^{-1} \mathbf{u}$$

Least squares smoothing interpolation

Case 1: ***n*** **>** ***m*** **(more equations than unknowns). No solution a exists**

Example 1:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x^2$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) = a_1 + a_2 x + a_3 x^2$$

Least squares smoothing interpolation

Case 1: ***n*** **>** ***m*** **(more equations than unknowns). No solution a exists**

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$$f_1 = f(x_1) = a_1 + a_2 x_1 + a_3 x_1^2$$
$$f_2 = f(x_2) = a_1 + a_2 x_2 + a_3 x_2^2$$
$$f_3 = f(x_3) = a_1 + a_2 x_3 + a_3 x_3^2$$
$$f_4 = f(x_4) = a_1 + a_2 x_4 + a_3 x_4^2$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution $\mathbf{a}$ exists**

Example 1:

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$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) = a_1 + a_2 x + a_3 x^2$$

$$\begin{aligned} f_1 &= f(x_1) = a_1 + a_2 x_1 + a_3 x_1^2 \\ f_2 &= f(x_2) = a_1 + a_2 x_2 + a_3 x_2^2 \\ f_3 &= f(x_3) = a_1 + a_2 x_3 + a_3 x_3^2 \\ f_4 &= f(x_4) = a_1 + a_2 x_4 + a_3 x_4^2 \end{aligned}$$

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \mathbf{F}\mathbf{a} + \mathbf{e} \qquad \mathbf{P} = \mathbf{I}$$

$$\hat{\mathbf{a}} = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\mathbf{f} = \mathbf{N}^{-1}\mathbf{u}$$

Least squares smoothing interpolation

Case 1: ***n* > *m*** **(more equations than unknowns). No solution a exists**

Example 1:

$$\varphi_1(x) = 1,\ \ \varphi_2(x) = x,\ \ \varphi_3(x) = x^2$$

$$f(x) = a_1 + a_2 x + a_3 x^2$$

$$\mathbf{N} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} = \begin{bmatrix} 4 & x_1 + x_2 + x_3 + x_4 & x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ x_1 + x_2 + x_3 + x_4 & x_1^2 + x_2^2 + x_3^2 + x_4^2 & x_1^3 + x_2^3 + x_3^3 + x_4^3 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 & x_1^3 + x_2^3 + x_3^3 + x_4^3 & x_1^4 + x_2^4 + x_3^4 + x_4^4 \end{bmatrix}$$

Least squares smoothing interpolation

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Example 1:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x^2$$

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$$\mathbf{u} = \mathbf{F}^T\mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} f_1 + f_2 + f_3 + f_4 \\ x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 \\ x_1^2 f_1 + x_2^2 f_2 + x_3^2 f_3 + x_4^2 f_4 \end{bmatrix}$$

$$\hat{\mathbf{a}} = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\mathbf{f} = \mathbf{N}^{-1}\mathbf{u}$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Example 1:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x^2$$

$$f(x) = a_1 + a_2 x + a_3 x^2$$

$$\mathbf{N} = \begin{bmatrix} 4 & x_1 + x_2 + x_3 + x_4 & x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ x_1 + x_2 + x_3 + x_4 & x_1^2 + x_2^2 + x_3^2 + x_4^2 & x_1^3 + x_2^3 + x_3^3 + x_4^3 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 & x_1^3 + x_2^3 + x_3^3 + x_4^3 & x_1^4 + x_2^4 + x_3^4 + x_4^4 \end{bmatrix} = \begin{bmatrix} 4 & \sum_{i=1}^{4} x_i & \sum_{i=1}^{4} x_i^2 \\ \sum_{i=1}^{4} x_i & \sum_{i=1}^{4} x_i^2 & \sum_{i=1}^{4} x_i^3 \\ \sum_{i=1}^{4} x_i^2 & \sum_{i=1}^{4} x_i^3 & \sum_{i=1}^{4} x_i^4 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} f_1 + f_2 + f_3 + f_4 \\ x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 \\ x_1^2 f_1 + x_2^2 f_2 + x_3^2 f_3 + x_4^2 f_4 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} f_i \\ \sum_{i=1}^{4} x_i f_i \\ \sum_{i=1}^{4} x_i^2 f_i \end{bmatrix}$$

$$\hat{\mathbf{a}} = \mathbf{N}^{-1}\mathbf{u}$$

$$\hat{f}(x) = \hat{a}_1 + \hat{a}_2 x + \hat{a}_3 x^2$$

Least squares smoothing interpolation

Case 1: ***n*** **>** ***m*** **(more equations than unknowns). No solution a exists**

Example 2:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = \cos(\omega x), \quad \varphi_3(x) = \sin(\omega x)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{b-a}$$

$$f(x) = a_1 + a_2 \cos(\omega x) + a_3 \sin(\omega x)$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Example 2:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = \cos(\omega x), \quad \varphi_3(x) = \sin(\omega x)$$

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$$f(x) = a_1 + a_2\cos(\omega x) + a_3\sin(\omega x)$$

$$\begin{aligned}
f_1 &= f(x_1) = a_1 + a_2\cos(\omega x_1) + a_3\sin(\omega x_1) \\
f_2 &= f(x_2) = a_1 + a_2\cos(\omega x_2) + a_3\sin(\omega x_2) \\
f_3 &= f(x_3) = a_1 + a_2\cos(\omega x_3) + a_3\sin(\omega x_3) \\
f_4 &= f(x_4) = a_1 + a_2\cos(\omega x_4) + a_3\sin(\omega x_4)
\end{aligned}$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Example 2:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = \cos(\omega x), \quad \varphi_3(x) = \sin(\omega x)$$

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$$f(x) = a_1 + a_2 \cos(\omega x) + a_3 \sin(\omega x)$$

$$\begin{aligned}
f_1 &= f(x_1) = a_1 + a_2 \cos(\omega x_1) + a_3 \sin(\omega x_1) \\
f_2 &= f(x_2) = a_1 + a_2 \cos(\omega x_2) + a_3 \sin(\omega x_2) \\
f_3 &= f(x_3) = a_1 + a_2 \cos(\omega x_3) + a_3 \sin(\omega x_3) \\
f_4 &= f(x_4) = a_1 + a_2 \cos(\omega x_4) + a_3 \sin(\omega x_4)
\end{aligned}$$

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 1 & \cos(\omega x_1) & \sin(\omega x_1) \\ 1 & \cos(\omega x_2) & \sin(\omega x_2) \\ 1 & \cos(\omega x_3) & \sin(\omega x_3) \\ 1 & \cos(\omega x_4) & \sin(\omega x_4) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \mathbf{Fa} + \mathbf{e}$$

$$\mathbf{P} = \mathbf{I}$$

$$\hat{\mathbf{a}} = (\mathbf{F}^T \mathbf{F})^{-1} \mathbf{F}^T \mathbf{f} = \mathbf{N}^{-1} \mathbf{u}$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Example2:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = \cos(\omega x), \quad \varphi_3(x) = \sin(\omega x)$$

$$f(x) = a_1 + a_2 \cos(\omega x) + a_3 \sin(\omega x)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{b-a}$$

$$\mathbf{N} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \cos(\omega x_1) & \cos(\omega x_2) & \cos(\omega x_3) & \cos(\omega x_4) \\ \sin(\omega x_1) & \sin(\omega x_2) & \sin(\omega x_3) & \sin(\omega x_4) \end{bmatrix} \begin{bmatrix} 1 & \cos(\omega x_1) & \sin(\omega x_1) \\ 1 & \cos(\omega x_2) & \sin(\omega x_2) \\ 1 & \cos(\omega x_3) & \sin(\omega x_3) \\ 1 & \cos(\omega x_4) & \sin(\omega x_4) \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & \sum_{i=1}^{4} \cos(\omega x_i) & \sum_{i=1}^{4} \sin(\omega x_i) \\ \sum_{i=1}^{4} \cos(\omega x_i) & \sum_{i=1}^{4} \cos^2(\omega x_i) & \sum_{i=1}^{4} \cos(\omega x_i)\sin(\omega x_i) \\ \sum_{i=1}^{4} \sin(\omega x_i) & \sum_{i=1}^{4} \cos(\omega x_i)\sin(\omega x_i) & \sum_{i=1}^{4} \sin^2(\omega x_i) \end{bmatrix}$$

Least squares smoothing interpolation

Case 1: ***n*** **>** ***m*** **(more equations than unknowns). No solution a exists**

Example 2:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = \cos(\omega x), \quad \varphi_3(x) = \sin(\omega x)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{b-a}$$

$$f(x) = a_1 + a_2 \cos(\omega x) + a_3 \sin(\omega x)$$

$$\mathbf{u} = \mathbf{F}^T \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \cos(\omega x_1) & \cos(\omega x_2) & \cos(\omega x_3) & \cos(\omega x_4) \\ \sin(\omega x_1) & \sin(\omega x_2) & \sin(\omega x_3) & \sin(\omega x_4) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} f_i \\ \sum_{i=1}^{4} f_i \cos(\omega x_i) \\ \sum_{i=1}^{4} f_i \sin(\omega x_i) \end{bmatrix}$$

Least squares smoothing interpolation

Case 1: ***n* > *m* (more equations than unknowns). No solution a exists**

Example2:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = \cos(\omega x), \quad \varphi_3(x) = \sin(\omega x)$$

$$f(x) = a_1 + a_2 \cos(\omega x) + a_3 \sin(\omega x)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{b-a}$$

$$\mathbf{N} = \begin{bmatrix} 4 & \sum_{i=1}^{4} \cos(\omega x_i) & \sum_{i=1}^{4} \sin(\omega x_i) \\ \sum_{i=1}^{4} \cos(\omega x_i) & \sum_{i=1}^{4} \cos^2(\omega x_i) & \sum_{i=1}^{4} \cos(\omega x_i) \sin(\omega x_i) \\ \sum_{i=1}^{4} \sin(\omega x_i) & \sum_{i=1}^{4} \cos(\omega x_i) \sin(\omega x_i) & \sum_{i=1}^{4} \sin^2(\omega x_i) \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \sum_{i=1}^{4} f_i \\ \sum_{i=1}^{4} f_i \cos(\omega x_i) \\ \sum_{i=1}^{4} f_i \sin(\omega x_i) \end{bmatrix}$$

$$\hat{\mathbf{a}} = (\mathbf{F}^T \mathbf{F})^{-1} \mathbf{F}^T \mathbf{f} = \mathbf{N}^{-1} \mathbf{u}$$

Least squares smoothing interpolation

Case 1: ***n*** **>** ***m*** **(more equations than unknowns). No solution a exists**

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution $\mathbf{a}$ exists**

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} = \begin{bmatrix} a_1 + a_2 x_1 + e_1 \\ a_2 + a_2 x_2 + e_2 \\ \vdots \\ a_1 + a_2 x_n + e_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \mathbf{Fa} + \mathbf{e}$$

Least squares smoothing interpolation

Case 1: ***n* > *m*** **(more equations than unknowns). No solution a exists**

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} = \begin{bmatrix} a_1 + a_2 x_1 + e_1 \\ a_2 + a_2 x_2 + e_2 \\ \vdots \\ a_1 + a_2 x_n + e_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \mathbf{Fa} + \mathbf{e}$$

$$\mathbf{P} = \mathbf{I}$$

$$\hat{\mathbf{a}} = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\mathbf{f} = \mathbf{N}^{-1}\mathbf{u}$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

$$\mathbf{N} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & x_1 + x_2 + \ldots + x_n \\ x_1 + x_2 + \ldots + x_n & x_1^2 + x_2^2 + \ldots + x_n^2 \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}$$

Least squares smoothing interpolation

Case 1: $n > m$ (more equations than unknowns). No solution $\mathbf{a}$ exists

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

$$\mathbf{N} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & x_1 + x_2 + \ldots + x_n \\ x_1 + x_2 + \ldots + x_n & x_1^2 + x_2^2 + \ldots + x_n^2 \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{F}^T\mathbf{f} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} f_1 + f_2 + \ldots + f_n \\ x_1 f_1 + x_2 f_2 + \ldots + x_n f_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} f_i \\ \sum_{i=1}^{n} x_i f_i \end{bmatrix}$$

$$\hat{\mathbf{a}} = \mathbf{N}^{-1}\mathbf{u}$$

Least squares smoothing interpolation

Case 1: ***n*** **>** ***m*** **(more equations than unknowns). No solution a exists**

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

$$\mathbf{N} = n\begin{bmatrix} 1 & \frac{1}{n}\sum_{i=1}^{n} x_i \\ \frac{1}{n}\sum_{i=1}^{n} x_i & \frac{1}{n}\sum_{i=1}^{n} x_i^2 \end{bmatrix} \equiv n\begin{bmatrix} 1 & \overline{x} \\ \overline{x} & m_x^2 \end{bmatrix}$$

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad m_x^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\mathbf{u} = n\begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} f_i \\ \frac{1}{n}\sum_{i=1}^{n} x_i f_i \end{bmatrix} \equiv n\begin{bmatrix} \overline{f} \\ m_{xf} \end{bmatrix}$$

$$\overline{f} = \frac{1}{n}\sum_{i=1}^{n} f_i \qquad m_{xf} = \frac{1}{n}\sum_{i=1}^{n} x_i f_i$$

$$\hat{\mathbf{a}} = \mathbf{N}^{-1}\mathbf{u}$$

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$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad m_x^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\bar{f} = \frac{1}{n}\sum_{i=1}^{n} f_i \qquad m_{xf} = \frac{1}{n}\sum_{i=1}^{n} x_i f_i$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

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$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad m_x^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \qquad s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = m_x^2 - \bar{x}^2$$

$$\bar{f} = \frac{1}{n}\sum_{i=1}^{n} f_i \qquad m_{xf} = \frac{1}{n}\sum_{i=1}^{n} x_i f_i \qquad s_{xf} = \sum_{i=1}^{n}(x_i - \bar{x})(f_i - \bar{f}) = m_{xf} - \bar{x}\bar{f}$$

Least squares smoothing interpolation

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$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad m_x^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \qquad s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 = m_x^2 - \overline{x}^2 \qquad m_x^2 = s_x^2 + \overline{x}^2$$

$$\overline{f} = \frac{1}{n}\sum_{i=1}^{n} f_i \qquad m_{xf} = \frac{1}{n}\sum_{i=1}^{n} x_i f_i \qquad s_{xf} = \sum_{i=1}^{n}(x_i - \overline{x})(f_i - \overline{f}) = m_{xf} - \overline{x}\overline{f} \qquad m_{xf} = s_{xf} + \overline{x}\,\overline{f}$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution $\mathbf{a}$ exists**

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$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

$$\mathbf{N} = n\begin{bmatrix} 1 & \bar{x} \\ \bar{x} & s_x^2 + \bar{x}^2 \end{bmatrix} \qquad \mathbf{u} = n\begin{bmatrix} \bar{f} \\ s_{xf} + \bar{x}\,\bar{f} \end{bmatrix} \qquad \hat{\mathbf{a}} = \mathbf{N}^{-1}\mathbf{u}$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\bar{f} = \frac{1}{n}\sum_{i=1}^{n} f_i \qquad s_{xf} = \sum_{i=1}^{n}(x_i - \bar{x})(f_i - \bar{f})$$

Least squares smoothing interpolation

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$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

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$$\hat{\mathbf{a}} = \mathbf{N}^{-1}\mathbf{u} = \begin{bmatrix} 1 & \overline{x} \\ \overline{x} & s_x^2 + \overline{x}^2 \end{bmatrix}^{-1} \begin{bmatrix} \overline{f} \\ s_{xf} + \overline{x}\,\overline{f} \end{bmatrix}$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

$$\mathbf{N} = n\begin{bmatrix} 1 & \bar{x} \\ \bar{x} & s_x^2 + \bar{x}^2 \end{bmatrix} \qquad \mathbf{u} = n\begin{bmatrix} \bar{f} \\ s_{xf} + \bar{x}\,\bar{f} \end{bmatrix}$$

$$\hat{\mathbf{a}} = \mathbf{N}^{-1}\mathbf{u} = \begin{bmatrix} 1 & \bar{x} \\ \bar{x} & s_x^2 + \bar{x}^2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{f} \\ s_{xf} + \bar{x}\,\bar{f} \end{bmatrix} = \frac{1}{s_x^2 + \bar{x}^2 - \bar{x}^2} \begin{bmatrix} s_x^2 + \bar{x}^2 & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix} \begin{bmatrix} \bar{f} \\ s_{xf} + \bar{x}\,\bar{f} \end{bmatrix} =$$

Use

$$\begin{bmatrix} a & c \\ d & b \end{bmatrix}^{-1} = \frac{1}{ab - cd}\begin{bmatrix} b & -c \\ -d & a \end{bmatrix}$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

$$\mathbf{N} = n\begin{bmatrix} 1 & \bar{x} \\ \bar{x} & s_x^2 + \bar{x}^2 \end{bmatrix} \qquad \mathbf{u} = n\begin{bmatrix} \bar{f} \\ s_{xf} + \bar{x}\,\bar{f} \end{bmatrix}$$

$$\hat{\mathbf{a}} = \mathbf{N}^{-1}\mathbf{u} = \begin{bmatrix} 1 & \bar{x} \\ \bar{x} & s_x^2 + \bar{x}^2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{f} \\ s_{xf} + \bar{x}\,\bar{f} \end{bmatrix} = \frac{1}{s_x^2 + \bar{x}^2 - \bar{x}^2} \begin{bmatrix} s_x^2 + \bar{x}^2 & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix} \begin{bmatrix} \bar{f} \\ s_{xf} + \bar{x}\,\bar{f} \end{bmatrix} =$$

$$= \frac{1}{s_x^2} \begin{bmatrix} (s_x^2 + \bar{x}^2)\bar{f} - \bar{x}(s_{xf} + \bar{x}\,\bar{f}) \\ -\bar{x}\bar{f} + (s_{xf} + \bar{x}\,\bar{f}) \end{bmatrix} = \frac{1}{s_x^2} \begin{bmatrix} s_x^2\bar{f} - \bar{x}s_{xf} \\ s_{xf} \end{bmatrix} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution a exists**

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) = a_1 + a_2 x$$

$$\mathbf{N} = n\begin{bmatrix} 1 & \overline{x} \\ \overline{x} & s_x^2 + \overline{x}^2 \end{bmatrix} \qquad \mathbf{u} = n\begin{bmatrix} \overline{f} \\ s_{xf} + \overline{x}\,\overline{f} \end{bmatrix}$$

$$\hat{\mathbf{a}} = \mathbf{N}^{-1}\mathbf{u} = \begin{bmatrix} 1 & \overline{x} \\ \overline{x} & s_x^2 + \overline{x}^2 \end{bmatrix}^{-1} \begin{bmatrix} \overline{f} \\ s_{xf} + \overline{x}\,\overline{f} \end{bmatrix} = \frac{1}{s_x^2 + \overline{x}^2 - \overline{x}^2} \begin{bmatrix} s_x^2 + \overline{x}^2 & -\overline{x} \\ -\overline{x} & 1 \end{bmatrix} \begin{bmatrix} \overline{f} \\ s_{xf} + \overline{x}\,\overline{f} \end{bmatrix} =$$

$$= \frac{1}{s_x^2} \begin{bmatrix} (s_x^2 + \overline{x}^2)\overline{f} - \overline{x}(s_{xf} + \overline{x}\,\overline{f}) \\ -\overline{x}\overline{f} + (s_{xf} + \overline{x}\,\overline{f}) \end{bmatrix} = \frac{1}{s_x^2} \begin{bmatrix} s_x^2 \overline{f} - \overline{x} s_{xf} \\ s_{xf} \end{bmatrix} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}$$

$$\hat{a}_2 = \frac{s_{xf}}{s_x^2} \qquad \hat{a}_1 = \frac{s_x^2 \overline{f} - \overline{x} s_{xf}}{s_x^2} = \overline{f} - \overline{x}\frac{s_{xf}}{s_x^2} = \overline{f} - \overline{x}\hat{a}_2$$

Least squares smoothing interpolation

Case 1: $n > m$ **(more equations than unknowns). No solution** $\mathbf{a}$ **exists**

Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

$$\hat{f}(x) = \hat{a}_1\varphi_1(x) + \hat{a}_2\varphi_2(x) = \hat{a}_1 + \hat{a}_2 x$$

$$\hat{a}_2 = \frac{s_{xf}}{s_x^2} \qquad \hat{a}_1 = \frac{s_x^2\,\overline{f} - \overline{x}s_{xf}}{s_x^2} = \overline{f} - \overline{x}\frac{s_{xf}}{s_x^2} = \overline{f} - \overline{x}\hat{a}_2$$

Least squares smoothing interpolation

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Example 3: Linear regression

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x$$

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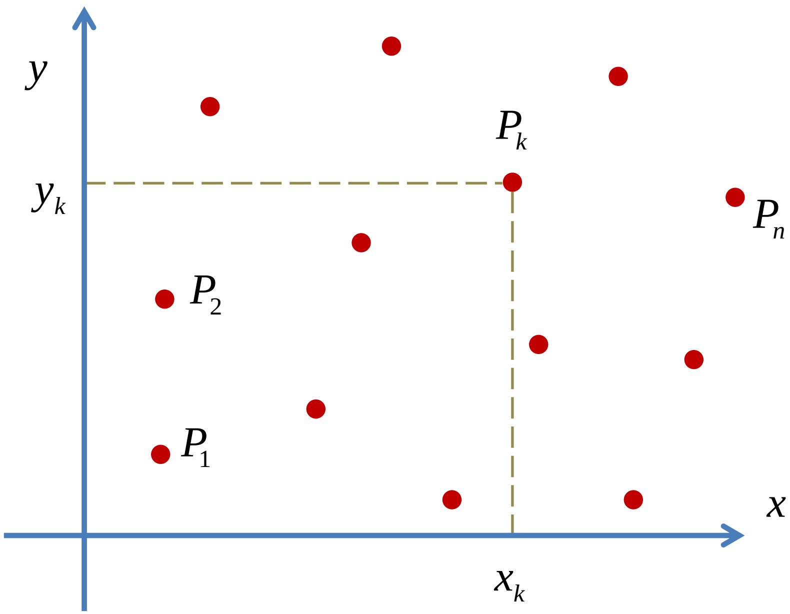
$$\hat{a}_2 = \frac{s_{xf}}{s_x^2} \qquad \hat{a}_1 = \frac{s_x^2 \bar{f} - \bar{x} s_{xf}}{s_x^2} = \bar{f} - \bar{x}\frac{s_{xf}}{s_x^2} = \bar{f} - \bar{x}\hat{a}_2$$

$$\hat{f}(x) = \bar{f} + \frac{s_{xf}}{s_x^2}(x - \bar{x})$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \bar{f} = \frac{1}{n}\sum_{i=1}^{n} f_i$$

$$s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \qquad s_{xf} = \sum_{i=1}^{n}(x_i - \bar{x})(f_i - \bar{f})$$

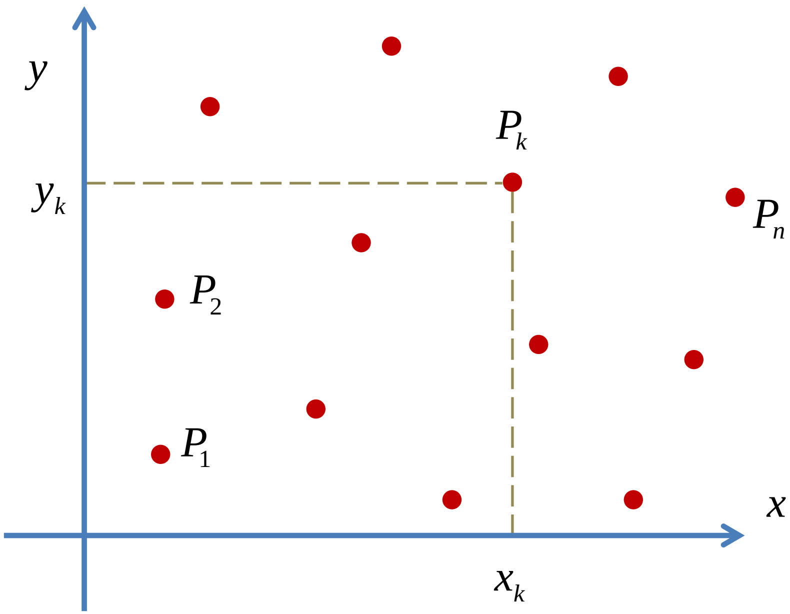
Least squares smoothing interpolation in 2 dimensions



Available data:

$$f_1 = f(P_1) = f(x_1, y_1)$$
$$\vdots$$
$$f_k = f(P_k) = f(x_k, y_k)$$
$$\vdots$$
$$f_n = f(P_n) = f(x_n, y_n)$$

Least squares smoothing interpolation in 2 dimensions



Available data:

$$f_1 = f(P_1) = f(x_1, y_1)$$
$$\vdots$$
$$f_k = f(P_k) = f(x_k, y_k)$$
$$\vdots$$
$$f_n = f(P_n) = f(x_n, y_n)$$

Base functions:

$$\varphi_1(x,y),\ \varphi_2(x,y),\ \ldots,\ \varphi_k(x,y),\ \ldots,\ \varphi_m(x,y)$$

Interpolating function:

$$f(x,y) = a_1\varphi_1(x,y) + a_2\varphi_2(x,y) + \ldots + a_k\varphi_k(x,y) + \ldots + a_m\varphi_m(x,y) = \boldsymbol{\varphi}(x,y)^T\,\mathbf{a}$$

Least squares smoothing interpolation in 2 dimensions

Satisfying the data values

$$f_i = f(x_i, y_i) = a_1\varphi_1(x_i, y_i) + a_2\varphi_2(x_i, y_i) + \ldots + a_k\varphi_k(x_i, y_i) + \ldots + a_m\varphi_m(x_i, y_i) + e_i$$

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} f(x_1, y_1) \\ f(x_2, y_2) \\ \vdots \\ f(x_i, y_i) \\ \vdots \\ f(x_n, y_n) \end{bmatrix} = \begin{bmatrix} a_1\varphi_1(x_1, y_1) + a_2\varphi_2(x_1, y_1) + \ldots + a_k\varphi_k(x_1, y_1) + \ldots + a_m\varphi_m(x_1, y_1) + e_1 \\ a_1\varphi_1(x_2, y_2) + a_2\varphi_2(x_2, y_2) + \ldots + a_k\varphi_k(x_2, y_2) + \ldots + a_m\varphi_m(x_2, y_2) + e_2 \\ \vdots \\ a_1\varphi_1(x_i, y_i) + a_2\varphi_2(x_i, y_i) + \ldots + a_k\varphi_k(x_i, y_i) + \ldots + a_m\varphi_m(x_i, y_i) + e_i \\ \vdots \\ a_1\varphi_1(x_n, y_n) + a_2\varphi_2(x_n, y_n) + \ldots + a_k\varphi_k(x_n, y_n) + \ldots + a_m\varphi_m(x_n, y_n) + e_n \end{bmatrix}$$

Least squares smoothing interpolation in 2 dimensions

Satisfying the data values ($n > m$) impossible

$$f_i = f(x_i, y_i) = a_1\varphi_1(x_i, y_i) + a_2\varphi_2(x_i, y_i) + \ldots + a_k\varphi_k(x_i, y_i) + \ldots + a_m\varphi_m(x_i, y_i) + e_i$$

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} \varphi_1(x_1, y_1) & \varphi_2(x_1, y_1) & \cdots & \varphi_k(x_1, y_1) & \cdots & \varphi_m(x_1, y_1) \\ \varphi_1(x_2, y_2) & \varphi_2(x_2, y_2) & \cdots & \varphi_k(x_2, y_2) & \cdots & \varphi_m(x_2, y_2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_i, y_i) & \varphi_2(x_i, y_i) & \cdots & \varphi_k(x_i, y_i) & \cdots & \varphi_m(x_i, y_i) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_n, y_n) & \varphi_2(x_n, y_n) & \cdots & \varphi_k(x_i, y_i) & \cdots & \varphi_m(x_n, y_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_i \\ \vdots \\ e_n \end{bmatrix}$$

$$F_{ik} = \varphi_k(x_i, y_i)$$

Least squares smoothing interpolation in 2 dimensions

Satisfying the data values ($n > m$) impossible

$$f_i = f(x_i, y_i) = a_1\varphi_1(x_i, y_i) + a_2\varphi_2(x_i, y_i) + \ldots + a_k\varphi_k(x_i, y_i) + \ldots + a_m\varphi_m(x_i, y_i) + e_i$$

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} \varphi_1(x_1, y_1) & \varphi_2(x_1, y_1) & \cdots & \varphi_k(x_1, y_1) & \cdots & \varphi_m(x_1, y_1) \\ \varphi_1(x_2, y_2) & \varphi_2(x_2, y_2) & \cdots & \varphi_k(x_2, y_2) & \cdots & \varphi_m(x_2, y_2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_i, y_i) & \varphi_2(x_i, y_i) & \cdots & \varphi_k(x_i, y_i) & \cdots & \varphi_m(x_i, y_i) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varphi_1(x_n, y_n) & \varphi_2(x_n, y_n) & \cdots & \varphi_k(x_i, y_i) & \cdots & \varphi_m(x_n, y_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_i \\ \vdots \\ e_n \end{bmatrix}$$

$$F_{ik} = \varphi_k(x_i, y_i)$$

$$\mathbf{f} = \mathbf{Fa} + \mathbf{e} \longrightarrow \boxed{\mathbf{e}^T\mathbf{Pe} = \min} \longrightarrow \hat{\mathbf{a}} = (\mathbf{F}^T\mathbf{PF})^{-1}\mathbf{F}^T\mathbf{Pf}$$

$$\hat{f}(x, y) = \hat{a}_1\varphi_1(x, y) + \hat{a}_2\varphi_2(x, y) + \ldots + \hat{a}_k\varphi_k(x, y) + \ldots + \hat{a}_m\varphi_m(x, y) = \boldsymbol{\varphi}(x, y)^T\,\hat{\mathbf{a}}$$

$$\hat{f}(x, y) = \boldsymbol{\varphi}(x, y)^T\,\hat{\mathbf{a}} = \boldsymbol{\varphi}(x, y)^T(\mathbf{F}^T\mathbf{PF})^{-1}\mathbf{F}^T\mathbf{Pf}$$