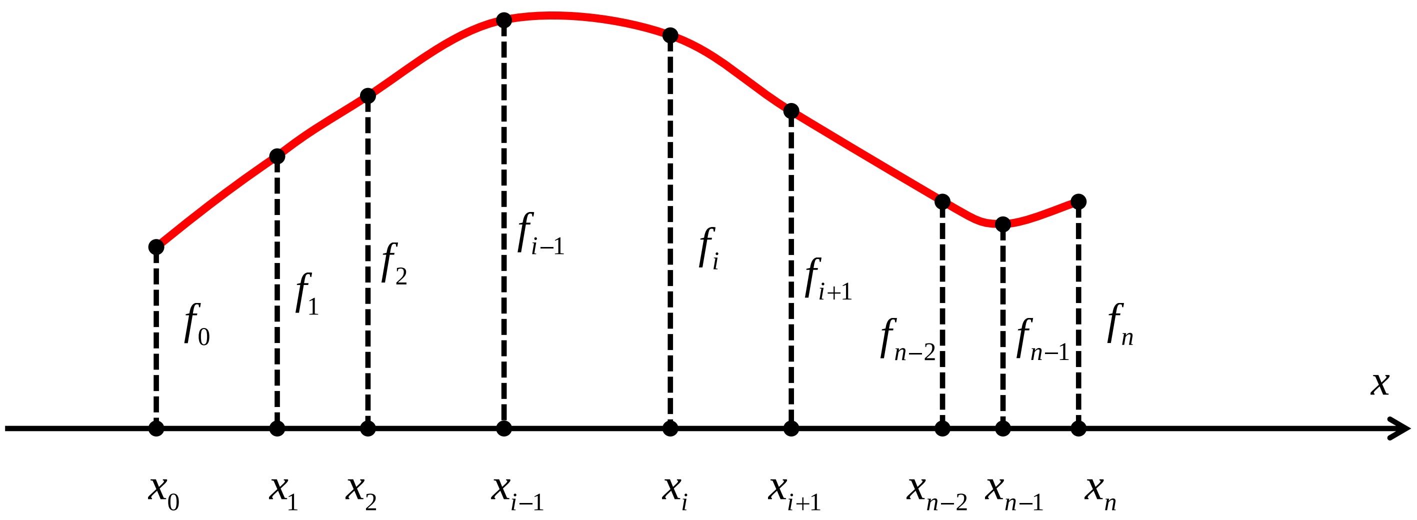
Exact interpolation by cubic splines

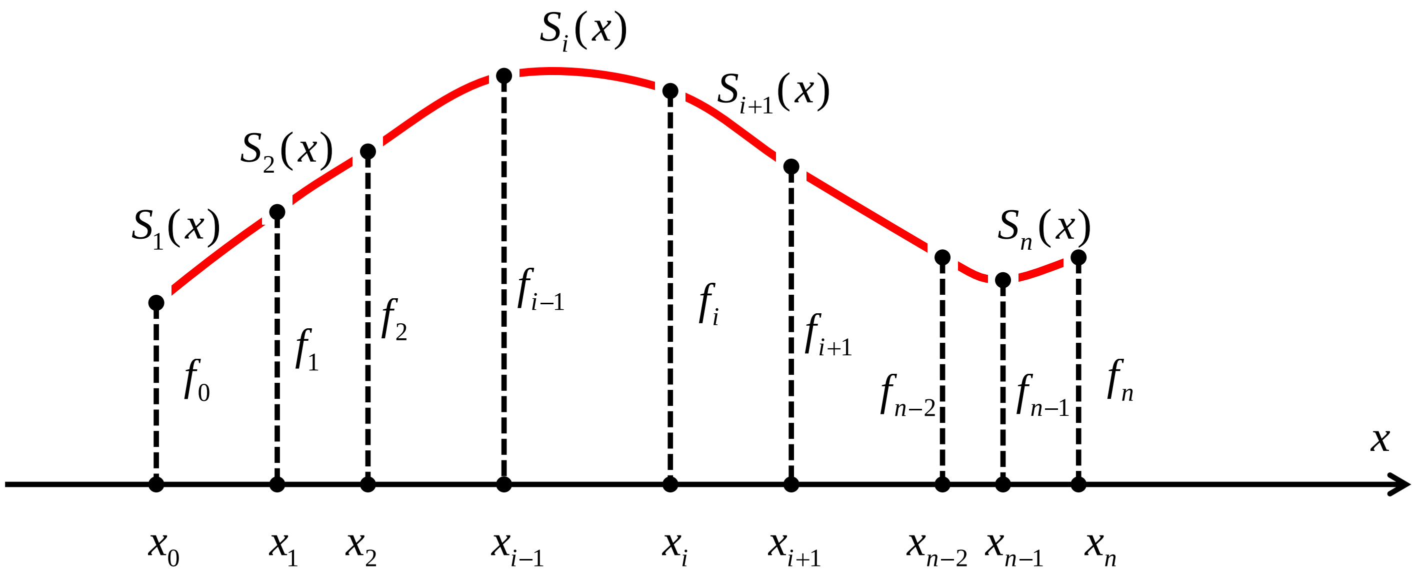
Problem statement



Interpolation: Given a set of $n+1$ values $f_0, f_1, \ldots, f_i, \ldots, f_{n\text{-}1}, f_n$
at $n+1$ points $x = x_0, x_1, \ldots, x_i, \ldots, x_{n\text{-}1}, x_n$
produce an interpolating function $f(x)$
which reproduces the given data:

$$f(x_i) = f_i, \quad i = 0, 1, \ldots, n$$

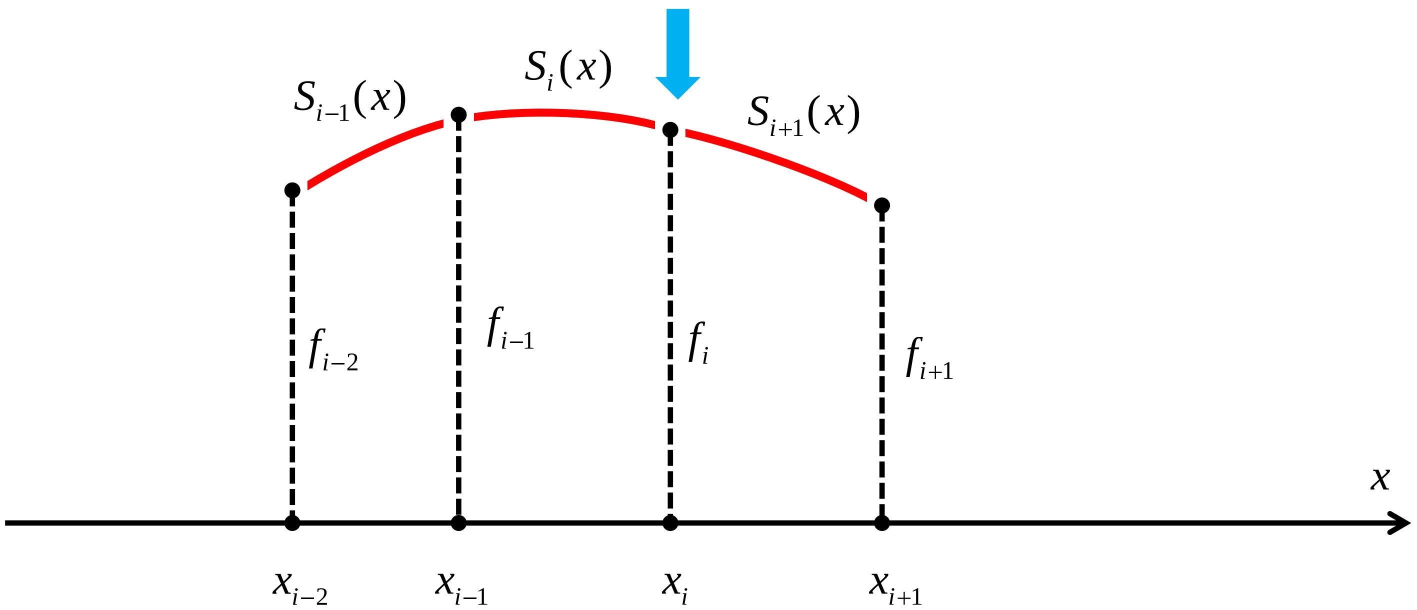
Problem statement



Piecewise Interpolation:

Use a different interpolating function $S_i\,(x)$ for each interval $[x_{i\text{-}1}, x_i\,]$.

Desired properties of piecewise interpolating functions at break (data) points

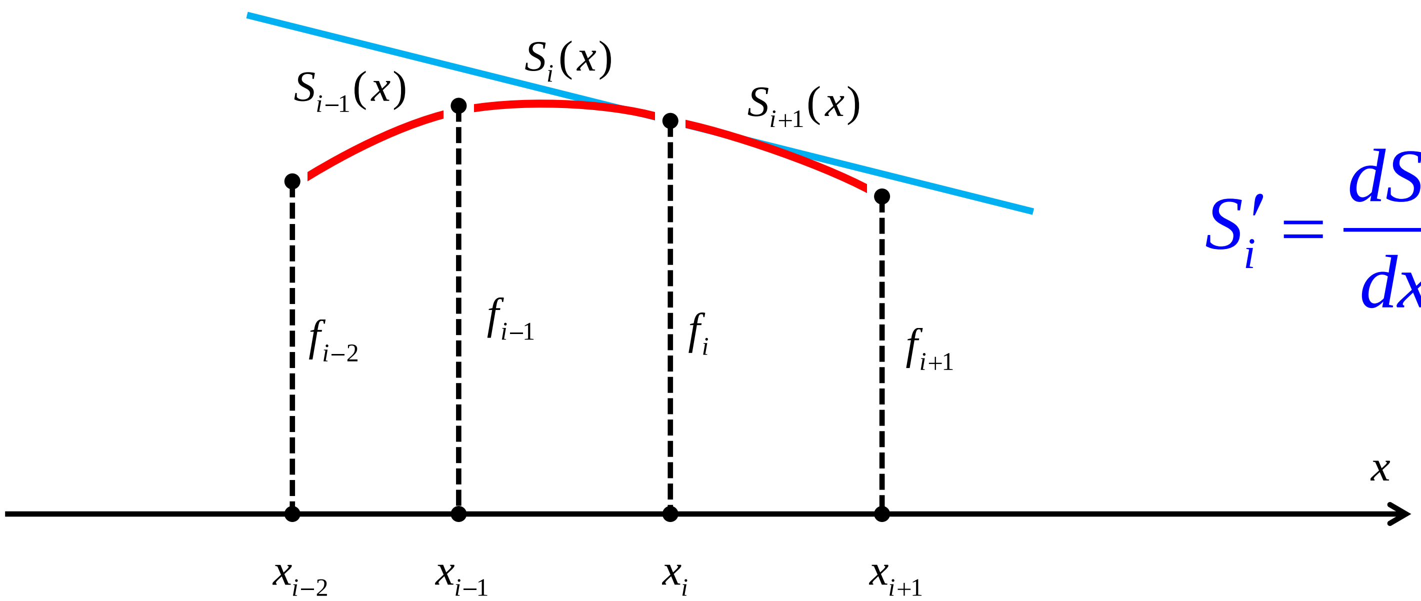


Continuity:

Already satisfied by the interpolating property:
Neighboring functions have same value at common point

$$S_i(x_i) = S_{i+1}(x_i) = f_i$$

Desired properties of piecewise interpolating functions at break (data) points



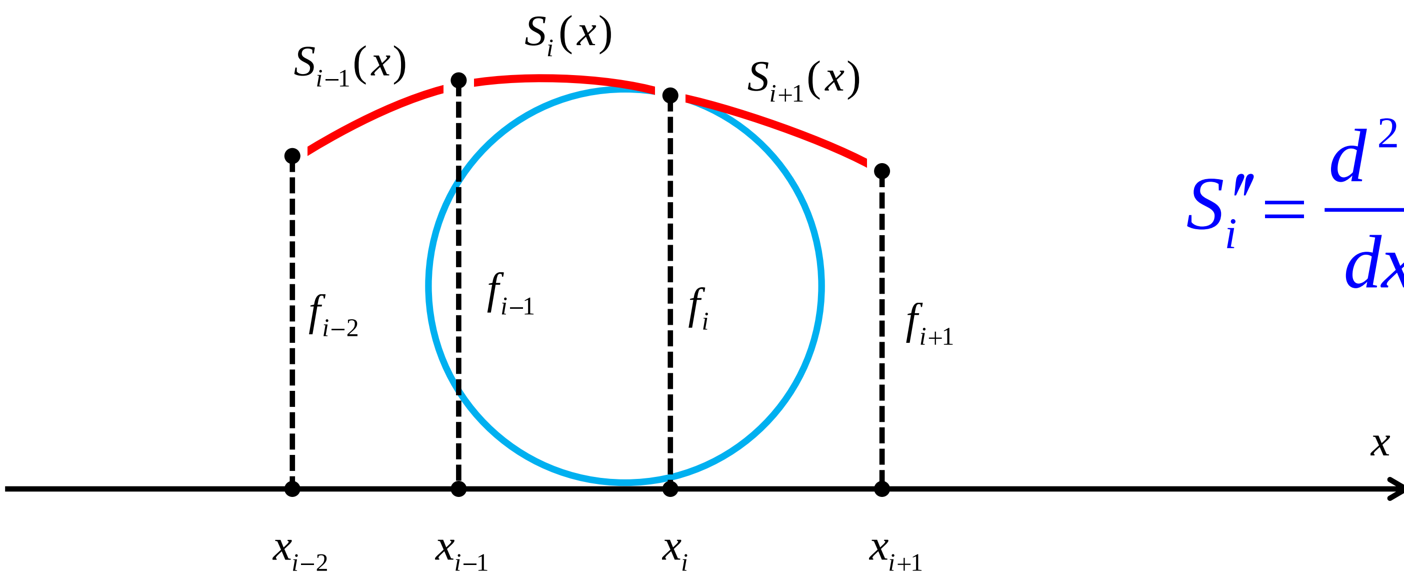
$$S_i' = \frac{dS_i}{dx}$$

Smoothness:

Neighboring functions have same tangent at common point =
= Same value of first derivative:

$$S_i'(x_i) = S_{i+1}'(x_i) = N_i$$

Desired properties of piecewise interpolating functions at break (data) points



$$S_i'' = \frac{d^2 S_i}{dx^2}$$

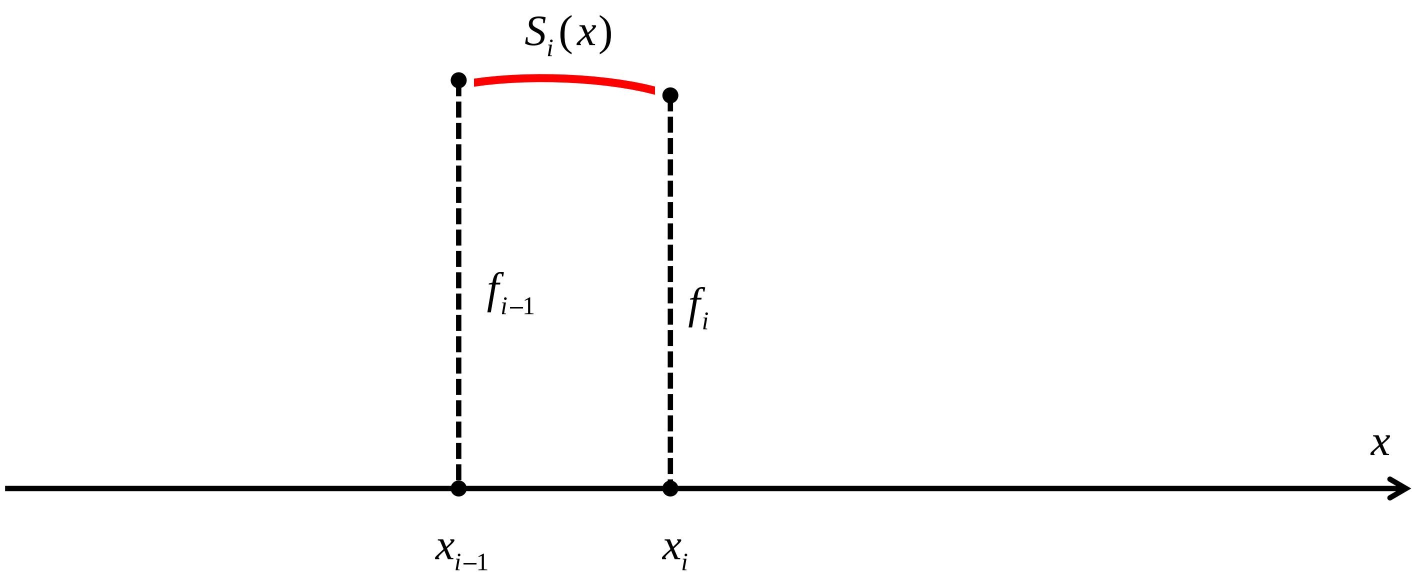
Smoothness:

Neighboring functions have same best fitting circle (same curvature) at common point =

= Same value of second derivative:

$$S_i''(x_i) = S_{i+1}''(x_i) = M_i$$

The cubic polynomial



Polynomial of third order - Newton form :

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3$$

4 unknown coefficients for each one of the n splines $S_i(x)$**,** $i = 1, 2, \ldots, n$

`Exact` interpolation by cubic splines

Problem statement:

Find a set of coefficients

$$c_0^1,\ c_1^1,\ c_2^1,\ c_3^1 \quad \cdots \quad c_0^i,\ c_1^i,\ c_2^i,\ c_3^i \quad \ldots \quad c_0^n,\ c_1^n,\ c_2^n,\ c_3^n$$

for the splines $S_1(x), \ldots, S_i(x), \ldots, S_n(x)$ **where**

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3$$

such that the following 3 sets of conditions are satisfied

$$S_1(x_0) = f_0,\quad S_1(x_1) = f_1 \qquad S_1'(x_1) = S_2'(x_1) \qquad S_1''(x_1) = S_2''(x_1)$$

$$S_2(x_1) = f_1,\quad S_2(x_2) = f_2 \qquad S_2'(x_2) = S_3'(x_1) \qquad S_2''(x_2) = S_3''(x_1)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$S_i(x_{i-1}) = f_{i-1},\quad S_i(x_i) = f_i \qquad S_i'(x_i) = S_{i+1}'(x_i) \qquad S_i''(x_i) = S_{i+1}''(x_i)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$S_n(x_{n-1}) = f_{n-1},\quad S_n(x_n) = f_n \qquad S_{n-1}'(x_{n-1}) = S_n'(x_{n-1}) \qquad S_{n-1}''(x_{n-1}) = S_n''(x_{n-1})$$

Exact interpolation by cubic splines

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3, \qquad i = 1, 2, \ldots, n$$

Unknown coefficients: **4n**

Conditions on coefficients:

$S_i(x_{i-1}) = f_{i-1}, \quad S_i(x_i) = f_i, \quad i = 1, 2, \ldots, n$ **2n**

$S'_{i-1}(x_i) = S'_i(x_i) \equiv N_i, \quad i = 2, \ldots, n$ **n–1**

$S''_{i-1}(x_i) = S''_i(x_i) \equiv M_i, \quad i = 2, \ldots, n$ **n–1**

Total number of conditions: **4n–2**

We need 4n equations to determine 4n unknowns

2 additional conditions are required! **e.g. fixing M_0 and M_n**

`Exact` interpolation by cubic splines

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3$$

$$S_i'(x) = \frac{dS_i}{dx} = c_1^i + 2c_2^i(x - x_{i-1}) + 3c_3^i(x - x_{i-1})^2$$

$$S_i''(x) = \frac{d^2S_i}{dx^2} = 2c_2^i + 6c_3^i(x - x_{i-1})$$

Border values: $h_i = x_i - x_{i-1}$

$$S_i(x_{i-1}) = c_0^i + c_1^i(x_{i-1} - x_{i-1}) + c_2^i(x_{i-1} - x_{i-1})^2 + c_3^i(x_{i-1} - x_{i-1})^3 = c_0^i = f_{i-1}$$

$$S_i(x_i) = c_0^i + c_1^i(x_i - x_{i-1}) + c_2^i(x_i - x_{i-1})^2 + c_3^i(x_i - x_{i-1})^3 = c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$$

$$S_i'(x_{i-1}) = c_1^i + 2c_2^i(x_{i-1} - x_{i-1}) + 3c_3^i(x_{i-1} - x_{i-1})^2 = c_1^i = N_{i-1}$$

$$S_i'(x_i) = c_1^i + 2c_2^i(x_i - x_{i-1}) + 3c_3^i(x_i - x_{i-1})^2 = c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$$

$$S_i''(x_{i-1}) = 2c_2^i + 3c_3^i(x_{i-1} - x_{i-1}) = 2c_2^i = M_{i-1}$$

$$S_i''(x_i) = 2c_2^i + 3c_3^i(x_i - x_{i-1}) = 2c_2^i + 3c_3^i h_i = M_i$$

`Exact` interpolation by cubic splines

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3$$

$$S_i'(x) = \frac{dS_i}{dx} = c_1^i + 2c_2^i(x - x_{i-1}) + 3c_3^i(x - x_{i-1})^2$$

$$S_i''(x) = \frac{d^2S_i}{dx^2} = 2c_2^i + 6c_3^i(x - x_{i-1})$$

Border values: $h_i = x_i - x_{i-1}$

$S_i(x_{i-1}) = f_{i-1}$: $\quad c_0^i = f_{i-1}$

$S_i(x_i) = f_i$: $\quad c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$

$S_i'(x_{i-1}) = N_{i-1}$: $\quad c_1^i = N_{i-1}$

$S_i'(x_i) = N_i$: $\quad c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$

$S_i''(x_{i-1}) = M_{i-1}$: $\quad 2c_2^i = M_{i-1}$

$S_i''(x_i) = M_i$: $\quad 2c_2^i + 3c_3^i h_i = M_i$

$\texttt{Exact}$ interpolation by cubic splines

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3$$

$$S_i'(x) = \frac{dS_i}{dx} = c_1^i + 2c_2^i(x - x_{i-1}) + 3c_3^i(x - x_{i-1})^2$$

$$S_i''(x) = \frac{d^2S_i}{dx^2} = 2c_2^i + 6c_3^i(x - x_{i-1})$$

Border values: $h_i = x_i - x_{i-1}$

$S_i(x_{i-1}) = f_{i-1}:$ $\quad \boxed{c_0^i = f_{i-1}}$

$S_i(x_i) = f_i:$ $\quad \boxed{c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i}$

$S_i'(x_{i-1}) = N_{i-1}:$ $\quad \boxed{c_1^i = N_{i-1}}$

$S_i'(x_i) = N_i:$ $\quad \boxed{c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i}$

$S_i''(x_{i-1}) = M_{i-1}:$ $\quad 2c_2^i = M_{i-1}$

$S_i''(x_i) = M_i:$ $\quad 2c_2^i + 3c_3^i h_i = M_i$

Exact interpolation by cubic splines

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3$$

$$S_i'(x) = \frac{dS_i}{dx} = c_1^i + 2c_2^i(x - x_{i-1}) + 3c_3^i(x - x_{i-1})^2$$

$$S_i''(x) = \frac{d^2 S_i}{dx^2} = 2c_2^i + 6c_3^i(x - x_{i-1})$$

Border values: $h_i = x_i - x_{i-1}$

$S_i(x_{i-1}) = f_{i-1}$: $\quad c_0^i = f_{i-1}$

$S_i(x_i) = f_i$: $\quad c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$

$S_i'(x_{i-1}) = N_{i-1}$: $\quad c_1^i = N_{i-1}$

$S_i'(x_i) = N_i$: $\quad c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$

$S_i''(x_{i-1}) = M_{i-1}$: $\quad 2c_2^i = M_{i-1}$

$S_i''(x_i) = M_i$: $\quad 2c_2^i + 3c_3^i h_i = M_i$

Choice A:

Replace coefficients

$c_0^i,\ c_1^i,\ c_2^i,\ c_3^i$

with

$f_{i-1}, f_i, N_{i-1}, N_i$

Determine N_{i-1}, N_i **from the conditions**

$$S_i''(x_i) = S_{i+1}''(x_i)$$

Exact interpolation by cubic splines

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3$$

$$S_i'(x) = \frac{dS_i}{dx} = c_1^i + 2c_2^i(x - x_{i-1}) + 3c_3^i(x - x_{i-1})^2$$

$$S_i''(x) = \frac{d^2 S_i}{dx^2} = 2c_2^i + 6c_3^i(x - x_{i-1})$$

Border values: $h_i = x_i - x_{i-1}$

$S_i(x_{i-1}) = f_{i-1}$: $\boxed{c_0^i = f_{i-1}}$

$S_i(x_i) = f_i$: $\boxed{c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i}$

$S_i'(x_{i-1}) = N_{i-1}$: $c_1^i = N_{i-1}$

$S_i'(x_i) = N_i$: $c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$

$S_i''(x_{i-1}) = M_{i-1}$: $\boxed{2c_2^i = M_{i-1}}$

$S_i''(x_i) = M_i$: $\boxed{2c_2^i + 3c_3^i h_i = M_i}$

Exact interpolation by cubic splines

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3$$

$$S_i'(x) = \frac{dS_i}{dx} = c_1^i + 2c_2^i(x - x_{i-1}) + 3c_3^i(x - x_{i-1})^2$$

$$S_i''(x) = \frac{d^2 S_i}{dx^2} = 2c_2^i + 6c_3^i(x - x_{i-1})$$

Border values: $h_i = x_i - x_{i-1}$

$S_i(x_{i-1}) = f_{i-1}$: $\quad c_0^i = f_{i-1}$

$S_i(x_i) = f_i$: $\quad c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$

$S_i'(x_{i-1}) = N_{i-1}$: $\quad c_1^i = N_{i-1}$

$S_i'(x_i) = N_i$: $\quad c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$

$S_i''(x_{i-1}) = M_{i-1}$: $\quad 2c_2^i = M_{i-1}$

$S_i''(x_i) = M_i$: $\quad 2c_2^i + 6c_3^i h_i = M_i$

Choice B:

Replace coefficients

$c_0^i,\ c_1^i,\ c_2^i,\ c_3^i$

with

$f_{i-1}, f_i, M_{i-1}, M_i$

Determine M_{i-1}, M_i **from the conditions**

$$S_i'(x_i) = S_{i+1}'(x_i)$$

`Exact` interpolation by cubic splines – Choice A:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ with $f_{i-1}, f_i, N_{i-1}, N_i$ - Determine N_{i-1}, N_i from $S_i''(x_i) = S_{i+1}''(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$c_0^i = f_{i-1}$$

$$c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$$

$$c_1^i = N_{i-1}$$

$$c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$$

`Exact` interpolation by cubic splines – Choice A:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ with $f_{i-1}, f_i, N_{i-1}, N_i$ - Determine N_{i-1}, N_i from $S_i''(x_i) = S_{i+1}''(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$\boxed{\begin{aligned} &c_0^i = f_{i-1} \\ &c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i \\ &c_1^i = N_{i-1} \\ &c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i \end{aligned}}$$

A system of 4 equations with 4 unknowns ($c_0^i, c_1^i, c_2^i, c_3^i$ **) has a unique solution**

$h_i, f_{i\text{-}1}, f_i$ **are known.** $N_{i\text{-}1}, N_i$ **are considered as known.**

`Exact` interpolation by cubic splines – Choice A:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ with $f_{i-1}, f_i, N_{i-1}, N_i$ - Determine N_{i-1}, N_i from $S_i''(x_i) = S_{i+1}''(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$c_0^i = f_{i-1}$$

$$c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$$

$$c_1^i = N_{i-1}$$

$$c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$$

System solution $\Longrightarrow$

$$c_0^i = f_{i-1}$$

$$c_1^i = N_{i-1}$$

$$c_{i2} = \frac{3(f_i - f_{i-1})}{h_i^2} - \frac{2N_{i-1} + N_i}{h_i}$$

$$c_{i3} = -\frac{2(f_i - f_{i-1})}{h_i^3} + \frac{N_{i-1} + N_i}{h_i^2}$$

Exact interpolation by cubic splines – Choice A:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ with $f_{i-1}, f_i, N_{i-1}, N_i$ - Determine N_{i-1}, N_i from $S_i''(x_i) = S_{i+1}''(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$c_0^i = f_{i-1}$$

$$c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$$

$$c_1^i = N_{i-1}$$

$$c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$$

System solution

$$c_0^i = f_{i-1}$$

$$c_1^i = N_{i-1}$$

$$c_{i2} = \frac{3(f_i - f_{i-1})}{h_i^2} - \frac{2N_{i-1} + N_i}{h_i}$$

$$c_{i3} = -\frac{2(f_i - f_{i-1})}{h_i^3} + \frac{N_{i-1} + N_i}{h_i^2}$$

Cubic polynomial in terms of border values and first derivatives

$$S_i(x) = f_{i-1} + N_{i-1}(x - x_{i-1}) + \left[\frac{3(f_i - f_{i-1})}{h_i^2} - \frac{2N_{i-1} + N_i}{h_i}\right](x - x_{i-1})^2 + \left[-\frac{2(f_i - f_{i-1})}{h_i^3} + \frac{N_{i-1} + N_i}{h_i^2}\right](x - x_{i-1})^3$$

`Exact` interpolation by cubic splines – Choice A:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ with $f_{i-1}, f_i, N_{i-1}, N_i$ - Determine N_{i-1}, N_i from $S_i''(x_i) = S_{i+1}''(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$c_0^i = f_{i-1}$$

$$c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$$

$$c_1^i = N_{i-1}$$

$$c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$$

System solution

$$c_0^i = f_{i-1}$$

$$c_1^i = N_{i-1}$$

$$c_{i2} = \frac{3(f_i - f_{i-1})}{h_i^2} - \frac{2N_{i-1} + N_i}{h_i}$$

$$c_{i3} = -\frac{2(f_i - f_{i-1})}{h_i^3} + \frac{N_{i-1} + N_i}{h_i^2}$$

$$\boxed{S_i''(x_i) = S_{i+1}''(x_i)}$$

`Exact` interpolation by cubic splines – Choice A:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ with $f_{i-1}, f_i, N_{i-1}, N_i$ - Determine N_{i-1}, N_i from $S_i''(x_i) = S_{i+1}''(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$\boxed{\begin{aligned} c_0^i &= f_{i-1} \\ c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 &= f_i \\ c_1^i &= N_{i-1} \\ c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 &= N_i \end{aligned}}$$

System solution

$$\boxed{\begin{aligned} c_0^i &= f_{i-1} \\ c_1^i &= N_{i-1} \\ c_{i2} &= \frac{3(f_i - f_{i-1})}{h_i^2} - \frac{2N_{i-1} + N_i}{h_i} \\ c_{i3} &= -\frac{2(f_i - f_{i-1})}{h_i^3} + \frac{N_{i-1} + N_i}{h_i^2} \end{aligned}}$$

$$\boxed{S_i''(x_i) = S_{i+1}''(x_i)}$$

$$h_{i+1}N_{i-1} + 2\left(h_i + h_{i+1}\right)N_i + h_i N_{i+1} = 3\frac{f_{i+1} - f_i}{h_{i+1}}h_i + 3\frac{f_i - f_{i-1}}{h_i}h_{i+1} \qquad i = 1,\ldots,n-1$$

Exact interpolation by cubic splines – Choice A:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ with $f_{i-1}, f_i, N_{i-1}, N_i$ - Determine N_{i-1}, N_i from $S_i''(x_i) = S_{i+1}''(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$c_0^i = f_{i-1}$$

$$c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$$

$$c_1^i = N_{i-1}$$

$$c_1^i + 2c_2^i h_i + 3c_3^i h_i^2 = N_i$$

System solution

$$c_0^i = f_{i-1}$$

$$c_1^i = N_{i-1}$$

$$c_{i2} = \frac{3(f_i - f_{i-1})}{h_i^2} - \frac{2N_{i-1} + N_i}{h_i}$$

$$c_{i3} = -\frac{2(f_i - f_{i-1})}{h_i^3} + \frac{N_{i-1} + N_i}{h_i^2}$$

$$\boxed{S_i''(x_i) = S_{i+1}''(x_i)}$$

$$h_{i+1}N_{i-1} + 2\left(h_i + h_{i+1}\right)N_i + h_i N_{i+1} = 3\frac{f_{i+1} - f_i}{h_{i+1}}h_i + 3\frac{f_i - f_{i-1}}{h_i}h_{i+1} \qquad i = 1,\dots,n-1$$

n–1 equations with **n+1** unknowns $N_0, N_1, \dots, N_n$: fix **2**, e.g. N_0, N_n

`Exact` interpolation by cubic splines – Choice A:

$$h_{i+1}N_{i-1} + 2\left(h_i + h_{i+1}\right)N_i + h_i N_{i+1} = 3\frac{f_{i+1} - f_i}{h_{i+1}}h_i + 3\frac{f_i - f_{i-1}}{h_i}h_{i+1} \qquad i = 1,\ldots,n-1$$

$$a_i N_{i-1} + b_i N_i + c_i N_{i+1} = d_i$$

$$a_1 N_0 + b_1 N_1 + c_1 N_2 = d_1$$

$$a_2 N_1 + b_2 N_2 + c_2 N_3 = d_2$$

$$a_3 N_2 + b_3 N_3 + c_3 N_4 = d_3$$

$$\vdots$$

$$a_{n-2} N_{n-3} + b_{n-2} N_{n-2} + c_{n-2} N_{n-1} = d_{n-2}$$

$$a_{n-1} N_{n-2} + b_{n-1} N_{n-1} + c_{n-1} N_n = d_{n-1}$$

Exact interpolation by cubic splines – Choice A:

$$h_{i+1}N_{i-1} + 2\left(h_i + h_{i+1}\right)N_i + h_i N_{i+1} = 3\frac{f_{i+1} - f_i}{h_{i+1}}h_i + 3\frac{f_i - f_{i-1}}{h_i}h_{i+1} \qquad i = 1,\ldots,n-1$$

$$a_i N_{i-1} + b_i N_i + c_i N_{i+1} = d_i$$

$$\begin{aligned}
&a_1 N_0 + b_1 N_1 + c_1 N_2 = d_1 \\
&a_2 N_1 + b_2 N_2 + c_2 N_3 = d_2 \\
&a_3 N_2 + b_3 N_3 + c_3 N_4 = d_3 \\
&\qquad\vdots \\
&a_{n-2} N_{n-3} + b_{n-2} N_{n-2} + c_{n-2} N_{n-1} = d_{n-2} \\
&a_{n-1} N_{n-2} + b_{n-1} N_{n-1} + c_{n-1} N_n = d_{n-1}
\end{aligned}$$

Fix N_0, N_n**:**

$$\begin{aligned}
&b_1 N_1 + c_1 N_2 = d_1 - a_1 N_0 \\
&a_2 N_1 + b_2 N_2 + c_2 N_3 = d_2 \\
&a_3 N_2 + b_3 N_3 + c_3 N_4 = d_3 \\
&\qquad\vdots \\
&a_{n-2} N_{n-3} + b_{n-2} N_{n-2} + c_{n-2} N_{n-1} = d_{n-2} \\
&a_{n-1} N_{n-2} + b_{n-1} N_{n-1} = d_{n-1} - c_{n-1} N_n
\end{aligned}$$

`Exact` interpolation by cubic splines – Choice A:

$$h_{i+1}N_{i-1} + 2\left(h_i + h_{i+1}\right)N_i + h_i N_{i+1} = 3\frac{f_{i+1} - f_i}{h_{i+1}}h_i + 3\frac{f_i - f_{i-1}}{h_i}h_{i+1} \qquad i = 1,\ldots,n-1$$

$$a_i N_{i-1} + b_i N_i + c_i N_{i+1} = d_i$$

$$\begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-3} & c_{n-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-2} & b_{n-2} & c_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ \vdots \\ N_{n-3} \\ N_{n-2} \\ N_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - a_1 N_0 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-3} \\ d_{n-2} \\ d_{n-1} - c_{n-1}N_n \end{bmatrix}$$

Solve the above system and use

$$S_i(x) = f_{i-1} + N_{i-1}(x - x_{i-1}) + \left\lfloor \frac{3(f_i - f_{i-1})}{h_i^2} - \frac{2N_{i-1} + N_i}{h_i} \right\rfloor (x - x_{i-1})^2 + \left\lfloor -\frac{2(f_i - f_{i-1})}{h_i^3} + \frac{N_{i-1} + N_i}{h_i^2} \right\rfloor (x - x_{i-1})^3$$

$$h_i = x_i - x_{i-1}$$

`Exact` interpolation by cubic splines – Choice B:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ **with** $f_{i-1}, f_i, M_{i-1}, M_i$ **- Determine** M_{i-1}, M_i **from** $S_i'(x_i) = S_{i+1}'(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$c_0^i = f_{i-1}$$

$$c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$$

$$2c_2^i = M_{i-1}$$

$$2c_2^i + 6c_3^i h_i = M_i$$

Exact interpolation by cubic splines – Choice B:

Replace $c_0^i,\, c_1^i,\, c_2^i,\, c_3^i$ **with** $f_{i-1}, f_i, M_{i-1}, M_i$ **- Determine** M_{i-1}, M_i **from** $S_i'(x_i) = S_{i+1}'(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$\boxed{\begin{aligned} &c_0^i = f_{i-1} \\ &c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i \\ &2c_2^i = M_{i-1} \\ &2c_2^i + 6c_3^i h_i = M_i \end{aligned}}$$

A system of 4 equations with 4 unknowns ($c_0^i,\, c_1^i,\, c_2^i,\, c_3^i$ **) has a unique solution**

h_i, f_{i-1}, f_i **are known.** M_{i-1}, M_i **are considered as known.**

$\texttt{Exact}$ interpolation by cubic splines – Choice B:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ **with** $f_{i-1}, f_i, M_{i-1}, M_i$ **- Determine** M_{i-1}, M_i **from** $S_i'(x_i) = S_{i+1}'(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$c_0^i = f_{i-1}$$

$$c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$$

$$2c_2^i = M_{i-1}$$

$$2c_2^i + 6c_3^i h_i = M_i$$

System solution

$$c_0^i = f_{i-1}$$

$$c_{i1} = \frac{f_i - f_{i-1}}{h_i} - \frac{(2M_{i-1} + M_i)h_i}{6}$$

$$c_{i2} = \frac{M_{i-1}}{2}$$

$$c_{i3} = \frac{M_i - M_{i-1}}{6h_i}$$

Exact interpolation by cubic splines – Choice B:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ **with** $f_{i-1}, f_i, M_{i-1}, M_i$ **- Determine** M_{i-1}, M_i **from** $S_i'(x_i) = S_{i+1}'(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$c_0^i = f_{i-1}$$

$$c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 = f_i$$

$$2c_2^i = M_{i-1}$$

$$2c_2^i + 6c_3^i h_i = M_i$$

System solution

$$c_0^i = f_{i-1}$$

$$c_{i1} = \frac{f_i - f_{i-1}}{h_i} - \frac{(2M_{i-1} + M_i)h_i}{6}$$

$$c_{i2} = \frac{M_{i-1}}{2}$$

$$c_{i3} = \frac{M_i - M_{i-1}}{6h_i}$$

Cubic polynomial in terms of border values and second derivatives

$$S_i(x) = f_{i-1} + \left[\frac{f_i - f_{i-1}}{h_i} - \frac{2h_i}{6}M_{i-1} - \frac{h_i}{6}M_i\right](x - x_{i-1}) + \frac{M_{i-1}}{2}(x - x_{i-1})^2 + \frac{M_i - M_{i-1}}{6h_i}(x - x_{i-1})^3$$

`Exact` interpolation by cubic splines – Choice B:

Replace $c_0^i, c_1^i, c_2^i, c_3^i$ **with** $f_{i-1}, f_i, M_{i-1}, M_i$ **- Determine** M_{i-1}, M_i **from** $S_i'(x_i) = S_{i+1}'(x_i)$

$$S_i(x) = c_0^i + c_1^i(x - x_{i-1}) + c_2^i(x - x_{i-1})^2 + c_3^i(x - x_{i-1})^3 \qquad h_i = x_i - x_{i-1}$$

$$\boxed{\begin{aligned} c_0^i &= f_{i-1} \\ c_0^i + c_1^i h_i + c_2^i h_i^2 + c_3^i h_i^3 &= f_i \\ 2c_2^i &= M_{i-1} \\ 2c_2^i + 6c_3^i h_i &= M_i \end{aligned}} \xrightarrow[\textbf{solution}]{\textbf{System}} \boxed{\begin{aligned} c_0^i &= f_{i-1} \\ c_{i1} &= \frac{f_i - f_{i-1}}{h_i} - \frac{(2M_{i-1} + M_i)h_i}{6} \\ c_{i2} &= \frac{M_{i-1}}{2} \\ c_{i3} &= \frac{M_i - M_{i-1}}{6h_i} \end{aligned}}$$

$$\boxed{S_i'(x_i) = S_{i+1}'(x_i)}$$

$$h_i M_{i-1} + 2(h_i + h_{i+1})M_i + h_{i+1}M_{i+1} = 6\frac{f_{i+1} - f_i}{h_{i+1}} - 6\frac{f_i - f_{i-1}}{h_i} \qquad i = 1, \ldots, n-1$$

n–1 equations with **n+1** unknowns $M_0, M_1, \ldots, M_n$: fix **2**, e.g. M_0, M_n

`Exact` interpolation by cubic splines – Choice B:

$$h_i M_{i-1} + 2(h_i + h_{i+1}) M_i + h_{i+1} M_{i+1} = 6\frac{f_{i+1} - f_i}{h_{i+1}} - 6\frac{f_i - f_{i-1}}{h_i} \qquad i = 1, \ldots, n-1$$

$$a_i M_{i-1} + b_i M_i + c_i M_{i+1} = d_i$$

$$a_1 M_0 + b_1 M_1 + c_1 M_2 = d_1$$

$$a_2 M_1 + b_2 M_2 + c_2 M_3 = d_2$$

$$a_3 M_2 + b_3 M_3 + c_3 M_4 = d_3$$

$$\vdots$$

$$a_{n-2} M_{n-3} + b_{n-2} M_{n-2} + c_{n-2} M_{n-1} = d_{n-2}$$

$$a_{n-1} M_{n-2} + b_{n-1} M_{n-1} + c_{n-1} M_n = d_{n-1}$$

Fix M_0, M_n**:** $\Rightarrow$

$$b_1 M_1 + c_1 M_2 = d_1 - a_1 M_0$$

$$a_2 M_1 + b_2 M_2 + c_2 M_3 = d_2$$

$$a_3 M_2 + b_3 M_3 + c_3 M_4 = d_3$$

$$\vdots$$

$$a_{n-2} M_{n-3} + b_{n-2} M_{n-2} + c_{n-2} M_{n-1} = d_{n-2}$$

$$a_{n-1} M_{n-2} + b_{n-1} M_{n-1} = d_{n-1} - c_{n-1} M_n$$

`Exact` interpolation by cubic splines – Choice B:

$$h_i M_{i-1} + 2(h_i + h_{i+1}) M_i + h_{i+1} M_{i+1} = 6\frac{f_{i+1} - f_i}{h_{i+1}} - 6\frac{f_i - f_{i-1}}{h_i} \qquad i = 1, \ldots, n-1$$

$$a_i M_{i-1} + b_i M_i + c_i M_{i+1} = d_i$$

$$\begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-3} & c_{n-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-2} & b_{n-2} & c_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - a_1 M_0 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-3} \\ d_{n-2} \\ d_{n-1} - c_{n-1} M_n \end{bmatrix}$$

Solve the above system and use

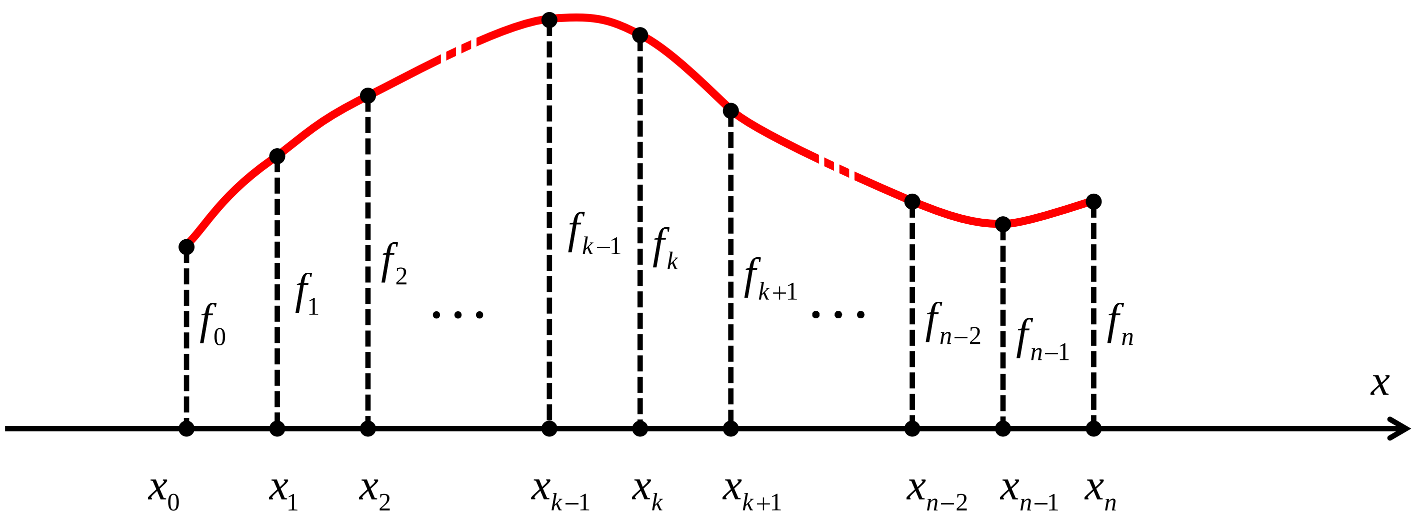
$$S_i(x) = f_{i-1} + \left[\frac{f_i - f_{i-1}}{h_i} - \frac{2h_i}{6} M_{i-1} - \frac{h_i}{6} M_i\right](x - x_{i-1}) + \frac{M_{i-1}}{2}(x - x_{i-1})^2 + \frac{M_i - M_{i-1}}{6h_i}(x - x_{i-1})^3$$

$$h_i = x_i - x_{i-1}$$

Special case:
regularly distributed observations

$$x_i - x_{i-1} = h = \text{const.}$$

$$x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ \ldots,\ x_k = x_0 + kh,\ \ldots,\ x_n = x_0 + nh$$



Choice A:

$\boxed{h = x_i - x_{i-1} = \text{const.}}$ $\Rightarrow$ $$N_{i-1} + 4N_i + N_{i+1} = 3\frac{f_{i+1} - f_{i-1}}{h} \qquad i = 1, \ldots, n-1$$

$$N_{i-1} + 4N_i + N_{i+1} = d_i$$

$$\begin{bmatrix} 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ \vdots \\ N_{n-3} \\ N_{n-2} \\ N_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - N_0 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-3} \\ d_{n-2} \\ d_{n-1} - N_n \end{bmatrix}$$

Solve the above system and use

$$S_i(x) = f_{i-1} + N_{i-1}(x - x_{i-1}) + \left[\frac{3(f_i - f_{i-1})}{h^2} - \frac{2N_{i-1} + N_i}{h} \right](x - x_{i-1})^2 + \left[-\frac{2(f_i - f_{i-1})}{h^3} + \frac{N_{i-1} + N_i}{h^2} \right](x - x_{i-1})^3$$

Choice B:

$\boxed{h = x_i - x_{i-1} = \text{const.}} \Rightarrow$

$$M_{i-1} + 4M_i + M_{i+1} = 6\frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} \qquad i = 1,\dots,n-1$$

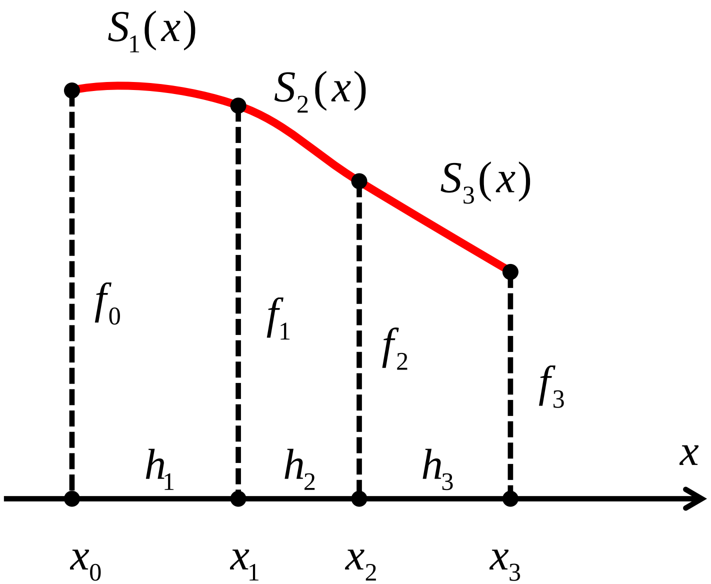
$$M_{i-1} + 4M_i + M_{i+1} = d_i$$

$$\begin{bmatrix} 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - M_0 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-3} \\ d_{n-2} \\ d_{n-1} - M_n \end{bmatrix}$$

Solve the above system and use

$$S_i(x) = f_{i-1} + \left[\frac{f_i - f_{i-1}}{h} - h\frac{2M_{i-1} + M_i}{6} \right](x - x_{i-1}) + \frac{M_{i-1}}{2}(x - x_{i-1})^2 + \frac{M_i - M_{i-1}}{6h}(x - x_{i-1})^3$$

An example



$$S_1(x) = c_0^1 + c_1^1(x - x_0) + c_2^1(x - x_0)^2 + c_3^1(x - x_0)^3$$

$$S_2(x) = c_0^2 + c_1^2(x - x_1) + c_2^2(x - x_1)^2 + c_3^2(x - x_1)^3$$

$$S_3(x) = c_0^3 + c_1^3(x - x_2) + c_2^3(x - x_2)^2 + c_3^3(x - x_2)^3$$

$$h_1 = x_1 - x_0 \qquad h_2 = x_2 - x_1 \qquad h_3 = x_3 - x_2$$

$$S_1(x) = f_0 + N_0(x - x_0) + \left[\frac{3(f_1 - f_0)}{h_1^2} - \frac{2N_0 + N_1}{h_1}\right](x - x_0)^2 + \left[-\frac{2(f_1 - f_0)}{h_1^3} + \frac{N_0 + N_1}{h_1^2}\right](x - x_0)^3$$

$$S_2(x) = f_1 + N_1(x - x_1) + \left[\frac{3(f_2 - f_1)}{h_2^2} - \frac{2N_1 + N_2}{h_2}\right](x - x_1)^2 + \left[-\frac{2(f_2 - f_1)}{h_2^3} + \frac{N_1 + N_2}{h_2^2}\right](x - x_1)^3$$

$$S_3(x) = f_2 + N_2(x - x_2) + \left[\frac{3(f_3 - f_2)}{h_3^2} - \frac{2N_2 + N_3}{h_3}\right](x - x_2)^2 + \left[-\frac{2(f_3 - f_2)}{h_3^3} + \frac{N_2 + N_3}{h_3^2}\right](x - x_2)^3$$

An example

$$\boxed{S_1'(x_1) = S_2'(x_1)} \Rightarrow \quad h_2 N_0 + 2\left(h_1 + h_2\right) N_1 + h_1 N_2 = 3\frac{f_2 - f_1}{h_2} h_1 + 3\frac{f_1 - f_0}{h_1} h_2 = d_1$$

$$\boxed{S_2'(x_2) = S_3'(x_2)} \Rightarrow \quad h_3 N_1 + 2\left(h_2 + h_3\right) N_2 + h_2 N_3 = 3\frac{f_3 - f_2}{h_3} h_2 + 3\frac{f_2 - f_1}{h_2} h_3 = d_2$$

$$\begin{bmatrix} 2(h_1 + h_2) & h_1 \\ h_3 & 2(h_2 + h_3) \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} d_1 - h_2 N_0 \\ d_2 - h_2 N_3 \end{bmatrix}$$

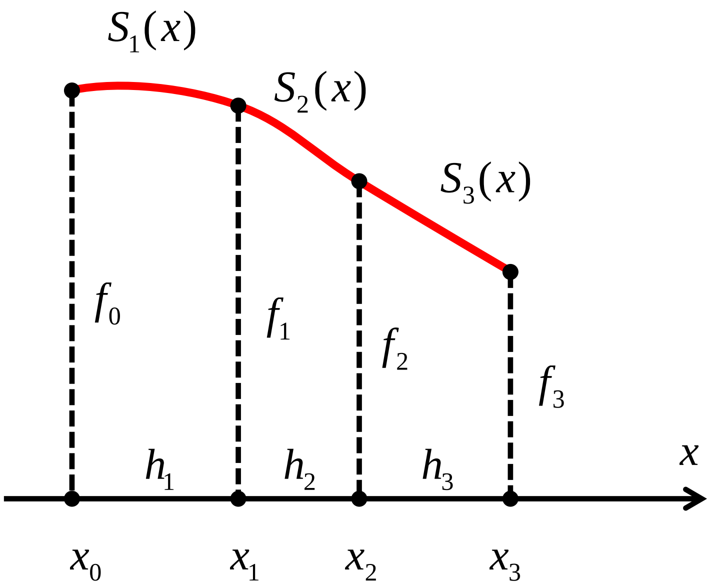
Solve the above system and use

$$S_1(x) = f_0 + N_0(x - x_0) + \left[\frac{3(f_1 - f_0)}{h_1^2} - \frac{2N_0 + N_1}{h_1} \right](x - x_0)^2 + \left[-\frac{2(f_1 - f_0)}{h_1^3} + \frac{N_0 + N_1}{h_1^2} \right](x - x_0)^3$$

$$S_2(x) = f_1 + N_1(x - x_1) + \left[\frac{3(f_2 - f_1)}{h_2^2} - \frac{2N_1 + N_2}{h_2} \right](x - x_1)^2 + \left[-\frac{2(f_2 - f_1)}{h_2^3} + \frac{N_1 + N_2}{h_2^2} \right](x - x_1)^3$$

$$S_3(x) = f_2 + N_2(x - x_2) + \left[\frac{3(f_3 - f_2)}{h_3^2} - \frac{2N_2 + N_3}{h_3} \right](x - x_2)^2 + \left[-\frac{2(f_3 - f_2)}{h_3^3} + \frac{N_2 + N_3}{h_3^2} \right](x - x_2)^3$$

An example



$$S_1(x) = c_0^1 + c_1^1(x - x_0) + c_2^1(x - x_0)^2 + c_3^1(x - x_0)^3$$

$$S_2(x) = c_0^2 + c_1^2(x - x_1) + c_2^2(x - x_1)^2 + c_3^2(x - x_1)^3$$

$$S_3(x) = c_0^3 + c_1^3(x - x_2) + c_2^3(x - x_2)^2 + c_3^3(x - x_2)^3$$

$h_1 = x_1 - x_0$ $\qquad$ $h_2 = x_2 - x_1$ $\qquad$ $h_3 = x_3 - x_2$

$$S_1(x) = f_0 + \left[\frac{f_1 - f_0}{h_1} - \frac{2h_1}{6} M_0 - \frac{h_1}{6} M_1 \right](x - x_0) + \frac{M_{1-1}}{2}(x - x_0)^2 + \frac{M_1 - M_0}{6h_1}(x - x_0)^3$$

$$S_2(x) = f_1 + \left[\frac{f_2 - f_1}{h_2} - \frac{2h_2}{6} M_1 - \frac{h_2}{6} M_2 \right](x - x_1) + \frac{M_1}{2}(x - x_1)^2 + \frac{M_2 - M_1}{6h_2}(x - x_1)^3$$

$$S_3(x) = f_2 + \left[\frac{f_3 - f_2}{h_3} - \frac{2h_3}{6} M_2 - \frac{h_3}{6} M_3 \right](x - x_2) + \frac{M_2}{2}(x - x_2)^2 + \frac{M_3 - M_2}{6h_3}(x - x_2)^3$$

An example

$$\boxed{S_1'(x_1) = S_2'(x_1)} \Rightarrow \quad h_1 M_0 + 2(h_1 + h_2) M_1 + h_2 M_2 = 6\frac{f_2 - f_1}{h_2} - 6\frac{f_1 - f_0}{h_1} = d_1$$

$$\boxed{S_2'(x_2) = S_3'(x_2)} \Rightarrow \quad h_2 M_1 + 2(h_2 + h_3) M_2 + h_3 M_3 = 6\frac{f_3 - f_2}{h_3} - 6\frac{f_2 - f_1}{h_2} = d_2$$

$$\begin{bmatrix} 2(h_1 + h_2) & h_2 \\ h_2 & 2(h_2 + h_3) \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} d_1 - h_1 M_0 \\ d_2 - h_3 M_3 \end{bmatrix}$$

Solve the above system and use

$$S_1(x) = f_0 + \left[\frac{f_1 - f_0}{h_1} - \frac{2h_1}{6} M_0 - \frac{h_1}{6} M_1 \right](x - x_0) + \frac{M_0}{2}(x - x_0)^2 + \frac{M_1 - M_0}{6h_1}(x - x_0)^3$$

$$S_2(x) = f_1 + \left[\frac{f_2 - f_1}{h_2} - \frac{2h_2}{6} M_1 - \frac{h_2}{6} M_2 \right](x - x_1) + \frac{M_1}{2}(x - x_1)^2 + \frac{M_2 - M_1}{6h_2}(x - x_1)^3$$

$$S_3(x) = f_2 + \left[\frac{f_3 - f_2}{h_3} - \frac{2h_3}{6} M_2 - \frac{h_3}{6} M_3 \right](x - x_2) + \frac{M_2}{2}(x - x_2)^2 + \frac{M_3 - M_2}{6h_3}(x - x_2)^3$$