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Lévy Processes and Applications in Finance

Stochastic Volatility for Lévy Processes

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Abstract

As part of our study of Lévy Processes and Applications in Finance, inspired by Schoutens' "Lévy Processes in Finance: Pricing Financial Derivatives," we investigated stochastic volatility for Lévy process. The aim was to gain a deeper understanding of their potential to capture complex financial asset dynamics, compare to the traditional Black-Scholes model.

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1 Introduction

Financial markets exhibit volatility due to various economic and behavioral factors. News announcements, investor sentiment, and other phenomena can all influence asset prices, posing limitations for traditional models like Black-Scholes in accurately reflecting real-world dynamics. Specifically, Black-Scholes assumes constant volatility and a normal distribution for returns, which often doesn't hold true.

To address these limitations and improve option pricing models, Levy processes offer a more flexible approach. These processes allow us to model the dynamic behavior of financial markets using stochastic processes. This project aims to explore one such process, the Normal Inverse Gaussian (NIG) model proposed by Barndorff-Nielsen (1995), to model stock price dynamics. Additionally, to capture time-varying volatility, we incorporate the Cox-Ingersoll-Ross (CIR) process introduced by Cox et al. (1985) for stochastic volatility modeling.

This project will involve studying the chosen Levy process, its properties, financial applications, and its behavior under different parameter settings. We will then simulate the model and use it to price call options on the S&P 500 Index, comparing the results with the Black-Scholes model and actual market prices. The analysis will aim to evaluate the effectiveness of the chosen Levy process in overcoming the limitations of traditional models like Black-Scholes.

2 Theoretical Concept

2.1 Brownian Motion

Brownian motion, also called Wiener Process is a fundamental stochastic process widely used in finance for option pricing typically in the Black Scholes model. It is suitable to model continuous time of stock price.

In order to ensure that the stock being strictly positive, we use the exponential of the brownian motion which is the Geometric Brownian Motion (GBM). A stock price S_t follows a Geometric Brownian Motion, described by the following stochastic differential equation (SDE):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \tag{1}$$

where:

- S_t = Stock price at time t
- μ = Drift rate (representing the expected return of the stock)
- σ = Volatility (the standard deviation of the stock's returns)
- dW_t = Increment of a Wiener process (Brownian motion)

2.2 Lévy Processes

The Black-Scholes model, a cornerstone in finance, assumes that stock prices follow a normal distribution. However, real-world stock prices often exhibit skewness (asymmetry) and excess kurtosis (fatter tails) than what a normal distribution can capture. Lévy processes, a more flexible class of stochastic processes, offer a solution to this limitation. They are characterized by independent and stationary increments and can be based on various infinitely divisible distributions. These distributions, unlike the normal distribution, can accommodate skewness and excess kurtosis. By using Lévy processes, financial models can better capture the complex behavior of asset prices, leading to more accurate pricing and hedging of derivative securities. Lévy processes, instead of using a pure Brownian motion to model asset prices, consider three parts: one deterministic part, a diffusion part, and a pure jump part. These components are known as the Lévy triplet (γ, σ, ν), where γ represents the drift, σ the diffusion part, and ν the Lévy measure associated with the jumps.

Let $X = \{X_t, t \geq 0\}$ be an infinitely divisible stochastic process. The process X_t is a Lévy process if $X_0 = 0$ and has independent and stationary increments. Using the Lévy-Itô decomposition, we can write the Lévy process X_t as:

$$X_t = \gamma t + \sigma W_t + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx) \tag{2}$$

where:

- γ is the drift term,
- W_t denotes the Brownian motion (or the Wiener process),
- $\tilde{N}(t, dx)$ represents the compensated Poisson random measure, and
- $N(t, dx)$ is the Poisson random measure.

By the Lévy-Khintchine formula, we can write the characteristic exponent of a Lévy process formula of the Lévy process:

$$\psi(u) = iu\gamma - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|<1}\right) \nu(dx) \tag{3}$$

Schoutens (2003) gives us the most popular of Lévy processes used in financial modeling, such as the Variance Gamma process (VG), Carr, Geman, Madan and Yor process (CGMY), and Normal Inverse Gaussian process (NIG) and many others. In our case of study, we will use the Normal Inverse Gaussian (NIG) process.

2.3 The Normal Inverse Gaussian (NIG) Process

As seen in limitations of the Black-Scholes model, in reality we can have heavy tail and skewness in financial markets, characteristics not captured by normal distribution. By extending this limitation, the Normal Inverse Gaussian (NIG) process provides us more accuracy for modelling stock prices.

Normal Inverse Gaussian (NIG) was introduced by Barndorff-Nielsen (1995) as part of the generalized hyperbolic distributions family. It takes into account the asymmetry, fits well high frequency financial data and allows us to capture the heavy tails and skewed observed in financial markets. This type of process is suitable for modelling real world financial data, as it avoids issues of infinite variance, thereby ensuring finite variance. The Normal Inverse Gaussian (NIG) distribution with parameters $\alpha > 0$, $-\alpha < \beta < \alpha$, and $\delta > 0$, denoted as $NIG(\alpha, \beta, \delta)$, has a characteristic function given by:

$$\phi_{NIG}(u; \alpha, \beta, \delta) = \exp\left(-\delta\left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right). \tag{4}$$

This is an infinitely divisible characteristic function and we can define the NIG process:

$$X^{(NIG)} = \{X_t^{(NIG)}, t \geq 0\}$$

with $X_0^{(NIG)} = 0$, stationary and independent NIG-distributed increments.

The NIG process is related to an Inverse Gaussian time-changed Brownian motion. Let $W = \{W_t, t \geq 0\}$ be a standard Wiener Process and let $I = \{I_t, t \geq 0\}$ be an Inverse Gaussian (IG) process with parameters $a = 1$ and $b = \frac{\delta}{\sqrt{\alpha^2 - \beta^2}}$, where $\alpha > 0$, $-\alpha < \beta < \alpha$, and $\delta > 0$. Then, stochastic process will be:

$$X_t = \beta\delta^2 I_t + \delta W_{I_t} \tag{5}$$

which is an NIG process with parameters α, β, and δ.

If a random variable X follows an $NIG(\alpha, \beta, \delta)$ distribution, then $-X$ is distributed with $NIG(\alpha, -\beta, \delta)$. If $\beta = 0$, the distribution is symmetric.

2.4 Stochastic Time Change

To incorporate stochastic volatility, we can manipulate the passage of time itself. In periods of high volatility, time effectively accelerates, compressing multiple days of returns into a single calendar day. Conversely, during low volatility periods, time slows down, stretching a single day into a fraction of a business day. This time-warping mechanism, known as stochastic time change, can amplify returns during high volatility and dampen them during low volatility, aligning with observed market behavior.

The concept of stochastic time change dates back to Clark (1973), who introduced the idea of subordinating a geometric Brownian motion with an independent Lévy subordinator. This approach involves modeling the rate of time change using various positive stochastic processes. A popular choice is the mean-reverting Cox-Ingersoll-Ross (CIR) process, which is based on Brownian motion.

CIR is considered to be a quite simple model to implement since it only depends on one factor, which intends to describe the path of interest rates using only one source of risk. Its most interesting feature is that it incorporates the fact that interest rates tend to revert to their average, unlike the behavior of stocks or other capital markets products.

The model equation is the following:

$$dy_t = \kappa(\eta - y_t)\,dt + \lambda\sqrt{y_t}\,dW_t, \tag{6}$$

where $\{y_t, t \geq 0\}$ is a positive process. Then, the new clock $\{Y_t, t \geq 0\}$ is defined as

$$Y_t = \int_0^t y_s\,ds. \tag{7}$$

Which is called Integrated CIR process. Since y is positive, Y is increasing. Also, in our work next following equation will play the most important part in risk-neutral price processing, what you will see next chapter. Before that the characteristic function of integrated CIR - Y_t given y_0 is explicitly known (see Cox et al. (1985) or Elliott and Kopp (1999, Theorem 9.6.3)).

$$E[\exp(iuY_t) \mid y_0] = \phi(u, t; \kappa, \eta, \lambda, y_0) = \frac{\exp(\kappa^2\eta t/\lambda^2)\exp(2y_0 iu/(\kappa + \gamma\coth(\gamma t/2)))}{(\cosh(\gamma t/2) + \kappa\sinh(\gamma t/2)/\gamma)^{2\kappa\eta/\lambda^2}}, \tag{8}$$

where

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}.$$

2.5 ARPE (Average Relative Percentage Error)

ARPE, Average Relative Percentage Error is statistic error measure, a metric used to evaluate the accuracy of a predictive model in general. Specifically, it can be adapted to assess the performance of pricing model. The formula for ARPE is:

$$ARPE = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{MarketPrice_i - ModelPrice_i}{MarketPrice_i}\right| \times 100 \tag{9}$$

Where:

- n = Number of options (or data points)
- $MarketPrice_i$ = Actual observed price of the option i in the market
- $ModelPrice_i$ = Predicted price of the option i from the pricing model

3 Option Pricing

In order to minimize the average relative percentage error in NIG_CIR process, the parameters are already calibrated (see: *Lévy Processes in Finance: Pricing Financial Derivatives* by Wim Schoutens).

The calibrated parameters are as follows:

- Mean-reversion speed: $\kappa = 0.5391$
- Long-term mean: $\eta = 1.5746$
- Volatility: $\lambda = 1.8772$
- Initial value: $y_0 = 1$
- Tail heaviness parameter for the NIG process: $\alpha = 18.4815$
- Asymmetry parameter for the NIG process: $\beta = -4.8412$
- Scale parameter for the NIG process: $\delta = 0.4685$

3.1 Risk-Neutral Price Process

Consider a stochastic process Y, representing the time evolution of our business model. Let $\varphi(u; t, y_0)$ denote the characteristic function of Y at time t, given an initial value y_0.

We model the price process S as:

$$S_t = S_0 \frac{\exp((r-q)t)}{E[\exp(X_{Y_t}) \mid y_0]} \exp(X_{Y_t}), \tag{10}$$

where:

- S_0 is the initial price.
- r is the risk-free interest rate.
- q is the dividend yield.
- X is a Lévy process with characteristic exponent $\psi_X(u)$.

The factor:

$$\frac{\exp((r-q)t)}{E[\exp(X_{Y_t}) \mid y_0]} \tag{11}$$

puts us immediately into the risk-neutral world by a mean-correcting argument.

If we look carefully at the denominator, it is similar to the equation (8). The difference is only X_{Y_t}, which is subordinated process and based on the mean-correcting martingale measure its characteristic function will be:

$$E[\exp(X_{Y_t}) \mid y_0] = \phi(-i\psi_{NIG}(-i), t; \kappa, \eta, \lambda, y_0) \tag{12}$$

Let calculate:

$$-i\psi_{NIG}(-i) = -i(-\delta(\sqrt{\alpha^2 - (\beta + i(-i))^2} - \sqrt{\alpha^2 - \beta^2})) = i\delta((\sqrt{\alpha^2 - (\beta^2 + 2\beta + 1)} - \sqrt{\alpha^2 - \beta^2})) := u*$$

we called it new $u*$ in characteristic function of Y_t, so we have $\mathrm{E}[\exp(iuY_t) \mid y_0] = \phi(u*, t; \kappa, \eta, \lambda, y_0)$.

This method are represented in Figure 1, where we simulate of possible trajectories for the S&P 500 index from the close of the market on April 18th 2002 (close price: 1124.47) to April 18th 2003.

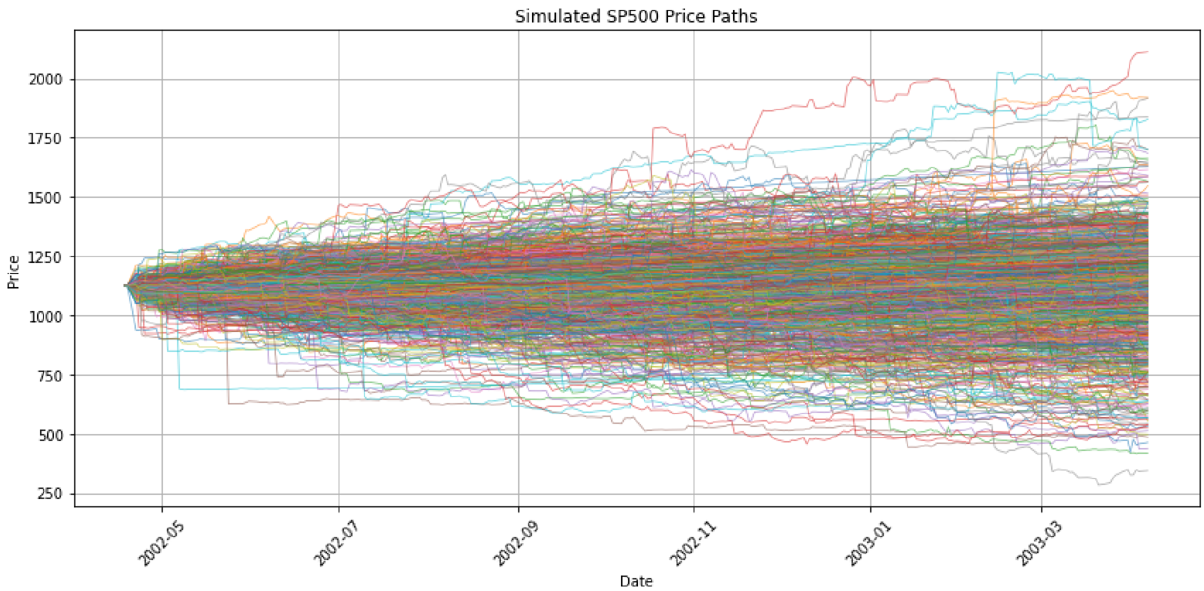

Figure 1: Simulation of trajectories.

3.2 Monte Carlo method

When closed-form solutions are unavailable for pricing options, the Monte Carlo method provides a useful estimation approach. This method involves generating a large number of potential price paths, or sample paths, for the underlying asset to approximate the expected payoff of the option. In particular, the procedure is the following:

1. **Simulating Price Paths**: For Lévy Stochastic Volatility (SV) models, simulating each price path requires capturing both the jump dynamics and time-varying volatility. The process includes:
   - **Calibrate Model Parameters**: First, calibrate the Lévy SV model parameters to observed market prices of vanilla options. This step identifies the risk-neutral parameters that minimize model error, often measured by the root mean square error (RMSE) relative to market prices.
   - **Path Simulation**: With calibrated parameters:
     - Simulate the rate of time change process y_t over the interval $[0, T]$.
     - Compute the cumulative time change $Y_t = \int_0^t y_s \, ds$ for $t \in [0, T]$.
     - Simulate the Lévy process X_t up to Y_T to account for jumps.
     - Calculate the time-changed Lévy process X_{Y_t} over $[0, T]$.
     - Derive the stock-price process S_t based on this time-changed Lévy process over $[0, T]$

   This process generates m simulated paths of the stock price, with each path producing a final payoff value, denoted as V_i for $i = 1, \ldots, m$.

2. **Estimating the Expected Payoff**: The expected payoff $\hat{V}$ is estimated by averaging the payoffs from all simulated paths:
$$\hat{V} = \frac{1}{m} \sum_{i=1}^{m} V_i. \tag{13}$$

3. **Discounting to Present Value**: To obtain the option price, the estimated payoff is discounted back to the present using the risk-free rate r and the option's maturity T:
$$OptionPrice = e^{-rT} \hat{V}. \tag{14}$$

The accuracy of the Monte Carlo estimate improves with the number of simulated sample paths, as the standard error of the estimate decreases with the square root of the number of paths. The standard error SE of the Monte Carlo estimate is calculated as follows:
$$SE = \sqrt{\frac{1}{(m-1)^2} \sum_{i=1}^{m} (\hat{V} - V_i)^2}. \tag{15}$$

4 Model Simulation

4.1 Simulation of Brownian Motion

To simulate a Brownian motion, we discretize time by taking ($\Delta t_i = t_i - t_{i-1}$, $i = 1, \ldots, N$) as the size of time steps, which is very small.

Then, we simulate the value of the Brownian motion at the time points $\{t_n\}, n = 0, 1, \ldots$. Remember, the Brownian motion has a normal distribution and independent increments.

The approximation of the process at each point is given by:
$$W_0 = 0, \quad W_{nt} = W_{(n-1)t} + \sqrt{\Delta t}\, v_n, \quad n = 1, 2, \ldots$$

where $\{v_n, n = 1, 2, \ldots\}$ is a series of standard normal random variables ($v_n \sim \mathcal{N}(0, 1)$).

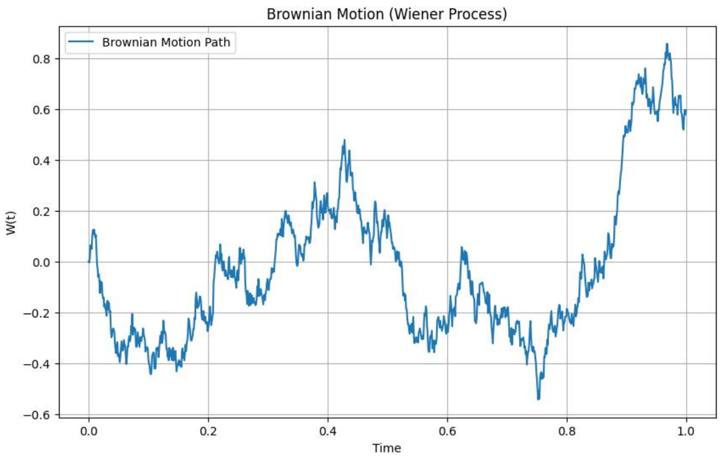

Figure 2: Brownian Motion

4.2 Simulate the CIR process.

Simulating a CIR process $y = \{y_t, t \geq 0\}$ is quite straightforward and classical. We discretize the SDE as follows:

$$dy_t = \kappa(\eta - y_t)\,dt + \lambda y_t^{1/2}\,dW_t, \quad y_0 \geq 0. \tag{16}$$

The sample path of the CIR process $y = \{y_t, t \geq 0\}$ at the time points $t = n\Delta t$, (n = 0, 1, 2, . . . , is given by

$$y_{n\Delta t} = y_{(n-1)\Delta t} + \kappa(\eta - y_{(n-1)\Delta t})\Delta t + \lambda y_{(n-1)\Delta t}^{1/2}\sqrt{\Delta t}\,v_n, \tag{17}$$

where $\{v_n, n = 1, 2, \ldots\}$ is a series of independent standard normal random numbers.

In Figure 3 is representation of simulating a CIR process with parameters κ, η, λ, initial value y_0, over a given maturity T and a specified number of time steps N. This CIR process serves as our time reference. By integrating the CIR process, we obtain the integrated CIR (ICIR), which will guide the time intervals for subsequent steps.

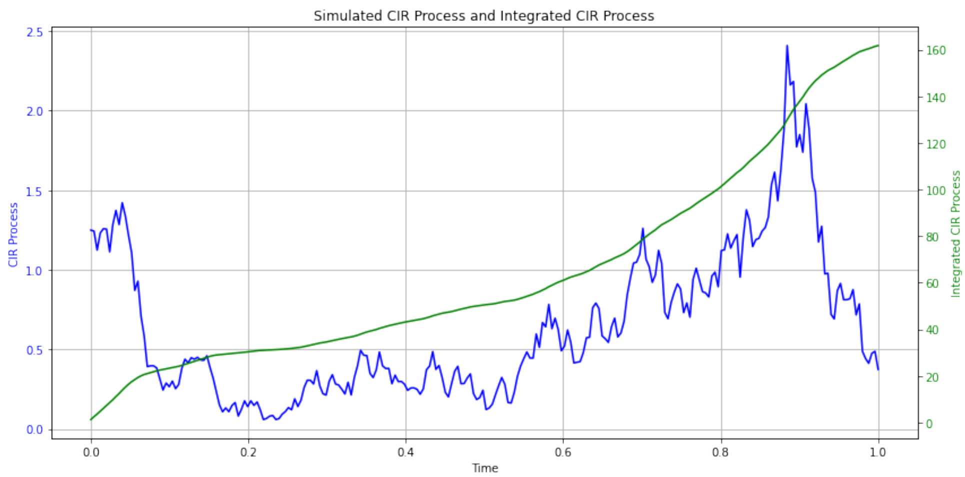

Figure 3: Simulation of CIR (blue line) and ICIR (green line).

4.3 Simulation of an IG Process

To simulate the Normal Inverse Gaussian (NIG) process using the Cox-Ingersoll-Ross (CIR) process as a time reference, in advance we should generate Inverse Gaussian (IG) process.

In order to to simulate IG process, we need IG random generator proposed by Michael *et al.* (1976). For the purpose of to sample from IG(a, b) distribution, we have to follow algorithm:

- Generate standard Normal random number ν.

- Set $y = \nu^2$
- $x = (a/b) + y/(2b^2) - \sqrt{4aby + y^2}/(2b^2$
- Generate a uniform random number u
- if $u \leq (a + xb)$, then return the number x as the $IG(a, b)$ random number, else return $a^2/(b^2x)$ as the IG(a, b) random number.

With the generated ICIR as a reference, we construct the Inverse Gaussian (IG) process, denoted by IG_{ICIR}. The parameters of this IG process are defined as $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$, where δ, α, and β are the parameters of the NIG process. For each of the N time steps, we generate an independent random variable following an Inverse Gaussian distribution with parameters $a \cdot \Delta t$ and b, where Δt at each step k is defined by the change $ICIR[k] - ICIR[k-1]$. The IG process is built incrementally:

- Start with $IG_{ICIR}[0] = 0$.
- For each step k, compute $IG_{ICIR}[k] = IG_{ICIR}[k-1] + i_k$, where i_k is the sampled random variable for that step.

We can see corelation in graphs, Figure 4, as times going fast in CIR process from 0.0 to 0.4, IG_CIR process has more fluctuation, also similar paths are repeating from 0.6. But when times are going slowly from 0.4 to 0.6 in CIR process, IG_CIR does not have fluctuation and roughly said it keeps same values.

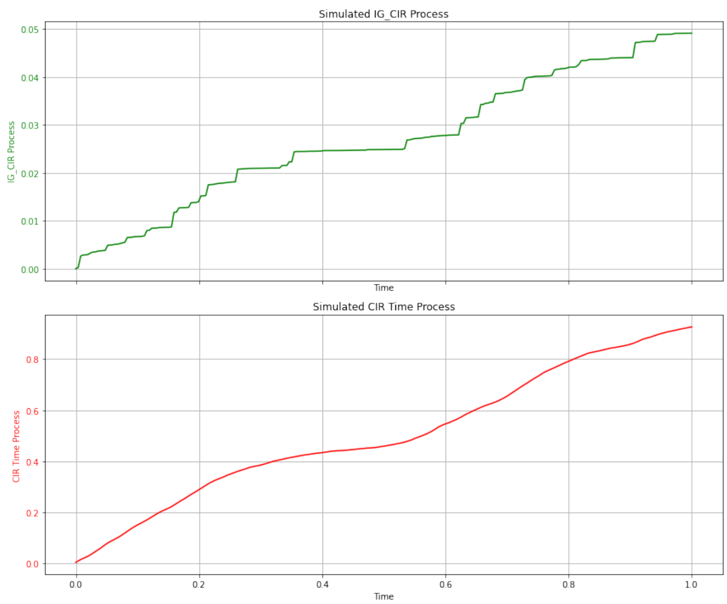

Figure 4: Simulation of IG process (green line), Simulation of ICIR (red line).

4.4 NIG-CIR Stochastic Volatility Simulation

Using the generated IG_{ICIR} process as a time reference, we simulate the NIG process with parameters α, β, and δ. We create a time-changed Brownian motion $W_{IG(a \cdot ICIR,b)}$ by using the IG_{ICIR} process as the time variable. The NIG process at each step k is then given by:

$$X_{NIG}[ICIR[k]] = \beta\delta^2 IG_{ICIR}[k] + \delta W_{IG(a \cdot ICIR,b)}[k]. \tag{18}$$

Based on empirical knowledge the evidence is insufficient to draw a firm conclusion about explicit same movement in NIG_CIR and CIR process, as we saw IG_CIR and CIR process, because of nature of the Brownian Motion and additional parameter in Normal Inverse Gaussian process, but in our charts allows us to say that, when times are going fast NIG_CIR process are more fluctuating as you can see in Figure 5. The most important role of this process is Brownian scaling property. This property links scale changes to time variations, allowing random volatility fluctuations to be captured through random temporal shifts.

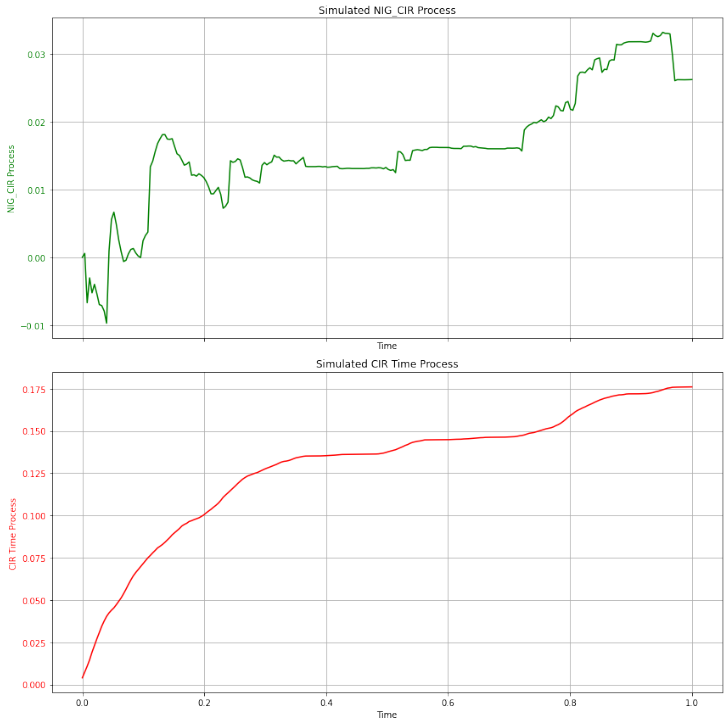

Figure 5: Simulation of NIG process.

To simulate trajectories of the underlying Lévy process of our NIG_CIR model, we employed a well-established algorithm, what we described in previous sections. Figure 6 depicts simulated NIG_CIR processes, which are used to efficiently generate sample paths. The implementation of this simulation algorithm allows us to gain a deeper understanding of the NIG_CIR model's behavior and its potential applications in financial modeling. We can assess the impact of parameter variations on the process's volatility, skewness, and kurtosis to analyze trajectories, which are given in next section.

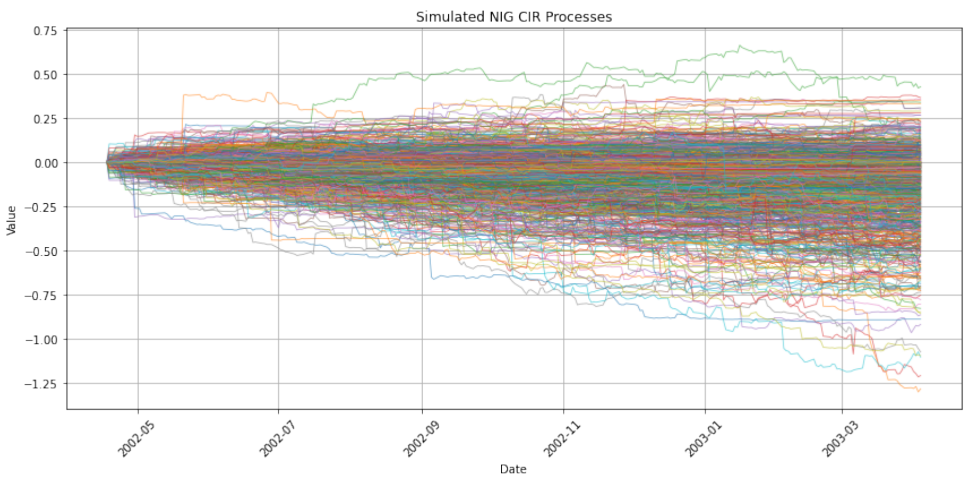

Figure 6: Simulate trajectories

4.5 Parameters effect

In this section we represent parameters effect in NIG_CIR. In the Figure 7, we see how simulated trajectories are effected by CIR parameters changing, in firs column we halved κ - mean-reversion speed, η - long-term mean, λ - volatility.

We can conclude that Increasing κ strengthens mean reversion, while reducing κ weakens it and the parameter η controls the target mean level of the process. In the chart we see left-hand side and right-hand side prices have high range when we changed either κ or η. Also in case of halved λ, the paths exhibit lower variability and smoother behavior. As we increase λ the paths display increased variability and more pronounced fluctuations, which is very intuitive.

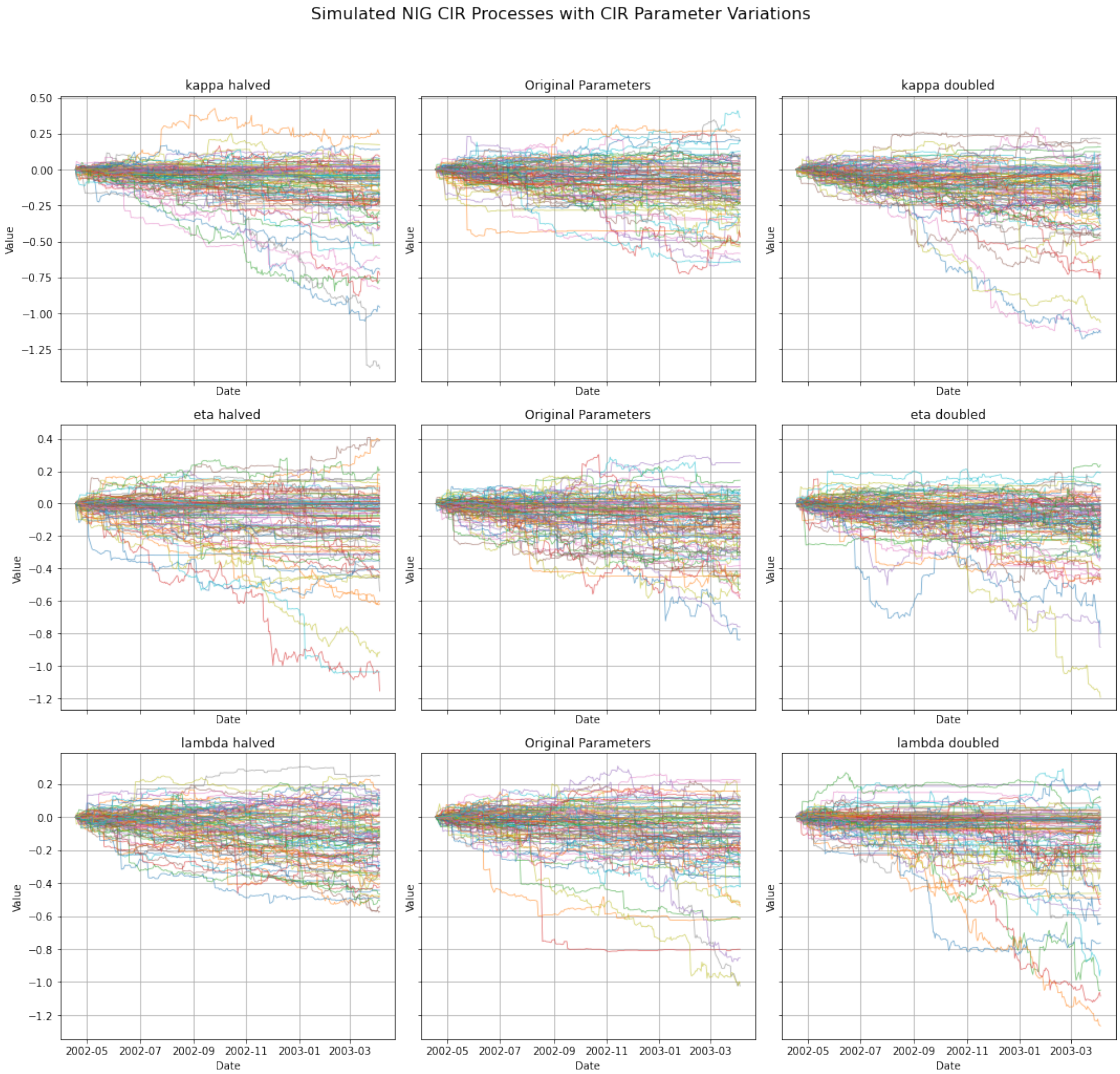

Figure 7: Variation of CIR parameters.

On the other hand, in Figure 8, we changed NIG prameters. Halved α - the paths display increased variability and fatter tails. Lower values of α allow for more extreme deviations from the mean, reflecting a heavier-tailed distribution. Doubled α - the paths are less volatile, and the tails of the distribution become thinner, meaning fewer extreme events. Asymmetry parameter β controls the skewness or asymmetry of the process. Original parameter are negative, therefore trajectories are tend to negative directions when we have doubled β. δ has same effect as *lambda* in CIR.

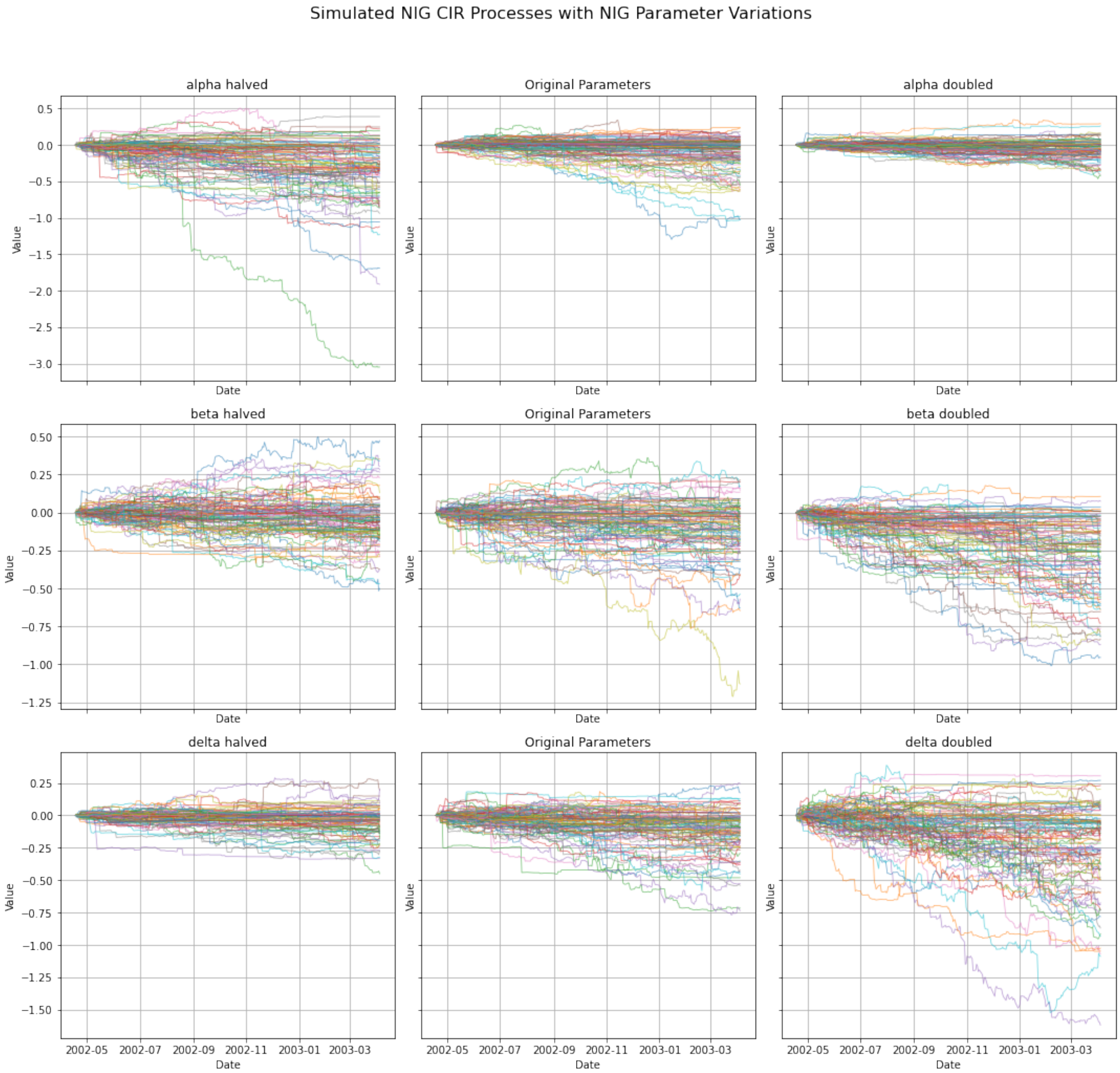

Figure 8: Variations in NIG parameters.

4.6 Implied Volatility

Implied volatility serves as a crucial metric in options trading, providing valuable insights into market participants' risk perceptions. By quantifying the market's expectation of future price volatility, implied volatility plays a pivotal role in option pricing and risk management. Traders and investors leverage this information to make informed decisions, construct optimal trading strategies, and assess potential market opportunities.

Mathematically we have:

$$V_c(t, S) = BS(\sigma, r, T, K, S_0) \tag{19}$$

where BS is monotonically increasing in σ (higher volatility corresponds to higher prices). Now, assume the existence of some inverse function

$$g_\sigma(\cdot) = BS^{-1}(\cdot) \tag{20}$$

so that

$$\sigma_{impl} = g_\sigma\left(V_c^{mkt}, r, T, K, S_0\right) \tag{21}$$

By computing the implied volatility for traded options with different strikes and maturities, we can test the Black-Scholes model.

How to find implied volatility?

The BS pricing function BS does not have a closed-form solution for its inverse $g_\sigma(\cdot)$. Instead, a root-finding technique is used to solve the equation:

$$BS(\sigma_{impl}, r, T, K, S_0) - V_c^{mkt} = 0. \tag{22}$$

There are many ways to solve this equation, one of the most popular methods are methods of "Newton" and "Brent". We used Brent's method.

5 European Call Option

We're looking at options tied to a specific S&P 500 stock Based on the data from the book (see: Schoutens). On April 18, 2002, this stock closed at $1124.47, risk-free interest rate of 1.9% per year and a dividend yield of 1.2% per year. For our calculations, we'll use a 252-day trading year. We're considering strike prices ranging from $1025 to $1350, increasing by $25 each time. Since stock options typically expire on the third Friday of the month, the option maturing in March 2003 will actually expire on March 21, 2003, which is 234 trading days from now.

Based on the data and model parameters, which are mentioned above, we compare our NIG_CIR model to the Black-Scholes and market real prices. In Figure 9, each model prices are represented for each strikes and and in Figure 10, chart shows how our model prices are closer to the real market than BS model's prices.

The Results

- Average Relative Percentage Error (ARPE) - Black-Scholes: 19.24%
- Average Relative Percentage Error (ARPE) - NIG_CIR: **1.26%**
- The NIG_CIR model is better by 17.98%.

Strike	Market Price	Black-Scholes Price	NIG_CIR Price	BS Relative Error	NIG_CIR Relative Error
1025	146.5	137.707	145.538	0.060	0.007
1100	96.2	92.911	95.997	0.034	0.002
1125	81.7	80.518	81.600	0.014	0.001
1150	68.3	69.370	68.308	0.016	0.000
1175	56.6	59.421	56.244	0.050	0.006
1200	46.1	50.611	45.545	0.098	0.012
1225	36.9	42.871	36.198	0.162	0.019
1250	29.3	36.120	28.350	0.233	0.032
1275	22.5	30.273	21.938	0.345	0.025
1300	17.2	25.246	16.899	0.468	0.018
1325	12.8	20.950	13.005	0.637	0.016

Figure 9: Table 1

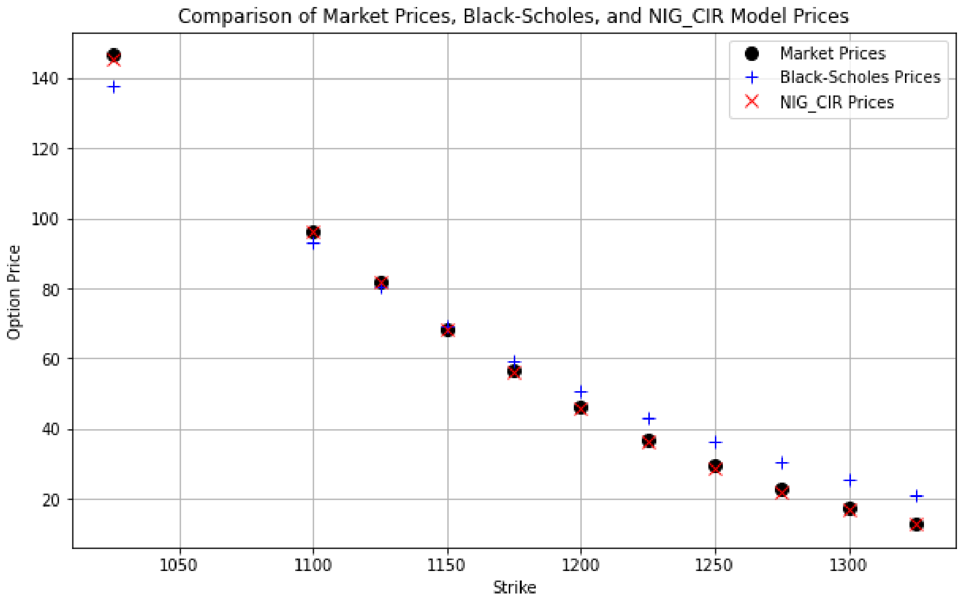

Figure 10: Comparison of models

Stochastic volatility models are highly effective for modeling financial time series because they incorporate the random nature of volatility and its dependence on time, providing a realistic framework to study and predict market behavior over different time horizons. For the purpose of strength the argument, our model are more accurate than Normal Inverse Gaussian process, we represent results in the table and charts for NIG (see: Figure 11 and 12), in this case we use Monte Carlo simulation for risk-neutrality instead of what implement in section 3.1.

The Results

- Average Relative Percentage Error (ARPE) - Black-Scholes: 19.24%
- Average Relative Percentage Error (ARPE) - NIG: **4.24%**
- The NIG model is better by 15.00%.

Strike	Market Price	Black-Scholes Price	NIG Price	BS Relative Error	NIG Relative Error
1025	146.5	137.707	147.413	0.060	0.006
1100	96.2	92.911	96.828	0.034	0.007
1125	81.7	80.518	82.216	0.014	0.006
1150	68.3	69.370	68.918	0.016	0.009
1175	56.6	59.421	57.010	0.050	0.007
1200	46.1	50.611	46.626	0.098	0.011
1225	36.9	42.871	37.757	0.162	0.023
1250	29.3	36.120	30.322	0.233	0.035
1275	22.5	30.273	24.134	0.345	0.073
1300	17.2	25.246	19.101	0.468	0.111
1325	12.8	20.950	15.086	0.637	0.179

Figure 11: Table 2

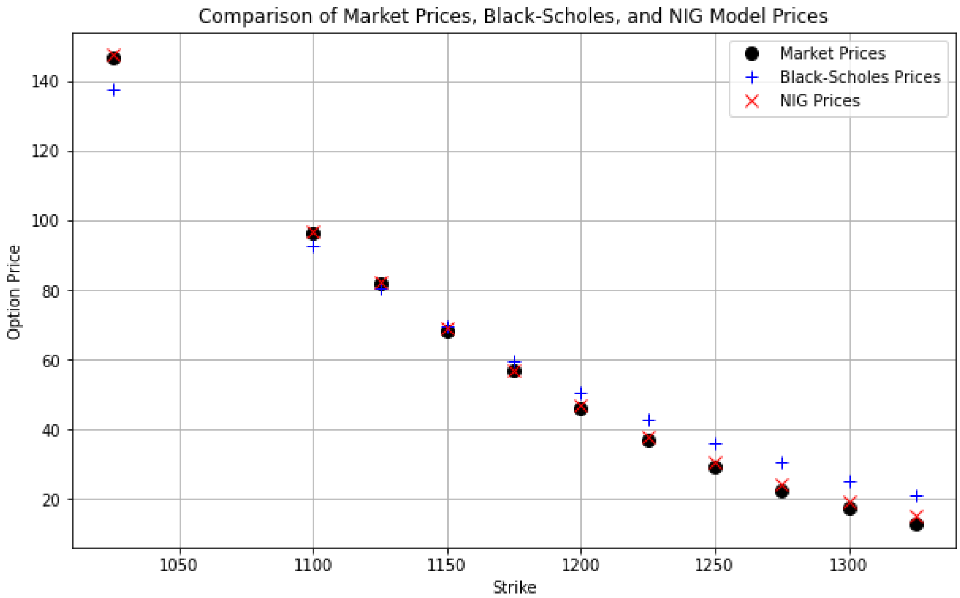

Figure 12: Comparison of models

5.1 Implied Volatility for SV

By using the Black-Scholes formula, and based on the prices calculated for the options by the NIG_CIR model, we calculated the implied volatilities against the strike, it is compared with the implied volatilities obtained from the real option prices for the maturity date March 2003. We can see numbers in Table 3 and visualization in Figure 14, how they are moving close to each other, which emphasizes NIG_CIR model accuracy.

	Strike Price	Market Price (1st Set)	Implied Volatility (1st Set)	NIG CIR Price (2nd Set)	Implied Volatility (2nd Set)	Implied Volatility Difference
0	1025.0	146.5	0.2076	145.538	0.2049	0.0027
1	1100.0	96.2	0.1904	95.997	0.1899	0.0005
2	1125.0	81.7	0.1853	81.6	0.185	0.0002
3	1150.0	68.3	0.1799	68.308	0.1799	-0.0
4	1175.0	56.6	0.1757	56.244	0.1749	0.0008
5	1200.0	46.1	0.1714	45.545	0.1701	0.0014
6	1225.0	36.9	0.1673	36.198	0.1655	0.0018
7	1250.0	29.3	0.1641	28.35	0.1615	0.0026
8	1275.0	22.5	0.1599	21.938	0.1582	0.0017
9	1300.0	17.2	0.1569	16.899	0.1558	0.001
10	1325.0	12.8	0.1536	13.005	0.1544	-0.0008

Figure 13: Table 3

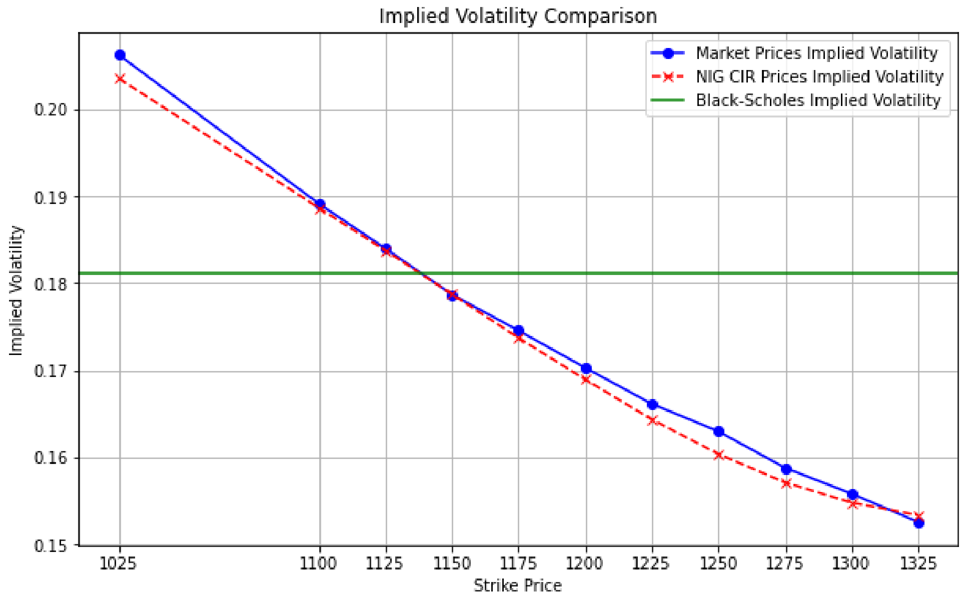

Figure 14: Implied Volatilities

6 Pricing Down-and-Out Barrier Call Options

At the moment, we used European call options; however, there are other types of options. Beyond the familiar European and American options, a diverse world of exotic options exists, offering tailored risk management and investment strategies. These options introduce unique features and complexities, often making them more flexible and potentially more profitable than traditional options.

Barrier Options are a prominent example of exotic options. They are contingent on whether the underlying asset's price reaches a specific level, known as the barrier. There are four primary types of barrier options:

- **Up-and-out (UOBC):** The option becomes worthless if the underlying asset's price rises above the barrier level at any time during its life.
- **Down-and-out (DOBC):** The option becomes worthless if the underlying asset's price falls below the barrier level at any time during its life.
- **Up-and-in (UIBC):** The option comes into existence only if the underlying asset's price rises above the barrier level at some point during its life.
- **Down-and-in (DIBC):** The option comes into existence only if the underlying asset's price falls below the barrier level at some point during its life.

In our work, we concentrate only on the DOBC. If contracts have a duration of T, we can denote the maximum and minimum processes as

$$X = \{X_t, 0 \leq t \leq T\}$$

by

$$M_t^X = \sup\{X_u; 0 \leq u \leq t\} \quad and \quad m_t^X = \inf\{X_u; 0 \leq u \leq t\}, \quad 0 \leq t \leq T. \tag{23}$$

Its initial price are given by

$$DOBC = \exp(-rT)E_Q\left[(S_T - K)^+ 1(m_T^S > H)\right]. \tag{24}$$

The fair price of a down-and-out barrier call option is the discounted expectation of its future payoff. We simulated different numbers of paths for underlying asset with Monte Carlo method, where the model prices are in risk-neutral world. If the price process does not breach the barrier. the payoff is calculated, otherwise it is zero. Time to maturity is 1 year, $K = S_0 = \$1124.47$, level of barrier is $H = 0.8 \times S_0$. We considered 500, 1000 and 2000 simulations of paths, which gave use interesting results.

- **500 Paths**:
  - Option Price: 96.77
  - Probability of Remaining Active: 83.6%
  - Conditional Probability of Positive Payoff if Remaining Active: 72.7%
- **1000 Paths**:
  - Option Price: 82.99
  - Probability of Remaining Active: 78.6%
  - Conditional Probability of Positive Payoff if Remaining Active: 72.3%
- **2000 Paths**:
  - Option Price: 86.13
  - Probability of Remaining Active: 78.2%
  - Conditional Probability of Positive Payoff if Remaining Active: 70.8%
- **The benchmark form book of Schoutens is**: 86.18

6.1 Interpretation of Results

The option price converges toward the benchmark value of 86.18 as the number of paths increases, confirming that the Monte Carlo method approaches the correct theoretical price. For instance, at 2000 paths, the estimated price stabilizes around 86.13, which is very close to the benchmark. This indicates that a higher number of paths improves the accuracy of the simulation by better capturing rare events like barrier breaches. In contrast, a 500-path simulation significantly overestimates the price relative to Schoutens' benchmark due to insufficient representation of these events.

Probability of Remaining Active

The probability of the barrier condition not being breached decreases as the number of paths increases:

$$83.6\% \ (500 \ paths) \ \rightarrow 78.6\% \ (1000 \ paths) \ \rightarrow 78.2\% \ (2000 \ paths).$$

This trend suggests that with more paths, the simulation better accounts for scenarios where the barrier is breached, reducing the estimated probability of remaining active.

Conditional Probability of Positive Payoff if Remaining Active

This probability also decreases with the number of paths:

$$72.7\% \ (500 \ paths) \ \rightarrow 72.3\% \ (1000 \ paths) \ \rightarrow 70.8\% \ (2000 \ paths).$$

The slight decline reflects improved modeling of active paths, which includes paths that may result in zero payoff due to unfavorable price movements even if the barrier condition is satisfied.

7 Conclusion

Financial markets are inherently volatile, influenced by economic and behavioral factors such as news announcements, investor sentiment, and other phenomena. These dynamics often challenge the assumptions of traditional models like Black-Scholes, which rely on constant volatility and normally distributed returns—assumptions that frequently fail in real-world markets.

To address these challenges, this project explored the use of Levy processes as a flexible alternative for modeling the behavior of financial markets. Specifically, we studied the Normal Inverse Gaussian (NIG) model proposed by Barndorff-Nielsen (1995) for modeling stock price dynamics, combined with the Cox-Ingersoll-Ross (CIR) process introduced by Cox et al. (1985) to account for stochastic volatility. These models were implemented to simulate stock price paths and price call options on the S&P 500 Index, with results compared against the Black-Scholes model and actual market prices.

The analysis demonstrated the effectiveness of the NIG and CIR processes in capturing market characteristics that traditional models struggle to address, such as heavy tails and volatility clustering. By accommodating these features, the chosen Levy process proved to be a more robust tool for understanding and pricing financial derivatives in complex market conditions.

Ultimately, this project highlighted the limitations of simplistic models and underscored the importance of advanced stochastic methods in modern financial analysis.

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