Aerial Manipulator Dynamics Modeling and Control

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1 Manipulator Kinematics

1.1 Forward Kinematics

To detailed analysis the kinematics and dynamics of the aerial manipulator, three frames of reference are defined:

- Special frame: a fixed inertial frame.
- Body frame: a body fixed frame that attached to the body principle axis.
- Tool frame: the end-effector frame on which the measurement devices (e.g. IMU) are mounted.

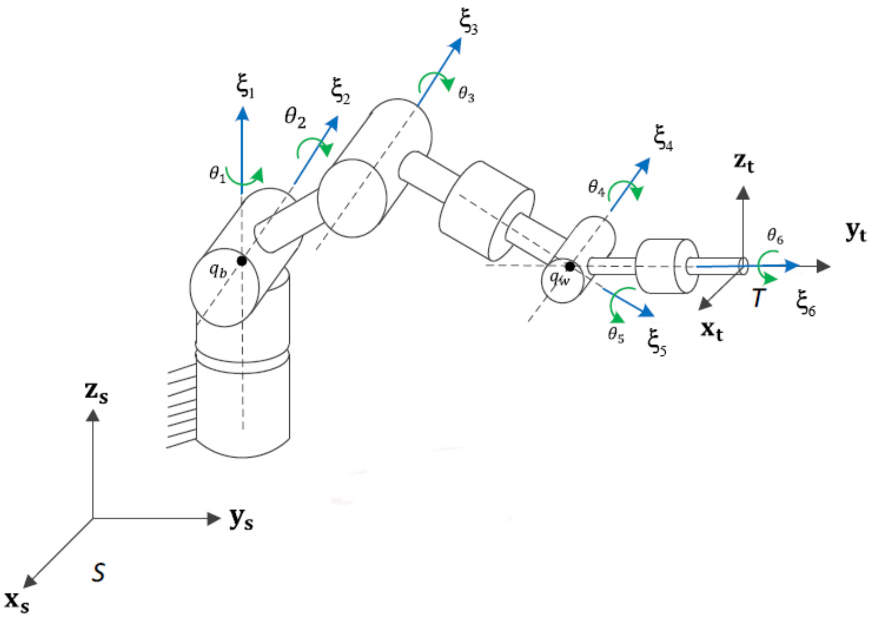

Figure 1: A 6-axis Industrial Robot

We define the body frame of the i-th link of the manipulator as the frame $\{i\}$, especially the link 0 is the base.

Define the configuration of frame $\{i\}$ in frame $\{j\}$ as $g_{i,j} \in SE(3)$, which can be interpreted as the transformation from frame $\{i\}$ to frame $\{j\}$. The configuration of frame $\{i\}$ in the spacial frame usually uses shortly as $g_i = g_{s,i}$.

$$g_{i,j} = g_i^{-1} g_j \tag{1}$$

The twist $\mathcal{A}_s = [\xi_s; u_s]$ and $\mathcal{A}_t = [\xi_t; u_t]$ are respectively respected in the initial special frame and tool frame, remaining unchanged both. The transformation of these two presentations is

$$\mathcal{A}_t = Ad^{-1}_{g_{st}(0)}\mathcal{A}_s \tag{2}$$

According to the Product of Exponential (POE) formula, the forward kinematics map $g_{sb}(k)$ is given by

$$g_t(k) = g_{st}(k) = g_0(k)g_{0t}(0)e^{\widehat{\mathcal{A}}_{t_1}\theta_1(k)}e^{\widehat{\mathcal{A}}_{t_2}\theta_2(k)}\cdots e^{\widehat{\mathcal{A}}_{t_n}\theta_n(k)} \tag{3}$$

where $g_{0t}(0)$ is the initial configuration of the tool frame in the body frame 0 (i.e. the base), when all the joint angles are defined at zero encoder readings.

1.2 The Manipulator Jacobian

The velocity of the end-effector expressed in the tool frame is given by

$$\begin{aligned}
\widehat{V}_{b_t} &= g_t^{-1}\dot{g}_t \\
&= g_t^{-1}\left(\left(\frac{d}{dt}g_0\right)g_{0t}(0)e^{\widehat{\mathcal{A}}_{t_1}\theta_1}e^{\widehat{\mathcal{A}}_{t_2}\theta_2}\cdots e^{\widehat{\mathcal{A}}_{t_n}\theta_n} + \sum_{i=1}^{n}\frac{\partial g_t}{\partial \theta_i}\right) \\
&= Ad^{-1}_{g_{0t}(0)e^{\widehat{\mathcal{A}}_{t_1}\theta_1}e^{\widehat{\mathcal{A}}_{t_2}\theta_2}\ldots e^{\widehat{\mathcal{A}}_{t_n}\theta_n}}\widehat{V}_{b_0} + \sum_{i=1}^{n}\left(g_t^{-1}\frac{\partial g_t}{\partial \theta_i}\right)\dot{\theta}_i
\end{aligned} \tag{4}$$

Because

$$\begin{aligned}
g_t^{-1}\frac{\partial g_t}{\partial \theta_i} &= e^{-\widehat{\mathcal{A}}_{t_n}\theta_n}\cdots e^{-\widehat{\mathcal{A}}_{t_1}\theta_1}g_{0t}^{-1}(0)g_0^{-1} \\
&\qquad g_0 g_{0t}(0)e^{\widehat{\mathcal{A}}_{t_1}\theta_1}\cdots e^{\widehat{\mathcal{A}}_{t_{i-1}}\theta_{i-1}}\left(\frac{\partial}{\partial \theta_i}e^{\widehat{\mathcal{A}}_{t_i}\theta_i}\right)e^{\widehat{\mathcal{A}}_{t_{i+1}}\theta_{i+1}}\cdots e^{\widehat{\mathcal{A}}_{t_n}\theta_n} && (5)\\
&= Ad^{-1}_{e^{\widehat{\mathcal{A}}_{t_{i+1}}\theta_{i+1}}\ldots e^{\widehat{\mathcal{A}}_{t_n}\theta_n}}\widehat{\mathcal{A}}_{t_i} && (6)
\end{aligned}$$

then we have

$$V_{b_t} = Ad^{-1}_{g_{0t}(0)e^{\widehat{\mathcal{A}}_{t_1}\theta_1}e^{\widehat{\mathcal{A}}_{t_2}\theta_2}\ldots e^{\widehat{\mathcal{A}}_{t_n}\theta_n}}V_{b_0} + J^b\dot{\theta} \tag{7}$$

where

$$J^b = \begin{bmatrix} Ad^{-1}_{e^{\widehat{\mathcal{A}}_{t_2}\theta_2}\ldots e^{\widehat{\mathcal{A}}_{t_n}\theta_n}}\mathcal{A}_{t_1} & \cdots & Ad^{-1}_{e^{\widehat{\mathcal{A}}_{t_n}\theta_n}}\mathcal{A}_{t_{n-1}} & \mathcal{A}_{t_n}\end{bmatrix} \tag{8}$$

2 Manipulator Dynamics

2.1 Inverse Dynamics

The effector of each joint is considered as a rigid body, and its inertia matrix in the body frame can be expressed in a rotated frame $\{C\}$ described by the rotation matrix R_{cb}.

$$\mathcal{I}_b = R_{cb}^T\mathcal{I}_c R_{cb} \tag{9}$$

Noting that the inertia matrix $\mathcal{I}_c$ is diagonalizable as

$$\mathcal{I}_c = P\Lambda P^{-1} \tag{10}$$

where P is a invertible matrix. We define $R_{cb} = P$, making the body inertia matrix $\mathcal{I}_b$ to be a diagonal matrix Λ. So the orientation R_b can be given as

$$R_b = R_c P \tag{11}$$

Given the configuration g_t and g_c, the configuration of the tool frame in the body frame now can be given as

$$g_{bt} = (g_b)^{-1} g_t \tag{12}$$

Therefore , screw axis of joint i expressed in tool frame as $\mathcal{A}_{t_i}$ and that expressed in the frame {i} are related by

$$\mathcal{A}_b = Ad_{g_{bt}} \mathcal{A}_t \tag{13}$$

where Ad_g is defined as

$$Ad_g = \begin{bmatrix} R & 0 \\ \widehat{p}R & R \end{bmatrix} \tag{14}$$

With definitions in **subsection 1.1**, $g_{i-1,i}$ can be calculated as

$$\begin{aligned} g_{i-1,i} &= g_{i-1}^{-1} g_i \\ &= \begin{bmatrix} R_{i-1}^T & -R_{i-1}^T p_{i-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_i & p_i \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_{i-1.i} & R_{i-1}^T p_{i-1,i} \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{15}$$

and therefore,

$$Ad_{g_{i-1,i}} = \begin{bmatrix} R_{i-1,i} & 0 \\ R_{i-1}^T \widehat{p}_{i-1,i} R_i & R_{i-1,i} \end{bmatrix} \tag{16}$$

and

$$Ad_{g_{i-1,i}}^{-1} = Ad_{g_{i-1,i}^{-1}} = \begin{bmatrix} R_{i-1,i}^T & 0 \\ -R_i^T \widehat{p}_{i-1,i} R_{i-1} & R_{i-1,i}^T \end{bmatrix} \tag{17}$$

Let $\mathcal{A}_{b_i}$ be the screw axis of joint i expressed in {i}. Then the configuration of frame {j} in {i} due to arbitrary joint angle θ_i is given as

$$g_{i-1,i} = g_{i-1,i}(0) e^{\widehat{\mathcal{A}}_{b_i} \theta_i} \tag{18}$$

where $g_{i-1,i}(0) = g_{i-1}(0)^{-1} g_i(0)$.

With these definitions, we can recursively calculate the velocity $V_{b_i} = [\omega_{b_i}; v_{b_i}]$ of each link in the frame {i} expressed by twist and joint rate $\dot{\theta}_i$:

$$\widehat{V}_{b_i} = g_i^{-1} \dot{g}_i \tag{19a}$$
$$= (g_{i-1} g_{i-1,i})^{-1} (\dot{g}_{i-1} g_{i-1,i} + g_{i-1} \dot{g}_{i-1,i}) \tag{19b}$$
$$= g_{i-1,i}^{-1} \widehat{V}_{b_{i-1}} g_{i-1,i} + \widehat{\mathcal{A}}_{b_i} \dot{\theta}_i \tag{19c}$$
$$= Ad_{g_{i-1,i}}^{-1} \widehat{V}_{b_{i-1}} + \widehat{\mathcal{A}}_{b_i} \dot{\theta}_i \tag{19d}$$

and thus,

$$V_{b_i} = Ad_{g_{i-1,i}}^{-1} V_{b_{i-1}} + \mathcal{A}_{b_i} \dot{\theta}_i \tag{20}$$

or written as

$$\begin{bmatrix} \omega_{b_i} \\ v_{b_i} \end{bmatrix} = \begin{bmatrix} R_{i-1,i}^T & 0 \\ -R_{i-1,i}^T \widehat{p}_{i-1,i} & R_{i-1,i}^T \end{bmatrix} \begin{bmatrix} \omega_{b_{i-1}} \\ v_{b_{i-1}} \end{bmatrix} + \begin{bmatrix} \xi_{b_i} \\ u_{b_i} \end{bmatrix} \dot{\theta}_i \tag{21}$$

The acceleration $\dot{V}_{b_i}$ can also be written recursively as

$$\dot{\widehat{V}}_{b_i} = \dot{g}_{i-1,i}^{-1} \widehat{V}_{b_{i-1}} g_{i-1,i} + g_{i-1,i}^{-1} \dot{\widehat{V}}_{b_{i-1}} g_{i-1,i} + g_{i-1,i}^{-1} \widehat{V}_{b_{i-1}} \dot{g}_{i-1,i} + \widehat{\mathcal{A}}_{b_i} \ddot{\theta}_i \tag{22a}$$
$$= -\widehat{A}_{b_i} \dot{\theta}_i Ad_{g_{i-1,i}}^{-1} \widehat{V}_{b_{i-1}} + Ad_{g_{i-1,i}}^{-1} \widehat{V}_{b_{i-1}} \widehat{A}_i \dot{\theta}_i + Ad_{g_{i-1,i}}^{-1} \dot{\widehat{V}}_{b_{i-1}} + \widehat{A}_{b_i} \ddot{\theta}_i \tag{22b}$$
$$= [Ad_{g_{i-1,i}}^{-1} \widehat{V}_{b_{i-1}}, \widehat{\mathcal{A}}_{b_i} \dot{\theta}_i] + Ad_{g_{i-1,i}}^{-1} \dot{\widehat{V}}_{b_{i-1}} + \widehat{\mathcal{A}}_{b_i} \ddot{\theta}_i \tag{22c}$$

where $[X, Y]$ is call the Lie bracket of skew-matrix X and Y.

Definition 2.1 Given a twist $\xi = [\omega, v]^T$, the hat operation of ξ is defined as follows:

$$\forall \xi = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6, \quad \widehat{\xi} \triangleq \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \tag{23}$$

Definition 2.2 The Lie bracket of X and Y is defined as

$$\forall X, Y \in \mathbb{R}^{n \times n}, \quad [X, Y] \triangleq XY - YX \tag{24}$$

Lemma 2.1 Given two twist $\xi_1 = [\omega_1, v_1]^T$ and $\xi_2 = [\omega_2, v_2]^T$, the Lie bracket of $\widehat{\xi}_1$ and $\widehat{\xi}_2$ can be written as

$$[\widehat{\xi}_1, \widehat{\xi}_2] = (\mathrm{ad}_{\xi_1}\xi_2)^\wedge = (-\mathrm{ad}_{\xi_2}\xi_1)^\wedge \tag{25}$$

where

$$\mathrm{ad}_\xi = \begin{bmatrix} \widehat{\omega} & 0 \\ \widehat{v} & \widehat{\omega} \end{bmatrix} \tag{26}$$

Therefore, the acceleration $\dot{V}_{b_i}$ can be derived from (22) as

$$\begin{aligned} \dot{V}_{b_i} &= [Ad^{-1}_{g_{i-1,i}}\widehat{V}_{b_{i-1}}, \widehat{\mathcal{A}}_{b_i}\dot{\theta}_i]^\vee + Ad^{-1}_{g_{i-1,i}}\dot{V}_{b_{i-1}} + \mathcal{A}_{b_i}\ddot{\theta}_i \\ &= \mathrm{ad}_{(Ad^{-1}_{g_{i-1,i}}V_{b_{i-1}})}\mathcal{A}_{b_i}\dot{\theta}_i + Ad^{-1}_{g_{i-1,i}}\dot{V}_{b_{i-1}} + \mathcal{A}_{b_i}\ddot{\theta}_i \end{aligned} \tag{27}$$

Substituting (20) into (27), we obtain the equivalent formula

$$\begin{aligned} \dot{V}_{b_i} &= \mathrm{ad}_{(V_{b_i} - \mathcal{A}_{b_i}\dot{\theta}_i)}\mathcal{A}_{b_i}\dot{\theta}_i + Ad^{-1}_{g_{i-1,i}}\dot{V}_{b_{i-1}} + \mathcal{A}_{b_i}\ddot{\theta}_i \\ &= \mathrm{ad}_{V_{b_i}}\mathcal{A}_{b_i}\dot{\theta}_i + Ad^{-1}_{g_{i-1,i}}\dot{V}_{b_{i-1}} + \mathcal{A}_{b_i}\ddot{\theta}_i \end{aligned} \tag{28}$$

or written as

$$\begin{bmatrix} \dot{\omega}_{b_i} \\ \dot{v}_{b_i} \end{bmatrix} = \begin{bmatrix} \widehat{\omega}_{b_i} & 0 \\ \widehat{v}_{b_i} & \widehat{\omega}_{b_i} \end{bmatrix} \begin{bmatrix} \xi_{b_i} \\ u_{b_i} \end{bmatrix} \dot{\theta}_i + \begin{bmatrix} R^T_{i-1,i} & 0 \\ -R^T_{i-1,i}\widehat{p}_{i-1,i} & R^T_{i-1,i} \end{bmatrix} \begin{bmatrix} \dot{\omega}_{b_{i-1}} \\ \dot{v}_{b_{i-1}} \end{bmatrix} + \begin{bmatrix} \xi_{b_i} \\ u_{b_i} \end{bmatrix} \ddot{\theta}_i \tag{29}$$

The Newton-Euler equations involve coupling forces and moments. Let $N_{i-1,i}$ be a three-dimensional vector representing the linear constraint force acting from link {i-1} to link {i}, expressed in the frame {i}. The balance of linear forces is given by

$$m_i(\dot{v}_{b_i} + \widehat{\omega}_{b_i}v_{b_i}) = N_{i-1,i} - Ad_{R_{i,i+1}}N_{i,i+1} + R_i^T m_i g \tag{30}$$

Next we derive the balance of moments. The moment applied to link {i} by link {i-1} is denoted $\mathcal{M}_{i-1,i}$. For the revolute joint, the moment $\mathcal{M}_{i-1,i}$ includes the electromagnetic torque with the joint axis and the coupling moment applied to the rotor from the stator. The balance of moments is thus given by

$$\mathcal{I}_{b_i}\dot{\omega}_{b_i} + \widehat{\omega}_{b_i}\mathcal{I}_{b_i}\omega_{b_i} = \mathcal{M}_{i-1,i} + \mathcal{M}_{i+1,i} \tag{31}$$

The moment $\mathcal{M}_{i-1,i}$ can be given by

$$\begin{aligned} \mathcal{M}_{i-1,i} &= \tau_i\xi_{b_i} + \oint_{B_i}(r_{ci,x} \times N_{i-1,i_x})dx \\ &= \tau_i\xi_{b_i} + \oint_{B_i}\left[(r_{ci,i} + r_{i,x}) \times N_{i-1,i_x}\right]dx \\ &= \tau_i\xi_{b_i} + r_{ci,i} \times N_{i-1,i} + \oint_{B_i}(r_{i,x} \times N_{i-1,i_x})dx \\ &= \tau_i\xi_{b_i} + r_{ci,i} \times N_{i-1,i} + \tau^\xi_{f_i}\xi_{b_i} + \tau^u_{f_i}u_{b_i} \\ &= (\tau_i + \tau^\xi_{f_i})\xi_{b_i} + \tau^u_{f_i}u_{b_i} + r_{ci,i} \times N_{i-1,i} \end{aligned} \tag{32}$$

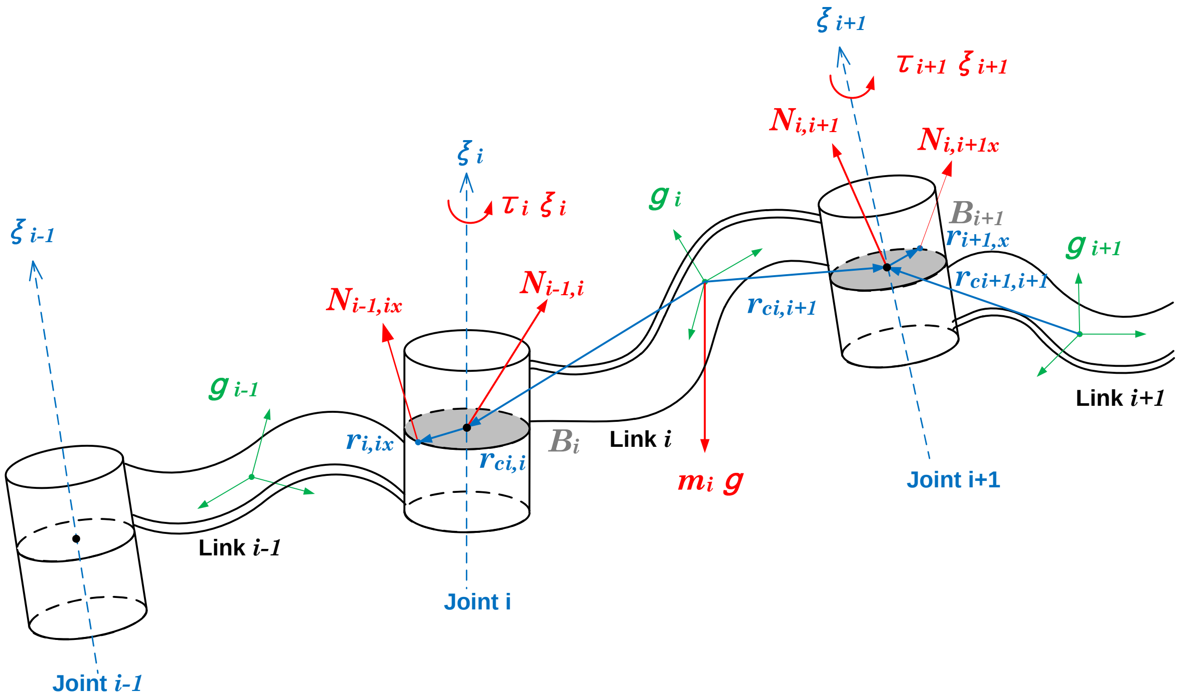

Figure 2: Forces and moments applying on the link i

where B is the contact area of the rotor and the stator of joint {i}.

The moment $\mathcal{M}_{i+1,i}$ can be given by

$$
\begin{aligned}
\mathcal{M}_{i+1,i} &= -Ad_{R_{i,i+1}}\left(\tau_{i+1}\xi_{b_{i+1}} + \oint_{B_{i+1}}(r_{ci,x}\times N_{i,i+1_x})dx\right) \\
&= -Ad_{R_{i,i+1}}\left(\tau_{i+1}\xi_{b_{i+1}} + r_{ci,i+1}\times N_{i,i+1} + \oint_{B_{i+1}}(r_{i+1,x}\times N_{i,i+1_x})dx\right) \\
&= -Ad_{R_{i,i+1}}\left(\mathcal{M}_{i,i+1} + r_{ci,i+1}\times N_{i,i+1} - r_{ci+1,i+1}\times N_{i,i+1}\right) \\
&= -Ad_{R_{i,i+1}}\left(\mathcal{M}_{i,i+1} + r_{ci,ci+1}\times N_{i,i+1}\right) \qquad (33)
\end{aligned}
$$

Therefore, (31) can be rewritten as

$$
\mathcal{I}_{b_i}\dot{\omega}_{b_i} + \widehat{\omega}_{b_i}\mathcal{I}_{b_i}\omega_{b_i} = \mathcal{M}_{i-1,i} - Ad_{R_{i,i+1}}\left(\mathcal{M}_{i,i+1} + r_{ci,ci+1}\times N_{i,i+1}\right) \qquad (34)
$$

Now we integrate (30) and (34) into the mapping as

$$
\begin{aligned}
&\begin{bmatrix}\mathcal{I}_{b_i} & 0\\ 0 & m_i I\end{bmatrix}\begin{bmatrix}\dot{\omega}_{b_i}\\ \dot{v}_{b_i}\end{bmatrix} + \begin{bmatrix}\widehat{\omega}_{b_i} & 0\\ 0 & \widehat{\omega}_{b_i}\end{bmatrix}\begin{bmatrix}\mathcal{I}_{b_i} & 0\\ 0 & m_i I\end{bmatrix}\begin{bmatrix}\omega_{b_i}\\ v_{b_i}\end{bmatrix} \\
&\qquad = \begin{bmatrix}\mathcal{M}_{i-1,i}\\ N_{i-1,i}\end{bmatrix} - \begin{bmatrix}R_{i,i+1} & R_{i,i+1}\widehat{r}_{ci,ci+1}\\ 0 & R_{i,i+1}\end{bmatrix}\begin{bmatrix}\mathcal{M}_{i,i+1}\\ N_{i,i+1}\end{bmatrix} + \begin{bmatrix}0\\ R_i^T m_i g\end{bmatrix}
\end{aligned} \qquad (35)
$$

Noting that $\widehat{v}_{b_i} v_{b_i} = 0$ and $r_{ci,ci+1} = R_{i+1}^T p_{i,i+1}$, we write (35) in the following form as

$$\begin{bmatrix} \mathcal{I}_{b_i} & 0 \\ 0 & m_i I \end{bmatrix} \begin{bmatrix} \dot{\omega}_{b_i} \\ \dot{v}_{b_i} \end{bmatrix} + \begin{bmatrix} \widehat{\omega}_{b_i} & \widehat{v}_{b_i} \\ 0 & \widehat{\omega}_{b_i} \end{bmatrix} \begin{bmatrix} \mathcal{I}_{b_i} & 0 \\ 0 & m_i I \end{bmatrix} \begin{bmatrix} \omega_{b_i} \\ v_{b_i} \end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{M}_{i-1,i} \\ N_{i-1,i} \end{bmatrix} - \begin{bmatrix} R_{i,i+1} & R_{i,i+1} R_{i+1}^T \widehat{p}_{i,i+1} R_{i+1} \\ 0 & R_{i,i+1} \end{bmatrix} \begin{bmatrix} \mathcal{M}_{i,i+1} \\ N_{i,i+1} \end{bmatrix} + Ad_{g_i}^{-1} \begin{bmatrix} 0 \\ m_i g \end{bmatrix} \tag{36a}$$

$$\begin{bmatrix} \mathcal{I}_{b_i} & 0 \\ 0 & m_i I \end{bmatrix} \begin{bmatrix} \dot{\omega}_{b_i} \\ \dot{v}_{b_i} \end{bmatrix} - \begin{bmatrix} \widehat{\omega}_{b_i} & 0 \\ \widehat{v}_{b_i} & \widehat{\omega}_{b_i} \end{bmatrix}^T \begin{bmatrix} \mathcal{I}_{b_i} & 0 \\ 0 & m_i I \end{bmatrix} \begin{bmatrix} \omega_{b_i} \\ v_{b_i} \end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{M}_{i-1,i} \\ N_{i-1,i} \end{bmatrix} - \begin{bmatrix} R_{i,i+1} & R_i^T \widehat{p}_{i,i+1} R_{i+1} \\ 0 & R_{i,i+1} \end{bmatrix} \begin{bmatrix} \mathcal{M}_{i,i+1} \\ N_{i,i+1} \end{bmatrix} + Ad_{g_i}^{-1} \begin{bmatrix} 0 \\ m_i g \end{bmatrix} \tag{36b}$$

or

$$\mathcal{G}_i \dot{V}_{b_i} - \mathrm{ad}_{V_{b_i}}^T \mathcal{G}_i V_{b_i} = \mathcal{F}_{i-1,i} - Ad_{g_{i,i+1}^{-1}}^T \mathcal{F}_{i,i+1} + Ad_{g_i}^{-1} G_{m_i} \tag{37}$$

where

$$\mathcal{G}_i = \begin{bmatrix} \mathcal{I}_{b_i} & 0 \\ 0 & m_i I \end{bmatrix}, \quad \mathcal{F}_{i-1,i} = \begin{bmatrix} \mathcal{M}_{i-1,i} \\ N_{i-1,i} \end{bmatrix}, \quad G_{m_i} = \begin{bmatrix} 0 \\ m_i g \end{bmatrix} \tag{38}$$

Then we achieve the torque τ_i and $\tau_{f_i}^u$ by (32)

$$\tau_i = \xi_{b_i}^T \left(\mathcal{M}_{i-1,i} - \widehat{r}_{ci,i} N_{i-1,i} \right) - \tau_{f_i}^{\xi} \tag{39}$$
$$\tau_{f_i}^u = u_{b_i}^T \left(\mathcal{M}_{i-1,i} - \widehat{r}_{ci,i} N_{i-1,i} \right) \tag{40}$$

Particularly, when $i = 0$, we have the dynamics equation for the manipulator's base as follows

$$\mathcal{G}_0 \dot{V}_{b_0} - \mathrm{ad}_{V_{b_0}}^T \mathcal{G}_0 V_{b_0} = \mathcal{F}_{f,0} - Ad_{g_{0,1}^{-1}}^T \mathcal{F}_{0,1} + Ad_{g_0}^{-1} G_{m_0} \tag{41}$$

where $\mathcal{F}_{f,0}$ denotes the wrench given from the vehicle to the base.

2.2 Closed-Form Dynamics Equations

Because of the lack of explicitly description for input-output relationship, the Newton-Euler equations we have derived are not in an appropriate form for use in dynamic analysis and control design. In this section, we modify the Newton-Euler equations so that the input joint torques can be separated from the constraint forces and moments.

The appropriate form of the dynamic equations therefore consists of equations described in terms of all independent position variables and input forces, i.e., joint torques, that are explicitly involved in the dynamic equations. Dynamic equations in such an explicit input-output form are referred to as closed-form dynamic equations.

We start by defining the following stacked vectors:

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix} \in \mathbb{R}^n \tag{42}$$

$$V_b = \begin{bmatrix} V_{b_1} \\ \vdots \\ V_{b_n} \end{bmatrix} \in \mathbb{R}^{6n} \tag{43}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} \in \mathbb{R}^n \tag{44}$$

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_{0,1} \\ \vdots \\ \mathcal{F}_{n-1,n} \end{bmatrix} \in \mathbb{R}^{6n} \tag{45}$$

Further define the following matrices:

$$\mathcal{A}_b = \begin{bmatrix} \mathcal{A}_{b_1} & 0 & \cdots & 0 \\ 0 & \mathcal{A}_{b_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_{b_n} \end{bmatrix} \in \mathbb{R}^{6n \times n} \tag{46}$$

$$S(\theta) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ Ad^{-1}_{g_{1,2}} & 0 & \cdots & 0 & 0 \\ 0 & Ad^{-1}_{g_{2,3}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & Ad^{-1}_{g_{n-1,n}} & 0 \end{bmatrix} \in \mathbb{R}^{6n \times 6n} \tag{47}$$

$$Ad^{-1}_g = \begin{bmatrix} Ad^{-1}_{g_1} & 0 & \cdots & 0 \\ 0 & Ad^{-1}_{g_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Ad^{-1}_{g_n} \end{bmatrix} \in \mathbb{R}^{6n \times 6n} \tag{48}$$

$$\tag{49}$$

$$\mathrm{ad}_{V_b} = \begin{bmatrix} \mathrm{ad}_{V_{b_1}} & 0 & \cdots & 0 \\ 0 & \mathrm{ad}_{V_{b_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{ad}_{V_{b_n}} \end{bmatrix} \in \mathbb{R}^{6n\times 6n} \tag{50}$$

$$\mathrm{ad}_{\mathcal{A}_b\dot{\theta}} = \begin{bmatrix} \mathrm{ad}_{\mathcal{A}_{b_1}\dot{\theta}_1} & 0 & \cdots & 0 \\ 0 & \mathrm{ad}_{\mathcal{A}_{b_2}\dot{\theta}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{ad}_{\mathcal{A}_{b_n}\dot{\theta}_n} \end{bmatrix} \in \mathbb{R}^{6n\times 6n} \tag{51}$$

$$K_1 = \begin{bmatrix} \begin{bmatrix} I_3 & 0 \end{bmatrix} & 0 & \cdots & 0 \\ 0 & \begin{bmatrix} I_3 & 0 \end{bmatrix} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{bmatrix} I_3 & 0 \end{bmatrix} \end{bmatrix} \in \mathbb{R}^{3n\times 6n} \tag{52}$$

$$K_2 = \begin{bmatrix} \begin{bmatrix} 0 & I_3 \end{bmatrix} & 0 & \cdots & 0 \\ 0 & \begin{bmatrix} 0 & I_3 \end{bmatrix} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{bmatrix} 0 & I_3 \end{bmatrix} \end{bmatrix} \in \mathbb{R}^{3n\times 6n} \tag{53}$$

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{G}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{G}_n \end{bmatrix} \in \mathbb{R}^{6n\times 6n} \tag{54}$$

$$\widehat{r}_{cj} = \begin{bmatrix} \widehat{r}_{c1,1} & 0 & \cdots & 0 \\ 0 & \widehat{r}_{c2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widehat{r}_{cn,n} \end{bmatrix} \in \mathbb{R}^{3n\times 3n} \tag{55}$$

$$\tag{56}$$

Finally, define the following vectors:

$$V_{base} = \begin{bmatrix} Ad_{g_{0,1}}^{-1} V_{b_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n} \tag{57}$$

$$\dot{V}_{base} = \begin{bmatrix} Ad_{g_{0,1}}^{-1} \dot{V}_{b_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n} \tag{58}$$

$$\mathcal{F}_{tip} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ Ad^T_{g^{-1}_{n,n+1}} \mathcal{F}_{n,n+1} \end{bmatrix} \in \mathbb{R}^{6n} \tag{59}$$

$$G_m = \begin{bmatrix} G_{m_1} \\ \vdots \\ G_{m_n} \end{bmatrix} \in \mathbb{R}^{6n} \tag{60}$$

With the above definitions, our earlier recursive inverse dynamics equations can be assembled into the following set of matrix equations:

$$\begin{aligned}
V_b &= S(\theta)V_b + \mathcal{A}_b\dot{\theta} + V_{base} && (61)\\
\dot{V}_b &= -\mathrm{ad}_{\mathcal{A}_b\dot{\theta}}\left(S(\theta)V_b + V_{base}\right) + S(\theta)\dot{V}_b + \mathcal{A}_b\ddot{\theta} + \dot{V}_{base} && (62)\\
\mathcal{F} &= \mathcal{G}\dot{V}_b - \mathrm{ad}_{V_b}^T\mathcal{G}V_b + S^T(\theta)\mathcal{F} - Ad_g^{-1}G_m + \mathcal{F}_{tip} && (63)\\
\tau &= \left(K_1\mathcal{A}_b\right)^T\left(K_1\mathcal{F} - \widehat{r}_{cj}K_2\mathcal{F}\right) - \tau_f^{\xi} && (64)\\
\tau_f^u &= \left(K_2\mathcal{A}_b\right)^T\left(K_1\mathcal{F} - \widehat{r}_{cj}K_2\mathcal{F}\right) && (65)
\end{aligned}$$

$S(\theta)$ has the property that $S^T(\theta) = 0$ (such a matrix is said to be nilpotent of order n), and one consequence verifiable through direct calculation is that $(I-S(\theta))^{-1} = I+S(\theta)+\cdots+S^{n-1}(\theta)$. Defining $\mathcal{L}(\theta) = (I - S(\theta))^{-1}$, it can further calculated as

$$\mathcal{L}(\theta) = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ Ad_{g_{1,2}}^{-1} & I & 0 & \cdots & 0 \\ Ad_{g_{1,3}}^{-1} & Ad_{g_{2,3}}^{-1} & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Ad_{g_{1,n}}^{-1} & Ad_{g_{2,n}}^{-1} & Ad_{g_{3,n}}^{-1} & \cdots & I \end{bmatrix} \in \mathbb{R}^{6n\times 6n} \tag{66}$$

The matrix equations above can now be reorganized as

$$\begin{aligned}
V_b &= \mathcal{L}(\theta)\left(\mathcal{A}_b\dot{\theta} + V_{base}\right) && (67)\\
\dot{V}_b &= \mathcal{L}(\theta)\left(-\mathrm{ad}_{\mathcal{A}_b\dot{\theta}}\left(S(\theta)V_b + V_{base}\right) + \mathcal{A}_b\ddot{\theta} + \dot{V}_{base}\right) && (68)\\
\mathcal{F} &= \mathcal{L}^T(\theta)\left(\mathcal{G}\dot{V}_b - \mathrm{ad}_{V_b}^T\mathcal{G}V_b - Ad_g^{-1}G_m + \mathcal{F}_{tip}\right) && (69)\\
\tau &= \left(K_1\mathcal{A}_b\right)^T\left(K_1\mathcal{F} - \widehat{r}_{cj}K_2\mathcal{F}\right) - \tau_f^{\xi} && (70)\\
\tau_f^u &= \left(K_2\mathcal{A}_b\right)^T\left(K_1\mathcal{F} - \widehat{r}_{cj}K_2\mathcal{F}\right) && (71)
\end{aligned}$$

The Lagrangian dynamics formulation of manipulator is written as the form

$$\tau = M(\theta)\ddot{\theta} + c(\theta,\dot{\theta}) + g(\theta) + \Phi(\theta)\mathcal{F}_{tip} - \tau_f^{\xi} \tag{72}$$

With the matrix equations above, we have

$$\begin{aligned}
M(\theta) &= (K_1\mathcal{A}_b)^T(K_1 - \widehat{r}_{cj}K_2)\mathcal{L}^T(\theta)\mathcal{G}\mathcal{L}(\theta)\mathcal{A}_b && (73a)\\
c(\theta,\dot{\theta}) &= (K_1\mathcal{A}_b)^T(K_1 - \widehat{r}_{cj}K_2)\mathcal{L}^T(\theta)\left(\mathcal{G}\mathcal{L}(\theta)\left(\dot{V}_{base} - \mathrm{ad}_{\mathcal{A}\dot{\theta}}V_{base}\right)\right.\\
&\quad \left. - \left(\mathcal{G}\mathcal{L}(\theta)\mathrm{ad}_{\mathcal{A}\dot{\theta}}S(\theta) + \mathrm{ad}_{V_b}^T\mathcal{G}\right)V_b\right) && (73b)\\
g(\theta) &= -(K_1\mathcal{A}_b)^T(K_1 - \widehat{r}_{cj}K_2)\mathcal{L}^T(\theta)Ad_g^{-1}G_m && (73c)\\
\Phi(\theta) &= (K_1\mathcal{A}_b)^T(K_1 - \widehat{r}_{cj}K_2)\mathcal{L}^T(\theta) && (73d)
\end{aligned}$$

2.3 Forward Dynamics

The inverse dynamics problem is to solve

$$\ddot{\theta} = \Gamma(\theta,\dot{\theta},\tau,\mathcal{F}_{n+1_n}) \tag{74}$$

where Γ denotes the forward-dynamics function.

By (72) we have

$$\ddot{\theta} = M^{-1}(\theta)\left(\tau + \tau_f^{\xi} - c(\theta,\dot{\theta}) - g(\theta) - \Phi(\theta)\mathcal{F}_{tip}\right) \tag{75}$$

3 Position Control for Three-Axis Aerial Manipulator

The Position controller is to generate a control action (e.g. joint torque) to bring the tool frame to the same value as the desired configuration, according to the feedback variable of the deviation between actual and desired configuration.

Because a three-axis aerial manipulator only has three rotational joints with total degree of freedom and we only care about the orientation of the tool frame, the given desired configuration of tool is often a orientation R_d.

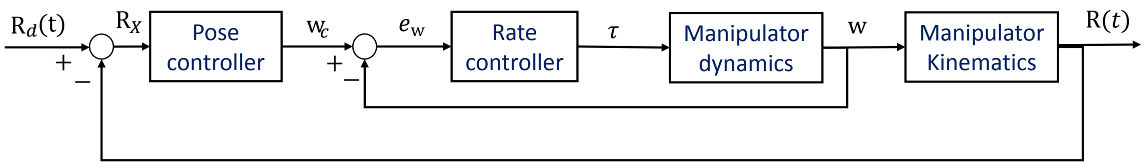

Figure 3: Cascaded attitude control closed-loop

3.1 Pose Controller

3.1.1 Pose Controller in Spacial Frame

Being the feedback variable of the controller, the orientation error R_e between actual and desired orientation is defined as

$$
\begin{aligned}
R_e &\triangleq R_d R_t^T &(76)\\
w_d &= \dot{R}_d R_d^T &(77)
\end{aligned}
$$

We further define the tracking error e_R and the error angular velocity ω_e in the homogeneous coordinates as follows

$$
\begin{aligned}
\widehat{e}_R &\triangleq \log R_e &(78)\\
\widehat{\omega}_e &\triangleq \dot{R}_e R_e^T &(79)
\end{aligned}
$$

Substituting (76) into (79), we have

$$\widehat{\omega}_e = \widehat{\omega}_d - R_e \widehat{\omega}_{s_t} R_e^T \tag{80}$$

therefore

$$\omega_e = \omega_d - R_e \omega_{s_t} \tag{81}$$

Theorem 3.1 Let $R(t)$ be a smooth curve on $SO(3)$, $\widehat{\psi}(t) = \log(R(t)) \in so(3)$ be the exponential coordinate of $R(t)$, $\widehat{\omega}_b = R^T \dot{R}$ the body velocity and $\widehat{\omega}_s = \dot{R}R^T$ the spatial velocity, then

$$
\begin{aligned}
\widehat{\omega}_b &= \int_0^1 Ad_{e^{-\lambda\widehat{\psi}}} \frac{d\widehat{\psi}}{dt} d\lambda &(82)\\
\widehat{\omega}_s &= \int_0^1 Ad_{e^{\lambda\widehat{\psi}}} \frac{d\widehat{\psi}}{dt} d\lambda &(83)
\end{aligned}
$$

With **Theorem 3.1**, the error angular velocity ω_e is given by

$$\omega_e = \int_0^1 e^{\lambda \widehat{e}_R} \dot{e}_R d\lambda \tag{84}$$

then

$$
\begin{aligned}
\dot{e}_R &= \left(\int_0^1 e^{\lambda\widehat{e}_R} d\lambda\right)^{-1} \omega_e \\
&= \left(\int_0^1 e^{\lambda\widehat{e}_R} d\lambda\right)^{-1} (\omega_d - R_e\omega_{s_t}) \qquad (85)
\end{aligned}
$$

Control the pose using synthetic control, the angular velocity ω_{s_t}, by assuming the actual velocity ω_{s_t} is instantaneously equal to the commanded value ω_c. We apply the pose control law to the system that renders the tracking error e_R asymptotically converging to zero

$$
\omega_c = R_e^T(\omega_d + Ke_R) \qquad (86)
$$

where the gain matrix K is positive definite.

Proof: Consider the natural candidate Lyapunov function

$$
V = \frac{1}{2} e_R^T e_R \qquad (87)
$$

The function V is globally positive definite. Evaluating $\dot{V}$ along the trajectories of

$$
\begin{aligned}
\dot{V} &= \frac{1}{2}(\dot{e}_R^T e_R + e_R^T \dot{e}_R) \\
&= e_R^T \dot{e}_R \\
&= -e_R^T \left(\int_0^1 e^{\lambda\widehat{e}_R} d\lambda\right)^{-1} Ke_R \\
&= -e_R^T e_R e_R^{-1} \left(\int_0^1 e^{\lambda\widehat{e}_R} d\lambda\right)^{-1} Ke_R \\
&= -e_R^T e_R \left(\int_0^1 e^{\lambda\widehat{e}_R} d\lambda e_R\right)^{-1} Ke_R \\
&= -e_R^T e_R \left(\int_0^1 e^{\lambda\widehat{e}_R} e_R d\lambda\right)^{-1} Ke_R \\
&= -e_R^T e_R \left(\int_0^1 \left(\sum_{n=0}^{\infty} \frac{(\lambda\widehat{e}_R)^n}{n!} e_R\right) d\lambda\right)^{-1} Ke_R \\
&= -e_R^T e_R \left(\int_0^1 e_R d\lambda\right)^{-1} Ke_R \\
&= -e_R^T e_R e_R^{-1} Ke_R \\
&= -e_R^T Ke_R \qquad (88)
\end{aligned}
$$

Because K is positive definite, $\dot{V}$ is negative. Therefore, this control law ensures the convergence of the position.

3.1.2 Pose Controller in Tool Frame

Being the feedback variable of the controller, the orientation error R_e between actual and desired orientation is defined as

$$
\begin{aligned}
R_e &\triangleq R_t^T R_d \qquad (89) \\
w_d &= R_d^T \dot{R}_d \qquad (90)
\end{aligned}
$$

We further define the tracking error e_R and the error angular velocity ω_e in the homogeneous coordinates as follows

$$\begin{aligned}\widehat{e}_R &\triangleq \log R_e &(91)\\ \widehat{\omega}_e &\triangleq R_e^T \dot{R}_e &(92)\end{aligned}$$

Substituting (89) into (92), we have

$$\widehat{\omega}_e = \widehat{\omega}_d - R_e^T \widehat{\omega}_{b_t} R_e \tag{93}$$

therefore

$$\omega_e = \omega_d - R_e^T \omega_{b_t} \tag{94}$$

The error angular velocity ω_e is given by

$$\omega_e = \int_0^1 e^{-\lambda \widehat{e}_R} \dot{e}_R d\lambda \tag{95}$$

then

$$\begin{aligned}\dot{e}_R &= \left(\int_0^1 e^{-\lambda \widehat{e}_R} d\lambda\right)^{-1} \omega_e \\ &= \left(\int_0^1 e^{-\lambda \widehat{e}_R} d\lambda\right)^{-1} \left(\omega_d - R_e^T \omega_{b_t}\right) &(96)\end{aligned}$$

Assume that the actual velocity ω_{b_t} is instantaneously equal to the commanded value ω_c. We apply the pose control law to the system that renders the tracking error e_R asymptotically converging to zero

$$\omega_c = R_e(\omega_d + K e_R) \tag{97}$$

where $K > 0$

Proof. Consider the natural candidate Lyapunov function

$$V = \frac{1}{2} e_R^T e_R \tag{98}$$

The function V is globally positive definite. Evaluating $\dot{V}$ along the trajectories of

$$\begin{aligned}\dot{V} &= \frac{1}{2}(\dot{e}_R^T e_R + e_R^T \dot{e}_R) \\ &= e_R^T \dot{e}_R \\ &= -e_R^T \left(\int_0^1 e^{-\lambda \widehat{e}_R} d\lambda\right)^{-1} K e_R \\ &= -e_R^T e_R e_R^{-1} \left(\int_0^1 e^{-\lambda \widehat{e}_R} d\lambda\right)^{-1} K e_R \\ &= -e_R^T e_R \left(\int_0^1 e^{-\lambda \widehat{e}_R} d\lambda e_R\right)^{-1} K e_R \\ &= -e_R^T e_R \left(\int_0^1 e^{-\lambda \widehat{e}_R} e_R d\lambda\right)^{-1} K e_R \\ &= -e_R^T e_R \left(\int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-\lambda \widehat{e}_R)^n}{n!} e_R\right) d\lambda\right)^{-1} K e_R \\ &= -e_R^T e_R \left(\int_0^1 e_R d\lambda\right)^{-1} K e_R \\ &= -e_R^T e_R e_R^{-1} K e_R \\ &= -e_R^T K e_R &(99)\end{aligned}$$

Because $K > 0$, $\dot{V}$ is negative. Therefore, this control law ensures the convergence of the position.

3.2 Rate Controller

Reminding the kinematics on SO(3) that

$$\begin{aligned} R_t &= R_0 R_{0t}(0) e^{\hat{\xi}_{t_1}\theta_1} e^{\hat{\xi}_{t_2}\theta_2} e^{\hat{\xi}_{t_3}\theta_3} && (100) \\ \widehat{\omega}_{b_t} &= R_t^T \dot{R}_t && (101) \end{aligned}$$

then we have

$$\begin{aligned} \omega_{b_t} &= J_\omega \dot{\theta} + Ad^{-1}_{R_{0t}(0) e^{\hat{\xi}_{t_1}\theta_1} e^{\hat{\xi}_{t_2}\theta_2} e^{\hat{\xi}_{t_3}\theta_3}} \omega_{b_0} \\ &= J_\omega \dot{\theta} + Ad^{-1}_{R_{0t}} \omega_{b_0} && (102) \end{aligned}$$

where the Jacobian matrix in tool frame for the angular velocity is given as

$$J_{\omega_t} = \begin{bmatrix} Ad^{-1}_{e^{\hat{\xi}_{t_2}\theta_2} e^{\hat{\xi}_{t_3}\theta_3}} \xi_{t_1} & Ad^{-1}_{e^{\hat{\xi}_{t_3}\theta_3}} \xi_{t_2} & \xi_{t_3} \end{bmatrix} \quad (103)$$

Further,

$$\dot{\omega}_{b_t} = \dot{J}_{\omega_t} \dot{\theta} + J_{\omega_t} \ddot{\theta} + \frac{d}{dt}\left(Ad^{-1}_{R_{0t}} \omega_{b_0} \right) \quad (104)$$

where

$$\dot{J}_{\omega_t} = \left[-\left(\widehat{\xi}_{t_3} \dot{\theta}_3 e^{-\widehat{\xi}_{t_3}\theta_3} e^{-\widehat{\xi}_{t_2}\theta_2} + e^{-\widehat{\xi}_{t_3}\theta_3} e^{-\widehat{\xi}_{t_2}\theta_2} \widehat{\xi}_{t_2} \dot{\theta}_2 \right) \xi_{t_1} \quad -e^{-\widehat{\xi}_{t_3}\theta_3} \widehat{\xi}_{t_3} \dot{\theta}_3 \xi_{t_2} \quad 0 \right] \quad (105)$$

$$\begin{aligned} \frac{d}{dt}\left(Ad^{-1}_{R_{0t}} \omega_{b_0} \right) &= -\left(\widehat{\xi}_{t_3} \dot{\theta}_3 e^{-\widehat{\xi}_{t_3}\theta_3} e^{-\widehat{\xi}_{t_2}\theta_2} e^{-\widehat{\xi}_{t_1}\theta_1} + e^{-\widehat{\xi}_{t_3}\theta_3} e^{-\widehat{\xi}_{t_2}\theta_2} \widehat{\xi}_{t_2} \dot{\theta}_2 e^{-\widehat{\xi}_{t_1}\theta_1} \right. \\ &\quad \left. + e^{-\widehat{\xi}_{t_3}\theta_3} e^{-\widehat{\xi}_{t_2}\theta_2} e^{-\widehat{\xi}_{t_1}\theta_1} \widehat{\xi}_{t_1} \dot{\theta}_1 \right) R_{0t}^{-1}(0) \omega_{b0} + Ad^{-1}_{R_{0t}} \dot{\omega}_{b_0} && (106) \end{aligned}$$

Because the gimbal system only involves position control, we do not consider the force on the tip of the gimbal system. Now the dynamics formulation in joint space (72) can be written in workspace as

$$\begin{aligned} \tau &= M(\theta) J_\omega^{-1} \left(\dot{\omega}_{b_t} - \dot{J}_\omega \dot{\theta} - \frac{d}{dt}\left(Ad^{-1}_{R_{0t}} \omega_{b_0} \right) \right) + c(\theta, \dot{\theta}) + g(\theta) - \tau_f^\xi \\ \tau &= \tilde{M}(\theta) \dot{\omega}_{b_t} + \tilde{h}(\theta, \dot{\theta}) - \tau_f^\xi && (107) \end{aligned}$$

where

$$\begin{aligned} \tilde{M}(\theta) &= M(\theta) J_\omega^{-1} && (108) \\ \tilde{h}(\theta, \dot{\theta}) &= c(\theta, \dot{\theta}) - M(\theta) J_\omega^{-1} \left(\dot{J}_\omega \dot{\theta} + \frac{d}{dt}\left(Ad^{-1}_{R_{0t}} \omega_{b_0} \right) \right) + g(\theta) && (109) \end{aligned}$$

Define the angular velocity error e_ω as

$$e_\omega \triangleq \omega_c - \omega_t \quad (110)$$

Since we assume that the commanded angular velocity ω_c is constant, we have the angular velocity error dynamics as

$$\begin{aligned} \dot{e}_\omega &= -\dot{\omega}_t \\ &= -\left(\tilde{M}(\theta) \right)^{-1} \left(\tau + \tau_f^\xi - \tilde{h}(\theta, \dot{\theta}) \right) && (111) \end{aligned}$$

The linearity of (111) suggests the following controlling law:

$$\tau = \tilde{h}(\theta, \dot{\theta}) + \tilde{M}(\theta) \left(K_p e_\omega + K_i \int e_\omega dt \right) \quad (112)$$

where K_p and K_i are constant and positive gain diagonal matrices.

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