

## NOTE OF WEEK 11

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ABSTRACT. In week 11, we proved a theorem relating the periodic orbit and values of products of linear forms.

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### 1. THE MAIN THEOREM

The main theorem we proved in this week is the following one

**Theorem 1.1.** *The orbit  $A\Lambda \subset SL_d(\mathbb{R})/SL_d(\mathbb{Z})$  is compact (periodic), if and only if  $N(\Lambda) > 0$  and the set  $\{N(v)|v \in \Lambda\}$  is discrete.*

**Remark 1.2.** Here  $N(v) = |x_1 \cdots x_d|$ , where  $v = (x_1, \dots, x_d)^t$ .

For one direction, the result follows from the classification of periodic orbits we proved in last few weeks. Those period orbits gives forms of degree  $d$  with integer coefficients (or multiplying a common constant), thus the set  $\{N(v)|v \in \Lambda\}$  is discrete. The result  $N(\Lambda) > 0$  follows from another lemma proved before, which is a variant of Mahler principle, saying that  $A\Lambda$  has compact closure if and only if  $N(\Lambda) > 0$ .

For another direction, the condition  $N(\Lambda) > 0$  already implies that the orbit has compact closure, thus it remains to show that it is a close orbit.

*Proof.* As is explained in the remark, it suffices to show that  $A\Lambda$  is closed to finish the proof. Denote  $G = SL_d(\mathbb{R}), \Gamma = SL_d(\mathbb{Z})$ . Since the topology on  $G/\Gamma$  is the quotient topology, it suffices to show the corresponding set  $Ag\Gamma$  is closed in  $G$ , where  $\Lambda = g\mathbb{Z}^n$ . This is equivalent to showing that  $Ag\Gamma$  is closed in  $A \backslash G$  under quotient topology. Using the transposition of Iwasawa decomposition, we know that  $G$  can be decomposed into  $G = ANK$ , here  $K = SO(d)$ , and  $N$  the lower triangular matrices. Thus  $A \backslash G$  has a nice description  $NK$ .

In order to use the condition  $\{N(v)|v \in \Lambda\}$  is discrete. We consider the map  $A \backslash \rightarrow P_d[x_1, \dots, x_d]$ , where  $P_d[x_1, \dots, x_d]$  denote polynomials of  $d$  variables with degree  $d$ . The mapping is given by  $g \mapsto \phi \circ g$ ,  $g$  acts on  $x_1, \dots, x_d$  by linear transformation, and  $\phi = x_1 \cdots x_d$ . The 'kernel' (inverse image of  $\phi$ ) of this map is not large, because the stabilizer of  $\phi$  under  $G$  is not large.

If  $\phi = \phi \circ g$ , where

$$\phi \circ g = (g_{1,1}x_1 + \cdots + g_{1,d}x_d)(g_{2,1}x_1 + \cdots + g_{2,d}x_d) \cdots (g_{d,1}x_1 + \cdots + g_{d,d}x_d)$$

Since  $P_d$  is a unique factorization domain, each factor in  $\phi \circ g = \phi$  should also look like  $c_i x_i$ , which means that  $g$  only has one nonzero element in each column and row. (This can be also seen from the fact that zero set of  $\phi \circ g$  is the union of  $d$  hyperplane  $\{x_i = 0\}$ , thus the zero set of each factor is one of the  $\{x_i = 0\}$ , also implying that  $g_{i,1}x_1 + \cdots + g_{i,d}x_d = c_j x_j$  for some  $j$ ).

Thus the stabilizer consists of permutation matrices, diagonal matrices and their products, and this implies that the inverse image of each point in  $A \setminus G$  is a finite set of cardinal at most  $2^d d!$ .

The map  $A \setminus G \rightarrow P_d$  is also a proper mapping. To show the map is proper, it suffices to show that any convergent sequence in the image set gives a convergent subsequence in  $A \setminus G$ . We know  $A \setminus G$  can be described as  $NK$ , thus it suffices to show if  $\phi \circ n_i k_i$  is convergent, then  $n_i k_i$  has a convergent subsequence. Since  $K$  is compact, passing to a subsequence, we only need to show, for convergent  $\phi \circ n_i$ ,  $n_i$  also converges.

If we can recover the entries of  $n_i$  from coefficients of  $\phi \circ n_i$ , then the result follows. This can be shown by induction on level of entries. For  $n = (n_{i,j})$  lower triangular,  $i - j$  is called the level. If the level is 1, then the coefficient of  $x_1 \cdots x_i x_i x_{i+2} \cdots x_d$  is exactly  $n_{i+1,i}$ , so the result is true for level 1 entries. Similarly, consider coefficients of  $x_1 \cdots x_i x_{i-1} x_{i+2} \cdots x_d$ , they are  $n_{i+1,i-1} + n_{i,i-1} n_{i+1,i}$ . Since entries of level 1 can be recovered, from this we know  $n_{i+1,i-1}$  can also be recovered from coefficients of  $\phi \circ n$ . Continuing on induction, we can show similarly that entries of higher level can also be recovered. Thus proving the map  $A \setminus G \rightarrow P_d$  is proper.

Then we want to encode the information of polynomials in  $P_d$  by its values on finitely many integer points, and the information about  $\{N(v) | v \in \Lambda\}$  can be used under this setting. More precisely, we want to find a finite set  $E \subset \mathbb{Z}^d$ , such that  $\forall f \in P_d$ , it is determined by its value on  $E$ . To find such a set  $E$ , we argue as follows. Fix a basis  $f_1, \dots, f_m$  of the vector space  $P_d$ , then choose a maximal subset  $E$  of  $\mathbb{Z}^d$ , such that  $(f_1(v), \dots, f_m(v)), v \in E$  is linearly independent.

We claim that  $E$  has exactly  $m$  elements (It has at most  $m$  elements). Suppose the contrary that it only has  $r < m$  elements, then by the choice of  $E$ , we know for every  $w \in \mathbb{Z}^d$ ,  $(f_1(w), \dots, f_m(w))$  can be written as linear combination of  $(f_1(v), \dots, f_m(v)), v \in E$ . With out loss of generality, we assume the matrix  $(f_i(v_j)), 1 \leq i \leq r, v_j \in E$  is invertible. But this implies that values of  $f_j, r + 1 \leq j \leq m$  on  $\mathbb{Z}^d$  is the same as some linear combination of  $f_i, 1 \leq i \leq r$  (the coefficient is determined by the linear dependence in the matrix  $(f_i(v_j))_{1 \leq i \leq m, 1 \leq j \leq r}$ ), implying that  $f_i, 1 \leq i \leq r$  gives a basis, thus we can only have  $r = m$ .

Then every polynomial  $f$  in  $P_d$  is determined by its value on  $E$ , since we can solve the coefficient of  $f$  under  $f_i$  by solving the full rank linear equation  $\sum a_i f_i(v) = f(v), v \in E$ , where the coefficient matrix is a  $m \times m$  invertible matrix.

Now we are ready to show that  $Ag\Gamma$  is closed in  $A \setminus G$ . It suffices to show if  $Ag\gamma_i$  converges in  $A \setminus G$ , then the limit is still in  $Ag\Gamma$ . It suffices to show  $\phi \circ g\gamma_i$  has limit in the image set, because the preimage of every point in  $P_d$  is discrete, this is enough to show  $Ag\Gamma$  is closed. Since  $f \in P_d$  is determined on its value on  $E$ , and  $\{|\phi \circ g\gamma_i(v)| | i \in \mathbb{Z}_{>0}, v \in E\} \subset \{N(v) | v \in \Lambda\}$  lies in this discrete set,  $\phi \circ g\gamma_i(v)$  eventually stabilizes after finite many terms. Since there are only finitely many

elements in  $E$ , after finitely many terms,  $\phi \circ g\gamma_i$  takes the same value on  $E$ , thus  $\phi \circ g\gamma_i$  eventually stabilizes. This proves that the image of  $Ag\Gamma$  in  $P_d$  is closed, also proves that  $Ag\Gamma$  is also closed, thus proving the theorem.  $\square$

## 2. THEOREM OF CASSELS AND SWINNERTON-DYER

In [4], Cassels and Swinnerton-Dyer proved the following theorem.

**Theorem 2.1.** *For  $g \in SL_3(\mathbb{R})$ ,  $Ag\Gamma$  a compact orbit in  $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$ , then for all  $(\delta_1, \delta_2) \subset \mathbb{R}$ , there exists a neighborhood  $U$  of  $e$  in  $SL_3(\mathbb{R})$ , such that for all  $h \in Ug \setminus Ag$ ,  $\phi \circ h(\mathbb{Z}^3 \setminus \{0\}) \cap (\delta_1, \delta_2) \neq \emptyset$ .*

We consider the following observation, for  $U_t$  the one parameter subgroup whose Lie algebra is  $E_{i,j}$ , for example,  $E_{1,2}$ , to be a root vector. Denote  $u_t$  to be

$$(2.2) \quad u_t = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then fix  $\Lambda$  be a lattice given by a full module  $I$  of a totally real cubic field  $K$ , i.e.  $\Lambda = \sigma(I)$ , where  $\sigma$  is the Minkowski embedding. Now we consider the vector  $x = (\sigma_1(a), \sigma_2(a), \sigma_3(a))^t$  in  $\Lambda$ , where  $a \in I$ . It gives an order  $\mathcal{O}$  isomorphic to  $\mathbb{Z}^3$ , which will give a  $\mathcal{O}^\times \cong \mathbb{Z}^2$  lies in the ring of integers.

Then we consider the image of  $u_t\sigma(sa)$ ,  $a \in \mathcal{O}^\times$  under the norm.

$$\begin{aligned} N(u_t\sigma(sa)) &= |(\sigma_1(sa) + t\sigma_2(sa))\sigma_2(sa)\sigma_3(sa)| \\ &= |(\sigma_1(a) + t\frac{\sigma_2(s)}{\sigma_1(s)}\sigma_2(a))\sigma_2(a)\sigma_3(a)| \end{aligned}$$

The second equation is due to the fact that  $|\sigma_1(s)\sigma_2(s)\sigma_3(s)| = 1$  for elements in the unit group. This gives a map from  $\mathcal{O}^\times \cong \mathbb{Z}^2$  to  $\mathbb{R}^+$ . Roughly, this means the image in  $\mathbb{R}_+$  is dense, since it maps  $\mathbb{Z}^2$  into  $\mathbb{R}_+$ . This argument shows that for any small perturbation given by a unipotent element in  $U_t$ , we get another lattice with dense image under taking norm. By the decomposition  $KNA$ , this might tell how to prove the theorem.

## 3. BIBLIOGRAPHY

### REFERENCES

- [1] J. Neukrich. Algebraic number theory. Springer-Verlag Berlin Heidelberg. 1999.
- [2] J. Milne. Algebraic number theory. <https://www.jmilne.org/math/CourseNotes/ANT.pdf> Version 3.08. 2020.
- [3] C. McMullen. Ergodic theory, geometry and dynamics. <https://people.math.harvard.edu/~ctm/papers/home/text/class/notes/erg/> 2020.
- [4] J. W. S. Cassels and H. P. F. Swinnerton-Dyer. On the product of three homogeneous linear forms and the indefinite ternary quadratic forms. Philos. Trans. Roy. Soc. London. Ser. A. 248 (1955), 73–96.
- [5] Anthony W. Knap. Lie Groups Beyond an Introduction. Second Edition. Birkhauser. 2002.