NOTE OF WEEK 13: RATNER'S MEASURE CLASSIFICATION THEOREM

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Abstract. In week 13, Ratner's theorem of measure classification is introduced. On the class, we consider the classification of ergodic measure of action of $SL(2, \mathbb{R})$. The proof of the key technical lemma is given in week 14, for completeness of the proof, it is included too.

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1. Introduction

We first state the main theorem that we gonna prove, the measure classification theorem proved by Ratner.

Theorem 1.1 (Ratner)**.** *For G a lattice, Γ is a discrete subgroup of G. For a H a subgroup of G generated by some one parameter Ad-unipotent subgroups. Then any H-invariant ergodic probability measure μ on $X = G/\Gamma$ is homogeneous.*

Remark 1.2. Here $g \in G$ is called Ad-unipotent if $Ad(g)$ is a unipotent matrix.

Notice here Γ is not necessarily a lattice, but we require the measure μ to be a probability measure.

In this note, we will prove a slightly weaker form of this theorem. We assume $H \cong SL(2, \mathbb{R})$ in this note. However, the argument given here shows the theorem for H semisimple without compact factor with little modification. From now on, we denote $H = SL(2, \mathbb{R})$, it is generated by two unipotent subgroup, the upper triangular group and lower triangular group.

The reason why assuming $H = SL(2, \mathbb{R})$, or semisimple, make things easy is that adjoint representation of semisimple group is completely reducible, since adjoint representation has finite dimension and all finite dimensional representations of semisimple groups are completely reducible.

We now sketch the outline of the proof, and clarify what we exactly gonna prove.

Although μ is invariant under H, it is possible that it is invariant under a larger group, we denote this group as S

$$S = \{g \in G | g_* \mu = \mu\},$$

and it is clear S forms a group. It is also closed by regularity of measure μ. This is because μ is a Borel measure on locally compact Hausdorff metric space G/Γ. Thus if $g_n \to g$, then $\mu(g_n^{-1}B) \to \mu(g^{-1}B)$, which implies the closeness of S.

Using results from Lie group theory, S is a Lie subgroup of G, which means S is also a Lie group, and has a compatible embedding into G.

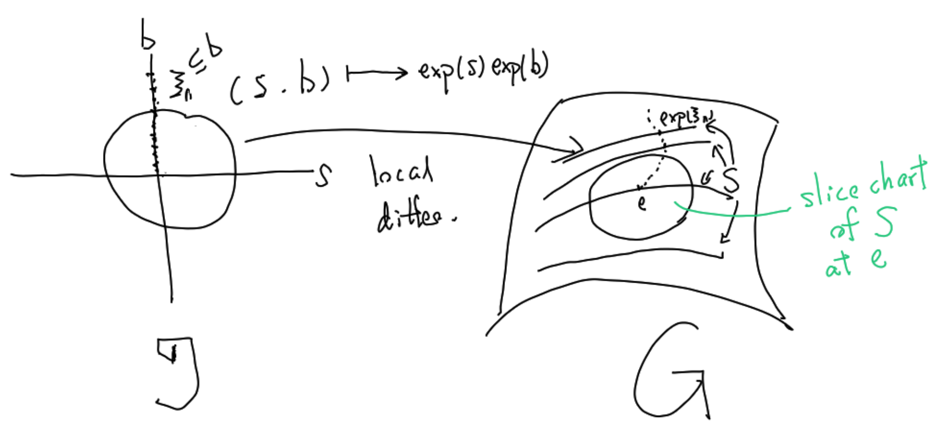
Since S contains all invariance of the measure μ, it is natural to guess that this group will give the homogeneous measure. More precisely, if $\mu(Sx) > 0$ for some $x \in G/\Gamma$, then Sx is also H-invariant. By ergodicity of μ under H, this implies that $\mu(Sx) = 1$. Then μ is supported on this homogeneous set Sx and also invariant under S, which implies that μ is a homogeneous measure and satisfies the theorem.

Otherwise, we assume $\mu(Sx) = 0$, and we gonna find other invariance not contained in S, contradicting the choice of S, *i.e.* find some $g \notin S$ with $g_* \mu = \mu$.

Considering the adjoint action of H on Lie algebra $\mathfrak{g}$. This representation has a invariant subspace, which is the Lie algebra of S, denoted as $\mathfrak{s}$. Using the representation is completely reducible, we can write $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{b}$, with $\mathfrak{b}$ also invariant under adjoint action of H. If we can find a sequence of vectors $\xi_n \to 0$ in $\mathfrak{b}$, such that for each n, $\exp(\xi_n)_* \mu = \mu$, then this give the new invariance leading contradiction. Because locally

$$\varphi(s, b) := \exp(s) \exp(b)$$

gives a diffeomorphism near identity, say $U' \to U$, and S has a local slice chart near identity, combine these two together, this shows for $\exp(\xi_n)$ lying in the intersection of the slice chart and U, it can not stay in S, as is shown in the figure below. Because only vectors with zero $\mathfrak{b}$ component can be mapped to S in this area, this means ξ_n will leave S and giving new invariance.



In rest of the note, we will find such a sequence, under the assumption $\mu(Sx) = 0$ for all $x \in G/\Gamma$. Actually, the arguments above are the only place we use semisimplicity of H, and in the following step we will reduce to the case H is a unipotent flow. After reducing to the one parameter unipotent group, we will find a suitable subset E with good properties (such as all points in E are generic, and Birkhoff ergodic theorem holds uniformally for points in E). Then we will show E has good recurrence properties that allow us to take a sequence ξ_n, with both x_n and $\exp(\xi_n)x_n$ in E. Finally, we will show $\exp(\xi_n)$ leave μ invariant using good properties of points in E.

We will carry out these steps in following sections.

2. Reducing H to one parameter unipotent action

In this section, we gonna reduce $H = SL(2,\mathbb{R})$ to the case $u_t = \left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right), t \in \mathbb{R}$.

Since μ is H-ergodic, and u_t is unbounded in H, by Moore ergodicity theorem, μ is still u_t ergodic. This completes the reduction.

3. Finding a large set E with good 'generic' property

In this section, we find a set E with positive measure satisfying good properties. More precisely, we gonna prove the following lemma.

Lemma 3.1. *There exists a measurable subset E, with $\mu(E) > 0$, such that with $\forall f \in \mathscr{C}_c(G/\Gamma)$ fixed,*

$$\frac{1}{T}\int_0^T f(u_t x)dt \to \int_X f d\mu$$

uniformly on E as a sequence of functions of x. Due to regularity of measure μ, we may assume E is compact upon passing to a compact subset in E with measure close enough to E.

Remark 3.2. The measure of $\mu(E)$ can be chosen such that it is arbitrarily close to 1.

Proof. By Birkhoff ergodic theorem and ergodicity of u_t action, we know that given $f \in \mu$-a.e. $x \in G/\Gamma$,

$$\frac{1}{T}\int_0^T f(u_t x)dt \to \int_X f d\mu \tag{3.3}$$

as $T \to \infty$. Thus for each fixed $f \in \mathscr{C}_c(G/\Gamma)$, the set of generic points is of full measure, which implies *a.e.* pointwise convergence of $\frac{1}{T}\int_0^T f(u_t x)dt$ to constant function $\int_X f d\mu$. Using Ergrov's theorem, after deleting a subset of small measure, we may assume the uniform convergence on the set left. Thus for f fixed, and $\epsilon > 0$ very small, we can find $E_{f,\epsilon}$, such that (3.2) holds uniformly on $E_{f,\epsilon}$.

Noticing the set $\mathscr{C}_c(G/\Gamma)$ has a countable, dense subset (since G/Γ is a locally compact hausdorff metric space), denoted as $\{f_n\}$, and for each f_n, choose $\epsilon_n > 0$ small enough such that $\sum_n \epsilon_n << 1$. Then $E := \cap_n E_{f_n,\epsilon_n}$ has positive measure, and every point in E satisfies the theorem. □

4. Recurrence property of E

This is the main technical lemma of the proof.

Lemma 4.1. *There exists $x_n, n \in \mathbb{Z}_{>0}$ in E (the E in Lemma3.1), $g_n = \exp(\xi_n)$, satisfying $\xi_n \in \mathfrak{b}$,$\xi_n \neq 0$, $\xi_n \to 0$, such that $g_n x_n \in E$.*

Proof. We fix a compact set E as in Lemma 3.1, and assume $\mu(E) > 0.9$. Then choose a small neighborhood O of identity in S, such that $O = O^{-1}$.

Since μ is invariant under S, for any $g \in S$, we have $\mu(gE) > 0.9$, thus $\mu(E \cap gE) > 0.8$. Denote m the Haar measure on S, and notice that

$$\int_E \chi_E(gx)d\mu(x) = \mu(E \cap gE),$$

we obtain the following inequality

$$\int_O \int_E \chi_E(gx)d\mu(x)dm(g) > 0.8m(O).$$

Then we interchange the order of integration to get

$$\int_E \int_O \chi_E(gx)dm(g)d\mu(x) > 0.8m(O). \tag{4.2}$$

Roughly speaking, (4.2) means that for many points in E, its left translation by many elements in O still lies in E. However, we need those elements not lying in S, thus we need a more careful analysis.

First, denote $E_1 = \{x \in E | \int_O \chi_E(gx)dm(g) > 0.7m(O)\}$, then

$$\mu(E_1)m(O) + \mu(E - E_1)0.7m(O) \geq \int_E \int_O \chi_E(gx)dm(g)d\mu(x) > 0.8m(O),$$

thus at least $\mu(E_1) > 0.1$ has positive measure. Using regularity of the measure μ, we may assume that E_1 is compact.

Fix $z \in supp\mu|_{E_1}$, then every open neighborhood of z has positive measure. Notice that we assume $\mu(Sz) = 0$ for every $z \in G/\Gamma$, as is stated in section 1. This implies that we can choose a sequence of $z_n \to z$, satisfying $z_n \notin Sz$ and $d(z_n, z) \leq \frac{1}{n}$. Denoting $z_n = h_n z$, then h_n tends to identity, and $h_n \notin S$.

Then we want to modify each $z_n = h_n z$, to let it lies in $\exp(\mathfrak{b})$. We may assume h_n are all close enough to identity, such that for each h_n fixed, gh_n can be decomposed as below for $g \in O$ (it can be done upon shrinking O)

$$gh_n = \varphi(gh_n)\psi(gh_n), \varphi(gh_n) \in \exp(\mathfrak{b}), \psi(gh_n) \in S.$$

Because g is taken from $O \subset S$ and is sufficiently close to identity, we have

$$\varphi(g) = e, \psi(g) = g.$$

Then using $h_n \to e$, this tells us that $\psi(\{g \in O | gz_n \in E\}h_n \cap O)$ tends to the set $\{g \in O | gz_n \in E\}$, which means

$$m(\psi(\{g \in O | gz_n \in E\}h_n) \cap O) > 0.6m(O), \tag{4.3}$$

because $z_n \in E_1$, and $m(\{g \in O | gz_n \in E\}) > 0.7m(O)$ by the definition of E_1.

Now we can choose $\tilde{g}_n \in \psi(\{g \in O | gz_n \in E\}h_n) \cap \{g \in O | gz_n \in E\}$ (this set is nonempty and occupies a positive proportion in O), with $\tilde{g}_n = \psi(g_n h_n)$ for some $g_n \in O$. Now taking

$$y_n = \tilde{g}_n z,$$

then $g_n z_n = g_n h_n z = \varphi(g_n h_n)\psi(g_n h_n)z = \varphi(g_n h_n)y_n$, and also $y_n \in E$ due to $\tilde{g}_n \in \{g \in O | gz_n \in E\}$.

Now we have found $y_n, \varphi(g_n h_n)y_n \in E$, with $\varphi(g_n h_n) \in \exp(\mathfrak{b})$ and h_n tending to identity. This sequence of y_n satisfies the lemma. $\square$

This recurrence property is the main ingredient of arguments in the next section.

5. Behavior of unipotent orbit

In this section, we finish the proof using results in section 3 and 4.

The argument below relies on polynomial behavior of unipotent orbit, which means we can control the behavior of unipotent orbit using the estimation on a positive proportion of time.

We first consider a simple case, when infinitely many g_n chosen in Lemma4.1 lies in $C_G(U)$ (*i.e.* commuting with $U = \{u_t | t \in \mathbb{R}\}$). Under this case,

$$\frac{1}{T}\int_0^T f(u_t g_n x_n)dt = \frac{1}{T}\int_0^T f \circ g_n(u_t x_n)dt \tag{5.1}$$

for any fixed f. Using $x_n, g_n x_n \in E$, we know the left hand side tends to $\int_X f d\mu$, while right hand side tends to $\int_X f \circ g_n d\mu$, then (5.1) implies that

$$\int_X f d\mu = \int_X f \circ g_n d\mu$$

for all fixed f continuous with compact support, meaning that μ is invariant under g_n. This means $\xi_n \to 0$ is the desired sequence in section 1, finishing the proof.

The remaining case, where we may assume all $g \notin C_G(U)$, is more tricky, and requires the polynomial property of unipotent orbit.

Unfortunately, we can not show $g_n = \exp(\xi_n)$ themselves give new invariance of measure μ. We need to properly dilate g_n in some direction and contract it in some direction, and let it tend to some element that gives new invariance, say g^*.

As stated in section 1, we need g^* not lying in S. If we take $g_n^* = u_{t_n} g_n u_{-t_n}$, then it is still of the form

$$g_n^* = \exp(Ad(u_{t_n})\xi_n), Ad(u_{t_n})\xi_n \in \mathfrak{b},$$

because $\mathfrak{b}$ is invariant under u_t by adjoint action. Thus if g_n is close to identity, using the slice chart of S, we can still derive that g_n^* is not in S. If $g^* = \lim g_n^*$, then $g^* = \exp(\xi^*)$ with $\xi^* = \lim Ad(u_{t_n})\xi_n$, and g^* is still close to identity. This implies $g^* \notin S$ too. Since we do not want g^* to be the identity, t_n should be chosen such that $Ad(u_{t_n})\xi_n$ is kept away from zero uniformly.

We fix a small open ball B_r containing identity, such that its closure is contained in a slice chart and itself is diffeomorphic to open ball $B_r' \subset \mathfrak{g}$ through exponential map $\exp(s)\exp(b)$. Then every point of the form $\exp(b), b \in \mathfrak{b}, b \neq 0$ in $\overline{B_r}$ does not belong to S.

For $g_n \notin C_G(U)$, $u_t g_n u_{-t} = \exp(Ad(u_t)\xi_n)$ is non-constant. Since u_t is unipotent, entries of $Ad(u_t)\xi_n$ are polynomials of t. The condition $g_n \notin C_G(U)$ means that there exists some entry which is a nontrivial polynomial, thus $Ad(u_t)\xi_n$ tends to infinity as $t \to \infty$. With out loss of generality, we may assume $g_n \in B_r$ for all n, the divergence of polynomial at infinity implies trajectory $Ad(u_t)\xi_n$ will meet $\partial B_r'$ at some time, say T_n. Denote $\xi_n^* = Ad(u_{T_n})\xi_n$ and $g_n^* = \exp(\xi_n^*)$. Next, we gonna show for a positive portion of time in $[0, T_n]$, $Ad(u_t)\xi_n$ is close to g_n^*. The precise statement is the following lemma.

Lemma 5.2. *For every $\epsilon_1 > 0$, there exists $\delta > 0$, such that for all $g \in B_r \backslash C_G(U)$, and $\forall t \in [(1-\delta)T_g, T_g]$,*

$$d_G(g^*, u_t g u_{-t}) < \epsilon_1. \tag{5.3}$$

Here $g^* \in \partial B_r$, T_g is the first time when $u_t g u_{-t}$ meet ∂B_r, and d_G is a right invariant metric on G.

Remark 5.4. The importance is that the δ depend only on the neighborhood B_r and ϵ_1. Once they are fixed, the estimation is uniform for all $g \in B_r \backslash C_G(U)$.

Remark 5.5. It is easy to see, if g tends to identity, then $T_g \to \infty$, since $g \to e$ means coefficients of $Ad(u_{T_n})\xi$ tends to zero, thus it takes more time to leave B_r. In particular, $T_n \to \infty$ for those T_n chosen for g_n before.

Proof. Denote $g = \exp(\xi)$, $q_g(t) := Ad(u_{tT_g})\xi, t \in [0,1]$. Then q_g maps the unit interval into $\overline{B'_r}$, and it is a non-constant polynomial, whose degree is bounded by a constant depending only on dimension of G. Since $q_g([0,1]) \subset B'_r$ is bounded, and we can recover polynomial's coefficients using its value on finitely many points (the precise number of points needed depends only on dimension), as is stated below,

$$(a_m, a_{m-1}, \cdots, a_0)\begin{pmatrix} x_1^m & \cdots & x_{m+1}^m \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = (f(x_1), \cdots, f(x_{m+1})), \tag{5.6}$$

the coefficients are also bounded, and the bound is independent of g.

This uniform bound implies that there exists a δ independent of g, such that $\|q_g(t) - q_g(1)\| < \epsilon$ for all $t \in [1-\delta, 1]$. Choose an ϵ such that $\exp(B_\epsilon(q_g(1))) \subset B_{\epsilon_1}(g^*)$ proves the lemma. $\square$

We can utilize this lemma to show

$$|\frac{1}{\delta T_g}\int_{(1-\delta)T_g}^{T_g} f(u_t g x)dt - \frac{1}{\delta T_g}\int_{(1-\delta)T_g}^{T_g} f(g^* u_t x)dt| < \epsilon, \tag{5.7}$$

where the ϵ_1 in Lemma 5.2 is chosen such that $|f(x)-f(y)| < \epsilon$ whenever $d_G(x,y) < \epsilon_1$ (since we mainly consider the case f has compact support, it is uniformally continuous). The following lemma enable us to replace the time average on $[0, T_g]$ with (5.7) in ergodic theorem.

Lemma 5.8. *For any $f \in \mathscr{C}_c(G/\Gamma)$ fixed, given $\epsilon, \delta > 0$, then there exists a $T_0 > 0$, such that $\forall T > T_0, x \in E$, we have*

$$|\frac{1}{\delta T}\int_{(1-\delta)T}^{T} f(u_t x)dt - \int_X f d\mu| < \epsilon$$

Proof. First use the choice of E to take $T_1 > 0$, such that $\forall x \in E, \forall T > T_1$,

$$|\frac{1}{T}\int_0^T f(u_t x)dt - \int_X f d\mu| < \frac{\delta\epsilon}{2}. \tag{5.9}$$

Choose $T_0 > 0$, such that $(1-\delta)T_0 > T_1$. Then for $T > T_0$,

$$\begin{gathered} |\int_0^{(1-\delta)T} f(u_t x)dt - (1-\delta)T\int_X f d\mu| < (1-\delta)T\frac{\delta\epsilon}{2} \\ |\int_0^T f(u_t x)dt - T\int_X f d\mu| < T\frac{\delta\epsilon}{2}. \end{gathered} \tag{5.10}$$

Subtract the second inequality with the first one, and dividing both side by δT, we obtain

$$|\frac{1}{\delta T}\int_{(1-\delta)T}^{T} f(u_t x)dt - \int_X f d\mu| < \epsilon\frac{2-\delta}{2} < \epsilon. \tag{5.11}$$

$\square$

With out loss of generality, assume g_n^* converges to g^*, by arguments before, g^* does not belong to S. Finally, we will show μ is invariant under g^*, that is

Lemma 5.12. *For all $f \in \mathscr{C}_c(G/\Gamma)$ fixed,*

$$\int_X f d\mu = \int_X f \circ g^* d\mu.$$

Proof. It suffices to show, for any $\epsilon > 0$ and $f \in \mathscr{C}_c(G/\Gamma)$ fixed,

$$|\int_X f d\mu - \int_X f \circ g^* d\mu| < \epsilon. \tag{5.13}$$

Since f is uniformly continuous, there exists some $\epsilon_1 > 0$, such that $d_G(g', g'') < \epsilon_1$ implies that $|f(g'x) - f(g''x)| < \frac{\epsilon}{10}$.

Use this ϵ_1 as the ϵ_1 in Lemma 5.2, we get a $\delta > 0$ satisfying Lemma 5.2. Then we apply Lemma 5.8 to get a T_0 for both $f, \delta, \frac{\epsilon}{10}$ and $f \circ g^*, \delta, \frac{\epsilon}{10}$.

Recall that $T_n \infty$ as $n \to \infty$ and $g_n x_n, x_n \in E$, thus for n large, $T_n > T_0$,

$$|\frac{1}{\delta T_n} \int_{(1-\delta)T_n}^{T_n} f(u_t g_n x_n) dt - \int_X f d\mu| < \frac{\epsilon}{10}.$$

Using $d_G(u_t g_n u_{-t}, g_n^*) < \epsilon_1$ for $t \in [(1-\delta)T_n, T_n]$, the uniform continuity of f implies

$$|\frac{1}{\delta T_n} \int_{(1-\delta)T_n}^{T_n} f(g_n^* u_t x_n) dt - \int_X f d\mu| < \frac{2\epsilon}{10}.$$

For large n, $d_G(g_n^*, g^*)$ also holds, using uniform continuity again, we obtain

$$|\frac{1}{\delta T_n} \int_{(1-\delta)T_n}^{T_n} f(g^* u_t x_n) dt - \int_X f d\mu| < \frac{3\epsilon}{10}. \tag{5.14}$$

However, from Lemma 5.8, we know

$$|\frac{1}{\delta T_n} \int_{(1-\delta)T_n}^{T_n} f \circ g^*(u_t x_n) dt - \int_X f \circ g^* d\mu| < \frac{\epsilon}{10}. \tag{5.15}$$

because $T_n > T_0$. Combining (5.14) and (5.15), we obtain the estimation (5.13) and proves the lemma. $\square$

As is stated in section 1, Lemma 5.12 leads to the contradiction, thus the theorem is proved.

6. Bibliography

References

[1] M.Ratner. On measure rigidity of unipotent subgroups of semisimple groups. Acta Math. 165 (1): 229–309. 1990.

[2] D.W.Morris. Ratner's Theorems on Unipotent Flows. Chicago Lectures in Mathematics. Chicago, IL: University of Chicago Press. 2005.