SOME GEOMETRY OF $PSL(2, \mathbb{R})$

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ABSTRACT. On the class, some properties of this real semisimple lie group $SL(2, \mathbb{R})$ and its discrete subgroup $SL(2, \mathbb{Z})$ with finite covolume is mentioned. In this note, we mainly consider some geometric structure on $PSL(2, \mathbb{R})$ and introduce two flows on it.

CONTENTS

1. $PSL(2, \mathbb{R})$ AS THE UNIT TANGENT BUNDLE OF HYPERBOLIC PLANE

We use the upper half plane model of hyperbolic plane. Its tangent bundle is trivial and can be identified with $\mathbb{H} \times \mathbb{C}$. We can endow it with a Riemannian metric defined by

$$\langle v, w \rangle_z = \frac{1}{y^2}(v \cdot w)$$

at the point $z = x + iy \in \mathbb{H}$ and $v, w \in T_z\mathbb{H}$. Here the $(v \cdots w)$ is just the usual Euclidean inner product on $\mathbb{C} = T_z\mathbb{H}$. It can be verified that the scalar curvature equal to constant -1, so it is called the *hyperbolic Riemannian metric* on $\mathbb{H}$ and its induced metric is called the *hyperbolic metric*.

$PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I_2\}$ acts on hyperbolic plane $\mathbb{H}$ by fractional linear transformation. This action preserves the hyperbolic Riemannian metric on $\mathbb{H}$, so it induces an action on unit tangent bundle of $\mathbb{H}^2$. Moreover, we have the following lemma from [2].

Lemma 1.1. *The action of $PSL(2, \mathbb{R})$ on unit tangent bundle $U\mathbb{H}$ is simply transitive*

Remark 1.2. Here simply transitive means transitive and the isotropy group of each point is trivial. This lemma allows us to identify $PSL(2, \mathbb{R})$ with unit tangent bundle $U\mathbb{H}$

Remark 1.3. We can also regard $\mathbb{H}$ as $SL(2, \mathbb{R})/SO(2)$. Then it admits a natural $SL(2, \mathbb{R})$ action. The measure on $\mathbb{H}$ induced by Haar measure $\frac{1}{2\pi}\frac{dxdy}{y^2}d\theta$ is exactly $\frac{dxdy}{y^2}$, and equals to the volume form induced by the hyperbolic Riemannian metric.

Remark 1.4. The identification here is given by

$$g \mapsto Dg(i, i)$$

Proof. First we show $PSL(2,\mathbb{R})$ acts by isometries. Fix $A \in PSL(2,\mathbb{R})$, and

$$Az = \frac{az+b}{cz+d}$$

A is a holomorphic map on upper half plane and its differential is

$$A'(z) = \frac{1}{(cz+d)^2}$$

So the action of tangent map DA on $U\mathbb{H}$ can be explicitly given by

$$DA(z,v) = (\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2}) \tag{1.5}$$

Now a direct calculation shows this action is an isometry.

$$\langle DA_z(v), DA_z(w)\rangle_{g(z)} = (\frac{y}{|cz+d|})^{-2}(\frac{(cz+d)^2}{v}, \frac{1}{(cz+d)^2}w) \tag{1.6}$$

since the Euclidean inner product on $\mathbb{C}$ is just $(v,w) = \Re(v\overline{w})$.

Then we show the action is simply transitive. From complex analysis, we know its action on $\mathbb{H}$ is transitive, see also Lemma 9.1 in [2]. It suffices to show $Stab_i(PSL(2,\mathbb{R})) = PSO(2)$ (stabilizer of its action on $\mathbb{H}$) acts transitively on $T_i\mathbb{H}$ and has trivial stabilizer.

Its action is transitive because the $A \in Stab_i(PSL(2,\mathbb{R}))$ action on $v \in T_i\mathbb{H} = \mathbb{C}$ is of the form

$$DA_i(v) = \frac{1}{(i\sin\theta + \cos\theta)^2}v = (\cos 2\theta - i\sin 2\theta)v \tag{1.7}$$

By varying θ, $DA_i(v)$ takes every vector of modulus one. If $DA_i(v) = v$, then $\theta \in \mathbb{Z}\pi$. Thus $A = \pm I_2$. This gives the result that stabilizer of $PSL(2,\mathbb{R})$ action on $U\mathbb{H}$ is trivial. □

Combine this lemma and the fact that a geodesic is determined by its initial point and initial velocity, it is clear that every geodesic starting at $z \in \mathbb{H}$ can be translated from a chosen standard geodesic by actions of $PSL(2,\mathbb{R})$.

Furthermore, we have the following proposition, also from [2]

Property 1.8. For any two points $z_0, z_1 \in \mathbb{H}$, there is a unique path

$$\phi : [0, d(z_0,z_1)] \to \mathbb{H}$$

of unit speed with $\phi(0) = z_0$ and $\phi(d(z_0,z_1)) = z_1$. Moreover, there is a unique isometry $g \in PSL(2,\mathbb{R})$ such that $\phi(t) = g(e^t i)$.

Proof. Now it suffices to show the existence and uniqueness between arbitrary two points in hyperbolic plane. And this means $\mathbb{H}$ is a complete Riemannian manifold and has injective radius ∞. These two properties follows directly from Cartan-Hadamard theorem for space forms.

The interesting part of this proposition is that we can give an explicit formula for geodesics in $\mathbb{H}$. It starts by first finding a geodesic in $\mathbb{H}$, that is $e^t i$.

Denote $\gamma(t) = e^t i$, and view it as a curve in $\mathbb{H} \subset \mathbb{C} = \mathbb{R}^2$, and substitute it into the geodesic equation

$$\frac{d^2\gamma^a}{dt^2} + \Gamma^a_{bc}\frac{d\gamma^b}{dt}\frac{d\gamma^c}{dt} = 0 \tag{1.9}$$

the only nontrivial equation is the following one, and it holds by a direct calculation.

$$\begin{aligned}
&\frac{d^2(e^t)}{dt^2}+\Gamma^2_{22}\frac{de^t}{dt}\frac{de^t}{dt}=0\\
\Gamma^2_{22}&=\frac{1}{2}g^22(g_{22,2}+g_{22,2}-g_{22,2})\\
&=\frac{1}{2}y^2\frac{\partial\frac{1}{y^2}}{\partial y}|_{y=e^t}=-e^t
\end{aligned}\tag{1.10}$$

After we have found this specific geodesic, other geodesics can be given by $PSL(2,\mathbb{R})$ translations. □

2. Geodesic flow and horocycle flow on $PSL(2,\mathbb{R})$

As in the [2], we define geodesic flow and horocycle flow.

In Riemannian geometry, the geodesic flow is a one parameter subgroup of the group of diffeomorphism of tangent bundle. It is defined by

$$g_t(z,v)=(\gamma(t),\dot\gamma(t))\tag{2.1}$$

Here $\gamma(t)$ is the geodesic starting at z with initial speed vector v. Because geodesic has constant speed, geodesic flow can be restricted to the unit tangent bundle.

In the previous section, we have proved the identification $U\mathbb{H}=PSL(2,\mathbb{R})$. The simply transitive action is given by the tangential map of $g\in PSL(2,\mathbb{R})$. Now we want to find the flow on $PSL(2,\mathbb{R})$ corresponds to geodesic flow on $U\mathbb{H}$.

Using Property 1.7, every geodesic looks like $g(e^ti)$, so its trajectory in tangent bundle is $Dg(e^ti,e^ti)$. An important observation is the geodesic e^ti can be viewed as the orbit of i under a one parameter subgroup $diag\{e^{\frac{t}{2}},e^{\frac{-t}{2}}\}$. Moreover, the following equation holds

$$D(\begin{smallmatrix} e^{\frac{t}{2}} & 0\\ 0 & e^{-\frac{t}{2}}\end{smallmatrix})(i,i)=(e^ti,e^ti)\tag{2.2}$$

Denote

$$a_t=(\begin{smallmatrix} e^{-\frac{t}{2}} & 0\\ 0 & e^{\frac{t}{2}}\end{smallmatrix})\tag{2.3}$$

Then the geodesic flow g_t satisfies the following relation

$$g_t((z,v))=Dg((e^ti,e^ti))=Dg(D(a_t^{-1})(i,i))=D(ga_t^{-1})(i,i)\tag{2.4}$$

This relation means the flow on $PSL(2,\mathbb{R})$ corresponding to the geodesic flow is *right multiplication by the inverse of matrix* a_t, and we denote it as R_{a_t}.

Notice the derivative action of $PSL(2,\mathbb{R})$ on $U\mathbb{H}=PSL(2,\mathbb{R})$ is the left multiplication. The associativity of this action is just the chain rule of differentiation.

Another important flow is the horocycle flow. The motivation is to consider the orbit of (i,i) under the upper triangular matrix

$$U^-=\{(\begin{smallmatrix}1 & s\\ 0 & 1\end{smallmatrix})|s\in\mathbb{R}\}\tag{2.5}$$

As a set, it is the line $\Im z=1$. Consider its trajectory in $U\mathbb{H}$, that is

$$D(\begin{smallmatrix}1 & -s\\ 0 & 1\end{smallmatrix})(i,i)=(i-s,i)\tag{2.6}$$

This trajectory has the property that the tangent vector of the curve is pointwise orthogonal to its associated tangent vector, just like the circle in Euclidean plane, thus it is called horocycle flow.

Using translations by $PSL(2,\mathbb{R})$, we can get horocycles at each point and each direction, giving a flow in $U\mathbb{H}$, denoted as $u^-(s)$. In order to make it a flow on $PSL(2,\mathbb{R})$, we proceed like the case of geodesic flow

$$u^-(s)((z,v)) := Dg(D(\begin{smallmatrix} 1 & -s \\ 0 & 1 \end{smallmatrix})(i,i)) = D(gu^-(-s))(i,i) \tag{2.7}$$

Here we denote $u^-(s) = (\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix})$.

We have just mentioned that these flow looks like circle because the tangent vector of curve is orthogonal to the associated tangent vector, now it is natural to ask where is the center of this horocircle. Actually the center of $(i-s,i)$ is the point of infinity in the upper half plane model, here the center means the point where all geodesics $g_t(i-s,i)$ converge to.

The case $g_t(i-s,i) = (e^t i - s, e^t i)$ can be shown explicitly, as on P.288[2]. Also all the points (z,v) in $U\mathbb{H}$ satisfying $\lim_{t\to\infty} d_{\mathbb{H}}(g_t(z,v), g_t(i,i)) = 0$ is contained in $u^-(s)(i,i)$, see P.305[2].

3. Bibliography

The structure of this note mainly follows [2], and I try to restate these proofs in [2] in a more geometric way, and added some understanding from the geometric side.

References

[1] Peter Petersen. Riemannian Geometry. Springer-Verlag New York. 2006.

[2] Manfred Einsiedler and Thomas Ward. Ergodic Theory: with a view towards Number Theory. Springer-Verlag London Limited. 2011.