A LEMMA ABOUT H-INVARIANT FUNCTION AND BIRKHOFF ERGODIC THEOREM FOR FLOW

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Abstract. In this week, by introducing the Siegel domain, we proved that $SL(n,\mathbb{Z})$ is a lattice in $SL(n,\mathbb{R})$, and started to study the dynamics on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ and ergodicity with respect to a subgroup action is defined. A lemma about essentially H-invariant functions are a.e. equal to H-invariant functions is stated. Also Birkhoff ergodic theorem for flow is mentioned.

Contents

1. Essentially H-invariant functions are almost H-invariant

This is an elementary technical lemma.

Lemma 1.1. *G is a locally compact topological group with left Haar measure μ, and H a topological subgroup of G. If f is an essentially H-invariant μ-measurable function, i.e. for $\mu - a.e.\ x \in X$, $f(hx) = f(x)$ holds for all $h \in H$, then there exists another measurable function F, such that $F = f$ holds almost everywhere, and F is H-invariant, i.e. $F(Hx) = F(x)$ holds for all $x \in X$.*

Proof. Choose μ_H to be the right Haar measure on H. Since it might not be a probability measure, we choose a strictly positive function ψ such that $\upsilon = \psi d\mu_H$ is a probability measure on H.

For example, we can choose the ψ as follows: Find a increasing exhausting sequence consisted of finite measure set of H, denoted as $E_n, n \in \mathbb{Z}_{\geq 0}$, $E_0 = \emptyset$. Then

$$\psi = \sum_{i=1}^{\infty} \frac{1}{2^i \mu(E_i \backslash E_{i-1})} \mathbb{1}_{E_i \backslash E_{i-1}} \tag{1.2}$$

An impotant property of υ is that any H-translation of υ-null subset of H still has zero measure. This is due to the fact that any υ-null set is also μ_H-null, and vise versa.

Define the subset of $H \times X$

$$Q = \{(h,x) | f(hx) \neq f(x)\} \tag{1.3}$$

According to definition of essentially H-invariant.

$$0 = \int_H \int_X \mathbb{1}_Q = \int_X \upsilon(\{h \in H | f(hx) \neq f(x)\}) \tag{1.4}$$

Thus for $\mu - a.e.$ x, $f(hx) = f(x)$ holds $\upsilon - a.e.$, meaning

$$X_0 = \{x \in X | f(hx) = f(x), \upsilon - a.e.\ h \in H\} \tag{1.5}$$

has full measure.

Define

$$\begin{aligned} F(x) &= \int_H f(hx) d\upsilon(h) \\ X_1 &= \{x \in X | f(hx) = F(x)\ \upsilon - a.e.\ h\} \\ \tilde{f}(x) &= \begin{cases} F(x) & x \in X_1 \\ \frac{1}{2} & x \in X \backslash X_1 \end{cases} \end{aligned} \tag{1.6}$$

Then X_1 is H-invariant. Because if $x \in X_1$, then $\upsilon(\{h | f(hx) = f(x)\}) = 0$, and notice $\{h | f(hh_1 x) = f(x)\} = \{h | f(hx) = f(x)\} h_1^{-1}$, so it also has zero measure, which implies $h_1 x \in X_1$. As a corollary $X \backslash X_1$ is invariant too.

Also $F|_{X_1}$ is H-invariant. Because $F(x)$ equals to 'most of the values' in $\{f(hx) | h \in H\}$, which equals to 'most of the values' in $\{f(hh_1 x) | h \in H\}$ (because right H-translation preserves υ-null set), and this equals to $F(h_1 x)$.

We notice that $X_0 \subset X_1$. Because for $x \in X_0$, f is $\upsilon - a.e.$ constant on Hx. So X_1 has full measure too.

So $\tilde{f}$ is H-invariant, then $\tilde{f}$ is the needed function, since it equals f on X_0. $\square$

2. Birkhoff ergodic theorem for flow

Theorem 2.1. *$a_t, t \geq 0$ is a flow preserving measure μ and ergodic. Assume f is integrable on X, and $f(a_t(x))$ is also an integrable function on $(t, x) \in [0, 1] \times X$. Then*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(a_t(x)) dt = \int_X f(y) d\mu(y) \tag{2.2}$$

holds for $\mu - a.e.$ $x \in X$.

Proof. With out loss of generality, we may assume $f \geq 0$. Define

$$\psi(x) = \int_0^1 f(a_t(x)) dt \tag{2.3}$$

Then $\psi \geq 0$, and is measurable on X, due to the Tonelli theorem. Also, its integral over X is equal to the integral of f over X, because

$$\begin{aligned} & \int_X \psi(x) d\mu \\ = & \int_X \int_0^1 f(a_t(x)) dt d\mu \\ = & \int_0^1 \int_X f(a_t(x)) d\mu dt \\ = & \int_0^1 \int_X f(x) d\mu dt \\ = & \int_X f(x) d\mu < \infty \end{aligned} \tag{2.4}$$

So ψ is finite $\mu - a.e.$ and ψ is integrable over X.

Apply the discrete version of Birkhoff ergodic theorem to ψ, we have for $\mu - a.e.\ x$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \psi(a_i(x)) = \int_X f(x) d\mu \tag{2.5}$$

But according to the definition of ψ, and semi-group property of the flow, the left hand side is just

$$\frac{1}{N} \int_0^N f(a_t(x)) dt \tag{2.6}$$

Similarly, define

$$\psi_k(x) = \int_0^{\frac{1}{2^k}} f(a_t(x)) dt \tag{2.7}$$

We have for $\mu - a.e.\ x$

$$\lim_{N\to\infty} \frac{1}{N/2^k} \int_0^{\frac{N}{2^k}} f(a_t(x)) dt = \int_X f(x) d\mu \tag{2.8}$$

holds for arbitrary k.

If f is bounded, then the proof is completed. For general $f \geq 0$, this follows from that if $\frac{N}{2^k} < T < \frac{N+1}{2^k}$

$$\frac{1}{(N+1)/2^k} \int_0^{\frac{N}{2^k}} f(a_t(x)) dt \leq \frac{1}{T} \int_0^T f(a_t(x)) dt \leq \frac{1}{N/2^k} \int_0^{\frac{N+1}{2^k}} f(a_t(x)) dt \tag{2.9}$$

□

3. BIBLIOGRAPHY

Thanks Liu Xuan for telling me the proof of Birkhoff ergodic theorem for the flow.

References

[1] Manfred Einsiedler and Thomas Ward. Ergodic Theory: with a view towards Number Theory. Springer-Verlag London Limited. 2011.