Week 1. Explicit calculation of Haar measure on $GL(n, \mathbb{R})$, N and $SL(2, \mathbb{R})$

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Abstract

Two important things mentioned on the class last week are the existence of Haar measure for locally compact topological groups, and how Haar measure decompose with respect to the decomposition of the group.

Using Riemannian geometry, we can directly construct the Haar measure on Lie groups. But the choice of such a metric is ambiguous, giving different behavior of curvature. Although their volume form induce the same measure, which is ensured by the uniqueness of left invariant measure.

1 Left invariant metric and corresponding volume form of $GL(n, \mathbb{R})$ and N

Recall that $SL(n, \mathbb{R})$ is a noncompact manifold embedded in $M_{n\times n}$, this can be derived from the fact that it is the level set of smooth function $det : M_{n\times n} \to \mathbb{R}$ at the regular value 1, using rank theorem.

So its tangent bundle can be embedded in $M_{n\times n} \times M_{n\times n}$, it looks like

$$T(SL(n, \mathbb{R})) = \{(A, AX) | A \in SL(n, \mathbb{R}), X \in \mathfrak{sl}(n, \mathbb{R})\} \tag{1}$$

This follows directly from the fact that tangent space is the linear space of all tangent vectors of curves passing that point, and all curves passing A can be obtained from left translating a curve passing I_n.

Choose $\{X_1, \cdots, X_{n^2-1}\}$ be a basis of $\mathfrak{sl}(n, \mathbb{R})$. Then $\{(A, AX_i) | A \in SL(n, \mathbb{R})\}$ gives $n^2 - 1$ linear dependent left invariant vector fields which gives the trivialization of tangent bundle. Here the push forward of a tangent vector (B, X) by left translation A is simply (AB, AX).

So it is clear to construct a Riemannian metric, it suffices to give a positive definite inner product on $\mathfrak{sl}(n, \mathbb{R})$. The negative of Killing form is not applicable here, since if it is positive definite, this implies $\mathfrak{sl}(n, \mathbb{R})$ would be a compact Lie algebra, a contradiction. There seems no canonical choice of such a metric. All such metric are parameterized by positive definite matrices of size $(n^2 - 1) \times (n^2 - 1)$, which forms a positive cone.

We actually construct left invariant metrics on Lie group, and then their left invariant volume forms give Haar measure, which is bi-invariant in the case of $SL(n, \mathbb{R})$ for its semisimplicity, but the metric may not be bi-invariant for most metrics. For $SL(n, \mathbb{R})$, such metrics is never bi-invariant, because bi-invariance of metric equals the Ad-invariance of inner product, see Exercise 1.6.25 of Peterson, which must be a multiple of its Killing form, which is not definite. For compact Lie groups, this means the metric given by the negative of its Killing forms is some kind of 'best metric'.

In order to get a explicit expression of the volume form, a local chart of $SL(n, \mathbb{R})$ is needed. But the local chart of $SL(n, \mathbb{R})$ is not canonically given. However, $GL(n, \mathbb{R})$ has a nice coordinate chart. And we first construct the left invariant volume form on $GL(n, \mathbb{R})$. Similarly, we have

$$T(GL(n, \mathbb{R})) = \{(A, AX) | A \in GL(n, \mathbb{R}), X \in \mathfrak{gl}(n, \mathbb{R})\} \tag{2}$$

$\mathfrak{gl}(n, \mathbb{R})$ has a canonical basis $E_{i,j}, 1 \leq i, j \leq n$, and they give a global left invariant frame of tangent bundle. We choose a metric such that these frames are orthogonal. The local chart is given by $x_{i,j}, 1 \leq i, j \leq n$, which are entries of matrices in $GL(n, \mathbb{R})$. Now we need to express volume form in the form $f(A) dx_{1,1} \wedge \cdots \wedge dx_{n,n}$.

The volume form is simply the wedge product of 1-forms dual to vector fields in a standard orthogonal frame, and the order of wedge product is chosen to be consistent with orientation of the frame.

Notice that $\frac{\partial}{\partial x_{i,j}}$ is exactly the vector field $\{(A, E_{i,j}) | A \in GL(n, \mathbb{R})\}$. So at point A, the relation between $E_{i,j}$ and $\frac{\partial}{\partial x_{i,j}}$ is

$$E_{i,j} = \sum_{k=1..n} A_{k,i} \frac{\partial}{\partial x_{k,j}} \tag{3}$$

Where $A_{i,k}$ is the (i, k) entry of matrix A. From this relationship, it is clear that

$$\omega_{i,j} = \sum_{k=1..n} (A^{-1})_{i,k} dx_{k,j} \tag{4}$$

Where $\omega_{i,j}$ is the dual 1-forms with respect to $E_{i,j}$, and $(A^{-1})_{k,i}$ is the (k, i) entry of A^{-1}. The volume form is

$$\begin{aligned} dvol &= \bigwedge_{i,j=1..n} \omega_{i,j} \\ &= f(A) \bigwedge_{i,j=1..n} dx_{i,j} \end{aligned} \tag{5}$$

From the basic properties of wedge product, the coefficient $f(A)$ is the determinant of the translation matrix between $\omega_{i,j}$ and $dx_{i,j}$. Here the translation matrix between $\omega_{i,j}$ and $dx_{i,j}$ is the $n^2 \times n^2$ block diagonal matrix

$$\begin{pmatrix} A^{-1} & & & \\ & A^{-1} & & \\ & & \cdots & \\ & & & A^{-1} \end{pmatrix} \tag{6}$$

The basis is respectively $\omega_{1,1}, \omega_{2,1}, \cdots, \omega_{n,1}, \omega_{1,2}, \cdots, \omega_{n,n}$ and $dx_{1,1}, dx_{2,1}, \cdots, dx_{n,1}, dx_{1,2}, \cdots, dx_{n,n}$. It is clear its determinant is $(detA)^{-n}$, so the volume form should be

$$dvol = (detA)^{-n} \bigwedge_{i,j=1..n} dx_{i,j} \tag{7}$$

on the part $GL^+(n, \mathbb{R})$, where all matrices have positive determinant. If we use the method above to construct a right invariant metric, the resulting volume form will still be the same. Since the matrix (6) at this time would be its transposition if we choose an appropriate order of the basis. So it tells us that $GL(n, \mathbb{R})$ is unimodular.

Similar technique can be applied to the calculation of left invariant volume form on the group of upper triangular matrices with diagonal elements equal to 1. In this case, denote the group be N, and $E_{i,j}$ still denote both left invariant vector fields and elements in Lie algebra, we have

$$\begin{aligned} TN &= \{(A, AX) | A \in N, X \in \mathfrak{n}\} \\ \mathfrak{n} &= \bigoplus_{i<j} E_{i,j} \\ E_{i,j} &= \sum_{k \le i} A_{k,i} \frac{\partial}{\partial x_{k,j}} \end{aligned} \tag{8}$$

So under the basis $E_{1,2}, E_{1,3}, \cdots, E_{1,n}, E_{2,3}, \cdots, E_{n-1,n}$ and $\frac{\partial}{\partial x_{1,2}}, \frac{\partial}{\partial x_{1,3}}, \cdots, \frac{\partial}{\partial x_{1,n}}, \frac{\partial}{\partial x_{2,3}}, \cdots, \frac{\partial}{\partial x_{n-1,n}}$, the transition matrix is lower triangular with all entries on diagonal equal to 1. So the transition matrix of its dual basis is also lower triangular with all entries on diagonal equal to 1. The same calculation as the case $GL(n, \mathbb{R})$ shows the left invariant volume form is just

$$dvol_N = \bigwedge_{1 \le i < j \le n} dx_{i,j} \tag{9}$$

Just as the case $GL(n, \mathbb{R})$, the right invariant form has the same form. N is also unimodular.

2 Explicit expression of Haar measure on $SL(2, \mathbb{R})$ using Iwasawa decomposition

First we calculate the modular function in the setting of Lie group. Then we use decomposition of Haar measure to give a Haar measure on $SL(2, \mathbb{R})$.

For Lie groups, their modular functions have nice expressions as in the following proposition. The proof is from Knapp.

Proposition. If G is a Lie group, then the modular function for G is given by $\Delta(t) = |detAd(t)|$.

Proof. Denote the volume form with respect to a chosen left-invariant metric as ω. The push forward of Haar measure through right multiplication by t corresponds to the pull back of volume form by right multiplication of t^{-1}. It suffices to show

$$(R_{t^{-1}}^{*}\omega)_p = (detAd(t))\omega_{pt^{-1}} \tag{10}$$

Because when $detAd(t) > 0$

$$\begin{aligned}(detAd(t))\int_G f(x)d_l x &= (detAd(t))\int_G f\omega = \int_G fR_{t^{-1}}^{*}\omega \\ &= \int_G (f\circ R_t)\omega = \int_G f(xt)d_l x \\ &= \int_G f(x)d_l(xt^{-1}) = \Delta(t)\int_G f(x)d_l x\end{aligned} \tag{11}$$

$d_l x$ denote the Haar measure induced by volume form. And the second line simply uses the change of variable. The third line uses the change of variable again and then definition of modular function $Delta$. So $\Delta(t) = |detAd(t)|$.

If $detAd(t) < 0$, then the change of variable of the measure is still valid, but the change of variable for the form on the second line is not. Here $detAd(t) < 0$ implies $R_{t^{-1}}$ changes the orientation, so the correct calculation should be

$$\begin{aligned}(detAd(t))\int_G f(x)d_l x &= (detAd(t))\int_G f\omega = \int_G fR_{t^{-1}}^{*}\omega \\ &= -\int_G (f\circ R_t)\omega = -\int_G f(xt)d_l x \\ &= -\int_G f(x)d_l(xt^{-1}) = -\Delta(t)\int_G f(x)d_l x\end{aligned} \tag{12}$$

So $\Delta(t) = |detAd(t)|$ also holds in this case.

Now it suffices to show that $R_{t^{-1}}^{*}\omega = (detAd(t))\omega$. Assume the Lie group has dimension m, thus ω is a m-form. It suffices to determine the action of $R_{t^{-1},*}$ (the push forward map) on a frame of tangent bundle, it is a linear action, its determinant is the needed coefficient.

More explicitly, we fix a left invariant frame $\{\tilde{X}_1, \cdots \tilde{X}_m\}$. The push forward action of $R_{t^{-1}}$ on $\tilde{X}$ is $Ad(t)$, because for any smooth function h on G

$$\begin{aligned}(R_{t^{-1}})_{*,p}(\tilde{X}_p)h &= \tilde{X}_p(h\circ R_{t^{-1}}) = \frac{d}{dr}h(p(\exp rX)t^{-1})|_{r=0} \\ &= \frac{d}{dr}h(pt^{-1}\exp(rAd(t)X))|_{r=0} = \widetilde{Ad(t)X}_{pt^{-1}}h\end{aligned} \tag{13}$$

The $\widetilde{Ad(t)X}$ is the left invariant vector field obtained by left translating the tangent vector $Ad(t)X \in T_{1_G}G$. The first equation on the second line uses the definition of Ad.

Therefore, for the left-invariant top form ω, we have

$$\begin{aligned}
&(R^*_{t^{-1}}\omega)_p((\tilde{X}_1)_p, \cdots (\tilde{X}_m)_p) \\
=&\omega_{pt^{-1}}((R_{t^{-1}})_{*,p}(\tilde{X}_1)_p, \cdots, (R_{t^{-1}})_{*,p}(\tilde{X}_m)_p) \\
=&\omega_{pt^{-1}}((\widetilde{Ad(t)X_1})_{pt^{-1}}, \cdots, (\widetilde{Ad(t)X_m})_{pt^{-1}}) \\
=&\omega_{pt^{-1}}((Ad(t)\tilde{X}_1)_{pt^{-1}}, \cdots, (Ad(t)\tilde{X}_m)_{pt^{-1}}) \\
=&(detAd(t))\omega_{pt^{-1}}((\tilde{X}_1)_{pt^{-1}}, \cdots, (\tilde{X}_m)_{pt^{-1}}) \\
=&(detAd(t))\omega_p((\tilde{X}_1)_p, \cdots, (\tilde{X}_m)_p)
\end{aligned} \tag{14}$$

The final equation uses the left-invariance of both ω and $\tilde{X}_i$. Combined with the discussion before, the proof is finished.

With this proposition and the theorem about decomposition of Haar measure mentioned on the class, we can calculate the Haar measure of $SL(2,\mathbb{R})$ using Iwasawa decomposition.

The Iwasawa decomposition says that $SL(2,\mathbb{R}) = KAN$, where $K = SO(2)$, $A = \{\left(\begin{smallmatrix} y & \\ & y^{-1}\end{smallmatrix}\right) | y > 0\}$, $N = \{\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right) | x \in \mathbb{R}\}$.

The Haar measure $d_l z$ of $SL(2,\mathbb{R})$ is

$$\begin{aligned}
d_l z &= \frac{\Delta_{AN}(an)}{\Delta_{SL(2,\mathbb{R})}(an)} d_l k \ d_l(an) \\
&= \Delta_{AN}(an) d_l k \ d_l a \ d_l n
\end{aligned} \tag{15}$$

In the second equation, we use the fact that $SL(2,\mathbb{R})$ and N are both unimodular here.

Denote $t = \left(\begin{smallmatrix} y & xy \\ & y^{-1}\end{smallmatrix}\right)$, then $t^{-1} = \left(\begin{smallmatrix} y^{-1} & -xy \\ & y\end{smallmatrix}\right)$. A basis of the Lie algebra of AN is $\left(\begin{smallmatrix} 1 & \\ & -1\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$. A simple calculation shows that $|detAd_{AN}(t)| = y^2$. So the Haar measure is

$$\begin{aligned}
d_l z &= y^2 d\theta \ \frac{dy}{y} \ dx \\
&= y d\theta \ dy \ dx
\end{aligned} \tag{16}$$

Making the change of variable $y' = y^{-2}$, now the measure looks like $-\frac{1}{2}\frac{1}{y'^2} d\theta \ dy' dx$, and up to a scalar, it is $\frac{1}{y'^2} d\theta \ dy' dx$. It takes the similar form of the hyperbolic metric of the upper half plane model. Since $SL(2,\mathbb{R})$ is just the isometry group of upper half plane $\mathbb{H}^2$, and is the symmetric space $SL(2,\mathbb{R})/SO(2)$ ($SO(2)$ is exactly the isotropy group of point $i \in \mathbb{C}$).