PERIODIC ORBITS OF $SL(d, \mathbb{R})$

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ABSTRACT. In week 8, we learn about classification results of periodic orbits in SL(d, R), which is closely related to totally real fields.

In this week(week 9), we finish the classification of periodic orbits in $SL(d, \mathbb{R})$, proving that all periodic orbits come from totally real fields. This note contains the contents of these two weeks.

CONTENTS

1. INTRODUCTION

In this note, we fix $G = SL(d, \mathbb{R})$, a standard Iwasawa decomposition $G = KAN$, and $\Gamma = SL(d, \mathbb{Z})$. It is natural to ask the following question

Question 1.1. For which point $x \in G/\Gamma$, the orbit $Ax\Gamma$ is periodic, $i.e.$ the stabilizer of $x\Gamma$ forms a lattice in A?

However more basic question is, does there exist any period A-orbit? The answer is closely related with totally real field. For every totally real field of deg d, we can actually associate a lattice in $\mathbb{R}^d$, which naturally corresponds to a point in G/Γ, which is the space of all lattices in $\mathbb{R}^d$ modulo homothety.

The first result is

Theorem 1.1. *For each K over $\mathbb{Q}$ a totally real field of degree d, under a Minkowski embedding $K \to \mathbb{R}^d$, the image of ring of integers $\mathcal{O}_K$ is a lattice in $\mathbb{R}^d$. Moreover, denote this lattice as $\Lambda(\mathcal{O}_K)$, the corresponding point $x \in G/\Gamma$ gives a periodic orbit $Ax\Gamma$.*

Combining this theorem with the following fact, we know that there are a lot of periodic A-orbits.

Theorem 1.2. *There are infinitely many totally real fields of degree d over $\mathbb{Q}$.*

Finally, almost every periodic A-orbits appear in this form, with a slight modification.

Theorem 1.3. *Fix a $\mathbb{Q}-$basis, $\{\alpha_1, \cdots, \alpha_d\}$, $\Gamma(span_{\mathbb{Z}}\{\alpha_1, \cdots, \alpha_d\})$ is a lattice in $\mathbb{R}^d$ and corresponds to a periodic orbit. Furthermore, all periodic orbits is of this form.*

This gives the classification of periodic A-orbits in $SL(d, \mathbb{R})$.

2. Lattices from ring of integers of totally real fields

In this section, we explain how to get lattices whose orbit is periodic from totally real fields.

For a field extension K over $\mathbb{Q}$ of degree d. It has exactly d different embedding into the algebraic closed field $\mathbb{C}$, mapping the primitive element to other roots of its minimal polynomial. Denote these embedding as $\{\sigma_1, \cdots, \sigma_d\}$. K is said to be totally real if all these embedding maps K into $\mathbb{R}$. For totally real field, we have a natural embedding

$$\sigma : x \mapsto (\sigma_1(x), \cdots, \sigma_d(x))^t, \tag{2.1}$$

such an embedding is called a Minkowski embedding.

Choose a $\mathbb{Q}$ basis $\alpha_1, \cdots, \alpha_d$, then the determinant of matrix $\{\sigma_i(\alpha_j)\}_{1\le i,j\le d}$ is nonzero and equal to $\pm d_K(\alpha_1, \cdots, \alpha_d)^{\frac{1}{2}}$ by definition. Here $d_K(\alpha_1, \cdots, \alpha_d)$ is the discriminant of field extension with respect to the $K|\mathbb{Q}$ basis $\alpha_1, \cdots, \alpha_d$. Recall that the discriminant is also the determinant of the matrix $\{Tr_{K|\mathbb{Q}}(\alpha_i\alpha_j)\}_{1\le i,j\le d}$, implying $Tr_{K|\mathbb{Q}}$ is a non-degenerated bilinear form.

This tells us $\sigma(\alpha_i) = (\sigma_1(\alpha_i), \cdots, \sigma_d(\alpha_i))^t, 1 \le i \le d$ forms a basis of $\mathbb{R}^d$, thus its $\mathbb{Z}$ span gives a lattice in $\mathbb{R}^d$. If $\alpha_1, \cdots, \alpha_d$ is chosen to be a $\mathbb{Z}-$basis of $\mathcal{O}_K$, which always exists, see for example [1] P12, then the $\mathbb{Z}-$span of $\sigma(\alpha_i)$ is exactly $\sigma(\mathcal{O}_K)$, so $\mathbb{Z}-$span of $\sigma(\alpha_i)$ is independent of the choice of $\mathbb{Z}-$basis of $\mathcal{O}_K$, they all give the same lattice, in latter context, we denote this lattice as $\Lambda(\mathcal{O}_K)$.

Now we consider the stabilizer of A-orbit $A\Lambda(\mathcal{O}_K)$. We identify $[I_d]$ with the lattice $\mathbb{Z}^d$ modulo homothety, and the $SL(d, \mathbb{R})$ action on lattices is induced by its canonical action on $\mathbb{R}^d$. More precisely, the action of $T = diag\{t_1, \cdots, t_d\} \in A$ is described as

$$T(\sigma(\alpha_1), \cdots, \sigma(\alpha_d)).$$

The condition T lies in the stabilizer of $\Lambda(\mathcal{O}_K)$ implies that

$$T(\sigma(\alpha_1), \cdots, \sigma(\alpha_d)) = (\sigma(\alpha_1), \cdots, \sigma(\alpha_d))\gamma, \tag{2.2}$$

where $\gamma \in \Gamma$, the left hand side describes how T acts on this lattice, while the right hand side shows after this action, it is still a $\mathbb{Z}-$basis of $\Lambda(\mathcal{O}_K)$.

In particular, by checking entries in the first column, (2.2) implies

$$t_i\sigma_i(\alpha_1) = \sigma_i(\beta_1)$$

where

$$(\beta_1, \cdots, \beta_d) = (\alpha_1, \cdots, \alpha_d)\gamma.$$

Now it is natural to define $a = \frac{\beta_1}{\alpha_1}$, then we have $t_i = \sigma_i(a)$.

Combine this with (2.2), we have the following equation holding in K

$$a(\alpha_1, \cdots, \alpha_d) = (\alpha_1, \cdots, \alpha_d)\gamma \tag{2.3}$$

since all the maps σ_i are field embedding. But (2.3) implies the equation $aI_d - \gamma$ has a left solution, so

$$\det(aI_d - \gamma) = 0, \tag{2.4}$$

which gives a monic polynomial with $\mathbb{Z}$ coefficient that annihilate a. This implies $a \in \mathcal{O}_K$. Thus every element in $Stab_A(\Lambda(\mathcal{O}_K))$ takes the form $diag\{\sigma_1(a), \cdots, \sigma_d(a)\}$ for some $a \in \mathcal{O}_K$. Also since T^{-1} also lies in the stabilizer, $T^{-1} = diag\{\sigma_1(b), \cdots, \sigma_d(b)\}$ for some $b \in \mathcal{O}_K$. This gives the inverse of a, so $a \in \mathcal{O}_K^\times$.

Moreover, it is clear that if $T \in A$ takes the form $diag\{\sigma_1(a), \cdots, \sigma_d(a)\}$ for some $a \in \mathcal{O}_K^\times$, it lies in the stabilizer, since $\{a\alpha_1, \cdots, a\alpha_d\}$ is still a $\mathbb{Z}-$basis of $\mathcal{O}_K$ and norm of elements in $\mathcal{O}_K^\times$ are ± 1. So up to now, we have proven the following result:

Lemma 2.5. *Consider the map*

$$\begin{aligned} f : \mathcal{O}_K^\times &\to A \\ x &\mapsto diag\{\sigma_1(x), \cdots, \sigma_d(x)\} \end{aligned}$$

Then the stabilizer $Stab_A(\Lambda(\mathcal{O}))$ can be identified with the image of

$$\{x \in \mathcal{O}_K^\times | \sigma_i(x) > 0, \forall i\}$$

under f.

According to the following Dirichlet unit theorem, we know the stabilizer is actually a lattice in A.

Theorem 2.6. *(Dirichlet) Suppose K is a number field with degree $d = r + 2s$, which means it has r real embedding $\{\rho_1, \cdots, \rho_r\}$, and $2s$ complex embedding that forms s pairs $\{\sigma_1, \overline{\sigma}_1, \cdots, \sigma_s, \overline{\sigma}_s\}$. Then the unit of ring of integers $\mathcal{O}_K^\times$ is isomorphic to direct sum of cyclic group μ_K and $\mathbb{Z}^{r+s-1}$. Moreover, under the map $l : \mathcal{O}_K^\times \to \mathbb{R}^{r+s}$*

$$l(x) = (\log|\rho_1(x)|, \cdots, \log|\rho_r(x)|, \log|\sigma_1(x)|^2, \cdots, \log|\sigma_s(x)|^2)^t, \tag{2.7}$$

the image of the $\mathbb{Z}^{r+s-1}$ part is mapped to a lattice in the hyperplane $\sum_{i=1}^{r+s} x_i = 0$.

For a proof, see for example [2] P87.

Thus we know all $A\Lambda(\mathcal{O}_K)$ are periodic orbits. Because the index of $\{x \in \mathcal{O}_K^\times | \sigma_i(x) > 0, \forall i\}$ in $\mathcal{O}_K^\times$ is finite, since $(\mathcal{O}_K^\times)^2 \subset \{x \in \mathcal{O}_K^\times | \sigma_i(x) > 0, \forall i\}$.

3. Classification of all periodic orbits

The main reference of this section is [3] Sec24.

For a general $\mathbb{Q}-$basis $\{\alpha_1, \cdots, \alpha_d\}$ of totally real field K, we can also assign a lattice in $\mathbb{R}^d$, under Minkowski embedding. Since all elements in K take the form $\mathcal{O}_K/\mathbb{Z}$, upon multiplying a large integer, we may assume this basis lies in $\mathcal{O}_K$.

Denote this lattice as Λ and the $\mathbb{Z}-$span of $\{\alpha_1, \cdots, \alpha_d\}$ as $\alpha \subset \mathcal{O}_K$. We first consider the $\mathbb{Z}$-algebra $End_A(\Lambda) := \{T \textit{ is a diagonal matrix} | T\Lambda \subset \Lambda\}$, then argue just as (2.2) by replacing $\gamma \in \Gamma$ with an $\gamma \in M_d(\mathbb{Z})$, we find out that $T \in End_A(\Lambda)$ takes the form $f(a)$, where f and a are the same as in Sec 2, but this time it is only mapped to a diagonal matrix, not necessarily have determinant 1. Now a still satisfies equation (2.4), this implies $a \in \mathcal{O}_K$.

Define

$$\mathcal{O} = \{x \in K | x\alpha_i \in \mathbb{Z} - span\{\alpha_1, \cdots, \alpha_d\}, 1 \le i \le d\}, \tag{3.1}$$

it is clear that $f(\mathcal{O}) \subset End_A(\Lambda)$. From argument above, we see $f(\mathcal{O}) = End_A(\Lambda)$ and $\mathcal{O} \subset \mathcal{O}_K$.

Now we want to show that it is an order, *i.e.* a subring of $\mathcal{O}$ isomorphic to $\mathbb{Z}^d$. First, since $\Lambda \subset \Lambda(\mathcal{O}_K)$ is also a lattice, it is of finite index in $\Lambda(\mathcal{O}_K)$, so there exist m such that $m\Lambda(\mathcal{O}_K) \subset \Lambda$. We claim that $m\mathcal{O}_K \subset \mathcal{O}$. This is because for $x \in \mathcal{O}_K$

$$mf(x)\Lambda \subset m\Lambda(\mathcal{O}_K) \subset \Lambda, \tag{3.2}$$

thus $f(x) \in End_A(\Lambda)$, which implies $x \in \mathcal{O}$. Now we have proved that

$$m\mathcal{O}_K \subset \mathcal{O} \subset \mathcal{O}_K \tag{3.3}$$

and by definition $\mathcal{O}$ is a subring, so $\mathcal{O}$ is an order.

Similarly, we can show

$$f(\mathcal{O}^\times) = Aut_A(\Lambda), \tag{3.4}$$

where $Aut_A(\Lambda) := \{T \ diagonal, \det(T) = \pm 1 | T\Lambda = \Lambda\}$, $Stab_A(\Lambda)$ is a subgroup of $Aut_A(\Lambda)$ of finite index.

Notice that if we can show $\mathcal{O}^\times$ has finite index in $\mathcal{O}_K^\times$, then due to the fact that $Stab_A(\Lambda)$ is a lattice in A, $Stab_A(\Lambda)$ is also a lattice in A, *i.e.* $A\Lambda$ is a periodic orbit. Now we prove this claim.

Denote $n = \#Aut(\Lambda/m\Lambda(\mathcal{O}_K))$, n is finite since automorphism group of a finite group is finite. For every $x \in \mathcal{O}_K^\times$, $f(x^n) = id_{\Lambda/m\Lambda(\mathcal{O}_K)}$. So

$$x^n(\alpha_1, \cdots, \alpha_d) = (\alpha_1, \cdots, \alpha_d) + (\alpha_1, \cdots, \alpha_d)\gamma. \tag{3.5}$$

Here γ is just a d-by-d matrix of integer entries, and $(\alpha_1, \cdots, \alpha_d)\gamma$ lies in $m\mathcal{O}_K$. Mapped into $\mathbb{R}^d$, this gives

$$f(x^n)(\sigma(\alpha_1), \cdots, \sigma(\alpha_d)) = (\sigma(\alpha_1), \cdots, \sigma(\alpha_d)) + (\sigma(\alpha_1), \cdots, \sigma(\alpha_d))\gamma. \tag{3.6}$$

Since $f(x)$ has determinant ± 1, taking determinant on both side of (3.6) tells us that $det(I_d + \gamma) = \pm 1$. This is equivalent to $f(x) \in Aut_A(\Lambda)$, and equivalent to $x \in \mathcal{O}^\times$. Now we have proved $(\mathcal{O}_K^\times)^n \subset \mathcal{O}^\times$, implying $\mathcal{O}^\times$ has finite index in $\mathcal{O}_K^\times$ (because the $\mathbb{Z}$-rank is finite). Now we have proved

Proposition 3.7. *Lattices Λ taking the above form give periodic A-orbits $A\Lambda$.*

Actually, all periodic orbits are of this form.

Theorem 3.8. *All periodic A-orbits are of the form $AD\Lambda$, where Λ is defined as above, and D is an invertible diagonal matrix.*

Remark 3.9. Adding the action of a invertible diagonal matrix does not affect the result much, since it commutes with A. Since we only consider lattices modulo homothety, the determinant of D also does not matter.

Proof. Fix a periodic orbit $A\Lambda$ in G/Γ, then $Stab_A(\Lambda) \cong \mathbb{Z}^{d-1}$ is a lattice in A. Take $T_1, \cdots, T_{d-1}$ to be a $\mathbb{Z}$-basis of $Stab_A(\Lambda)$, and $\{v_1, \cdots, v_d\}$ a $\mathbb{Z}$-basis of Λ. Due to definition of stabilizer, there exists $\gamma_i \in SL(d, \mathbb{Z})$ such that

$$T_i(v_1, \cdots, v_d) = (v_1, \cdots, v_d)\gamma_i. \tag{3.10}$$

Denote $g \in SL(d, \mathbb{R})$, such that $ge_i = v_i$. Then $T_i = g^{-1}\gamma_i g$, so T_i and γ_i have the same characteristic polynomial. According to the Cayley-Hamilton theorem, the characteristic polynomial annihilate the matrix, and the characteristic polynomial of γ_i is a monic polynomial with integer coefficient, thus T_i is also a 'root' of some monic integer coefficient polynomial. Also, due to that all eigenvalues of T_i are real, such monic integer coefficient polynomials all only have real roots.

Now we naturally have a commutative algebra $K = \mathbb{Q}[T_1, \cdots, T_{d-1}]$, and a lattice $T_1^{\mathbb{Z}} \cdots T_{d-1}^{\mathbb{Z}} \subset Stab_A(\Lambda)$. In previous case, $Stab_A(\Lambda)$ is a subring of $\mathcal{O}_K^\times$. In order to recover the field from it, we need to find a primitive element in $Stab_A(\Lambda)$.

First we find an element in stabilizer with distinct eigenvalues. This is possible since after taking logarithm,

$$T_1^{\mathbb{Z}} \cdots T_{d-1}^{\mathbb{Z}} \to a\ lattice\ in\ H \tag{3.11}$$

Here H is the hyperplane in $\mathbb{R}^d$ with $\sum_i x_i = 0$. Define the proper subspace $H_{i,j} := \{(x_k) \in H | x_i = x_j\}, i \neq j$ of H. We can not choose a matrix in $Stab_A(\Lambda)$ with distinct eigenvalues, then its image in H lies in the union of finitely many proper subspace $H_{i,j}$. So does the $\mathbb{Q}$-span of this lattice in H. Because there are only finitely many $H_{i,j}$, and every vector in $\mathbb{R}$-span of the lattice is the limit of some sequence in $\mathbb{Q}$-span, we see that the $\mathbb{R}$-span of lattice, which is H, lies in the finite union of $H_{i,j}$ (This follows from each $H_{i,j}$ is closed). But it is a standard fact that a vector space can not be written as a finite union of proper subspace, we derive a contradiction. Thus we can choose a $T \in Stab_A(\Lambda)$ with distinct eigenvalues.

Then we want to show its characteristic polynomial is irreducible. Denote f_T as its characteristic polynomial. From discussion above, we know it is a monic polynomial with integer coefficients and of degree d. We argue by contradiction. By Gauss lemma, it suffices to show f_T is irreducible over $\mathbb{Z}$.

Assume f_T is reducible in $\mathbb{Z}[x]$, with $f_T = gh$ such that g is a monic irreducible integer polynomial of degree $m < d$. Denote l the splitting field of f_T, it is clearly a Galois extension over $\mathbb{Q}$. Denote its Galois group as $Gal(l/\mathbb{Q})$. The polynomial g has roots $\alpha_1, \cdots, \alpha_m$ in l. They are eigenvalues of T and $g^{-1}Tg$. Recall that $g^{-1}Tg \in SL(d,\mathbb{Z})$.

Choose v_1 an row eigenvector of α_1 of $g^{-1}Tg$ in l^d, satisfying

$$v_1 g^{-1} T g = \alpha_1 v_1. \tag{3.12}$$

Choose elements σ_i in $Gal(l/\mathbb{Q})$ such that $\alpha_i = \sigma_i(\alpha_1)$ ($\sigma_1 = id$). Then $\sigma_i(v_1)$ gives the eigenvector of $\sigma_i(\alpha_1)$. Here the action is just given by acting on each entry, and note that the matrix is unchanged under Galois group action because it has integer entries. Also $\{\sigma_i(\alpha_1)\} = \{\tau(\alpha_1) | \tau \in Gal(l/\mathbb{Q})\}$ because $\tau(\alpha_1)$ must have the same minimal polynomial as α_1.

We have obtained a subspace $W = span_{\mathbb{R}}\{\tau(v_1) | \tau \in Gal(l/\mathbb{Q})\}$. It is clearly stable under $Gal(l/\mathbb{Q})$ and $g^{-1}Tg$, and it has dimension m because every generator $\tau(v_1)$ is a eigenvector of some α_i, each eigenvalue has multiplicity 1.

According to a standard fact of Galois action on vector spaces, the fact $W \subset l^d$ is stable under $Gal(l/\mathbb{Q})$ implies that it has a basis consisting of vectors in $\mathbb{Q}^d$, thus also a basis consisting of vectors in $\mathbb{Z}^d$. Denote this basis as $w_1, \cdots, w_m$. According to structure theorem of finitely generated Abelian group, it is possible to extend it to a $\mathbb{Z}$-basis $\{w_1, \cdots, w_d\}$ of $\mathbb{Z}^d$ (By considering W as a finitely generated free $\mathbb{Z}$-module of $\mathbb{Z}^d$, it suffices to show for a basis look like $m_1 e_1, \cdots, m_{d-1} e_{d-1}$, it can be extended to a $\mathbb{Z}^d$ basis. But this case is easy).

Under $\{w_1, \cdots, w_d\}$, $g^{-1}Tg$ is still a matrix of integer entries, and it is now block upper triangular because W is an invariant subspace. Write

$$g^{-1}Tg = (\begin{matrix} A_m & * \\ 0 & C_{d-n} \end{matrix}) \tag{3.13}$$

in the block uppertriangular form. We notice that since $g^{-1}T^{-1}g$ commutes with T, it also takes this form, and it also has integer entries. This shows that A_m has its inverse with also integer entries, *i.e.* $A_m \in SL(m, \mathbb{Z}^d)$. Because all eigenvalues of T are positive by definition of A_m, we have $\alpha_i > 0, 1 \le i \le m$, thus $\det A_m = 1$.

Since $g^{-1}Hg$ has integer entries for every $H \in Stab_A(\Lambda)$, and they all commutes with T, the same argument shows they all take the form like (3.12) and with $\det A_m = 1$. But this adds an additional constrain that the product of first m eigenvalues of matrices in $Stab_A(\Lambda)$ equal to 1, contradicting the fact that it is a lattice in A isomorphic to $\mathbb{Z}^{d-1}$. Thus we have shown that f_T is irreducible.

Denote a root of f_T as α_1, and $T = diag\{\alpha_1, \cdots, \alpha_d\}$.Then $K = \mathbb{Q}[\alpha_1]$ is a totally real field of degree d because f_T only has real roots (T only have real eigenvalues). Denote $g^{-1}Tg = \gamma \in SL(d, \mathbb{Z})$. Write g in the form

$$g = \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_d \end{pmatrix} \tag{3.14}$$

Now $g^{-1}Tg = \gamma$ implies $\alpha_i u_i = u_i \gamma$. Choose $v_1 \in K^d$ be a row eigenvector of α_1 (it exists since $\alpha_1 \in K$), then from arguments before, we know there exists d different embedding of K, $\sigma_i, 1 \le i \le d, \sigma_1 = id$, such that $\alpha_i = \sigma_i(\alpha_1)$, and $\sigma_i(v_1)$ is a eigenvector of α_i. Since u_i is also an eigenvector of α_i, we have $u_i = h_i\sigma_i(v_1)$ for some nonzero $h_i \in \mathbb{R}$. Denote $v_1 = \{\beta_1, \cdots, \beta_d\} \in K^d$.

Notice the matrix $V = \begin{pmatrix} v_1 \\ \sigma_2(v_1) \\ \cdots \\ \sigma_d(v_1) \end{pmatrix}$ gives exactly the lattice

$$V\mathbb{Z}^d = span_{\mathbb{Z}}\{\sigma(\beta_1), \cdots, \sigma(\beta_d)\},$$

which is the form we need (Here β_i is automatically a basis of K because V is invertible due to the fact that eigenvectors of T are linearly independent). Thus the lattice $\Lambda = g\mathbb{Z}^n = HV\mathbb{Z}^d$, where H is the diagonal matrix $H = diag\{h_1, \cdots, h_d\}$, which takes the form we need in the statement of theorem.

□

It remains to show when two lattices are the same. For $\mathfrak{a}_1$, $\mathfrak{a}_2$ two full module in K_1 and K_2. They determine two lattices $\Lambda(\mathfrak{a}_1)$, $\Lambda(\mathfrak{a}_2)$. If they give the same periodic orbit, then they have the same stabilizer. Since the stabilizer correspond to an order, and from the argument above, we see it contains a primitive element, this implies that K_1 and K_2 have primitive element with same minimal polynomial (both correspond to the characteristic polynomial of correspondent matrix in stabilizer). So with out loss of generality, we may assume $K_1 = K_2$.

Now it suffices to consider when two full modules give the same periodic orbit. A straight forward argument shows $A\mathfrak{a}_1 = A\mathfrak{a}_2$ if and only if there exists a β, with $\sigma_i(\beta) > 0, \forall i$, such that $\mathfrak{a}_2 = \beta\mathfrak{a}_1$.

4. DISCRIMINANT AND REGULATOR

For a periodic orbit $A\Lambda$, we can assign several invariant to it. For example the order

$$End_A(L) = End(L) \cap \mathbb{R}[A],$$

the regulator

$$R = vol(A\Lambda) = vol(A/Stab_A(\Lambda)),$$

and the discriminant

$$D = vol(A\Lambda/End_A(L)).$$

When Λ comes from a full module of totally real field K, with order $\mathcal{O}$. Then

$$R = 2^m Reg(\mathcal{O}), \tag{4.1}$$

where $2^m = |\mathcal{O}^\times/(\pm\mathcal{O}_+^\times)|$ and $Reg(\mathcal{O})$ is just the usual regulator. Here $\mathcal{O}_+^\times$ are elements in $\mathcal{O}^\times$ with positive images under all σ_i. Since we are considering $Stab_A(\Lambda)$, we need to replace some elements in the $\mathbb{Z}$-basis $\{u_1, \cdots, u_{d-1}\}$ of $\mathcal{O}^\times$ with u_i^2, producing the coefficient 2^m.

5. Norm of periodic A-orbit

We define the norm $N : \mathbb{R}^d \to \mathbb{R}$, by

$$N(x) = |x_1 \cdots x_d|,$$

where x_i are the d components of vector x. For a given Minkowski embedding $\sigma : K \to \mathbb{R}^d$, $N \circ \sigma$ is just the absolute value of norm.

It behaves well under A-action, since A has positive entries and determinant 1. We have

$$N(Ax) = N(x).$$

From the AM-GM Inequality, it is clear that

$$\sqrt{d}N(x)^{1/d} \leq \|x\|, \tag{5.1}$$

where the right hand side is the usual Euclidean norm. Using Mahler's principle, we have the following theorem:

Theorem 5.2. *The orbit $A\Lambda$ has compact closure if and only if $N(\Lambda) > 0$. Here $N(\Lambda) = \inf\{N(x) : x \in \Lambda, x \neq 0\}$.*

Proof. If $N(\Lambda) > 0$, for all $a \in A$, $N(a\Lambda) = N(\Lambda) > 0$ uniformly. Thus from (5.1) and Mahler's principle, we deduce that it has compact closure.

If it has compact closure, then from Mahler's principle, $L(A\Lambda) > 0$, where $L(A\Lambda) = \inf\{\|x\| : x \in a\Lambda, a \in A, x \neq 0\}$. So there exists a positive real number r, such that $\|ax\| > r$ for all $a \in A$ and nonzero $x \in \Lambda$.

For each nonzero $x \in \Lambda$, we can find a suitable $a \in A$, such that ax is of the form $(x_1, x_2, \cdots, x_d)^t$ with all $|x_i|$ equal. Thus

$$N(x) = N(ax) = (\frac{\|ax\|}{\sqrt{d}})^d \geq (\frac{r}{\sqrt{d}})^d > 0 \tag{5.3}$$

holds, which implies $N(\Lambda) > 0$. □

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