# THE VANISHING THEOREM OF HOWE AND MOORE

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Abstract. In this week, basic properties of dynamics on Lie groups and their homogeneous spaces are introduced, concepts like stable and unstable subgroup and submanifold is introduced under the setting of homogeneous dynamics. Finally, notion of strong mixing is introduced in the language of unitary representation. In this note, we consider the proof of vanishing theorem of matrix coefficients due to Howe and Moore, and the proof follows from chapter 3 of [1].

## Contents



## 1. Preliminaries in Lie theory

For $G$ a semisimple real Lie group, denote its Lie algebra as $\mathfrak{g}$, choosing a Cartan involution, we can get a Cartan decomposition, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Corresponds to this decomposition, we can get a decomposition of the Lie group, also called Cartan decomposition, here $K$ is the analytic subgroup corresponds to $\mathfrak{k}$. If $G$ has finite center, then $K$ is the maximal compact subgroup of $G$, see for example Thm 6.31 [3]

$$G = K \exp \mathfrak{p} \tag{1.1}$$

From now on, we assume the Lie group $G$ has finite center, so $K$ is compact. It makes many arguments easier.

Choose $\mathfrak{a}$ the maximal abelian subalgebra of $\mathfrak{p}$, it gives a relative root decomposition, giving a set of relative roots, denoted as $\Sigma$. We can choose a positivity on $\Sigma$, getting a set of positive roots $\Sigma^+$.

Denote the positive Weyl chamber as $\mathfrak{a}^+ = \{H|\lambda(H) > 0, \forall H \in \Sigma^+\}$, let

$$A^+ := \{expH|H \in \mathfrak{a}^+\} \tag{1.2}$$

From the fact $\mathfrak{p} = \cup_{k\in K} Ad(k)\mathfrak{a}$ and Cartan decomposition, we have the KAK decomposition

$$G = K \cdot \overline{A^+} \cdot K \tag{1.3}$$

**Example 1.4.** For the real split Lie algebra $G = SL(n,\mathbb{R})$, $K = SO(n)$, and $\overline{A^+}$ can be chosen as diagonal matrices $diag\{a_1, \cdots, a_n\}$ with $a_1 \geq a_2 \geq \cdots \geq a_n$.

We also need to consider stable and unstable manifold under one-parameter diagonal flow, it has its correspondent decomposition on the Lie algebra level.

The importance of KAK decomposition is that it tells us in order to prove the vanishing of matrix coefficients at infinity, it suffices to show the vanishing at infinity for $\overline{A}^+$, which is the following lemma.

**Lemma 1.5.** *Let $(\pi, \mathcal{H}_\pi)$ be a strongly continuous unitary representation of $G$, i.e. $g \mapsto \pi(g)v$ is continuous for every $v \in \mathcal{H}_\pi$. Suppose that, for all matrix coefficients $\varphi_{\xi\eta}(g) := \langle \eta, \xi\pi(g)\rangle_{\mathcal{H}_\pi}$, and for all sequences $\{a_n\}$ in $\overline{A^+}$ with $\lim_{n\to\infty} a_n = \infty$, we have*

$$\lim_{n\to\infty} \varphi_{\xi\eta}(a_n) = 0 \tag{1.6}$$

*Then all matrix coefficients vanish at infinity on $G$.*

*Proof.* We argue by contradiction. We fix a matrix coefficient $\varphi_{\xi\eta}$. If the result does not hold, then there exists a sequence $\{g_n\}$ in $G$ tends to infinity, such that

$$\lim_{n\to\infty} \varphi_{\xi\eta}(g_n) \neq 0 \tag{1.7}$$

If the limit does not exist, then pass to a subsequence that tends to infinity. Using KAK decomposition, $g_n = k_n a_n h_n$, where $k_n, h_n \in K, a_n \in \overline{A^+}$. Since K is compact, $\lim_{n\to\infty} a_n = \infty$ follows immediately from $\lim_{n\to\infty} g_n = \infty$.

As $K$ is compact, by passing to a subsequence, we may assume that $\pi(h_n)\xi = \overline{\xi}$ and $\pi(k_n^{-1})\eta = \overline{\eta}$ under norm topology. Here the $\pi(k_n^{-1})$ is just the adjoint of $\pi(k_n)$ since it is unitary. Then we have

$$\begin{aligned} \varphi_{\xi\eta}(k_n a_n h_n) = (&\varphi_{\xi\pi(k_n^{-1})\eta}(a_n h_n) - \varphi_{\xi\overline{\eta}}(a_n h_n)) \\ &+ (\varphi_{\pi(h_n)\xi\overline{\eta}}(a_n) - \varphi_{\overline{\xi}\overline{\eta}}(a_n)) + \varphi_{\overline{\xi}\overline{\eta}}(a_n) \end{aligned} \tag{1.8}$$

But the first two terms both tend to zero, so

$$\lim_{n\to\infty} \varphi_{\overline{\xi}\overline{\eta}}(a_n) = \lim_{n\to\infty} \varphi_{\xi\eta}(g_n) \neq 0, \tag{1.9}$$

which contradicts our condition. □

Another notation from Lie theory needed is the Lie algebra counterpart of local composition into stable and unstable subgroup on the Lie group level.

We have the relative root decomposition

$$\begin{aligned} \mathfrak{g} &= \sum_{\lambda\in\Sigma} \mathfrak{g}_\lambda \oplus \mathfrak{g}_0 \\ \mathfrak{g}_0 &= \mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a}) \end{aligned} \tag{1.10}$$

For $b = exp_G H, H \in \mathfrak{a}$, define

$$\mathfrak{g}_b^- := \sum_{\lambda(H)<0} \mathfrak{g}_\lambda,\ \mathfrak{g}_b^0 := \sum_{\lambda(H)=0} \mathfrak{g}_\lambda,\ \mathfrak{g}_b^+ := \sum_{\lambda(H)>0} \mathfrak{g}_\lambda. \tag{1.11}$$

Which decomposes $\mathfrak{g} = \mathfrak{g}_b^- \oplus \mathfrak{g}_b^0 \oplus \mathfrak{g}_b^+$ with respect to signs of eigenvalue of $ad(H)$ action.

## 2. Vanishing theorem of Howe and Moore

The statement of the theorem is as follows

**Theorem 2.1.** *(Howe-Moore) Let $G$ a connected semisimple Lie group with finite center. Let $(\pi, \mathcal{H}_\pi)$ be a strongly continuous unitary representation of $G$. Assume that the restriction of $\pi$ to any non-compact simple factor $S_i$ of $G$ has no non-trivial invariant vector. Then all the matrix coefficients of $\pi$ vanish at infinity.*

Roughly speaking, the proof proceeds in two steps, the first step is the Mautner lemma, which states that a limit point of sequence $\{\pi(a_n)\}$ under weak topology is fixed by the closed subgroup generated by unstable manifold. The second step is to show if a vector in such unitary representation is fixed by such a closed subgroup, then it is fixed by the whole group or a non-compact factor of the group, then by the assumption, such vector can only be zero. If the two step are correct, then it shows for any diverging $\{a_n\}$, all the accumulation points of the weak compact set $\{\pi(a_n)\xi\}$ are zero, thus it must tends to zero vector weakly, showing the vanishing of matrix coefficients at infinity.

Now we state the key lemma in the first step.

**Theorem 2.2.** *Let $G$ be a locally compact group and $(\pi, \mathcal{H}_\pi)$ a strongly continuous unitary representation of $G$. Let $\alpha = \{a_n\}_n$ be a sequence in $G$ and let $\xi, \xi_0 \in \mathcal{H}_\pi$ such that under weak topology,*

$$\lim_{n\to\infty} \pi(a_n)\xi = \xi_0.$$

*Then*

$$\pi(x)\xi_0 = \xi_0$$

*for all $x \in N_\alpha^+$. Here $N_\alpha^+$ is the closed subgroup generated by $U_\alpha^+$, where*

$$U_\alpha^+ := \{g \in G | e \text{ is an accumulation point of } \{a_n^{-1} g a_n\}_n\}$$

**Remark 2.3.** The $U_\alpha^+$ here is larger than the $G_a^+ := \{\lim_{n\to\infty} a^{-n} g a^n = e\}$ defined on Friday's class when $\alpha = \{a\}_n$, where $G_a^+$ itself is already a closed subgroup.

*Proof.* Fix $x \in N_\alpha^+$. Upon passing to a subsequence, we may assume $\lim_{n\to\infty} a_n^{-1} x a_n = e$. Then

$$\begin{aligned} |\langle \pi(x)\xi_0, \eta\rangle - \langle \xi_0, \eta\rangle| &= \lim_{n\to\infty} |\langle \pi(x a_n)\xi, \eta\rangle - \langle \pi(a_n)\xi, \eta\rangle| \\ &= \lim_{n\to\infty} |\langle \pi(a_n^{-1} x a_n)\xi, \pi(a_n)^{-1}\eta\rangle - \langle \xi, \pi(a_n)^{-1}\eta\rangle| \\ &\le \lim_{n\to\infty} \|(\pi(a_n^{-1} x a_n) - Id_{\mathcal{H}_\pi})\xi\| \|\eta\| = 0 \end{aligned} \tag{2.4}$$

The last step uses the Cauchy-Schwarz identity, and uses strong continuity to show $\|(\pi(a_n^{-1} x a_n) - Id_{\mathcal{H}_\pi})\xi\|$ tends to zero (the continuity in strong continuity is under norm topology, but for the limit of $a_n\xi$ here, weak continuity is sufficient). □

Now we have seen limit points of every $\alpha = \{a_n\}$ is fixed by a closed subgroup $N_\alpha^+$ associated to it. But it remains unclear whether the subgroup $N_\alpha^+$ is large enough so that all $N_\alpha^+$ fixed vectors are also fixed by a non-compact component of $G$. The next lemma answers this question, it tells us that when $\{a_n\}_n$ diverges, this closed subgroup is non-discrete.

For $\alpha = \{b^n\}_n$, we write $U_b^+$ and $N_b^+$ instead of $U_\alpha^+$ and $N_\alpha^+$.

**Lemma 2.5.** *(i) For $b \in A\backslash\{e\}$, let $G_b$ be the closed subgroup generated by $N_b^+$ and $N_{b^{-1}}^+$. Then $G_b$ is a non-discrete normal subgroup of $G$.*
*(ii) For every sequence $\alpha$ in $\overline{A^+}$ converging to infinity, there exists $b \in \overline{A^+}\backslash\{e\}$ with $N_\alpha^+ = N_b^+$.*

*Proof.* For (i), we recall the decomposition $\mathfrak{g} = \mathfrak{g}_b^- \oplus \mathfrak{g}_b^0 \oplus \mathfrak{g}_b^+$ of the Lie algebra introduced at the end of section 1. According to the property

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu} \quad \forall \lambda, \mu \in \Sigma \tag{2.6}$$

of relative root spaces, we have $[\mathfrak{g}_b^0, \mathfrak{g}_b^\pm] \subset \mathfrak{g}_b^\pm$, and the fact that this is a decomposition of $\mathfrak{g}$ into subalgebras. So take $\mathfrak{g}_b$ to be the subalgebra generated by $\mathfrak{g}_b^+$ and $\mathfrak{g}_b^-$, it is also an ideal.

We want to prove the non-discreteness of $G_b$ by showing that $\exp_G(\mathfrak{g})$ generates $G_b$.

First, $\exp_G(\mathfrak{g}_b) \subset G_b$, because $\exp_G(\mathfrak{g}_b^+) \subset N_b^+$ and $\exp_G(\mathfrak{g}_b^-) \subset N_{b^{-1}}^+$. This is due to the following equation

$$b^n \exp_G X b^{-n} = \exp_G(Ad(b)^n X). \tag{2.7}$$

Note that $\exp_G(\mathfrak{g}_b)$ is contained in the closed subgroup generated by $\exp_G(\mathfrak{g}_b^+)$ and $\exp_G(\mathfrak{g}_b^-)$.

Second, we can use canonical chart of second type with respect to $\mathfrak{g} = \mathfrak{g}_b^- \oplus \mathfrak{g}_b^0 \oplus \mathfrak{g}_b^+$, see for example section 1.10 of [3], to show that there exists a diffeomorphism

$$U = U^- \cdot U^0 \cdot U^+, \tag{2.8}$$

where $U$ is a neighborhood of $e$ in $G$, and $U^+$ is a open neighborhood of $e$ in $\exp_G(\mathfrak{g}_b^+)$, $U^0$ and $U^-$ are chosen similarly.

For $x \in U_b^+$, there exists a sequence $a_k$, such that $b^{-a_k} x b^{a_k}$ tends to $e$, so it finally lies in $U$. But $b^{-a_k} x b^{a_k} \in U_b^+$ always holds, so it lies in $U_b^+ \cap U$. We may choose these neighborhoods smaller so that $U$ is contained in $\exp_G(\mathfrak{g})$, so at this time $b^{-n} x b^n = \exp_G(X)$ for some $X \in \mathfrak{g}$ and n. Now the condition $b^{-n} x b^n \in U_b^+$ implies that $X \in \mathfrak{g}_b^+$. Using (2.7), this shows $x \in \exp_G(\mathfrak{g}_b^+)$, *i.e.*$U_b^+ \subset \exp_G(\mathfrak{g}_b^+)$. Similarly $U_{b^{-1}}^+ \subset \exp_G(\mathfrak{g}_b^-)$. Combined with $\exp_G(\mathfrak{g}_b) \subset G_b$, this shows $\exp_G(\mathfrak{g})$ generates $G_b$.

Finally $\mathfrak{g}_b \neq 0$. Otherwise it lies in the center of $\mathfrak{g}$, but the center of $\mathfrak{g}$ is trivial since it is semisimple. So the proof of non-discreteness of $G_b$ is finished.

For (ii), assume $\alpha = \{\exp_G(a_n)\}_n$ with $a_n \in \overline{\mathfrak{a}^+}$, and $\lim_{n\to\infty} a_n = \infty$. Because $\Sigma$ is a abstract root system, we can choose a basis consisting of simple roots under the positivity that gives $\mathfrak{a}^+$, denoted as $R$, so all the relative roots are $\mathbb{Z}-$combination of simple roots with all coefficients non-negative or non-positive. Then there exists some simple root $\lambda \in R$ such that

$$\limsup_{n\to\infty} \lambda(a_n) = \infty. \tag{2.9}$$

Otherwise $\{a_n\}$ lies in a bounded region in $\overline{\mathfrak{a}^+}$ ($0 \le \lambda(a_n)$ always holds due to the definition of $\mathfrak{a}^+$), contradicting the assumption.

Let $R_\alpha$ be the set of all $\lambda \in R$ such that (2.9) holds. We can define a unique $H \in \overline{\mathfrak{a}^+}$ satisfying

$$\lambda(H) = \begin{cases} 1 & if\ \lambda \in R_\alpha \\ 0 & if\ \lambda \in R\backslash R_\alpha, \end{cases} \tag{2.10}$$

such $H$ can be easily constructed using the dual basis of $R$. Define $b := \exp_G H$. It lies in $\overline{A^+}$ and does not equal to identity.

Now $\lambda(H) > 0$ for $\lambda \in \Sigma$ if and only if $\lambda \in \Sigma^+$ and is not contained in the subspace generated by $R \backslash R_\alpha$.

We want to show $\exp_G(\mathfrak{g}_b^+) = U_\alpha^+$ to finish the proof, since we have seen $\exp_G(\mathfrak{g}_b^+) = U_b^+$.

For one side, $\exp_G(\mathfrak{g}_b^+) \subset U_\alpha^+$. Because if $\lambda(H) > 0$, *i.e.* $\mathfrak{g}_\lambda \subset \mathfrak{g}_b^+$, then for $X = \sum X_\lambda \in \mathfrak{g}_b^+$.

$$\begin{aligned} &\exp_G(a_n)^{-1} \exp_G X \exp_G(a_n) \\ =& \exp_G(-ad(a_n)X) \\ =& \exp_G(-ad(a_n)(\sum X_\lambda)) \\ =& \exp_G(-\sum \lambda(a_n)), \end{aligned} \tag{2.11}$$

the term on the exponential tends to $-\infty$, so it lies in $U_\alpha^+$.

For $U_\alpha^+ \subset \exp_G(\mathfrak{g}_b^+)$, we argue as in the proof of (i). Choose a decomposition

$$U = U^- \cdot U^0 \cdot U^+, \tag{2.12}$$

as before.

Fix $x \in U_\alpha^+$, since $\lim_{n\to\infty} \exp_G(a_n)^{-1} x \exp_G(a_n) = e$, for $n$ large enough, there exists $X_n^\pm \in \mathfrak{g}_b^\pm$ and $X_n^0 \in \mathfrak{g}_b^0$ with $\lim_{n\to\infty} X_n^\pm = \lim_{n\to\infty} X_n^0 = 0$,such that

$$\exp_G(a_n)^{-1} x \exp_G(a_n) = \exp X_n^- \exp X_n^0 \exp X_n^+. \tag{2.13}$$

Then for $n$ large,

$$x = \exp_G(ad(a_n)X_n^-) \exp_G(ad(a_n)X_n^0) \exp_G(ad(a_n)X_n^+) \tag{2.14}$$

Because $ad(a_n)$ is bounded on $\mathfrak{g}_b^0$ and $\mathfrak{g}_b^-$ (its eigenvalues are bounded), $ad(a_n)X_n^0$ and $ad(a_n)X_n^-$ tends to zero. Thus we have

$$x = \lim_{n\to\infty} \exp_G(ad(a_n)X_n^+) \tag{2.15}$$

which shows $x$ lies in the closure of $\exp_G(\mathfrak{g}_b^+)$, which is sufficient to show $U_\alpha^+$ and $\exp_G(\mathfrak{g}_b^+)$ generate the same closed subgroup. □

To finish the proof of the theorem, we need to show vectors fixed by $N_b^+$ is fixed by a large enough subgroup of $G$, and such vectors arise as weak limit points in orbits of vector under some divergence diagonal action $\alpha = \{a_n\}_n$ by Mautner's theorem. In chapter 3 of [1], this is done by first proving the theorem for $SL(2,\mathbb{R})$, then it is true for all $SL(2,\mathbb{R})$ copies in general semisimple group with finite center, and the theorem follows.

I do not have enough time to finish typing this final part, and will be finished later.

## 3. Bibliography

### References

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[3] Anthony W.Knapp. Lie Groups Beyond an Introduction. Second Edition. Birkhauser. 2002.