NOTE OF WEEK 10

RONGHAN YUAN

Abstract. In week 11, we proved a theorem relating the periodic orbit and values of products of linear forms.

Contents

1. The main theorem

The main theorem we proved in this week is the following one

Theorem 1.1. *The orbit $A\Lambda \subset SL_d(\mathbb{R})/SL_d(\mathbb{Z})$ is compact (periodic), if and only if $N(\Lambda) > 0$ and the set $\{N(v)|v \in \Lambda\}$ is discrete.*

Remark 1.2. Here $N(v) = |x_1 \cdots x_d|$, where $v = (x_1, \cdots, x_d)^t$.

For one direction, the result follows from the classification of periodic orbits we proved in last few weeks. Those period orbits gives forms of degree d with integer coefficients (or multiplying a common constant), thus the set $\{N(v)|v \in \Lambda\}$ is discrete. The result $N(\Lambda) > 0$ follows from another lemma proved before, which is a variant of Mahler principle, saying that $A\Lambda$ has compact closure if and only if $N(\Lambda) > 0$.

For another direction, the condition $N(\Lambda) > 0$ already implies that the orbit has compact closure, thus it remains to show that it is a close orbit.

Proof. As is explained in the remark, it suffices to show that $A\Lambda$ is closed to finish the proof. Denote $G = SL_d(\mathbb{R}), \Gamma = SL_d(\mathbb{Z})$. Since the topology on G/Γ is the quotient topology, it suffices to show the corresponding set $Ag\Gamma$ is closed in G, where $\Lambda = g\mathbb{Z}^n$. This is equivalent to showing that $Ag\Gamma$ is closed in $A\backslash G$ under quotient topology. Using the transposition of Iwasawa decomposition, we know that G can be decomposed into $G = ANK$, here $K = SO(d)$, and N the lower triangular matrices. Thus $A\backslash G$ has a nice description NK.

In order to use the condition $\{N(v)|v \in \Lambda\}$ is discrete. We consider the map $A\backslash \to P_d[x_1, \cdots, x_d]$, where $P_d[x_1, \cdots, x_d]$ denote polynomials of d variables with degree d. The mapping is given by $g \mapsto \phi \circ g$, g acts on $x_1, \cdots, x_d$ by linear transformation, and $\phi = x_1 \cdots x_d$. The 'kernel' (inverse image of ϕ) of this map is not large, because the stabilizer of ϕ under G is not large.

If $\phi = \phi \circ g$, where

$$\phi \circ g = (g_{1,1}x_1 + \cdots + g_{1,d}x_d)(g_{2,1}x_1 + \cdots + g_{2,d}x_d)\cdots(g_{d,1}x_1 + \cdots + g_{d,d}x_d)$$

Since P_d is a unique factorization domain, each factor in $\phi \circ g = \phi$ should also look like $c_i x_i$, which means that g only has one nonzero element in each column and row. (This can be also seen from the fact that zero set of $\phi \circ g$ is the union of d hyperplane $\{x_i = 0\}$, thus the zero set of each factor is one of the $\{x_i = 0\}$, also implying that $g_{i,1}x_1 + \cdots + g_{i,d}x_d = c_j x_j$ for some j).

Thus the stabilizer consists of permutation matrices, diagonal matrices and their products, and this implies that the inverse image of each point in $A\backslash G$ is a finite set of cardinal at most $2^d d!$.

The map $A\backslash G \to P_d$ is also a proper mapping. To show the map is proper, it suffices to show that any convergent sequence in the image set gives a convergent subsequence in $A\backslash G$. We know $A\backslash G$ can be described as NK, thus it suffices to show if $\phi \circ n_i k_i$ is convergent, then $n_i k_i$ has a convergent subsequence. Since K is compact, passing to a subsequence, we only need to show, for convergent $\phi \circ n_i$, n_i also converges.

If we can recover the entries of n_i from coefficients of $\phi \circ n_i$, then the result follows. This can be shown by induction on level of entries. For $n = (n_{i,j})$ lower triangular, $i-j$ is called the level. If the level is 1, then the coefficient of $x_1 \cdots x_i x_i x_{i+2} \cdots x_d$ is exactly $n_{i+1,i}$, so the result is true for level 1 entries. Similarly, consider coefficients of $x_1 \cdots x_i x_{i-1} x_{i+2} \cdots x_d$, they are $n_{i+1,i-1} + n_{i,i-1}n_{i+1,i}$. Since entries of level 1 can be recovered, from this we know $n_{i+1,i-1}$ can also be recovered from coefficients of $\phi \circ n$. Continuing on induction, we can show similarly that entries of higher level can also be recovered. Thus proving the map $A\backslash G \to P_d$ is proper.

Then we want to encode the information of polynomials in P_d by its values on finitely many integer points, and the information about $\{N(v)|v \in \Lambda\}$ can be used under this setting. More precisely, we want to find a finite set $E \subset \mathbb{Z}^d$, such that $\forall f \in P_d$, it is determined by its value on E. To find such a set E, we argue as follows. Fix a basis $f_1, \cdots, f_m$ of the vector space P_d, then choose a maximal subset E of $\mathbb{Z}^d$, such that $(f_1(v), \cdots, f_m(v)), v \in E$ is linearly independent.

We claim that E has exactly m elements (It has at most m elements). Suppose the contrary that it only has $r < m$ elements, then by the choice of E, we know for every $w \in \mathbb{Z}^d$, $(f_1(w), \cdots, f_m(w))$ can be written as linear combination of $(f_1(v), \cdots, f_m(v)), v \in E$. With out loss of generality, we assume the matrix $(f_i(v_j)), 1 \le i \le r, v_j \in E$ is invertible. But this implies that values of $f_j, r+1 \le j \le m$ on $\mathbb{Z}^d$ is the same as some linear combination of $f_i, 1 \le i \le r$ (the coefficient is determined by the linear dependence in the matrix $(f_i(v_j))_{1 \le i \le m, 1 \le j \le r}$) , implying that $f_i, 1 \le i \le r$ gives a basis, thus we can only have $r = m$.

Then every polynomial f in P_d is determined by its value on E, since we can solve the coefficient of f under f_i by solving the full rank linear equation $\sum a_i f_i(v) = f(v), v \in E$, where the coefficient matrix is a $m \times m$ invertible matrix.

Now we are ready to show that $Ag\Gamma$ is closed in $A\backslash G$. It suffices to show if $Ag\gamma_i$ converges in $A\backslash G$, then the limit is still in $Ag\Gamma$. It suffices to show $\phi \circ g\gamma_i$ has limit in the image set, because the preimage of every point in P_d is discrete, this is enough to show $Ag\Gamma$ is closed. Since $f \in P_d$ is determined on its value on E, and $\{|\phi \circ g\gamma_i(v)||i \in \mathbb{Z}_{>0}, v \in E\} \subset \{N(v)|v \in \Lambda\}$ lies in this discrete set, $\phi \circ g\gamma_i(v)$ eventually stabilizes after finite many terms. Since there are only finitely many elements in E, after finitely many terms, $\phi \circ g\gamma_i$ takes the same value on E, thus $\phi \circ g\gamma_i$ eventually stabilizes. This proves that the image of $Ag\Gamma$ in P_d is closed, also proves that $Ag\Gamma$ is also closed, thus proving the theorem. $\square$

2. Theorem of Cassels and Swinnerton-Dyer

In [4], Cassels and Swinnerton-Dyer proved the following theorem.

Theorem 2.1. *For $g \in SL_3(\mathbb{R})$, $Ag\Gamma$ a compact orbit in $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$, then for all $(\delta_1, \delta_2) \subset \mathbb{R}$, there exists a neighborhood U of e in $SL_3(\mathbb{R})$, such that for all $h \in Ug \backslash Ag$, $\phi \circ h(\mathbb{Z}^3 \backslash \{0\}) \cap (\delta_1, \delta_2) \neq \emptyset$.*

We consider the following observation, for U_t the one parameter subgroup whose Lie algebra is $E_{i,j}$, for example, $E_{1,2}$, to be a root vector. Denote u_t to be

$$u_t = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

Then fix Λ be a lattice given by a full module I of a totally real cubic field K, *i.e.* $\Lambda = \sigma(I)$, where σ is the Minkowski embedding. Now we consider the vector $x = (\sigma_1(a), \sigma_2(a), \sigma_3(a))^t$ in Λ, where $a \in I$. It gives an order $\mathcal{O}$ isomorphic to $\mathbb{Z}^3$, which will give a $\mathcal{O}^\times \cong \mathbb{Z}^2$ lies in the ring of integers.

Then we consider the image of $u_t\sigma(sa), a \in \mathcal{O}^\times$ under the norm.

$$\begin{aligned} N(u_t\sigma(sa)) &= |(\sigma_1(sa) + t\sigma_2(sa))\sigma_2(sa)\sigma_3(sa)| \\ &= |(\sigma_1(a) + t\frac{\sigma_2(s)}{\sigma_1(s)}\sigma_2(a))\sigma_2(a)\sigma_3(a)| \end{aligned}$$

The second equation is due to the fact that $|\sigma_1(s)\sigma_2(s)\sigma_3(s)| = 1$ for elements in the unit group. This gives a map from $\mathcal{O}^\times \cong \mathbb{Z}^2$ to $\mathbb{R}^+$. Roughly, this means the image in $\mathbb{R}_+$ is dense, since it maps $\mathbb{Z}^2$ into $\mathbb{R}_+$. This argument shows that for any small perturbation given by a unipotent element in U_t, we get another lattice with dense image under taking norm. By the decomposition KNA, this might tells how to prove the theorem.

3. bibliography

References

[1] J.Neukrich. Algebraic number theory. Springer-Verlag Berlin Heidelberg. 1999.

[2] J.Milne. Algebraic number theory. https://www.jmilne.org/math/CourseNotes/ANT.pdf Version 3.08. 2020.

[3] C.McMullen. Ergodic theory, geometry and dynamics. https://people.math.harvard.edu/ ctm/papers/home/text/class/notes/erg 2020.

[4] J. W. S. Cassels and H. P. F. Swinnerton-Dyer. On the product of three homogeneous linear forms and the indefinite ternary quadratic forms. Philos. Trans. Roy. Soc. London. Ser. A. 248 (1955), 73–96.

[5] Anthony W.Knapp. Lie Groups Beyond an Introduction. Second Edition. Birkhauser. 2002.