NOTE OF WEEK 10

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ABSTRACT. In week 10, the relationship between dynamics and quadratic forms, products of linear forms is introduced. This note collect some results that is only mentioned but not proved in the class.

CONTENTS

1. $SO(2,1)$ IS GENERATED BY UNIPOTENT ELEMENTS

This fact is used in one proof of famous Oppenheim conjecture using Ratner's theorem about classification of measures under unipotent action.

First we note $SO(1,1)_0$, *i.e.* the connected component of $SO(1,1)$ containing identity, has Lie algebra

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, a \in \mathbb{R}, \tag{1.1}$$

which is also a Cartan subalgebra of $SL(2,\mathbb{R})$. Since it has real eigenvalue $a, -a$, it can be conjugated to the standard diagonal subalgebra via $Int(\mathfrak{sl}(2,\mathbb{R}))$ action. This implies it is conjugate to A in $SL(2,\mathbb{R})$, which can not be generated by unipotent elements.

Now we consider the real semisimple Lie group $SO(2,1)$, whose Lie algebra is a split real form of $SO(3,\mathbb{C})$. It has a standard relative root decomposition, see for example Section 6.4 of [4]. For this specific example, the decomposition can be given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

Clearly, they form a basis of $\mathfrak{so}(2,1)$, and they satisfy the following bracket relationship

$$[A,B] = B, [A,C] = -C, [B,C] = 2A. \tag{1.3}$$

Which means B, C are relative root vectors with respect to the maximal Abelian subalgebra $\mathbb{R}A$ of $\mathfrak{p}$ in the Cartan decomposition determined by $\theta = -(\cdot)^t$.

Both B and C are nilpotent matrices, and taking the image under exponential, we get a one-parameter unipotent subgroup of $SO(2,1)$.

The closed subgroup generated by B and C is still a Lie subgroup, ensured by basic Lie theory, and its Lie algebra clearly contains B, C. Since it is closed under Lie bracket, it also contain A, that means it is three dimensional too. So the closed subgroup generated by B and C have the same dimension as $SO(2,1)$, thus it is both open (having same tangent space at identity, being locally diffeomorphic via exponential map) and closed. It must equal to $SO(2,1)_0$.

Recall for $SL(n,\mathbb{R})$, using Heisenberg pairs, we can directly show $SL(n,\mathbb{R})$ are generated by one parameter subgroups $exp(tE_{i,j}), i \neq j$, without taking closure. The same trick can be done here too. Since the A, B, C here forms a $\mathfrak{sl}(2,\mathbb{R})$-triple, with the correspondence

$$A \to \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, B \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.4}$$

Which give a local Lie group homomorphism from $SO(2,1)$ to $SL(2,\mathbb{R})$, so the Heisenberg trick can be done here too. So $SO(2,1)$ is generated by unipotent element too.

This correspondence can also be derived from the fact that they both have an action on hyperbolic plane. The group $SO(2,1)$ has a natural action on the pseudo sphere in Lorentz space with sign $(- - +)$, while $SL(2,\mathbb{R})$ act by fractional linear transformation on upperhalf plane model. They are both isometry group of hyperbolic plane, up to the quotient of a finite subgroup.

2. Products of linear forms

Quadratic forms are homogeneous polynomials of degree 2. A natural analogy is to consider homogeneous polynomials of higher degree. One kind of such object is products of linear forms, *i.e.* $L_1 \cdots L_n$, where $L_i : \mathbb{R}^m \mapsto \mathbb{R}$ is a linear function.

For such a product $L_1 \cdots L_n$, it is said to be rational if up to multiplying a constant, the coefficients are integers. Note that we are not requiring each L_i to have integer coefficients, but its product has integer coefficients. A product $L_1 \cdots L_n$, which is a polynomial function of $x^1, \cdots, x^m$, is said to be represent zero if there exists integers $x^1, \cdots, x^m$ not all being zero, such that $L_1 \cdots L_n(x^1, \cdots, x^m) = 0$.

We want to study product of n linear form depending on n variables. A typical one is $\phi = x^1 \cdots x^n$. More generally, we can consider

$$\prod_{i=1..n} \left(\sum_{j=1..n} g^i_j x^j \right), \tag{2.1}$$

it corresponds to a matrix $g = (g^i_j)$, and we denote it as $\phi \circ g$. It is clear for any $h \in A \subset SL(n,\mathbb{R})$, where A is diagonal subgroup with positive entries, $\phi \circ hg = \phi \circ g$. It means such form is invariant under A-action. The following theorem relates rationality of such form and periodicity of the A-orbit.

Theorem 2.2. *For $d \geq 2$, $g \in SL(n,\mathbb{R})$. If the orbit $Ag\Gamma$ in $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ is compact, then the form $\phi \circ g$ is rational and does not represent 0. If $d = 3$, the converse holds too.*

Remark 2.3. For $d \geq 3$, the converse always hold, here for convenience we prove the case $d = 3$ first.

Proof. If $Ag\Gamma$ is a compact orbit, then it is a periodic orbit. Since all periodic orbit come from totally real fields (up to a diagonal matrix with determinant 1), so the form looks like

$$\prod_{i=1..n}(\sum_{j=1..n}\sigma_i(\alpha_j)x^j), \tag{2.4}$$

where α_j forms a basis of totally real filed K of degree n. Then its all coefficients are fixed by Galois group action, thus lies in $\mathbb{Q}$, meaning the form is rational. It does not represent zero because for x^i integer, not all being zero, the value is just the norm of number $\sum_i \alpha_i x^i$, which is nonzero.

For the converse, we consider $n = 3$ for simplicity. There always exists some $h \in A$, such that hg takes the following form (up to some scalar)

$$\begin{pmatrix} 1 & g_2^1 & g_3^1 \\ 1 & g_2^2 & g_3^2 \\ 1 & g_2^3 & g_3^3 \end{pmatrix} \tag{2.5}$$

Such a form can always be obtained since we assume that the form does not represent zero, there must be some column with every entry nonzero.

Denote $f = \phi \circ g$. Then

$$f(x,1,0) = (x + g_2^1)(x + g_2^2)(x + g_2^3) \tag{2.6}$$

Since it does not represent zero and rational, this polynomial lies in $\mathbb{Q}[x]$ (up to multiplying a constant in $\mathbb{R}$) and has no rational solution, thus is irreducible over $\mathbb{Q}$. Its roots are all real, and they are $-g_2^1, -g_2^2, -g_2^3$. Denote $\alpha = -g_2^1$.

Similarly, we consider the polynomial $f(x,0,1)$, and denote $\beta = -g_3^1$. Then we claim that $\mathbb{Q}[\alpha] = \mathbb{Q}[\beta]$, and this gives a totally real field of degree 3. We denote the form as $L_1L_2L_3$. Since it is rational, it is stable under all the embedding in algebraic closure of rationals $Hom_{\mathbb{Q}}(\mathbb{Q}(\alpha,\beta), \overline{\mathbb{Q}})$. The action just swap three linear forms (we can check it must swap these linear forms, instead of by mapping to a different decomposition into linear forms, since the ring of polynomial of several variables is a UFD) , and has at most three different embedding since the action is determined by its action on α and β. This shows $[\mathbb{Q}[\alpha,\beta] : \mathbb{Q}] \leq 3$, which implies $\mathbb{Q}[\alpha] = \mathbb{Q}[\beta]$.

This implies $L_1L_2L_3$ can only take the form

$$(x^1 - \alpha x^2 - \beta x^3)(x^1 - \sigma_2(\alpha)x^2 - \sigma_2(\beta)x^3)(x^1 - \sigma_3(\alpha)x^2 - \sigma_3(\beta)x^3) \tag{2.7}$$

and $1, \alpha, \beta$ is a $\mathbb{Q}$-basis of totally real field $\mathbb{Q}[\alpha,\beta]$. From results last week, we know it is a periodic orbit. □

3. Bibliography

References

[1] J.Neukrich. Algebraic number theory. Springer-Verlag Berlin Heidelberg. 1999.

[2] J.Milne. Algebraic number theory. https://www.jmilne.org/math/CourseNotes/ANT.pdf Version 3.08. 2020.

[3] C.McMullen. Ergodic theory, geometry and dynamics. https://people.math.harvard.edu/ ctm/papers/home/text/class/notes/erg 2020.

[4] Anthony W.Knapp. Lie Groups Beyond an Introduction. Second Edition. Birkhauser. 2002.