NOTE OF WEEK 12: ISOLATION OF COMPACT A-ORBITS

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Abstract. In week 12, we proved a theorem of Cassels and Swinnerton-Dyer about isolation of compact orbits.

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1. Theorem of Cassels and Swinnerton-Dyer

In [1], Cassels and Swinnerton-Dyer proved the following theorem.

Theorem 1.1. *For $g \in SL_3(\mathbb{R})$, $Ag\Gamma$ a compact orbit in $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$, then for all $(\delta_1, \delta_2) \subset \mathbb{R}$, there exists a neighborhood U of e in $SL_3(\mathbb{R})$, such that for all $h \in Ug \backslash Ag$, $\phi \circ h(\mathbb{Z}^3 \backslash \{0\}) \cap (\delta_1, \delta_2) \neq \emptyset$.*

This note mainly consists of the proof of this theorem. Intuitively, this theorem says that for a compact orbit, for any compact subset of $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$, there exist a neighborhood Ug of this orbit Ag, such that for all $h \in Ug - Ag$, Ah does not lie in this chosen compact subset. This is because there exists a exhaustion of compact subsets that take the form $\{g | N(g\mathbb{Z}^n - \{0\}) > \epsilon\}$.

Basically, the theorem is proved in two steps. We fix a compact A-orbit $Ag\Gamma$. The first step is to show that we can approximate the orbits $Ah\Gamma, h \in Ug - Ag$ by those orbits took the form $Aexp(\lambda E_{i,j})g\Gamma$, for an appropriately chosen neighborhood U. The second step is to show that for orbits $Aexp(\lambda E_{i,j})g\Gamma$, the image of correspondent lattice under taking the norm is dense in $\mathbb{R}$. Combine the two steps, we can show the theorem.

We prove the second step in the following lemma.

Lemma 1.2. *Fix $Ag\Gamma$ a compact orbit in $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$, $d \geq 3$, then for $h = exp(\lambda E_{i,j}), i \neq j, \lambda \neq 0$, the image set*

$$N(hg\mathbb{Z}^d) = \{N(hgv) | v \in \mathbb{Z}^d\}$$

is dense in $\mathbb{R}_{>}0$. Here we just take $N(x) = |x_1 \cdots x_d|$.

Proof. According to classification result of periodic orbits, see note of Week 8 and 9, we know the lattice $\Lambda = g\mathbb{Z}^d$ is of the form

$$\sigma(\mathfrak{a}).$$

Where σ is the Minkowski embedding of a totally real field K of degree d, and $\mathfrak{a}$ is a full module of K. The corresponding stabilizer $Stab_A(\Lambda)$ of this lattice can be regarded as a subgroup of $\mathcal{O}_K^\times$ isomorphic to $\mathbb{Z}^{d-1}$, which is also a lattice in A.

Consider $\phi(x_1, \cdots, x_d) = x_1 \cdots x_d$. Under this notation,

$$\phi \circ g(\mathbb{Z}^d) = \{\sigma_1(x) \cdots \sigma_d(x) | x \in \mathfrak{a}\},$$

and this set equal to $N(g\mathbb{Z}^d)$ after taking absolute value. A direct calculation shows that for $h = exp(\lambda E_{i,j})$

$$\phi \circ hg(\mathbb{Z}^d) = \{\sigma_1(x) \cdots (\sigma_i(x) + \lambda\sigma_j(x)) \cdots \sigma_d(x) | x \in \mathfrak{a}\}. \tag{1.3}$$

Obviously, it contains a subset

$$\begin{aligned} &\{\sigma_1(sx) \cdots (\sigma_i(sx) + \lambda\sigma_j(sx)) \cdots \sigma_d(sx) | s \in Stab_A(\Lambda)\} \\ =&\{N(s)\sigma_1(x) \cdots (\sigma_i(x) + \lambda\frac{\sigma_j(s)}{\sigma_i(s)}\sigma_j(x)) \cdots \sigma_d(x) | s \in Stab_A(\Lambda)\} \\ =&\{\sigma_1(x) \cdots (\sigma_i(x) + \lambda\frac{\sigma_j(s)}{\sigma_i(s)}\sigma_j(x)) \cdots \sigma_d(x) | s \in Stab_A(\Lambda)\}. \end{aligned} \tag{1.4}$$

Here $N(s)$ means the norm of $s \in K$, it is always 1, since norm of elements in ring of integers has norm ± 1, and the condition that it lies in A says it must have norm 1.

Since all σ_i are embedding of field, take any nonzero x ensures that the above set is of the form

$$\{c_1 + c_2\frac{\sigma_j(s)}{\sigma_i(s)} | s \in Stab_A(\Lambda)\}$$

with $c_2 \neq 0$. If we can show $\{\frac{\sigma_j(s)}{\sigma_i(s)} | s \in Stab_A(\Lambda)\}$ in dense in $\mathbb{R}$, the result follows. However, all elements in $Stab_A(\Lambda)$ are positive under all σ_i, it can not be dense in $\mathbb{R}$. We will show that it is dense in $\mathbb{R}_{>0}$. Then by varying the choice of x, we see that $\phi \circ hg(\mathbb{Z}^d)$ is dense in $\mathbb{R}$. Since $\sigma(x), x \in \mathfrak{a}$ forms a lattice in $\mathbb{R}^d$, it is possible to choose $(x_1, \cdots, x_d) \in \Lambda$ satisfying

$$x_1 \cdots x_d < 0, x_ix_j < 0 \tag{1.5}$$

or

$$x_1 \cdots x_d > 0, x_ix_j < 0. \tag{1.6}$$

When $\lambda > 0$ and assume that $\{\frac{\sigma_j(s)}{\sigma_i(s)} | s \in Stab_A(\Lambda)\}$ is dense in $\mathbb{R}_{>0}$, then a $x = (x_1, \cdots, x_d)$ satisfying (1.5), gives the set $\{x_1 \cdots (x_i + \lambda\frac{\sigma_j(s)}{\sigma_i(s)}x_j) \cdots x_d\}$ whose closure contains $\mathbb{R}_{>0}$, and similarly choose a x satisfying (1.6) gives a set dense in $\mathbb{R}_{<0}$. The argument for $\lambda < 0$ is similar. Thus we only remain to show that $\{\frac{\sigma_j(s)}{\sigma_i(s)} | s \in Stab_A(\Lambda)\}$ is dense in $\mathbb{R}_{>0}$.

We want to show the kernel of the multiplicative homomorphism $\frac{\sigma_j(s)}{\sigma_i(s)}$ has a kernel smaller than $\mathbb{Z}^{d-2}$, thus giving a injective map from at least a $\mathbb{Z}^2$ to $\mathbb{R}_{>0}$, which must has a dense image.

First, since σ_i and σ_j are different embedding, the kernel can not isomorphic to $\mathbb{Z}^{d-1}$, because we can always find a primitive element in such a subgroup, as is in the proof in the classification of periodic orbits. Assume that the kernel of $\{\frac{\sigma_j(s)}{\sigma_i(s)}$ is isomorphic to $\mathbb{Z}^{d-2}$, then this implies the ring of integer of the subfield $L = \{x \in K | \sigma_i(x) = \sigma_j(x)\}$ contains a subgroup isomorphic to $\mathbb{Z}^{d-2}$. According to the Dirichlet unit theorem, this implies the degree $[L:\mathbb{Q}]$ is at least $d-1$. On the other hand, since L is a proper subfield of K, its degree is at most $\frac{d}{2}$. But the condition $d \geq 3$ implies $d-1 > \frac{d}{2}$ always hold, leading to contradiction. Thus the kernel is at most $\mathbb{Z}^{d-3}$. This means it maps $Stab_A(\Lambda)/Kernel$ injectively into $\mathbb{R}_{>0}$, and $Stab_A(\Lambda)/Kernel$ contains a subgroup isomorphic to $\mathbb{Z}^2$, thus the image is dense in $\mathbb{R}_{>0}$. □

The main reference of the proof above is the Theorem 25.3 in the course note of McMullen [2] and the Corollary 3.3 of Shapira's paper [3].

For the first step of the proof, we need to use orbits $Aexp(\lambda E_{i,j})g\Gamma$ to approximate elements in $Ug - Ag$ for some neighborhood U of e.

The theorem 1.1 has a corollary, which is the theorem 25.3 in McMullen's note [2].

Theorem 1.7. *Let $T = Ag\Gamma$ be a compact orbit, suppose $X = \overline{Ah\Gamma}$ meets T. Then either $X = T$, or $N(h\Gamma)$ is dense in $\mathbb{R}_{>0}$. Here $N(x) = |x_1 \cdots x_d|$.*

This theorem can be deduced from theorem 1.1, since if $\overline{Ah\Gamma}$ meets $T = Ag\Gamma$, then in every open neighborhood U of e, Ug contains some point in $Ah\Gamma$, thus $N(h\Gamma)$ intersects every interval (δ_1, δ_2), *i.e.* it is dense.

We give a proof of theorem 1.7, which is also from [2].

Proof. Denote the set $V = \{g | g \in SL_d(\mathbb{R}), X \cap gT \neq \emptyset\}$. Since X is closed and T is compact, it follows that V is closed.

V is stable under both left and right A action. Since if $a_1, a_2 \in A$, $g \in B$, then $X \cap a_1 g a_2 T = a_1(X \cap gT) \neq \emptyset$.

Assume $X \neq T$, then there exists some element $g \in V, g \notin A$. By the invariance of V under left and right A action, we know $aga^{-1} \in V, \forall a \in A$. By choosing appropriate a_n, we can let $a_n g a_n^{-1}$ tend to $A(I + \lambda E_{i,j})$. For example, in the $d = 3$ case, let

$$h = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ \epsilon_{2,1} & 1 & \\ \epsilon_{3,1} & \epsilon_{3,2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon_{1,2} & \epsilon_{1,3} \\ & 1 & \epsilon_{2,3} \\ & & 1 \end{pmatrix}. \tag{1.8}$$

Assume that $\epsilon_{3,1}$ has the largest absolute value among those $\epsilon_{i,j}$. We can choose appropriate $a \in A$, such that

$$aha^{-1} = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ e^{t_1}\epsilon_{2,1} & 1 & \\ e^{t_1+t_2}\epsilon_{3,1} & e^{t_2}\epsilon_{3,2} & 1 \end{pmatrix} \begin{pmatrix} 1 & e^{-t_1}\epsilon_{1,2} & e^{-t_1-t_2}\epsilon_{1,3} \\ & 1 & e^{-t_2}\epsilon_{2,3} \\ & & 1 \end{pmatrix}. \tag{1.9}$$

By choosing proper $t_1, t_2 > 0$, we can let h be arbitrarily close to

$$\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ \lambda & & 1 \end{pmatrix} \tag{1.10}$$

for any given $\lambda \neq 0$. That is $I + \lambda E_{i,j}$ lies in V, since V is closed.

By definition of V, this implies $X \cap (I + \lambda E_{i,j})T \neq \emptyset$, say $h_0\Gamma \in X \cap (I + \lambda E_{i,j})T$. Then $Ah_0\Gamma = A(I + \lambda E_{i,j})g\Gamma$, by the previous lemma, $N(h_0\Gamma)$ has dense image. Since $h_0\Gamma \in X = \overline{Ah\Gamma}$, $N(h\Gamma)$ also has dense image. □

This weak version of theorem 1.1 is enough to show the results in the next section.

We notice that $N(hg\Gamma) = N(ahg\Gamma) = N(aha^{-1}ag\Gamma) = N(aha^{-1}g\Gamma)$ for $a \in Stab_A(g\Gamma)$. So if we can use elements in stabilizer to approximate $A(I + \lambda E_{i,j})$ with given $h \in U$, for some suitable U, then the theorem 1.1 follows from Lemma 1.2.

This strategy seems plausible, since the stabilizer forms a lattice in A, but more careful analysis is needed. For the $d = 3$ case, assume that h takes the form as in (1.8), with $\epsilon_{3,1}$ the largest among other $\epsilon_{i,j}$, we want to find $a \in Stab_A(g\Gamma)$ satisfying

$$\begin{aligned} |e^{t_1+t_2}\epsilon_{3,1} - \lambda| &< \delta \\ |e^{t_1}\epsilon_{2,1}| &< \delta \\ |e^{t_2}\epsilon_{3,2}| &< \delta \\ t_1 > 0, t_2 > 0, |\epsilon_{i,j}| &< \delta \end{aligned} \tag{1.11}$$

These inequalities above tells us how small the neighborhood U should be in order to satisfy the theorem.

2. relationship with Littlewood's conjecture

In this section, we use the result proved in section 1 to show that Margulis's conjecture shows Littlewood's conjecture. We first state these two conjectures.

Conjecture 2.1 (Littlewood)**.** *For any $\alpha, \beta \in \mathbb{R}$, we have*

$$\inf_{n>0} \inf_{p,q\in\mathbb{Z}} n|n\alpha - p||n\beta - q| = 0 \tag{2.2}$$

Conjecture 2.3 (Margulis)**.** *Every bounded A-orbit in $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$, $d \geq 3$ is closed.*

Theorem 2.4. *Margulis's conjecture implies Littlewood's conjecture.*

Proof. If there exists α, β as the counterexample of Littlewood's conjecture, then we will construct a orbit that is bounded but cannot be compact.

We first define the lattice

$$L_0 = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3, e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix} \tag{2.5}$$

Denote $M_0 = \mathbb{Z}e_1 + \mathbb{Z}e_2$, then $N(M_0) = 0$, and α, β do not satisfy Littlewood's conjecture implies that

$$N(L_0 - M_0) = \inf_{n\neq 0, p,q\in\mathbb{Z}} |n||n\alpha - p||n\beta - q| \geq c > 0 \tag{2.6}$$

is bounded below by a positive constant. This implies that $N(L_0)$ is not dense, it has a gap at $(0, c)$.

Choose $a_t = diagt, t, t^{-2}$, then as $t \to \infty$, $M_t = a_tM_0$ will be pushed to infinity, while the image under norm remains unchanged. Also notice that $L_t = a_tL_0$ lies in a compact subset of $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$, because vectors in M_t has length greater than $\sqrt{2}$, and length of vectors in $L_t - M_t$ is also bounded from below since the norm is bounded from below. Upon passing to a subsequence, it converges to some L_∞. Since all vectors in M_t are pushed to infinity, every vector in L_∞ is the limit of some $v_n \in L_{t_n} - M_{t_n}$, thus the norm is also bounded below by constant c. This means that $N(L_\infty) > 0$.

According to a variant of Mahler principle, $N(L_\infty) > 0$ implies that AL_∞ is a bounded orbit. If the Margulis's conjecture is true, then AL_∞ is a compact orbit, and $L_\infty \in \overline{AL_0}$. By Theorem 1.7, this implies $N(L_0)$ is dense or $L_0 \in AL_\infty$, but $N(L_0)$ is not dense, so $L_0 \in AL_\infty$. However, lattices in AL_∞ can not have nonzero vector with zero norm, and this leads to a contradiction. Now we have proved that Margulis's conjecture implies Littlewood's conjecture. □

3. BIBLIOGRAPHY

References

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