DIVERGENT ORBITS OF MAXIMAL TORI IN $SL(n, \mathbb{R})$

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Abstract. In this week, the topic of classifying orbits is introduced, For unipotent orbits, this is the famous result of Ratner. But for actions of non-unipotent subgroups, less is known. One question is what does orbits of Cartan subgroup look like. Result of Margulis about divergent orbits is mentioned on the class. This note collects some relevant results about classification of orbits of Cartan subgroup. The main reference of this note is Appendix A of [2].

Contents

1. Orbits in $SL(2, \mathbb{R})$

In this section, we consider orbits of Cartan subgroup $A = \{diag\{a, a^{-1}\} | a > 0\}$ in $SL(2, \mathbb{R})$. We denote $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$ and $e = I_2$ in this section.

First we consider periodic A-orbits. Because all discrete subgroups of A are isomorphic to $\mathbb{Z}$, to construct periodic orbits, it suffices to find $g \in G$, $a \in A \backslash \{I_2\}$, such that $ag\Gamma = g\Gamma$, *i.e.* $g^{-1}ag \in \Gamma$. This implies trace of a belongs to $\mathbb{Z}$. Since determinant and trace determine the eigenvalues of a 2 by 2 matrix, there are only countably many possibilities of a, corresponding to integers greater than 2.

For a fixed, with $Tr(a) = n$, $n \geq 3, n \in \mathbb{Z}$. It suffices to find a matrix X in Γ with trace n. Then there must be some $g \in G$, such that $g^{-1}ag = X$, since X has two different eigenvalues in $\mathbb{R}$. Such X always exists, because

$$\begin{aligned} a + d &= n \\ ad - bc &= 1 \end{aligned} \tag{1.1}$$

have many integer solution. Then for each X as above, and each $g \in \mathbb{R}$ such that $g^{-1}ag = X$, they gives a periodic orbit $g\Gamma$, which is homeomorphic to the circle $A/\{a^k | k \in \mathbb{Z}\}$.

But telling which orbit is closed is not obvious. The case $A\Gamma$ is clear, since its trajectory in the hyperbolic plane (consider the action as fractional linear transformation on the upper half plane) is just the line $ai, a \in \mathbb{R}_+$. This line connects two cusps of the standard fundamental domain $\{z \in \mathbb{C} || \Re(z)| \leq \frac{1}{2}, |z| \geq 1\}$, one is the cusp at i, another is the cusp at infinity. Its image in $SO(2) \backslash SL(2, \mathbb{R}) / \Gamma$ is closed, so this orbit in $SL(2, \mathbb{R})/\Gamma$ is also closed.

A more general fact is the following

Proposition 1.2. *Suppose $\boldsymbol{H}$ is a reductive $\mathcal{Q}-$subgroup of G. Then there is a $\mathbb{Q}-$representation $\rho : \boldsymbol{G} \to \boldsymbol{GL}(\boldsymbol{V})$ and $v \in \boldsymbol{V}(\mathbb{Q})$ such that $\boldsymbol{H} = \{g \in \boldsymbol{G} : \rho(g)v = v\}$. In particular, $\rho(\boldsymbol{G}(\mathbb{Z}))v$ is closed in V and $H\pi(e)$ is a closed orbit, where $\pi : \boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{G}(\mathbb{Z})$*

A reference is [1] Prop7.7. But it is written in French.

2. Classification of divergent orbits of Cartan subgroup

Denote $\pi : G \to G/\Gamma$. The main theorem to prove in this section is

Theorem 2.1. *(Margulis) Let $G = SL(n, \mathbb{R}), \Gamma = SL(n, \mathbb{Z})$, and let T be the group of all diagonal matrices. Then $T\pi(g)$ is divergent if and only if $g^{-1}Tg$ is a real $\mathbb{Q}-$split $\mathbb{Q}-$torus.*

Before proving the theorem, we need to state two lemmas that helps us to find the compact set that contain the non-divergent orbits.

Lemma 2.2. *There is a ball $W \subset \mathbb{R}^n$, centered at 0, a finite set $F \subset T$, and $c > 1$ such that for every $g \in G$ there is $f \in F$ such that for every $w \in g\mathbb{Z}^n \cap W$ we have:*

$$\|fw\| \geq c\|w\| \tag{2.3}$$

Proof. First $W \cap g\mathbb{Z}^n$ should not contain a basis of this lattice, since a matrix of determinant 1 must shorten some vectors while stretching other vectors. So we should find a W such that for every $g \in G$, $W \cap g\mathbb{Z}^n$ never contains a basis. This can be done since determinant of g is 1, the volume of fundamental domain of the lattice (in $\mathbb{R}^n$) is preserved. Taking W to be a ball of radius strictly less than 1 satisfies this condition. If $W \cap g\mathbb{Z}^n$ contains a basis, then all vectors in this basis has length less than 1, so the volume of the fundamental domain is strictly less than 1, contradicting that g preserves volume.

Now it suffices to find a finite subset F of T, such that for every proper subspace $V \subset \mathbb{R}^n$, there exists a $t \in F$ satisfying

$$\|tv\| > c\|v\|, \ \forall v \in V \tag{2.4}$$

It is clear that for a fixed proper subspace V, there always exist such t. It suffices to show this result for all codimension 1 subspaces, which is the solution space of some

$$a_1x_1 + \cdots + a_nx_n = 0 \tag{2.5}$$

With out loss of generality, assume $a_1 \neq 0$, then x_1 can be written as a linear combination of $x_2, \cdots, x_n$. The diagonal matrix $diag\{e^{-(n-1)t}, e^t, \cdots, e^t\}$ for $t > 0$ large satisfy the need.

Since for t fixed

$$\inf_{\|v\|=1, v \in V} \frac{\|tv\|}{\|v\|} \tag{2.6}$$

varies continuously as V varies in the Grassmannian. By the compactness of Grassmannian, and the fact that for each V such t exists, passing to a finite subcover, we can find a finite subset $F \subset T$ satisfying the condition (2.4), which proves the lemma. □

Lemma 2.7. *If $g \in G$ and $g \notin TSL(n,\mathbb{Q})$ then for any neighborhood W of 0 in $\mathbb{R}^n$, any finite $J \subset g\mathbb{Z}^n - \{0\}$ and any compact $C \subset T$, there is $t \in T - C$ such that*

$$tJ \cap W = \emptyset \tag{2.8}$$

Proof. Observe that if $g \notin TSL(n,\mathbb{Q})$, then there must be some e_i in the standard basis of $\mathbb{R}^n$, such that

$$\mathbb{R}e_i \cap g\mathbb{Z}^n = \{0\} \tag{2.9}$$

If not, then for every e_i, $\mathbb{R}e_i \cap g\mathbb{Z}^n$ contains a nontrival element. Upon multiplying some diagonal matrix, we may assume such nontrival vectors in $\mathbb{R}e_i \cap g\mathbb{Z}^n$ all has rational coefficients. Thus g (up to a multiplication of diagonal matrix on the left) maps n linearly independent vectors in $\mathbb{Z}^n$ to n vectors in $g\mathbb{Z}^n$ that take the form $\frac{p}{q}e_i$. It is clear $g \in SL(n,\mathbb{Q})$. So for $g \notin TSL(n,\mathbb{Q})$, we can take such e_i.

Then take $\alpha_i(t)$ to be the diagonal matrix with i-th entry equal to $e^{-(n-1)t}$ and all other entries equal to e^t. Then for all $w \in g\mathbb{Z} - \{0\}$,

$$a_i(t)w \to \infty \text{ as } t \to \infty \tag{2.10}$$

So for any finite subset $J \subset g\mathbb{Z}^n - \{0\}$, $a_i(t)$ for t sufficiently large satisfies the lemma.

□

Proof. (theorem 2.1) According to Prop 1.2, the condition $g^{-1}Tg$ is a real $\mathbb{Q}-$split $\mathbb{Q}-$torus is sufficient.

To show necessary, we need to find a compact subset K of $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$, such that the orbit $Tg\Gamma$ returns to this compact set infinitely often. More precisely, for every compact subset $C \subset T$, there exists a $t \in T - C$, such that $tg\Gamma \in K$.

Choose W, F, c as in Lemma 2.1. With out loss of generality, we assume $1 \in F$, and define

$$W_0 = \cap_{f \in F^{-1}} fW \tag{2.11}$$

Since F is a finite subset, W_0 is still a neighborhood of 0. According to Mahler's principle,

$$K = \pi(\{g \in G | g\mathbb{Z}^n \cap W_0 = \{0\}\}) \tag{2.12}$$

is a compact subset of G/Γ.

We will then show this K is the compact subset we need. Fix an arbitrary compact subset $C \subset T$. Denote $J = g\mathbb{Z}^n \cap C^{-1}W$, this is a finite subset since $C^{-1}W$ is compact. According to Lemma 2.7, there exists a $t_0 \in T - C$, such that

$$t_0 J \cap W = \{0\} \tag{2.13}$$

If $t_0 g\mathbb{Z}^n \cap W_0 = \{0\}$ already holds, then $\pi(t_0 g) \in K$ satisfies the condition. If not, we want to find a sequence of t_k that dilates vectors in the lattice, finally making its intersection with W_0 trivial.

For example, take $t_1 \in F$, such that

$$w \in W \cap t_0 g\mathbb{Z}^n \implies \|t_1 w\| \geq c\|w\| \tag{2.14}$$

Then

$$W_0 \cap t_1 t_0 g\mathbb{Z}^n \subset t_1(W \cap t_0 g\mathbb{Z}^n) \tag{2.15}$$

due to the choice of W_0. This implies that all vectors in $W_0 \cap t_1 t_0 g\mathbb{Z}^n$ is at least c times longer than vectors in $W \cap t_0 g\mathbb{Z}^n$.

We can continue this operation. If $t_0, \cdots, t_{k-1}$ have been chosen, and $W_0 \cap t_{k-1} \cdots t_0 g\mathbb{Z}^n \neq \{0\}$, we can take $t_k \in F$ satisfying

$$w \in W \cap t_{k-1} \cdots t_0 g\mathbb{Z}^n \implies \|t_k w\| \geq c\|w\| \tag{2.16}$$

Then all vectors in $W_0 \cap t_k \cdots t_0 g\mathbb{Z}^n$ are at least c^k times longer than the shortest nonzero vector in $W_0 \cap t_0 g\mathbb{Z}^n$, which tends to infinity very quickly, implying that $W_0 \cap t_k \cdots t_0 g\mathbb{Z}^n = \{0\}$ for k sufficiently large.

Take k to be the smallest k such that $W_0 \cap t_k \cdots t_0 g\mathbb{Z}^n = \{0\}$. We claim that $t_k \in T - C$. If $k = 0$, this holds by the choice of t_0. If $k \geq 1$, since k is minimal, there exists nonzero $v \in W_0 \cap t_{k-1} \cdots t_0 g\mathbb{Z}^n$. Assuming the converse that $t_k \cdots t_0 \in C$. In [2], it uses an induction, to show $v = t_{k-1} \cdots t_0 v_0$ for some $v_0 \in W_0$. But it did not write down any detail, I can not understand why this induction should work. □

3. BIBLIOGRAPHY

References

[1] A.Borel. Introduction aux-groupes arithmetiques. Hermann, Paris. 1969.

[2] G.Tomanov and B.Weiss. Closed orbits for actions of maximal tori on homogeneous spaces. Duke Math. J. 119(2): 367-392 (15 August 2003).