7.7 问题

P1. (a) 略

(b) A, A^{-1} 为 Hermite 阵.

(c) 略

P2、$M^2 = \begin{bmatrix} A & A \\ I-A & I-A \end{bmatrix} \begin{bmatrix} A & A \\ I-A & I-A \end{bmatrix}$

$= \begin{bmatrix} A & A \\ I-A & I-A \end{bmatrix}$

当 M 为 Hermite 阵. $M^* = M$

$\begin{bmatrix} A^* & I-A^* \\ A^* & I-A^* \end{bmatrix} = \begin{bmatrix} A & A \\ I-A & I-A \end{bmatrix}$

$\Rightarrow A^* = A \quad I = A^* + A \Rightarrow A = \frac{1}{2} I$

P3. $\begin{bmatrix} I & A \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ A^* & 0 \end{bmatrix} \Rightarrow A = 0.$

P4. 略

P5、(a) $Pv \in \operatorname{ran} P$, $P(v - Pv) = Pv - Pv = 0$,

$v - Pv \in \ker P$, $v = Pv + v - Pv$,

所以 $V = \operatorname{ran} P \oplus \ker P$.

(b) 设 $v_1, v_2 \in V$，$u_1, u_2 \in \operatorname{ran} P$，$w_1, w_2 \in \ker P$

$v_1 = u_1 + w_1 \quad v_2 = u_2 + w_2$

$\langle Pv_1, v_2 \rangle = \langle u_1, u_2 + w_2 \rangle = \langle u_1, u_2 \rangle =$

$\langle u_1 + w_1, u_2 \rangle = \langle v_1, Pv_2 \rangle = \langle v_1, P^* v_2 \rangle$

所以 $P = P^*$

$\langle Pv, v - Pv \rangle = (v^* - v^* P^*) Pv = v^* Pv - v^* P^* Pv$

若 $P^* = P$，上式 $= v^* Pv - v^* P^2 v = v^* Pv - v^* Pv = 0$ 有

$\operatorname{ran} P \perp \ker P$

P6. $P_u = uu^* = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

$P_u^{\perp} = I - P_u = \begin{bmatrix} 1-a^2 & -ab & -ac \\ -ab & 1-b^2 & -bc \\ -ac & -bc & 1-c^2 \end{bmatrix}$

$P_u = 0$

P7. (a) v 到 P 的距离为 $\|P_P^{\perp}(v - x_0)\|_2$，而 $P_P^{\perp}$ 实为 $\operatorname{col} A^T$ 的正交射影，距离为 $v - x_0$ 在 $\operatorname{col} A^T$ 上的射影（A 与平面垂直）

(b) 显然

(c) 令 u 为 $\operatorname{col} A^T$ 的标准正交基，$x_0 = [x_1, x_2, x_3]^T \in R^3$

$d = |\langle u, v - x_0 \rangle| = (a^2 + b^2 + c^2)^{-1/2} |a(v_1 - x_1) + b(v_2 - x_2) + c(v_3 - x_3)|$

$$=(a^2+b^2+c^2)^{-1/2}|au_1+bv_2+cv_3+d|.$$

P8. $\|x(t)\|_2 = |(-3+t)^2+(3-2t)^2+t^2|$

$$=|6t^2-18t+18|$$

$\min\|x(t)\|_2=\frac{3\sqrt{2}}{2}$，结果一样

P9. $x=[-1,1,0]$. 相容.

$$AA^* = \begin{bmatrix}1&2&1\\2&4&2\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&0\\2&4&1\\1&2&0\end{bmatrix}=\begin{bmatrix}6&12&2\\12&24&4\\2&4&1\end{bmatrix}$$

$AA^*u=[1,2,1]^T$ 其中一解为 $[-\frac{5}{2},1,2]^T=u_0$

$$A^*u_0=[-\tfrac{1}{2},1,-\tfrac{1}{2}]^T$$

P10. A 行满秩，$r=m\le n$

又极小范数解为 $s=A^*u_0$ 其中 $AA^*u_0=y$，AA^* 非异.

$$u_0=(AA^*)^{-1}\cdot y=(R^*Q^*QR)^{-1}y$$

$$=R^{-1}R^{-*}y$$

$A^*u_0=QRR^{-1}R^{-*}y=QR^{-*}y$. 得证

P11. (a) 展示显然 对称阵 $A^T=A$ 反对称阵 $A^T=-A$ 两者交集为 $\{0\}$. $A+A^T$ 为对称阵. $A-A^T$ 为反对称阵. 任何 $A\in M_n(\mathbb{R})$ 均能由一个对称阵和一个反对称阵线性组合得出.

(b) 设 $M_1\in U_-$, $M_2\in U_+$

$$\langle M_1, M_2\rangle = \sum_{1\le i\le j\le n} M_{1ij}M_{2ij}\begin{cases}0, & i=j\\ -M_{ji}M_{ji}, & i\neq j\end{cases} = 0.$$

得证

(c) $A = P_{U_+}A + P_{U_-}A$ 且 $A+A^T\in U_+$. $A-A^T\in U_-$.

P12. $U_1\cap U_2=\{0\}$ 任意 $V\in M_n(\mathbb{R})$ 都可以表示为一个上三角+一个严格下三角, $\langle U_1, U_2\rangle = 0$, 得证

P13. $P(P(M))=P(M)$. 令 $X=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$ $Y=\begin{bmatrix}E & F\\ G & H\end{bmatrix}$

$$\langle P(X), Y\rangle = \left\langle \begin{bmatrix}A & \\ & 0\end{bmatrix}, \begin{bmatrix}E & F\\ G & H\end{bmatrix}\right\rangle = A^TE$$

$$= \langle X, P(Y)\rangle = \left\langle \begin{bmatrix}A & B\\ C & D\end{bmatrix}, \begin{bmatrix}E & 0\\ 0 & 0\end{bmatrix}\right\rangle$$

P14. 自伴随 $\langle T(x), y\rangle = \langle Ax, y\rangle = \langle x, A^*y\rangle$

$= \langle x, Ay\rangle = \langle x, T(y)\rangle$

$= \langle x, T(y) \rangle$

幂等. $T^2(x) = T(T(x)) = A^2 x = Ax = T(x)$

$\Rightarrow T^2 = T$

P15. 设 $\dim \mathrm{col} P = r$. $u_1 \cdots u_r$ 为 $\mathrm{col} P$ 的一组基, 有 $Pu_1 = u_1$, $Pu_2 = u_2 \cdots$ $Pu_r = u_r$. 可见是其特征向量, 对应特征值为1, 又设 $u_{r+1} \cdots u_n$ 为 $\mathrm{col} P^{\perp}$ 的一组基, 有 $Pu_{r+1} = 0$ ... $Pu_n = 0$. 亦是P的特征向量, 特征值为0, 有 $\mathrm{tr} P = r = \dim \mathrm{col} P$

P16. $(PQ)^* = Q^* P^* = QP = PQ$ Hermite阵

$(PQ)^2 = PQPQ = PPQQ = PQ$ 幂等

① $\mathrm{col} PQ \subset \mathrm{col} P$ 又 $PQ = QP$ $\mathrm{col} QP \subset \mathrm{col} Q$.

$\mathrm{col} PQ \subset \mathrm{col} P \cap \mathrm{col} Q$

② 设 $x \in \mathrm{col} P \cap \mathrm{col} Q$. 则存在 $u, v \in R^n$ 使

$x = Pu = Qv \Rightarrow Px = PQv \Rightarrow x = PQv$. 即

$\mathrm{col} P \cap \mathrm{col} Q \subset \mathrm{col} PQ$

①②有 $\mathrm{col} P \cap \mathrm{col} Q = \mathrm{col} PQ$. PQ为 $\mathrm{ran} P \cap \mathrm{col} Q$ 上的正交射影

P17. (a)(b)(c) 互推. ⇒(d) $PQ=0$ $QP=0$ 得证

⇒(e) $(P+Q)^* = P+Q$ Hermite阵.

$(P+Q)^2 = P+Q+QP+PQ = P+Q$ 幂等. 是正交投影.

(e)⇒(d) 亦可

(d)⇒(c) $QP = -PQ = -PQP$, QP 是 Hermite 矩阵.

$PQ+QP = PQ+(QP)^* = PQ+P^*Q^* = 2PQ = 0$

$PQ=0$

P18. 设 $v \in \operatorname{ran}P \cap \operatorname{ran}Q$. $u \in \ker P + \ker Q$.

$\langle u, v\rangle = 0$, 有 $\ker P + \ker Q \subseteq (\operatorname{ran}P \cap \operatorname{ran}Q)^\perp$

设 $u \in \ker P$, $w \in \ker Q$. $v \in (\ker P + \ker Q)^\perp$.

$\langle u, v\rangle = 0$, $\langle w, v\rangle = 0$ $v \in (\ker P)^\perp = \operatorname{ran}P$.

$v \in \operatorname{ran}Q$ 则 $v \in \operatorname{ran}P \cap \operatorname{ran}Q$. 有

$(\ker P + \ker Q)^\perp \subseteq \operatorname{ran}P \cap \operatorname{ran}Q$.

综上, $\ker P + \ker Q = (\operatorname{ran}P \cap \operatorname{ran}Q)^\perp$

P19 令 $v = Pv + v - Pv$. $Pv \in \operatorname{ran}P$.

$P(v - Pv) = 0 \Rightarrow v - Pv \in \ker P$

设 $u \in \operatorname{ran}P$ 且 $u \in \ker P$ $\exists y$ $u = Py$, $Pu = 0$

设 $u \in \operatorname{ran}P$ 且 $u \in \ker P$ ...

$P^2 y = Py = u = Pu = 0$ $\ker P \cap \operatorname{ran} P = \{0\}$.

P20 设 $x, y \in V$

$\langle Px, y\rangle = \langle Px, Py + y - Py\rangle = \langle Px, Py\rangle$

$\langle x, Py\rangle = \langle Px + x - Px, Py\rangle = \langle Px, Py\rangle$

$\langle Px, y\rangle = \langle x, Py\rangle$，自伴随.

P21 $\|Pv\| \le \|v\| \Rightarrow \|cPv\| \le \|cv\|$

$\Rightarrow \|cPv\| \le \|cPv + c(v - Pv)\|$

若 $cPv = w$ $v - Pv = u$，$\|w\| \le \|w + cu\|$

则 P4.10，可得 $\langle u, w\rangle = 0$，即 $\langle Pv, v - Pv\rangle = 0$.

$Pv \in \operatorname{ran}P$ $v - Pv \in \ker P$. $\ker P \in (\operatorname{ran}P)^\perp$

则 P21. $\Rightarrow$ 成立，反之易证.

P22. $Px = \lambda x$. $P^2 x = P(Px) = \lambda P x = \lambda^2 x = \lambda x$

$\lambda = 1/0$. $\operatorname{tr}(P) = 0$. 所有特征值为0.

即 $Pv = P(c v_1 + \dots c v_n) = 0$，$v \in F^n$. $P = 0$

P225. $Ax=y$ 相容，又 $\operatorname{rank} A = n$，解唯一。

$QRx=y$，$Q^*QRx=Q^*y \Rightarrow Rx=Q^*y$

$\Rightarrow x=R^{-1}Q^*y$.

P224. $A(A^*A)^{-1}A^* = QR(R^*Q^*QR)^{-1}R^*Q^*$

$= QRR^{-1}R^{-*}R^*Q^* = QQ^*$

P225. $Gx=0 \Rightarrow x^*Gx = \|\sum_{i=1}^{n} x_i u_i\|^2 = 0$ ①

若①成立，$\sum_{i=1}^{n} x_i u_i = 0$，又 $u_1, \dots, u_n$ 线性无关

仅当 $x=0$ 成立，即 $\operatorname{null}(G)=0$，G 可逆。

P226. (a) $d(v,u)^2 = \|v-Pv\|^2$

$= \|v\|^2 + \|Pv\|^2 - \langle v, Pv\rangle - \langle Pv, v\rangle$

$= \|v\|^2 + \langle Pv, Pv\rangle - \langle 2v, Pv\rangle$

$= \|v\|^2 + \langle Pv-v, Pv\rangle - \langle \sum_{i=1}^{n} c_i u_i, v\rangle$

$= \|v\|^2 - \sum_{i=1}^{n} c_i \langle u_i, v\rangle$

(b) 构成方程组

$\begin{bmatrix} \langle u_1, u_1\rangle & \cdots & \langle u_n, u_1\rangle & 0 \end{bmatrix} \begin{bmatrix} c_1 \end{bmatrix} \begin{bmatrix} \langle v, u_1\rangle \end{bmatrix}$

$$\begin{bmatrix} & \ddots & \vdots & \vdots \\ \langle u_1, u_n\rangle & \cdots & \langle u_n, u_n\rangle & 0 \\ \langle u_1, v\rangle & \cdots & \langle u_n, v\rangle & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ c_n \\ d \end{bmatrix} = \begin{bmatrix} \vdots \\ \langle v, u_n\rangle \\ \langle v, v\rangle \end{bmatrix}$$

根据克拉默法则 $d = \dfrac{g(u_1 \cdots u_n, v)}{\det M}$

所以最后一列展开 $d = \dfrac{g(v, u_1 \cdots u_n)}{g(u_1 \cdots u_n)}$

(1) $d = \left\| v - \dfrac{uu^*}{\|u\|^2} v \right\| = \left\| v - \dfrac{\langle v, u\rangle u}{\|u\|^2} \right\|$

$$= \|v\|^2 + \frac{|\langle v, u\rangle|^2}{\|u\|^4} \cdot \|u\|^2 - \frac{\langle v, u\rangle}{\|u\|^2} \langle v, u\rangle$$

$$- \frac{\langle v, u\rangle}{\|u\|^2} \langle u, v\rangle$$

$$= \|v\|^2 - \frac{|\langle v, u\rangle|^2}{\langle u, u\rangle}$$

$$= \frac{\langle u, u\rangle\langle v, v\rangle - \langle v, u\rangle\langle u, v\rangle}{\langle u, u\rangle}$$

$$= \frac{g(v, u)}{g(u)} \geq 0 \quad \text{又 } g(u) > 0$$

即 $g(v,u) \geq 0$

$$\langle v,u\rangle\langle u,v\rangle = \|\langle u,v\rangle\|^2 \leq \|u\|^2\|v\|^2$$

$$\|\langle u,v\rangle\| \leq \|u\|\|v\|$$

P27、(a) 设 $v \in V$ 且 $v \in \ker T$. $Tv = 0$

$T^*Tv = 0$ $\ker T \subseteq \ker T^*T$

设 $v' \in V$ 且 $v' \in \ker T^*T$.

$T^*Tv' = 0$ $v'^*T^*Tv' = 0$ $(Tv')^*Tv' = 0$

$Tv' = 0$ $\ker T^*T \subseteq \ker T$ 得证.

(b) $\dim \ker T^*T = \dim \ker T = \dim V - \dim \operatorname{ran} T$

$= 0$.

(c) 幂等: $P^2 = T(T^*T)^{-1}T^*T(T^*T)^{-1}T^*$

$= T(T^*T)^{-1}T^* = P$.

自伴随: $P^* = T(T^*T)^{-*}T^* = T(T^*T)^{-1}T^* = P$.

(d) 因为 $y \in \operatorname{ran} T$, 设 $y = Tu$, 则 $\|Tx - y\| = \|T(x-u)\|$

极小值, 即 $x - u \in \ker T$. 即 $T(x-u) = 0$. 又由 $\ker T = 0$

$x - u = 0$ 即 $x = u$ 唯一, 此时 $T^*T(x-u) = 0$

$T^*Tx = T^*Tu = T^*y$.

P.28. 略

P.29. 方程组 $\begin{bmatrix} (-2)^5 & -2 & 1 \\ & \vdots & \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ \vdots \\ 1 \end{bmatrix}$

P.30. $A^*Ax = A^*y$

$$\begin{bmatrix} mS_{x^2} & mS_x \\ mS_x & m \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} mS_{xy} \\ mS_y \end{bmatrix}$$ 解得 $a = \dfrac{S_{xy} - S_x S_y}{S_{x^2} - S_x^2}$, $b = \dfrac{S_y \cdot S_{x^2} - S_x S_{xy}}{S_{x^2} - S_x^2}$

P.31 (a) $\operatorname{rank} A = n \Rightarrow m \geq n$. $Ax = 0 \Leftrightarrow A^*Ax = 0$

$\dim \operatorname{null} A = 0 \Leftrightarrow \dim \operatorname{null} A^*A = 0$ A^*A 可逆

P 良好定义

(b) 略

(c) 设 $v \in \operatorname{col} P$, $v = Pu = A(A^*A)^{-1}A^*u$, $\operatorname{col} P \subseteq \operatorname{col} A$

设 $v \in \operatorname{col} A$, $v = Au$, $Pv = A(A^*A)^{-1}A^*Au = Au = v$

$\operatorname{col} A \subseteq \operatorname{col} P$ $\quad \operatorname{col} A = \operatorname{col} P$.

P.32. $A = QR \Rightarrow Q^*A = Q^*QR = R$

$\Rightarrow R_{kk} = Q_k^* A_{*k}$ 即 a_k 正交

化后为 Q_k 与 a_k 的内积 $\langle a_k, Q_k\rangle$ 且非负实数.

考虑 $d = \| a_k - P a_k \| = \| \underbrace{a_k - \sum_{i=1}^{k-1} \langle a_k, Q_i\rangle Q_i}_{①} \|$

①实为 a_k 在 Q_k 上的射影 $d = \| \langle a_k, Q_k\rangle Q_k \|$
$= |\langle a_k, Q_k\rangle|$

设 $u_1 \cdots u_n$ 为 U 的一组基,

构造 $A = [u_1 \cdots u_n v]$ 进行 QR 分解.

$R_{n+1\,n+1}$ 就是极小化距离.

P23. (a). $\|P_\varepsilon - f\|^2 = \int_0^1 |f(t) - P_\varepsilon(t)|^2 dt$

$\leq \int_0^1 |f(t) - P_\varepsilon(t)|^2 dt \leq \varepsilon^2.$

又 $\|f\|^2 \leq \|P_\varepsilon\|^2 + \|f\|^2 = \|P_\varepsilon - f\|^2 = \varepsilon^2$. 得证.

(b) 设 $|f(t)| \leq \rho$ $\|f\| = \left|\int_0^1 |f(t)|^2 dt\right|^{\frac{1}{2}} \leq \rho \leq \varepsilon$. 对

任意 $\varepsilon > 0$ 成立. 则 $\rho = 0$ $f = 0$. $f^\perp = \{0\}$,

(p) [illegible]

P34 $Ax_0 = y \Rightarrow y \in \operatorname{col} A$

$A^* z = 0 \Rightarrow z \in \operatorname{null} A^* = (\operatorname{col} A)^\perp$

$\langle y, z\rangle = z^* y = 0.$ 反之易证.

P35. $A = SB$ 有 $A^* = B^* S^*$ S^* 可逆

$\Leftrightarrow \operatorname{col} A^* = \operatorname{col} B^*$, $(\operatorname{col} A^*)^\perp = (\operatorname{col} B^*)^\perp$

$\Leftrightarrow \operatorname{null} A = \operatorname{null} B.$