

## 7.7 问题

P1. (a) 略

(b)  $A, A^*$  是 Hermite 阵.

(c) 略

$$P2. \quad M = \begin{bmatrix} A & A \\ I-A & I-A \end{bmatrix} \begin{bmatrix} A & A \\ I-A & I-A \end{bmatrix}$$

$$= \begin{bmatrix} A & A \\ I-A & I-A \end{bmatrix}$$

当  $M$  是 Hermite 阵.  $M^* = M$

$$\begin{bmatrix} A^* & I-A^* \\ A^* & I-A^* \end{bmatrix} = \begin{bmatrix} A & A \\ I-A & I-A \end{bmatrix}$$

$$\Rightarrow A^* = A \quad I \in A^* + A \Rightarrow A = \frac{1}{2}I$$

$$P3. \quad \begin{bmatrix} I & A \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ A^* & 0 \end{bmatrix} \Rightarrow A = 0$$

P4. 略

$$P5. (a) \quad Pv \in \text{ran } P, \quad P(v - Pv) = Pv - Pv = 0,$$

$$v - Pv \in \ker P, \quad v = Pv + v - Pv,$$

$$\text{从而 } V = \text{ran } P \oplus \ker P.$$

$$\text{从而 } v = Pv + (v - Pv), \quad v - Pv \in \ker P.$$

(b) 设  $u_1, u_2 \in V$ ,  $u_1, u_2 \in \text{rank } P$ ,  $w_1, w_2 \in \text{ker } P$

$$v_1 = u_1 + w_1 \quad v_2 = u_2 + w_2$$

$$\langle Pu_1, v_2 \rangle = \langle u_1, u_2 + w_2 \rangle = \langle u_1, u_2 \rangle =$$

$$\langle u_1 + w_1, u_2 \rangle = \langle v_1, Pu_2 \rangle = \langle v_1, P^T v_2 \rangle$$

$$\text{所以 } P = P^T$$

$$\langle Pv, v - Pv \rangle = (v^T - v^T P^T) Pv = v^T Pv - v^T P^T Pv$$

$$\text{若 } P^T = P, \text{ 上式} = v^T Pv - v^T P^2 v = v^T Pv - v^T Pv = 0 \text{ 有}$$

$$\text{rank } P \perp \text{ker } P$$

P6,  $P_u = uu^T = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

$$P_u = I - P_u = \begin{bmatrix} 1-a^2 & -ab & -ac \\ -ab & 1-b^2 & -bc \\ -ac & -bc & 1-c^2 \end{bmatrix}$$

$$P_u = 0$$

P7. (a)  $v$  到  $P$  的距离为  $\|P^\perp(v - x_0)\|_2$  而  $P^\perp$  实为  $\omega A^T$  的正交投影, 距离为  $v - x_0$  在  $\omega A^T$  上的射影 ( $A$  与平面垂直)

(b) 显然

(c)  $u$  为  $\omega A^T$  的标准正交基,  $x_0 = [x_1, x_2, x_3]^T \in \mathbb{R}^3$

$$d = |\langle u, v - x_0 \rangle| = (a^2 + b^2 + c^2)^{-1/2} \begin{vmatrix} a(v_1 - x_1) + b(v_2 - x_2) + c(v_3 - x_3) \end{vmatrix}$$

$$= (a^2 + b^2 + c^2)^{-1/2} |au + bv + cw + d|$$

P8.  $\|x(t)\|_2 = |(-3+t)^2 + (3-2t)^2 + t^2|$

$$= |6t^2 - 18t + 18|$$

$$\min \|x(t)\|_2 = \frac{3\sqrt{2}}{2}, \text{ 结果 - 17}$$

P9.  $x = [-1, 1, 0]^T$ . 矩阵

$$AA^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 2 \\ 12 & 24 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

$$AA^T u = [1, 2, 1]^T \text{ 其中一个解为 } [-\frac{5}{2}, 1, 2]^T = u_0$$

$$A^T u_0 = [-\frac{1}{2}, 1, -\frac{1}{2}]^T$$

P10.  $A$  满秩,  $r = m \leq n$

又最小范数解为  $s = A^T u_0$  其中  $AA^T u_0 = y$ ,  $AA^T$  可逆

$$u_0 = (AA^T)^{-1} \cdot y = (R^* Q^* Q R)^{-1} y$$

$$= R^{-1} R^{-*} y$$

$$A^T u_0 = Q R R^{-1} R^{-*} y = Q R^* y. \text{ 得证}$$

P11. (a) 展讲, 显然 对称阵  $A^T = A$  反对称阵  $A^T = -A$   
 两者交集为  $\{0\}$ .  $A + A^T$  为对称阵,  $A - A^T$  为反对  
 称阵. 任何  $A \in M_n(\mathbb{R})$  均能由一个对称阵和一个反  
 对称阵线性组合得出.

(b) 设  $M_1 \in U^-, M_2 \in U^+$

$$\langle M_1, M_2 \rangle = \sum_{1 \leq i \leq j \leq n} M_{1,ij} M_{2,ij} \begin{cases} 0, & i=j \\ M_{ji} M_{ij}, & i \neq j \end{cases} = 0.$$

得证

(c)  $A = P_{U^+} A + P_{U^-} A$  且  $A + A^T \in U^+$ ,  $A - A^T \in U^-$ .

P12.  $U_1 \cap U_2 = \{0\}$  任意  $v \in M_n(\mathbb{R})$  都可以表示为  
 一个上三角 + 一个严格下三角,  $\langle u_1, u_2 \rangle = 0$ , 得证

P13.  $P(P(U)) = P(U)$ .  $Ax = \begin{bmatrix} A & B \\ C & D \end{bmatrix} y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$

$$\langle P(x), y \rangle = \left\langle \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} E & F \\ G & H \end{bmatrix} \right\rangle = A^T E$$

$$= \langle x, P(y) \rangle = \left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \right\rangle$$

P14. 自伴算子  $\langle T(x), y \rangle = \langle Ax, y \rangle = \langle x, A^* y \rangle$

$$= \langle x, A^{-1} y \rangle = \langle x, T(y) \rangle$$



易知.  $T^2(x) = T(T(x)) = A^2x = Ax = T(x)$

$\Rightarrow T^2 = T$

15. 设  $\dim \text{col } P = r$ .  $u_1 \dots u_r$  为  $\text{col } P$  的一组

基, 有  $Pu_1 = u_1, Pu_2 = u_2 \dots Pu_r = u_r$ . 可见其特  
征向量为对应特征值为 1, 又设  $u_{r+1} \dots u_n$  为  $\text{col } P^\perp$  的

一组基, 有  $Pu_{r+1} = 0 \dots Pu_n = 0$ . 亦是  $P$  的特征向量,  
特征值为 0, 有  $\text{tr } P = r = \dim \text{col } P$

16.  $(PQ)^T = Q^T P^T = QP = PQ$  Hermite 阵

$(PQ)^2 = P(QP)Q = P(PQ)Q = PQ$  幂等

①  $\text{col } PQ \subset \text{col } P$  又  $PQ = QP$   $\text{col } QP \subset \text{col } Q$ .  
 $\text{col } PQ \subset \text{col } P \cap \text{col } Q$

② 设  $x \in \text{col } P \cap \text{col } Q$ . 则存在  $u, v \in R^n$  使

$x = Pu = Qv \Rightarrow Px = PQv \Rightarrow x = PQv$ . 即

$\text{col } P \cap \text{col } Q \subset \text{col } PQ$

①②有  $\text{col } P \cap \text{col } Q = \text{col } PQ$ .  $PQ$  为  $\text{col } P \cap \text{col } Q$  上的正交投影

P17. (a) (b) (c) 互推.  $\Rightarrow$  (d)  $PQ=0$   $QP=0$  证

$\Rightarrow$  (e)  $(P+Q)^* = P+Q$  Hermite 阵.

$$(P+Q)^* = P+Q + QP + PQ = P+Q \quad \text{等等. 是正交阵}$$

(e)  $\Rightarrow$  (d) 亦可

(d)  $\Rightarrow$  (c)  $QP^* - PQ = -PQP$ ,  $QP$  是 Hermite 阵.

$$PQ + QP = PQ + (QP)^* = PQ + P^*Q^* = 2PQ = 0$$

$$PQ = 0$$

P18. 设  $V \in \text{ran } P \cap \text{ran } Q$ .  $U \in \ker P + \ker Q$ .

$$\langle U, V \rangle = 0, \quad \text{有 } \ker P + \ker Q \subseteq (\text{ran } P \cap \text{ran } Q)^\perp$$

设  $U \in \ker P$ ,  $W \in \ker Q$ .  $V \in (\ker P + \ker Q)^\perp$ .

$$\langle U, V \rangle = 0, \quad \langle W, V \rangle = 0 \quad V \in (\ker P)^\perp = \text{ran } P.$$

$V \in \text{ran } Q$  则  $V \in \text{ran } P \cap \text{ran } Q$ . 有

$$(\ker P + \ker Q)^\perp \subseteq \text{ran } P \cap \text{ran } Q.$$

$$\text{综上. } \ker P + \ker Q = (\text{ran } P \cap \text{ran } Q)^\perp$$

P19 设  $V = PV + V - PV$ .  $PV \in \text{ran } P$ .

$$P(V - PV) = 0 \Rightarrow V - PV \in \ker P.$$

设  $U \in \text{ran } P \cap \text{ran } Q$  且  $U \in \ker P$  则  $U = PV$ .  $PV = 0$ .

$$P^2 y = Py = u = Pu = 0 \quad \{\ker P \wedge \text{ran } P = \{0\}\}.$$

$$P \neq 0 \quad \forall x, y \in V$$

$$\langle Px, y \rangle = \langle Px, Py + y - Py \rangle = \langle Px, Py \rangle$$

$$\langle x, Py \rangle = \langle Px + x - Px, Py \rangle = \langle Px, Py \rangle$$

$$\langle Px, y \rangle = \langle x, Py \rangle, \text{ 自伴性.}$$

$$P21 \quad \|Pv\| \leq \|v\| \Rightarrow \|Pv\| \leq \|Pv + (-\frac{1}{c}(v - Pv))\|$$

$$\text{令 } Pv = w, \quad \frac{1}{c}(v - Pv) = u, \text{ 有}$$

$$\|w\| \leq \|w + cu\|$$

$$\text{取 } P4.10, \text{ 可得 } \langle u, w \rangle = 0, \text{ 即 } \langle Pv, v - Pv \rangle = 0.$$

$$Pv \in \text{ran } P, \quad v - Pv \in \ker P. \quad \ker P \subseteq (\text{ran } P)^\perp$$

$$R21. \Rightarrow \text{成立, 反之亦证.}$$

$$P22. \quad Px = \lambda x. \quad P^2 x = P(\lambda x) = \lambda Px = \lambda^2 x = \lambda x$$

$$\lambda = 1/0. \quad \text{tr } (P) = 0. \quad \text{无特征值 } 0.$$

$$\text{若 } Pv = P(c_1 v_1 + \dots + c_n v_n) = 0, \quad v \in \mathbb{R}^n. \quad P=0$$



Pr2.  $Ax=y$  相容.  $\text{rank } A=n$ . 解唯一.

$$QRx=y, Q^T QRx=Q^T y \Rightarrow Rx=Q^T y$$

$$\Rightarrow x=R^{-1}Q^T y.$$

$$\text{Pr3. } A(A^T A)^{-1}A^T = QR(R^T Q^T QR)^{-1}R^T Q^T \\ = QR^{-1}R^T R^T Q^T = QQ^T$$

$$\text{Pr5. } Gx=0 \Rightarrow x^T Gx = \left\| \sum_{i=1}^n x_i u_i \right\|^2 = 0 \quad \textcircled{1}$$

由①成立.  $\sum_{i=1}^n x_i u_i = 0$ , 又  $u_1, \dots, u_n$  线性无关.

故  $x=0$  成立. 即  $\text{null}(G)=0$ ,  $G$  可逆.

$$\text{Pr6. (a) } d(v, u)^2 = \|v - Pv\|^2 \\ = \|v\|^2 + \|Pv\|^2 - \langle v, Pv \rangle - \langle Pv, v \rangle \\ = \|v\|^2 + \langle Pv, Pv \rangle - (2\langle v, Pv \rangle) \\ = \|v\|^2 + \langle Pv - v, Pv \rangle - \left\langle \sum_{i=1}^n G u_i, v \right\rangle \\ = \|v\|^2 - \sum_{i=1}^n G \langle u_i, v \rangle$$

(b) 构成正交组.

$$[\langle u_1, u_1 \rangle \quad \dots \quad \langle u_n, u_1 \rangle \quad 0 \quad \dots \quad \langle u_1, u_n \rangle \quad \dots \quad \langle u_n, u_n \rangle]$$



$$\begin{bmatrix} \langle u_1, u_1 \rangle & \dots & \langle u_{n-1}, u_{n-1} \rangle & 0 \\ \vdots & & \vdots & \vdots \\ \langle u_1, v \rangle & \dots & \langle u_{n-1}, v \rangle & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ c_n \\ d \end{bmatrix} = \begin{bmatrix} \vdots \\ \langle v, u_n \rangle \\ \langle v, v \rangle \end{bmatrix}$$

根据克拉默法则  $d = \frac{g(u_1, \dots, u_{n-1}, v)}{\det M}$

所以最后一列展开  $d = \frac{g(v, u_1, \dots, u_{n-1})}{g(u_1, \dots, u_{n-1})}$

(v)  $d = \left\| v - \frac{uu^*}{\|u\|^2} v \right\| = \left\| v - \frac{\langle v, u \rangle u}{\|u\|^2} \right\|$

$$= \|v\|^2 + \frac{|\langle v, u \rangle|^2}{\|u\|^4} \cdot \|u\|^2 - \frac{\overline{\langle v, u \rangle}}{\|u\|^2} \langle v, u \rangle$$

$$- \frac{\langle v, u \rangle}{\|u\|^2} \langle u, v \rangle$$

$$= \|v\|^2 - \frac{|\langle v, u \rangle|^2}{\langle u, u \rangle}$$

$$= \frac{\langle u, u \rangle \langle u, v \rangle - \langle v, u \rangle \langle u, v \rangle}{\langle u, u \rangle}$$

$$= \frac{g(v, u)}{g(u)} \geq 0 \quad \text{又 } g(u) > 0$$

$$\text{if } g(v, u) \neq 0$$

$$\langle v, u \rangle \langle u, v \rangle = |\langle u, v \rangle|^2 \leq \|u\|^2 \|v\|^2$$

$$\|\langle u, v \rangle\| \leq \|u\| \|v\|$$

p27. (a) 设  $v \in V$  且  $v \in \ker T$ .  $Tv = 0$

$$T^*Tv = 0 \quad \ker T \subseteq \ker T^*T$$

设  $v' \in V$  且  $v' \in \ker T^*T$ .

$$T^*Tv' = 0 \quad v'^*T^*Tv = 0 \quad (Tv)^*Tv = 0$$

$$Tv = 0 \quad \ker T^*T \subseteq \ker T \quad \text{证毕.}$$

$$(b) \dim \ker T^*T = \dim \ker T = \dim V - \dim \text{ran } T = 0$$

$$(c) \text{ 证明: } P^2 = T(T^*T)^{-1}T^*T(T^*T)^{-1}T^* \\ = T(T^*T)^{-1}T^* = P.$$

$$\text{自伴性: } P^* = T(T^*T)^{-*}T^* = T(T^*T)^{-1}T^* = P.$$

(d) 因为  $y \in \text{ran } T$ , 设  $y = Tu$ . 且  $\|Tx - y\| = \|T(x - u)\|$

最小, 即  $x - u \in \ker T$ . 即  $T(x - u) = 0$ . 又 (b)  $\ker T = 0$

$x - u = 0$  即  $x = u$  所以  $T^*T(x - u) = 0$

$$T^*Tx = T^*Tu = T^*y.$$

P.28. 证

$$P.29. \text{ 方程组 } \begin{bmatrix} c_1x^1 & \dots & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ \vdots \\ 1 \end{bmatrix}$$

$$P.30. A^*Ax = A^*y$$

$$\begin{bmatrix} mS_x^2 & mS_x \\ mS_x & m \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} mS_{xy} \\ mS_y \end{bmatrix}$$

解得

$$a = \frac{S_{xy} - S_x S_y}{S_x^2 - S_x^2}$$

$$b = \frac{S_y(S_x^2 - S_x S_{xy})}{S_x^2 - S_x^2}$$

$$P.31 (a) \text{ rank } A = n \Rightarrow m \geq n. \quad Ax=0 \Leftrightarrow A^*Ax=0$$

$$\dim \text{null } A = 0 \Leftrightarrow \dim \text{null } A^*A = 0 \quad A^*A \text{ 正定}$$

P 证法

(b) 证

$$(c) \text{ 设 } v \in \text{col } P, v = Pu = A(A^*A)^{-1}A^*u, \text{ 而 } P \subseteq \text{col } A$$

$$\text{设 } v \in \text{col } A, v = Au, Pv = A(A^*A)^{-1}A^*Au = Au = v$$

$$\text{col } A \subseteq \text{col } P \quad \text{col } A = \text{col } P.$$

$$P.32. A = QR \Rightarrow Q^*A = Q^*QR = R$$

$$\Rightarrow R_{kk} = Q_k^* A_k \quad \text{即 } a_k \text{ 正交}$$

化后为  $Q_k$  与  $a_k$  的内积  $\langle a_k, Q_k \rangle$  且非负实数.

$$\text{为 } d = \|a_k - P a_k\| = \left\| a_k - \underbrace{\sum_{i=1}^{k-1} \langle a_k, Q_i \rangle Q_i}_{\text{①}} \right\|$$

$$\text{① 实为 } a_k \text{ 在 } Q_k \text{ 上的投影} \quad d = \|\langle a_k, Q_k \rangle Q_k\| \\ = |\langle a_k, Q_k \rangle|$$

设  $u_1, \dots, u_n$  为  $U$  的一组基,

构造  $A = [u_1, \dots, u_n]^T$  进行 QR 分解.

$R_{n+1, n+1}$  就是极小化距离.

$$P_{2n+1}(u), \|P_\varepsilon - f\|^2 = \int_0^1 |f(t) - P_\varepsilon(t)|^2 dt$$

$$\leq \int_0^1 |f(t) - P_\varepsilon(t)|^2 dt \leq \varepsilon^2.$$

$$\text{又 } \|f\|^2 \leq \|P_\varepsilon\|^2 + \|f\|^2 = \|P_\varepsilon - f\|^2 = \varepsilon^2. \text{ 得证.}$$

$$(b) \text{ 设 } |f(t)| \leq \rho \quad \|f\| = \left( \int_0^1 |f(t)|^2 dt \right)^{\frac{1}{2}} \leq \rho \leq \varepsilon. \text{ 对}$$

$$\text{任意 } \varepsilon > 0 \text{ 成立. 则 } \rho = 0 \quad f = 0. \quad f' = \{0\},$$

$$\text{故 } f \text{ 在 } [0, 1] \text{ 上恒为 } 0.$$



$$(P^{-1})^T \text{ null } A^T = \text{null } A^T \text{ or } \neq \text{null } A^T$$

p34  $Ax_0 = y \Rightarrow y \in \text{col } A$

$$A^T z = 0 \Rightarrow z \in \text{null } A^T = (\text{col } A)^\perp$$

$$\langle y, z \rangle = z^T y = 0. \quad \text{反之易证.}$$

p38.  $A = SB$  有  $A^T = B^T S^T \quad S^T \text{ 可逆}$

$$\Leftrightarrow \text{col } A^T = \text{col } B^T, (\text{col } A^T)^\perp = (\text{col } B^T)^\perp$$

$$\Leftrightarrow \text{null } A = \text{null } B.$$