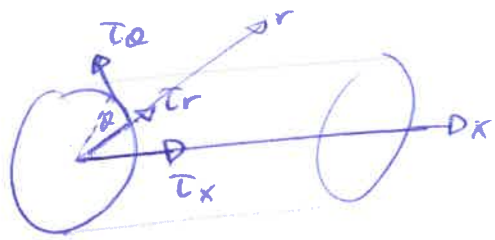
axisymmetric Vector-Laplace

$u = u_x \cdot \tau_x + u_r \cdot \tau_r + u_\theta \cdot \tau_\theta$

$v = v_x \cdot \tau_x + v_r \cdot \tau_r + v_\theta \, \tau_\theta$



$$\nabla_{xyz} \cdot = \tau_x \frac{\partial}{\partial x}\cdot + \boxed{\tau_y \frac{\partial}{\partial y}\cdot + \tau_z \frac{\partial}{\partial z}\cdot} \quad *1$$

$$= \tau_x \frac{\partial}{\partial x}\cdot + \boxed{\tau_r \frac{\partial}{\partial r}\cdot + \frac{1}{r}\tau_\theta \cdot \frac{\partial}{\partial \theta}\cdot}$$

$$\frac{\partial}{\partial x} u = \frac{\partial u_x}{\partial x}\cdot \tau_x + \frac{\partial u_r}{\partial x}\tau_r + \underbrace{\frac{\partial u_\theta}{\partial x}}_{=0}\cancel{\tau_\theta} \qquad \text{da} \quad \frac{\partial \tau_x}{\partial x} = \frac{\partial \tau_r}{\partial x} = \frac{\partial \tau_\theta}{\partial x} = 0$$

$$\frac{\partial}{\partial r} u = \frac{\partial u_x}{\partial r}\tau_x + \frac{\partial u_r}{\partial r}\tau_r + \underbrace{\frac{\partial u_\theta}{\partial r}}_{=0}\cancel{\tau_\theta} \qquad \text{da} \quad \frac{\partial \tau_x}{\partial r} = \frac{\partial \tau_r}{\partial r} = \frac{\partial \tau_\theta}{\partial r} = 0$$

$$\frac{\partial}{\partial \theta} u = \underbrace{\frac{\partial u_x}{\partial \theta}}_{=0}\cancel{\tau_x} + \underbrace{\frac{\partial u_r}{\partial \theta}}_{=0}\cancel{\tau_r} + u_r \underbrace{\frac{\partial \tau_r}{\partial \theta}}_{\tau_\theta} + \underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0}\cancel{\tau_\theta} + u_\theta \underbrace{\cancel{\frac{\partial \tau_\theta}{\partial \theta}}}_{=0} \qquad \text{da} \quad \frac{\partial \tau_x}{\partial \theta} = 0$$

$$= u_r \cdot \tau_\theta$$

*1:

$$\tau_y \, \partial_y = \overbrace{\begin{pmatrix} 0 \\ \cos\theta \\ 0 \end{pmatrix}}^{\tau_y \cdot \frac{\partial r}{\partial y}} \partial_r + \overbrace{\begin{pmatrix} 0 \\ -\frac{\sin\theta}{r} \\ 0 \end{pmatrix}}^{\tau_y \cdot \frac{\partial \theta}{\partial y}} \partial_\theta$$

$$\tau_z \, \partial_z = \underbrace{\begin{pmatrix} 0 \\ 0 \\ \sin\theta \end{pmatrix}}_{\tau_z \cdot \frac{\partial r}{\partial z}} \partial_r + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{\cos\theta}{r} \end{pmatrix}}_{\tau_z \cdot \frac{\partial \theta}{\partial z}} \partial_\theta$$

$$\overset{(+)}{\Rightarrow} \tau_y \partial_y + \tau_z \partial_z = \tau_r \partial_r + \frac{1}{r}\tau_\theta \partial_\theta$$

$$\nabla_{xyz} u : \nabla_{xyz} v = \nabla_{xr} u : \nabla_{xr} v + \frac{1}{r^2} u_r v_r$$

$$\mathrm{div}_{xyz}\, u = \mathrm{div}_{xr}(u) + \frac{1}{r} u_r = \frac{1}{r}\, \mathrm{div}_{xr}(u \cdot r)$$

$$\nabla_{xr}(u) = \frac{1}{r}\left(\nabla_{xr}(u \cdot r) - \frac{u}{r} \otimes \tau_r\right)$$