Abstract Algebra by Pinter, Chapter 17

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Abstract

Chapter 17 on Rings

Contents

1 A. Examples of Rings

Prove that the following are commutative rings with unity.

Indicate the zero element, the unity and the negative for an a.

Ring axioms:

1. $a \oplus b = b \oplus a$

2. $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
3. $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

Commutative:

1. $a \otimes b = b \otimes a$

With unity:

1. $\exists 1' \in A : a \otimes 1' = a$

1.1 Q1

$$a \oplus b = a + b - 1 \qquad a \otimes b = ab - (a + b) + 2$$

Axiom 1 is self evident.

Using sage, we prove axioms 2 and 3.

```
sage: a = var('a')
sage: b = var('b')
sage: c = var('c')
sage: ab = a*b - (a + b) + 2
sage: ab_c = ab*c - (ab + c) + 2
sage: bc = b*c - (b + c) + 2
sage: a_bc = a*bc - (a + bc) + 2
sage: ab_c.full_simplify()
-(a - 1)*b + ((a - 1)*b - a + 1)*c + a
sage: a_bc.full_simplify()
-(a - 1)*b + ((a - 1)*b - a + 1)*c + a
```

```
sage: def mul(a, b):
....:     return a*b - (a + b) + 2
....:
sage: def add(a, b):
....:     return a + b - 1
....:
sage: mul(a, add(b, c)).full_simplify()
(a - 1)*b + (a - 1)*c - 2*a + 3
sage: add(mul(a, b), mul(a, c)).full_simplify()
(a - 1)*b + (a - 1)*c - 2*a + 3
```

To calculate zero and unity:

$$a \oplus 0' = a$$
$$a + b - 1 = a$$
$$b = 1 = 0'$$

$$a \otimes 1' = a$$
$$ab - (a + b) + 2 = a$$
$$b = 2 = 1'$$

Lastly for the negative:

$$a \oplus b = 0'$$
$$a + b - 1 = 1$$
$$b = -a$$

1.2 Q2

$$a \oplus b = a + b + 1 \qquad a \otimes b = ab + a + b$$

```
sage: def add(a, b):
....:     return a + b + 1
....:
sage: def mul(a, b):
....:     return a*b + a + b
```

Axiom 1: $a \oplus b = b \oplus a$

Self-evident

Axiom 2: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$

```
sage: bool(mul(mul(a, b), c) == mul(a, mul(b, c)))
True
```

Axiom 3: $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

```
sage: bool(mul(a, add(b, c)) == add(mul(a, b), mul(a, c)))
True
```

Commutative: $a \otimes b = b \otimes a$

Self-evident

Zero:

```
sage: solve(add(a, b) - a, b)
[b == -1]
sage: add(a, -1)
a
```

Unity:

```
sage: solve(mul(a, b) - a, b)
[b == 0]
sage: mul(a, 0)
a
```

Negative a:

```
sage: solve(add(a, b) + 1, b)
[b == -a - 2]
sage: add(a, -a -2)
-1
```

1.3 Q3

$$(a, b) \oplus (c, d) = (a + c, b + d)$$

$$(a, b) \otimes (c, d) = (ac - bd, ad + bc)$$

```
sage: c = var('c')
sage: d = var('d')
sage: e = var('e')
sage: f = var('f')
sage: def add(ab, cd):
....:     a, b = ab
....:     c, d = cd
....:     return (a + c, b + d)
....:
sage: def mul(ab, cd):
....:     a, b = ab
....:     c, d = cd
....:     return (a*c - b*d, a*d + b*c)
....:
```

Axiom 1: $a \oplus b = b \oplus a$

```
sage: bool(add((a, b), (c, d)) == add((c, d), (a, b)))
True
```

Axiom 2: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$

```
sage: bool(mul(mul((a, b), (c, d)), (e, f)) == mul((a, b), mul((c, d), (e, f))))
True
```

Axiom 3: $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

```
sage: bool(mul((a, b), add((c, d), (e, f))) == add(mul((a, b), (c, d)), mul((a, b), (e, f))))
True
```

Commutative: $a \otimes b = b \otimes a$

Self-evident

Zero:

```
sage: ab_plus_cd = add((a, b), (c, d))
sage: solve(ab_plus_cd[0] - a, c)
[c == 0]
sage: solve(ab_plus_cd[1] - b, d)
[d == 0]
sage: add((a, b), (0, 0))
(a, b)
```

Unity:

```
sage: ab_mul_cd = mul((a, b), (c, d))
sage: solve([ab_mul_cd[0] - a, ab_mul_cd[1] - b], c, d)
[[c == 1, d == 0]]
sage: mul((a, b), (1, 0))
(a, b)
```

Negative a:

Since $0' = (0, 0)$ then the negative for (a, b) is simply $(-a, -b)$.

1.4 Q4

$$A = \{x + y\sqrt{2} : x, y \in \mathbb{Z}\}$$

Since normal algebraic operations are defined on A, then 1, 2 and 3 pass. It is also commutative.

Zero: 0

Unity: 1

Negative: $-x - y\sqrt{2}$

1.5 Q5

Prove the ring in part 1 is an integral domain.

We show that it has the cancellation property.

Assume $a \otimes b = a \otimes c$.

$$ab - (a + b) + 2 = ac - (a + c) + 2$$
$$ab - b = ac - c$$

Therefore $b = c$, and the ring has the cancellation property.

1.6 Q6

Prove the ring in part 2 is a field.

A field is a commutative ring with unity in which every nonzero element is invertible.

$$0' = -1$$
$$1' = 0$$

Thus

$$a \otimes b = 1'$$
$$ab + a + b = 0$$

We solve for b as follows

```
sage: def mul(a, b):
....:         return a*b + a + b
....:
sage: solve(mul(a, b), b)
[b == -a/(a + 1)]
```

(Excluding the $0'$ element which is equal to -1)

1.7 Q7

Find the inverse for the ring in part 3.

```
sage: def mul(ab, cd):
....:     a, b = ab
....:     c, d = cd
....:     return (a*c - b*d, a*d + b*c)
....:
sage: ab_mul_cd = mul((a, b), (c, d))
sage: solve([ab_mul_cd[0] - 1, ab_mul_cd[1]], c, d)
[[c == a/(a^2 + b^2), d == -b/(a^2 + b^2)]]
```

2 B. Ring of Real Functions

2.1 Q1

Let $a, b \in \mathcal{F}(\mathbb{R})$

Ring axioms:

1. $ab = ba$
2. $(ab)c = a(bc)$
3. $a(b + c) = ab + ac$

Commutative:

1. ab = ba

Zero: $f(x) = 0$

Unity: $f(x) = 1$

Negative: $-f(x)$

2.2 Q2

Divisors of zero, are any two functions which when $f(x) \neq 0$ then $g(x) = 0$ but in general $f(x) \neq 0$ and $g(x) \neq 0$.

See more here

2.3 Q3

Any functions which are one to one and have an inverse. That is $f(x) = x^3$ but not $f(x) = x^2$.

2.4 Q4

A field must have every element invertible. So the ring is not a field.

Ring has divisors of zero, so it does not have the cancellation property $\implies$ ring is not an integral domain.

3 C. Ring of 2×2 Matrices

3.1 Q1

```
sage: a = var('a')
sage: b = var('b')
sage: c = var('c')
sage: d = var('d')
sage: r = var('r')
sage: s = var('s')
sage: t = var('t')
sage: u = var('u')
sage: w = var('w')
sage: x = var('x')
sage: y = var('y')
sage: z = var('z')
sage:
sage: def add(abcd, rstu):
....:     a, b, c, d = abcd
....:     r, s, t, u = rstu
....:     return (a + r, b + s, c + t, d + u)
....:
sage: def mul(abcd, rstu):
....:     a, b, c, d = abcd
....:     r, s, t, u = rstu
....:     return (a*r + b*t, a*s + b*u, c*r + d*t, c*s + d*u)
```

Axiom 1:

Self evident.

Axiom 2:

```
sage: bool(mul((a,b,c,d), mul((r,s,t,u), (w,x,y,z))) == mul(mul((a,b,c,d), (r,s,t,u)), (w,x,y,z
....: )))
True
```

Axiom 3:

```
sage: bool(mul((a,b,c,d), add((r,s,t,u), (w,x,y,z))) == add(mul((a,b,c,d), (r,s,t,u)), mul((a,b
....: ,c,d), (w,x,y,z))))
True
```

3.2 Q2

```
sage: bool(mul((a,b,c,d), (r,s,t,u)) == mul((r,s,t,u), (a,b,c,d)))
False
```

Unity: $(a, b, c, d)(r, s, t, u) = (a, b, c, d)$

```
sage: solve([x_mul_y[0] - a, x_mul_y[1] - b, x_mul_y[2] - c, x_mul_y[3] - d], r,s,t,u)
[[r == 1, s == 0, t == 0, u == 1]]
```

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.3 Q3

Matrices don't have the cancellation property.

For example $ar_1 + bt_1 = ar_2 + bt_2$ does not imply that $r_1 = r_2$ and $t_1 = t_2$.

Thus is not an integral domain.

Not all matrices are invertible, for example when $det(A) = 0$. See more info here. Hence they $\mathcal{M}_2(\mathbb{R})$ is not a field either.

4 D. Rings of Subsets of a Set

$$A + B = (A - B) \cup (B - A)$$
$$AB = A \cap B$$

4.1 Q1

Ring axioms:

1.
$$\begin{aligned} A + B &= (A - B) \cup (B - A) \\ &= B + A \end{aligned}$$

2.
$$(AB)C = (A \cap B) \cap C = A \cap (B \cap C) = A(BC)$$

3.
$$\begin{aligned} A(B + C) &= A \cap [(B - C) \cup (C - B)] \\ &= [A \cap (B - C)] \cup [A \cap (C - B)] \\ &= (AB - AC) \cup (AC - AB) \\ AB + AC &= (AB - AC) \cup (AC - AB) \end{aligned}$$

Commutativity:

$$AB = A \cap B = BA$$

Unity:

$$AB = A \implies B = D$$

Zero:

$$A + B = A \implies B = \varnothing$$

4.2 Q2

All elements of P_D with non-overlapping regions are divisors of zero.

$$X \in P_D, X^2 = \varnothing$$

4.3 Q3

$$1' = D$$

$$AB = D \implies A \cap B = D$$

Thus $A = D$ and $B = D$

4.4 Q4

There exist non-zero non-invertible elements in P_D, hence it is *not* a field.

$AB = AC$ does not imply $B = C$, hence cancellation property does not hold, and P_D is not an integral domain.

4.5 Q5

$$
\begin{aligned}
e &= \varnothing \\
a &= \{a\} \\
b &= \{b\} \\
c &= \{c\} \\
ab &= \{a, b\} \\
ac &= \{a, c\} \\
bc &= \{b, c\} \\
abc &= \{a, b, c\}
\end{aligned}
$$

$\oplus$	e	a	b	c	ab	ac	bc	abc
e	e	a	b	c	ab	ac	bc	abc
a	a	e	ab	ac	b	c	abc	bc
b	b	ab	e	bc	a	abc	c	ac
c	c	ac	bc	e	abc	a	b	ab
ab	ab	b	a	abc	e	bc	ac	c
ac	ac	c	abc	a	bc	e	ab	b
bc	bc	abc	c	b	ac	ab	e	a
abc	abc	bc	ac	ab	c	b	a	e

$\otimes$	e	a	b	c	ab	ac	bc	abc
e	e	a	b	c	ab	ac	bc	abc
a	a	a	ab	ac	ab	ac	abc	abc
b	b	ab	b	bc	ab	abc	bc	abc
c	c	ac	bc	e	abc	a	b	abc
ab	ab	ab	ab	abc	ab	abc	abc	abc
ac	ac	ac	abc	ac	abc	ac	abc	abc
bc	bc	abc	bc	bc	abc	abc	bc	abc
abc	abc	abc	abc	abc	abc	abc	abc	abc

5 E. Ring of Quaternions

5.1 Q1

Unity:

```
sage: a = var('a')
sage: b = var('b')
sage: c = var('c')
sage: d = var('d')
sage: matrix([[a + b*I, c + d*I], [-c + d*I, a - b*I]])
[ a + I*b  c + I*d]
[-c + I*d  a - I*b]
sage: alpha = matrix([[a + b*I, c + d*I], [-c + d*I, a - b*I]])
sage: matrix([[1, 0], [0, 1]]) * alpha
[ a + I*b  c + I*d]
[-c + I*d  a - I*b]
```

Distributive law:

```
sage: bb = var('e f g h')
sage: cc = var('i j k l')
sage: def make_matrix(xx):
....:     return matrix([[xx[0] + I*xx[1], xx[2] + xx[3]*I], [-xx[2] + xx[3]*I, xx[0] - xx[1]*I]])
```

```
....:
sage: bool(alpha*(make_matrix(bb) + make_matrix(cc)) == (alpha*make_matrix(bb) + alpha*make_matrix(cc))
True
```

Non-commutative:

```
sage: bool(alpha*make_matrix(bb) == make_matrix(bb)*alpha)
False
```

5.2 Q2

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\begin{aligned} \alpha &= a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \\ &= \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \end{aligned}$$

5.3 Q3

For the formula $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$

```
sage: ii = matrix([[I, 0], [0, -I]])
sage: ii*ii
[-1  0]
[ 0 -1]
sage: -ii*ii
[1 0]
[0 1]
sage: jj = matrix([[0, 1], [-1, 0]])
sage: jj*jj
[-1  0]
[ 0 -1]
sage: kk = matrix([[0, I], [I, 0]])
sage: kk*kk
[-1  0]
[ 0 -1]
sage: bool(ii**2 == jj**2)
True
sage: bool(ii**2 == kk**2)
True
```

$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$

```
sage: bool(ii*jj == -jj*ii)
True
sage: bool(ii*jj == kk)
True
```

$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$

```
sage: bool(jj*kk == -kk*jj)
True
sage: bool(jj*kk == ii)
True
```

$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$

```
sage: bool(kk*ii == -ii*kk)
True
sage: bool(kk*ii == jj)
True
```

5.4 Q4

$$\bar{\alpha} = \begin{pmatrix} a - bi & -c - di \\ c - di & a + bi \end{pmatrix}$$

$$||\alpha|| = a^2 + b^2 + c^2 + d^2 = t$$

Show that

$$\bar{\alpha}\alpha = \alpha\bar{\alpha} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

```
sage: alpha
[ a + I*b  c + I*d]
[-c + I*d  a - I*b]
sage: alpha_bar = matrix([[a - b*I, -c - d*I], [c - d*I, a + b*I]])
sage: bool(alpha_bar*alpha == alpha*alpha_bar)
True
sage: alpha_bar*alpha
[(a + I*b)*(a - I*b) + (c + I*d)*(c - I*d)                                         0]
[                                        0 (a + I*b)*(a - I*b) + (c + I*d)*(c - I*d)]
```

Note that $(a + ib)(a - ib) = a^2 + b^2$ and the same for c and d.

Earlier we found the identity is

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus the multiplicative inverse (both on the left and right) such that $\alpha\beta = \beta\alpha = \mathbf{1}$ is given by $(1/t)\bar{\alpha}$.

5.5 Q5

From part 4 we show there is a multiplicative inverse. Thus by the definition, $\mathcal{L}$ is a skew field.

6 F. Ring of Endomorphisms

6.1 Q1

Let $f, g, h \in End(G)$

1. $f + g = g + f$
2. $(f \cdot g) \cdot h = f \cdot (g \cdot h)$
3. $f \cdot (g + h) = f \cdot g + f \cdot h$

6.2 Q2

For a homomorphism $f(0) = 0$

Applying the rule $f(a + b) = f(a) + f(b)$

$$e = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$a = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

$$b = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{pmatrix}$$

$$c = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

+	e	a	b	c
e	a	b	c	e
a	b	c	e	a
b	c	e	a	b
c	e	a	b	c

×	e	a	b	c
e	e	a	b	c
a	a	c	a	c
b	b	a	e	c
c	c	c	c	c