Abstract Algebra by Pinter, Chapter 16

Amir Taaki

Abstract

Chapter 16 on Fundamental Homomorphism Theorem

Contents

1 A. Examples of FHT

Use the FHT to prove that the two given groups are isomorphic. Then display their tables.

1.1 Q1

$\mathbb{Z}_5$ and $\mathbb{Z}_{20}/\langle 5\rangle$.

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$$K = \{0, 5, 10, 15\} = \langle 5\rangle$$

$$f : \mathbb{Z}_{20} \underset{\langle 5\rangle}{\twoheadrightarrow} \mathbb{Z}_5$$

$$\mathbb{Z}_5 \cong \mathbb{Z}_{20}/\langle 5\rangle$$

1.2 Q2

$\mathbb{Z}_3$ and $\mathbb{Z}_6/\langle 3\rangle$.

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

$$K = \{0, 3\} = \langle 3\rangle$$

$$f : \mathbb{Z}_6 \underset{\langle 3\rangle}{\twoheadrightarrow} \mathbb{Z}_3$$

$$\mathbb{Z}_3 \cong \mathbb{Z}_6/\langle 3\rangle$$

1.3 Q3

$\mathbb{Z}_2$ and $S_3/\{\epsilon, \beta, \delta\}$.

$$f = \begin{pmatrix} \epsilon & \alpha & \beta & \gamma & \delta & \kappa \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$K = \{\epsilon, \beta, \delta\}$$

$$f : S_3 \underset{\{\epsilon,\beta,\delta\}}{\twoheadrightarrow} \mathbb{Z}_2$$

$$\mathbb{Z}_2 \cong S_3/\{\epsilon, \beta, \delta\}$$

1.4 Q4

From Chapter 3, part C (at the end):

$$P_D = \{A : A \subseteq D\}$$

If A and B are any two sets, their symmetric difference is the set $A + B$ defined as follows:

$$A + B = (A - B) \cup (B - A)$$

$A - B$ represents the set obtained by removing from A all the elements which are in B.

$$P_3 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

Consider the function $f(C) = C \cap \{a, b\}$

$$P_2 = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$$

The kernel is $\{\varnothing, \{c\}\}$

Using the kernel we create the quotient cosets:

$$\begin{aligned}
K &= \{\varnothing, \{c\}\} \\
&= K + \{c\} \\
K + \{a\} &= \{\{a\}, \{a,c\}\} \\
&= K + \{a,c\} \\
K + \{b\} &= \{\{b\}, \{b,c\}\} \\
&= K + \{b,c\} \\
K + \{a,b\} &= \{\{a,b\}, \{a,b,c\}\} \\
&= K + \{a,b,c\}
\end{aligned}$$

Applying the function to the cosets, we get:

$$\begin{aligned}
f(K) &= \{\varnothing\} \\
f(K \cap \{a\}) &= \{\{a\}\} \\
f(K \cap \{b\}) &= \{\{b\}\} \\
f(K \cap \{a,b\}) &= \{\{a,b\}\}
\end{aligned}$$

Thus,

$$f : P_3 \underset{\{\varnothing,\{c\}\}}{\twoheadrightarrow} P_2$$

$$P_2 \cong P_3/\{\varnothing, \{c\}\}$$

1.5 Q5

$\mathbb{Z}_3$ and $(\mathbb{Z}_3 \times \mathbb{Z}_3)/K$ where $K = \{(0,0),(1,1),(2,2)\}$

Consider $f : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3$ by:

$$f(a,b) = a - b$$

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$$

$$\begin{aligned}
\bar{(0,0)} &= K + (0,0) = K + (1,1) = K + (2,2) \\
\bar{(0,1)} &= K + (0,1) = K + (1,2) = K + (2,0) \\
\bar{(0,2)} &= K + (0,2) = K + (1,0) = K + (2,1)
\end{aligned}$$

Applying the function to any element k from the cosets we get:

$$\begin{aligned}
f(0,0) &= f(1,1) = f(2,2) = 0 \\
f(0,1) &= f(1,2) = f(2,0) = 2 \\
f(0,2) &= f(1,0) = f(2,1) = 1
\end{aligned}$$

Thus,

$$f : \mathbb{Z}_3 \times \mathbb{Z}_3 \underset{K}{\twoheadrightarrow} \mathbb{Z}_3$$

$$\mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3/\{(0,0),(1,1),(2,2)\}$$

2 B. Example of the FHT Applied to $F(\mathbb{R})$

2.1 Q1

Let $\alpha : F(\mathbb{R}) \to \mathbb{R}$ be:

$$\alpha(f) = f(1)$$

Let $\beta : F(\mathbb{R}) \to \mathbb{R}$ be:

$$\beta(f) = f(2)$$

Prove α and β are homomorphisms from $F(\mathbb{R})$ onto $\mathbb{R}$.

Let $g, h \in F(\mathbb{R})$, then:

$$\begin{aligned}
f(g+h) &= (g+h)(1) \\
&= g(1) + h(1)
\end{aligned}$$

Likewise for β

The functions are onto because the range of each function are $f(1)$ and $f(2)$ respectively.

2.2 Q2

$$J = \{f : f(1) = 0, \forall f \in F(\mathbb{R})\}$$
$$K = \{f : f(2) = 0, \forall f \in F(\mathbb{R})\}$$

The cosets of $F(\mathbb{R})$ for α are:

$$J + g, \forall g \in F(\mathbb{R})$$

And for β:

$$K + g, \forall g \in F(\mathbb{R})$$

2.3 Q3

For any arbitrary $g, h \in F(\mathbb{R})$ and $k_1, k_2 \in J$,

$$\begin{aligned} f((k_1 + g) + (k_2 + h)) &= (k_1 + g + k_2 + h)(1) \\ &= f(k_1 + g) + f(k_2 + h) \end{aligned}$$

Thus $J + g$ and $K + g$ are valid quotient groups.

J and K have the same cardinality under $F(\mathbb{R})$ and so:

$$F(\mathbb{R})/J \cong F(\mathbb{R})/K$$

3 C. Example of FHT with Abelian Groups

3.1 Q1

Let $a, b \in G$

$$f(ab) = (ab)^2$$

But G is abelian, so:

$$\begin{aligned} (ab)^2 &= a^2b^2 \\ &= f(a)f(b) \end{aligned}$$

And $H = \{x^2 : x \in G\}$

So f is a homomorphism of G onto H

3.2 Q2

$ker(f)$ is defined as:

$$\begin{aligned} K &= \{x \in G : f(x) = e\} \\ &= \{x \in G : x^2 = e\} \end{aligned}$$

3.3 Q3

$f : G \to H$ is a homomorphism of G onto H, with a kernel K, $f : G \underset{K}{\twoheadrightarrow} H$ So therefore,

$$H \cong G/K$$

4 D. Group of Inner Automorphisms

See also the videos by Elliot724 on YouTube about automorphisms.

4.1 Q1

For $Aut(G) \subseteq S_G$, prove $Aut(G) \leq S_G$.

We must prove that $Aut(G)$ obeys the group axioms.

Definition of $Aut(G)$:

$$Aut(G) = \{f \in S_G : f(g_1 g_2) = f(g_1) f(g_2), \forall g_1, g_2 \in G\}$$

Therefore for any $f_1, f_2 \in Aut(G)$, it is true that:

$$\forall g_1, g_2 \in G, f_1(f_2(g_1 g_2)) = f_1(f_2(g_1)) f_1(f_2(g_2))$$

Set obeys **closure** property.

Secondly there is an **identity** element $f_e \in S_G$ such that $f_e : g \to g, \forall g \in G$. Thus $f_e \in Aut(G)$.

Lastly $\forall f \in Aut(G), \forall g_1, g_2 \in G$, that there exists:

$$\begin{aligned} f(\bar{g_1}) &= g_1 \\ f(\bar{g_2}) &= g_2 \end{aligned}$$

Because f is bijective, in particular from the surjective property, we can compose elements in the domain.

$$\begin{aligned} f(\bar{g_1}\bar{g_2}) &= f(\bar{g_1}) f(\bar{g_2}) \\ &= g_1 g_2 \end{aligned}$$

Now because know that:

$$f^{-1}(g_1 g_2) = f^{-1}(g_1) f^{-1}(g_2)$$

Substituting in the values of g_1 and g_2, we get:

$$\begin{aligned} f^{-1}(f(\bar{g_1}\bar{g_2})) &= f^{-1}(f(\bar{g_1})) f^{-1}(f(\bar{g_2})) \\ \bar{g_1}\bar{g_2} &= \bar{g_1}\bar{g_2} \end{aligned}$$

Thus group has an **inverse**.

$$Aut(G) \leq S_G$$

4.2 Q2

ϕ_a denotes an inner automorphism of G:

$$\text{for every } x \in G \qquad \phi_a(x) = axa^{-1}$$

Prove every inner automorphism is an automorphism of G.

$$\phi_a(x) = axa^{-1}$$

Show homomorphic property:

$$\phi_a(xy) = axya^{-1}$$

But $e = a^{-1}a$, so:

$$\phi_a(xy) = ax(a^{-1}a)ya^{-1} = \phi_a(x)\phi_a(y)$$

So ϕ_a is homomorphic.

Also $\phi_e(x) = x \quad \forall x \in G$

4.3 Q3

Likewise from above:

$$\phi_a \cdot \phi_b = \phi_{ab}$$

Because $a(bxb^{-1})a^{-1} = (ab)x(ab)^{-1}$

For the inverse, we note that:

$$\begin{aligned}\phi_a(x)\phi_b(x) &= \phi_e(x) = x \\ &= (ab)x(ab)^{-1}\end{aligned}$$

It therefore follows that the inverse automorphism of ϕ_a is:

$$(\phi_a)^{-1} = \phi_{a^{-1}}$$

4.4 Q4

$I(G) = \{\phi_a : a \in G\}$. Prove $I(G) \leq Aut(G)$.

Closure: for any $\phi_a, \phi_b \in I(G)$, then $\phi_a \cdot \phi_b \in I(G)$ because $\phi_a \cdot \phi_b = \phi_{ab}$

Identity: ϕ_e is the identity because $eae^{-1} = a$, so $\phi_e \in I(G)$.

Inverses: $\forall \phi_a \in I(G)$, there is an $\phi_{a^{-1}} \in I(G)$ because $\phi_a \cdot \phi_{a^{-1}} = \phi_{aa^{-1}} = \phi_e$, thus $\phi_{a^{-1}} = (\phi_a)^{-1}$

4.5 Q5

$$C = \{a \in G : ax = xa \text{ for every } x \in G\}$$

Let $a \in C$. Then for every $x \in G$:

$$ax = xa \text{ or } axa^{-1} = x$$

4.6 Q6

Let $h : G \to I(G)$ be a function defined by $h(a) = \phi$. Prove that h is a homomorphism from G *onto* $I(G)$ and that C is its kernel.

We can see that $h(ab) = \phi_{ab} = \phi_a \cdot \phi_b = h(a)h(b)$. Lastly the function is surjective (onto) because for every ϕ, there is a corresponding $a \in G$ (possibly multiple if for example the group is abelian), so the mapping is well defined.

The kernel is defined by:

$$K = \{x \in G : f(x) = e\}$$

In our case this is:

$$K = \{a \in G : h(a) = \phi_e\}$$

The center is defined as:

$$C = \{a \in G : axa^{-1} = x \text{ for every } x \in G\}$$

Which is also the same as writing:

$$K = \{a \in G : h(a) = \phi_e\}$$

4.7 Q7

Lastly using the FHT, we note that:

$$h : G \underset{C}{\twoheadrightarrow} I(G)$$

$$I(G) \cong G/C$$

5 E. FHT Applied to Direct Products of Groups

5.1 Q1

Let G and H be groups.

Suppose $J \trianglelefteq G$ and $K \trianglelefteq H$

$$f(x, y) = (Jx, Ky)$$

Assuming $x \in G$ and $y \in H$, then Jx and Ky form the cosets for G and H.

That is for every value from G and H maps onto $(G/J) \times (H/K)$ because:

$$x \in J\bar{x} \iff Jx = J\bar{x}$$
$$y \in K\bar{y} \iff Ky = K\bar{y}$$

$$f : G \times H \to (G/J) \times (H/K)$$

5.2 Q2

$$ker f = \{(x, y) \in G \times H : f(x, y) = (J, K)\} = J \times K$$

5.3 Q3

$$f : G \times H \underset{J \times K}{\twoheadrightarrow} (G/J) \times (H/K)$$

$$(G \times H)/(J \times K) \cong (G/J) \times (H/K)$$

6 F. First Isomorphism Theorem

6.1 Q1

$K \leq G, H \trianglelefteq G$

Both H and K are closed subgroups, so an element in both must by definition remain within $H \cap K$.

Let $h \in H \cap K$, then $\forall x \in G, xax^{-1} \in H$. This also applies to K. Therefore $H \cap K$ is a normal subgroup of K.

6.2 Q2

$HK = \{xy : x \in H \text{ and } y \in K\}$. Prove HK is a subgroup of G.

Let $a, b \in HK$, then $ab = (h_1k_1)(h_2k_2) = h_1(k_1h_2k_1^{-1})k_1k_2$ which is another element in HK.

6.3 Q3

H is a normal subgroup of HK.

Since HK is a subgroup of G then every element of H conjugated with elements from HK also lay within H.

$H \trianglelefteq HK$

6.4 Q4

Let $x \in HK$ then $x = hk$ for some $h \in H, k \in K$. Form the coset $Hx = H(hk) = Hk$.

Thus HK/H may be written as Hk for some $k \in K$.

6.5 Q5

Prove $f(k) = Hk$ is a homomorphism $f : K \to HK/H$, and its kernel is $H \cap K$

Since $Hk_1 = Hk_2$ for k_1, k_2 in the same coset, then any member of the quotient group HK/H is equal to H multiplied by a representative from that member.

To find the kernel, we need every $x \in K$ such that $f(x) = H$, the identity coset. That is $x \in H$. But since we are mapping from K, then $x \in K$ and $x \in H$. In other words, $ker f = H \cap K$.

6.6 Q6

$$f : K \underset{H\cap K}{\twoheadrightarrow} HK/H$$

$$K/(H \cap K) \cong HK/H$$

7 G. Sharper Cayley Theorem

7.1 Q1

To prove ρ_a is a permutation of X, we must show it is a bijective mapping from X to X.

To show it is injective, let $x_1, x_2 \in X$ and $a \in G$. Suppose $\rho_a(x_1H) = \rho_a(x_2H)$. Since $a \in G$ and G is a group, then $a^{-1} \in G$. Then $(ax_1)H = (ax_2)H$ and,

$$x_1H = a^{-1}ax_2H = x_2H$$

Therefore ρ_a is injective.

To show it is surjective, consider $g \in G$ such that $\rho_a(x) = gH$. But we note that $\rho_a(x) = (ax)H$, so:

$$gH = axH \text{ or } xH = a^{-1}gH$$

Thus ρ_a is both injective and surjective and is therefore a bijective mapping from $X \to X$.

7.2 Q2

Prove $h : G \to S_X$ defined by $h(a) = \rho_a$ is a homomorphism.

Definition of ρ_a:

$$\rho_a(xH) = (ax)H$$

Let $a, b \in G$, then $\forall x \in X$:

$$h(ab) = \rho_{ab}$$

$$\rho_{ab}(x) = (abx)H = (a(bxH)) = (\rho_a \cdot \rho_b)(x)$$

Therefore:

$$h(ab) = h(a) \cdot h(b)$$

7.3 Q3

Let ρ_e denote an identity permutation which leaves the coset unchanged.

$$\rho_e(xH) = xH$$

$$h(a) = \rho_a \implies \forall x \in G \qquad \rho_a(xH) = axH = xH$$

But because ρ_a is an identity permutation then $axH = xH$. That is,

$$xax^{-1}H = H$$

Thus the kernel of h is:

$$kerf = \{a \in H : xax^{-1} \in H, \forall x \in G\}$$

7.4 Q4

Since h is a homomorphism by:

$$f : G \underset{kerf}{\twoheadrightarrow} S_x$$

$$G/kerf \cong \bar{S} \leq S_X$$

If group is a normal subgroup then $\forall a \in A$ and $x \in G$, $xax^{-1} \in A$, which is contained in the kernel of f from the last exercise.

If H contains no normal subgroup of G except $\{e\}$ then:

$$kerf = \{e\}$$

So the quotient group $G/kerf$ is simply G, so we have:

$$G \cong \bar{S} \leq S_X$$

Since S_X is a permutation representation, for which we only define permutations depending on the elements in G. This is why the identity is an homomorphism and not an isomorphism.

8 H. Quotient Groups Isomorphic to the Circle Group

8.1 Q1

Cosine and sine identities:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\begin{aligned}
\operatorname{cis}(x + y) &= (\operatorname{cis} x)(\operatorname{cis} y) \\
&= \cos(x + y) + i\sin(x + y) = (\cos x + i\sin x)(\cos y + i\sin y) \\
&= \cos(x + y) + i\sin(x + y) = \operatorname{cis}(x + y)
\end{aligned}$$

8.2 Q2

$$T = \{\operatorname{cis} x : x \in \mathbb{R}\}$$

Properties of a group:

1. Closure
2. Associativity
3. Identity
4. Inverses

Let $u, v \in T$, then the group operation is multiplication and $u = \operatorname{cis} x$ for some $x \in \mathbb{R}$ and $v = \operatorname{cis} y$ for some $y \in \mathbb{R}$.

Then $u \cdot v = (\operatorname{cis} x)(\operatorname{cis} y) = \operatorname{cis}(x + y)$, where $x + y \in \mathbb{R}$ and so $u \cdot v \in T$ which obeys closure property.

Since the result of cis is a complex number, we conclude the group obeys associativity property.

For the identity, we must test whether 1 lies in T. That is $\exists x \in R : \operatorname{cis} x = 1 = \cos x + i\sin x$. Setting $x = 0$, we get $\operatorname{cis} x = 1$, so group obeys identity property.

For inverses, we know 1 lies in the group so:

$$|z| = 1 \implies \frac{1}{|z|} = 1 = |\frac{1}{z}|$$

So the value $\frac{1}{z}$ is also in the unit square.

8.3 Q3

Let $x, y \in \mathbb{R}$

$$\begin{aligned}
f(x + y) &= \operatorname{cis}(x + y) \\
&= (\operatorname{cis} x)(\operatorname{cis} y) \\
&= f(x)f(y)
\end{aligned}$$

Thus f is a homomorphism $f : \mathbb{R} \to T$

8.4 Q4

$$\begin{aligned}
\ker f &= \{x \in \mathbb{R} : f(x) = 1\} \\
&= \{2n\pi : n \in \mathbb{Z}\} = \langle 2\pi \rangle
\end{aligned}$$

8.5 Q5

$$f : \mathbb{R} \underset{\langle 2\pi \rangle}{\twoheadrightarrow} T$$

$$T \cong \mathbb{R}/\langle 2\pi \rangle$$

8.6 Q6

$$g(x) = \operatorname{cis} 2\pi x$$

$$\begin{aligned} g(x+y) &= \operatorname{cis}(2\pi x + 2\pi y) \\ &= g(x)g(y) \end{aligned}$$

$\ker g = \mathbb{Z}$ because $\operatorname{cis}(2\pi n) = 1$

8.7 Q7

$$g : \mathbb{R} \underset{\mathbb{Z}}{\twoheadrightarrow} T$$

9 I. Second Isomorphism Theorem

$$H \trianglelefteq G \qquad K \trianglelefteq G \qquad H \subseteq K$$

$$\phi : G/H \to G/K$$

$$\phi(Ha) = Ka$$

9.1 Q1

\$Ha = Hb \$ so $a \in Hb$, hence $a = hb$ for some $h \in H$

$$\phi(Ha) = \phi(Hhb) = \phi(Hb)$$

If $a = he$ then $\phi(Ha) = \phi(H)$ so ϕ has an identity.

9.2 Q2

Because H is a normal subgroup then $Ha = aH$ so $HaHb = Hab$. We can see this by:

$$\begin{aligned} h_1 a h_2 b &= h_1 a h_2 a^{-1} ab \\ &= h_1 \bar{h_2} ab \end{aligned}$$

$$\begin{aligned} \phi(HaHb) = \phi(Hab) &= Kab \\ = Kab &= KaKb & = \phi(Ha)\phi(Hb) \end{aligned}$$

9.3 Q3

Let there be a Ka, then $\phi(Ha)$ maps to that value. That is for a set Ka, let $x = ka$, then $a = xk^{-1}$. Thus function is surjective.

9.4 Q4

$$K/H = \{He, Ha, Hb, \dots\}$$

$$\begin{aligned}\ker \phi &= \{aH : Ka = K, \forall a \in G\} \\ &= \{aH : a \in K, \forall a \in G\}\end{aligned}$$

But $K \leq G$ so:

$$\ker \phi = \{aH : a \in K\}$$

9.5 Q5

$$\phi : G/H \underset{K/H}{\twoheadrightarrow} G/K$$

$$(G/H)/(K/H) \cong G/K$$

10 Correspondence Theorem

$$f : G \underset{K}{\twoheadrightarrow} H$$

$$S \leq H$$

$$S^* = \{x \in G : f(x) \in S\}$$

10.1 Q1

Prove $S^* \leq G$

Let $x, y \in S^*$, then $f(x) \in S$ and $f(y) \in S$

Since f is a homomorphism then $f(xy) = f(x)f(y) \in S$

So $xy \in S^*$

10.2 Q2

Prove $K \subseteq S^*$

$$K = \{x \in G : f(x) = e_H\}$$

$e_H \in S$ because S is a group.

Thus $K \subseteq S^*$

10.3 Q3

Let g be the restriction of f to S^*. That is, $g(x) = f(x)$ for every $x \in S^*$ and S^* is the domain of g. Prove g is a homomorphism from S^* onto S and $K = \ker g$.

$$S \leq H$$

Let $s \in S$, then $g(x) = S$, but definition of $S^* = \{x : f(x) \in S\}$, thus $x \in S^*$ and g is a homomorphism from S^* onto S.

$K = \ker g$ because $K \subseteq S^*$ and $g(x) = f(x)$

10.4 Q4

$$g : S^* \xrightarrow[K]{} S$$

$$S \cong S^*/K$$

11 K. Cauchy's Theorem

See also proof in this video.

$|G| = k$ and p is a prime divisor. Assume G is not abelian. Let C be the center of G and C_a be the centralizer of a for each $a \in G$.

Let $k = c + k_s + \cdots + k_t$ be the class equation.

Show G has at least one element of order p.

11.1 Q1

Prove: if p is a factor of $|C_a|$ for any $a \in G$ where $a \notin C$, we are done.

$$C_a = \{x \in G : xa = ax\}$$

Since C_a is subgroup, then this implies there is an element of order p inside C_a by Lagrange's theorem.

11.2 Q2

Prove that for any $a \notin C$ in G, if p is not a factor of $|C_a|$ then p is a factor of $(G : C_a)$.

From orbit-stabilizer theorem, orbits are conjugacy classes and stabilizers are centralizers, considering the group acting on itself through conjugation.

$$O(u) = \{g(u) : g \in G\}$$

$$G_u = \{g \in G : g(u) = u\}$$

$$C_a = \{x \in G : xax^{-1} = a\}$$

$$[a] = \{xax^{-1} : x \in G\}$$

Let the group action $g(u)$ be conjugation gug^{-1} then C_a is equivalent to G_u, and $O(u)$ equivalent to conjugacy class $[a]$. Thus,

$$(G : C_a) = \frac{|G|}{|C_a|} = |[a]|$$

Since p divides G but not C_a, then p divides $(G : C_a)$.

11.3 Q3

As shown above, the size of the conjugacy class $[a]$ is $(G : C_a)$

$$k_i = \frac{|G|}{|C_a|}$$

Where $|G|$ has a prime divisor p.

But $k = c + k_s + \cdots + k_t$ where k and all k_i are factors of p, so c is a factor of p.

12 L. Subgroups of p-Groups (Prelude to Sylow)

A *p-group* is any group whose order is a power of p.

If $|G| = p^k$ then G has a normal subgroup of order p^m for every m between 1 and k.

12.1 Q1

Prove there is an element in C such that $\text{ord}(a) = p$

$$|G| = p^k \implies |C| \text{ is a multiple of } p$$

Thus there is an $a \in C$ such that $\text{ord}(a) = p$

Let $x \in C$ st $\langle x \rangle = C$, then $x^{tp} = e$ and then $a = x^t$

12.2 Q2

Prove $\langle a \rangle$ is a normal subgroup of G.

Definition of normal subgroup:

$$\forall a \in H, \forall x \in G, xax^{-1} \in H$$

The center is a normal subgrop.

$\langle a \rangle \subseteq C$, thus $\langle a \rangle$ is a nromal subgroup of G

12.3 Q3

Explain why it may be assumed that $G/\langle a \rangle$ has a normal subgroup of order p^{m-1}

$$|G| = p^k \qquad |\langle a \rangle| = p$$

$$\text{ord}(G/\langle a \rangle) = p^{k-1}$$

Thus for m from 1 to k, there is a normal quotient subgroup of order p^{m-1}.

Note:

$$\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \iff \gcd(m, n) = 1$$

Because $\text{ord}((a, b)) = \text{lcm}(m, n) = \frac{mn}{gcd(m,n)} = mn$

12.4 Q4

Use J4 to prove that G has a normal subgroup of order p^m.

Correspondence theorem:

$$f : G \underset{K}{\twoheadrightarrow} H$$

$$S^* = \{x \in G : f(x) \in S\}$$

$$S \cong S^*/K$$

Use the natural homomorphism $f : G \to G/\langle a \rangle$ with kernel $\langle a \rangle$

Let S be a the normal subgroup of $G/\langle a \rangle$ whose order is p^{m-1}

Show S^* is a normal subgroup of G and its order is p^m

Since the order of $\langle a \rangle$ is p, and the order of S is p^{m-1} then the order of S^* is p^m

Both S and K are normal subgroups, thus S^* is normal.

13 M. p-Sylow Subgroups

13.1 Q1

Cauchy's theorem states: If G is a group and p is any prime divisor of $|G|$, then G has at least one element of order p.

If q is a prime that divides $|G|$ then there would be an element of order q. Thus the order of any p-group is a power of p.

13.2 Q2

Prove every conjugate of a p-Sylow subgroup of G is a p-Sylow subgroup of G.

gHg^{-1} is an inner automorphism hence $|H| = |gHg^{-1}|$

13.3 Q3

Let $a \in N$ and suppose the order of Ka in N/K is a power of p. Let $S = \langle Ka \rangle$ be the cyclic subgroup of N/K generated by Ka. Prove that N has a subgroup S^* such that S^*/K is a p-group.

$$N = N(K) = \{g \in G : gK = Kg\}$$

$$f : N \to N/K$$
$$f(a) = Ka$$

Let $x, y \in S^*$ then $f(xy) = f(x)f(y) \in S$

Hence $xy \in S^*$ and $S^* \leq N$. By J4:

$$S \cong S^*/K$$

$|S|$ is a power of p.

$$|S^*/K| = (S^* : K) = \frac{|S^*|}{|K|} = |S|$$

13.4 Q4

Prove that S^* is a p-subgroup of G, then explain why $S^* = K$ and why it follows that $Ka = K$.

$$S = \langle Ka \rangle$$
$$S^* = \{x \in N : Kx \in S\}$$

$$S^* \leq N \text{ and } a \in N$$

$K \leq N$ because normalizer contains the group itself

Let $x \in K$, then $Kx = K \in S$ thus $x \in S^*$, so $K \leq S^*$ but K is maximal, hence $S^* = K$ and it follows $Ka = K$.

13.5 Q5

$$S \cong S^*/K$$

Hence $S = \{K\}$

Any $Ka \in N/K$ with order p is equivalent to K the identity.

13.6 Q6

$$\text{ord}(a) = p^k \implies a^{p^k} = e$$

$Ka^{p^k} = K$, thus order of Ka in N/K is a power of p.

If $\text{ord}(a)$ is a power of p then $a \in K$

13.7 Q7

If $aKa^{-1} = K$ then $a \in N$

$\text{ord}(a)$ is a power of p then $a \in K$

14 N. Sylow's Theorem

Let G be a finite group and K a p-Sylow subgroup of G.

Let X be the set of all the conjugates of K.

If $C_1, C_2 \in X$, let $C_1 \sim C_2$ iff $C_1 = aC_2a^{-1}$ for some $a \in K$

14.1 Q1

Prove $\sim$ is an equivalence relation on X.

$$X = \{aKa^{-1}, \forall a \in G$$

$$C_1, C_2 \in X$$
$$C_1 \sim C_2 \text{ iff } C_1 = aC_2a^{-1} \text{ for an } a \in K$$

Let $u \in X$ st $u \sim C_1$ and $u \sim C_2$

$$\begin{aligned} u = a_1C_1a_1^{-1} &= a_2C_2a_2^{-1} \\ a_1C_1a_1^{-1} &= a_2C_2a_2^{-1} \\ C_1 &= a_1^{-1}a_2C_2a_2^{-1}a_1 \\ &= (a_1^{-1}a_2)C_2(a_1^{-1}a_2)^{-1} \\ &= \bar{a}C_2\bar{a}^{-1} \end{aligned}$$

Thus $C_1 \sim C_2$

14.2 Q2

For each $C \in X$, prove the number of elements in $[C]$ is a divisor of $|K|$.

Conclude that for each $C \in X$, the number of elemenbts in $[C]$ is either 1 or a power of p.

From orbit-stablizer:

$$O(C) = \{aCa^{-1} : a \in K\} = [C]$$
$$G_C = \{a \in K : aCa^{-1} = C\} = N(C) = N$$

$$|[C]| = (K : N)$$

Let $\phi : N^* \to [C]$ by $\phi(Na) = aCa^{-1}$

Thus $|O(C)| = |[C]| = \frac{|K|}{|N|}$ and the number of elements in $[C]$ is either 1 or a power of p.

Alternative: from M2, every conjugate of K is also a p-Sylow subgroup of G. Hence from Chapter 14 I10, number of elements in $X_C = [C]$ is a divisor of $|K|$.

14.3 Q3

Prove the only class with a single element is $[K]$ (using exercise M7).

$$\begin{aligned} [K] &= \{aKa^{-1} : a \in K\} \\ &= \{K\} \end{aligned}$$

If $|[C]| = 1$ then $C = aCa^{-1} \quad \forall a \in K$ which means $C = K$.

14.4 Q4

Prove the number of elements in X is $kp + 1$ usings parts 2 and 3.

$$X = \{K, C_2, C_3, \dots\}$$

$$X = \bigcup_i [C_i]$$

Where $[C_i] \cap [C_j] = \varnothing$ or $[C_i] = [C_j]$

But $|[K]| = 1$ while all other C_i is a positive power of p.

Thus $|X| = 1 + kp$

14.5 Q5

Prove that $(G : N)$ is not a multiple of p.

$(G : N)$ is the number of equivalency classes that partition G, which divides $kp + 1$ (number of elements in X). It does not divide p, hence $(G : N)$ is a not a multiple of p.

14.6 Q6

Prove that $(N : K)$ is not a multiple of p.

$(N : K) = \frac{|N|}{|K|}$ but K is a p-Sylow subgroup so $(N : K)$ is not a multiple of p.

$$(G : K) = (G : N)(N : K)$$

We know $(G : K)$ is not a factor of p, because p is a factor of $|K|$ (from K2), and M5 states no element of N/K has order a power of p.

$\therefore (N : K)$ is not a multiple of p.

14.7 Q7

Prove $(G : K)$ is not a multiple of p.

$$(G : K) = (G : N)(N : K)$$

14.8 Q8

Let G be a finite group of order $p^k m$ where p is not a factor of m. Conclude every p-Sylow subgroup K of G has order p^k

The only class with a single element is $[K]$ since $aKa^{-1} = K$, all elements where the order is a power of p are in K.

15 P. Decomposition of a Finite Abelian Group into p-Groups

Let G be an abelian group of order $p^k m$ where p^k and m are relatively prime.

Let G_{p^k} be the subgroup of G consisting of all elements whose order divides p^k.

Let G_m be the subgroup of G consisting of all elements whose order divides m.

15.1 Q1

Prove $\forall x \in G$ and integers s and t, $x^{sp^k} \in G_m$ and $x^{tm} \in G_{p^k}$.

p^k and m are coprime. Thus $sp^k + tm = gcd(p^k, m) = 1$

G_{p^k} and G_m are subgroups of order p^k and m respectively because $|G| = p^k m$

$(x^{sp^k})^m = e$ thus $\mathrm{ord}(x^{sp^k})|m$ and $x^{sp^k} \in G_m$

15.2 Q2

Let $x \in G$, then because p^k and m are coprime $sp^k + tm = 1$.

Thus $x = x^{sp^k} x^{tm} \in G$

But $x^{sp^k} \in G_m$ and $x^{tm} \in G_{p^k}$. Thus,

$$
\begin{aligned}
x &= yz \\
&= (x^{tm})(x^{sp^k})
\end{aligned}
$$

15.3 Q3

By Lagrange's theorem $G_{p^k} \cap G_m \leq G_{p^k}$ and also G_m.

Thus $|G_{p^k} \cap G_m|$ divides $|G_{p^k}|$ and $|G_m| \implies |G_{p^k} \cap G_m|$ divides $\gcd(|G_{p^k}|, |G_m|) = 1$

$$\therefore |G_{p^k} \cap G_m| = 1 = \{e\}$$

15.4 Q4

G_{p^k} and G_m are normal subgroups because G is abelian. $G_{p^k} \cap G_m = \{e\}$ and so $G = G_{p^k} G_m$

$$\forall x \in G \qquad \exists y \in G_{p^k} \quad \exists z \in G_m : x = yz$$

Let $\phi : G_{p^k} \times G_m \to G$ by,

$$\phi(y, z) = yz$$

Thus,

$$G \cong G_{p^k} \times G_m$$

16 Q. Basis Theorem for Finite Abelian Groups

16.1 Q1

$$\begin{aligned} G' &= \{a_2^{l_2} \cdots a_n^{l_n} : l_i \in \mathbb{Z}, 2 \le i \le n\} \\ &= [a_2, \dots, a_n] \end{aligned}$$

$\forall x, y \in G'$ then $xy \in G'$

Also by D2, $a_1^{l_1} = a_2^{l_2} = \cdots = a_n^{l_n} = e$, thus contains the identity.

G' contains inverses. Thus $G' \le G$

16.2 Q2

Prove:

$$G \cong \langle a_1 \rangle \times G'$$
$$a_1^{k_1} \in \langle a_1 \rangle$$

See also this question

From Chapter 14, H: if H and K are normal subgroups of G, such that $H \cap K = \{e\}$ and $G = HK$, then $G \cong H \times K$

Firstly all subgroups of G are normal since the group is abelian.

Lastly we have to prove that $\langle a \rangle \cap G' = \{e\}$

By Lagrange's theorem $\langle a \rangle \cap G' \le \langle a \rangle$ and also G'.

Thus $|\langle a \rangle \cap G'|$ divides $|\langle a \rangle|$ and $|G'| \implies |\langle a \rangle \cap G'|$ divides $\gcd(|\langle a \rangle|, |G'|) = 1$

$$\therefore |\langle a \rangle \cap G'| = 1 = \{e\}$$

16.3 Q3

Explain why we may assume that $G/H = [Hb_1, \dots, Hb_n]$ for some $b_1, \dots, b_n \in G$

Page 149 Theorem 4 from Quotient Groups: "G/H is a homomorphic image of G"

$$f : G \to G/H$$
$$f(x) = Hx$$

Let $x \in G$, then $x = a^{k_0} b_1^{k_1} \cdots b_n^{k_n}$ for some $a, b_1, \dots, b_n \in G$

$$\begin{aligned} f(x) &= f(ab_1^{k_1} \cdots b_n^{k_n}) \\ &= H(a \cdot b_1^{k_1} \cdots b_n^{k_n}) = H(b_1^{k_1} \cdots b_n^{k_n}) \qquad \text{(because } a \in H) \\ &= (Hb_1)^{k_1} \cdots (Hb_n)^{k_n} \end{aligned}$$

Now,

$$\begin{aligned} G/H &= \{f(x) : \forall x \in G\} \\ &= \{(Hb_1)^{k_1} \cdots (Hb_n)^{k_n} : k_i \in \mathbb{Z}, 1 \le i \le n\} \\ &= [Hb_1, \dots, Hb_n] \end{aligned}$$

16.4 Q4

$x \in G \implies x \in Hx$

But $H = \langle a \rangle$ and $G = [Hb_1, \dots, Hb_n]$.

Thus $x = a^{k_0} b_1^{k_1} \cdots b_n^{k_n}$

16.5 Q5

Prove that if $a^{l_0} b_1^{l_1} \cdots b_n^{l_n} = e$, then $a^{l_0} = b_1^{l_1} = \cdots = b_n^{l_n} = e$. Conclude that $G = [a, b_1, \ldots, b_n]$.

$$x = a^{l_0} b_1^{l_1} \cdots b_n^{l_n} = e$$
$$G \cong G_1 \times G_2 \times \cdots \times G_n$$
$$G/H \cong G_1/H \times G_2/H \times \cdots \times G_n/H$$

$$\begin{aligned} Hx &= (Ha^{l_0})(Hb_1^{l_1}) \cdots (Hb_n^{l_n}) \\ &= (Hb_1^{l_1}) \cdots (Hb_n^{n}) \end{aligned}$$

Chapter 10, E4: "If m and n are relatively prime, then $\text{ord}(ab) = mn$"

Also $\gcd(a, b) = 1 \implies \gcd(a^i, b^j) = 1$

$$\text{ord}(Hx) = \text{ord}(Hb_1^{l_1}) \cdots \text{ord}(Hb_n^{l_n})$$

Since $\text{ord}(Hx) = 1$, this means $\text{ord}(Hb_i^{l_i}) = 1$ and because $\text{ord}(b_i) = \text{ord}(Hb_i)$, thus $\text{ord}(b_i^{l_i}) = 1 \implies b_i = e$.

Lastly $a^{l_0} \cdot e = e \implies a = e$

16.6 Q6

If $|G|$ has the following factorization into primes: $|G| = p_1^{k_1} \cdots p_n^{k_n}$, then $G \cong G_1 \times \cdots \times G_n \cong \langle a_1 \rangle \times \cdots \times \langle a_n \rangle$.

As shown in previous exercise, the order of G is the product of the order of each generator for the subgroups.

Lastly chapter 10, E3 showed that is m and n are relatively prime, then the products $a^i b^j (0 \leq i \leq m, 0 \leq j \leq n)$ are all distinct. Thus the products of a and b can be decomposed as unique factors.