Abstract Algebra by Pinter, Chapter 16

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Abstract

Chapter 16 on Fundamental Homomorphism Theorem

Contents

1 A. Examples of FHT

Use the FHT to prove that the two given groups are isomorphic. Then display their tables.

1.1 Q1

$\mathbb{Z}_5$ and $\mathbb{Z}_{20}/\langle 5\rangle$.

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$$K = \{0, 5, 10, 15\} = \langle 5\rangle$$

$$f : \mathbb{Z}_{20} \underset{\langle 5\rangle}{\twoheadrightarrow} \mathbb{Z}_5$$

$$\mathbb{Z}_5 \cong \mathbb{Z}_{20}/\langle 5\rangle$$

1.2 Q2

$\mathbb{Z}_3$ and $\mathbb{Z}_6/\langle 3\rangle$.

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

$$K = \{0, 3\} = \langle 3\rangle$$

$$f : \mathbb{Z}_6 \underset{\langle 3\rangle}{\twoheadrightarrow} \mathbb{Z}_3$$

$$\mathbb{Z}_3 \cong \mathbb{Z}_6/\langle 3\rangle$$

1.3 Q3

$\mathbb{Z}_2$ and $S_3/\{\epsilon, \beta, \delta\}$.

$$f = \begin{pmatrix} \epsilon & \alpha & \beta & \gamma & \delta & \kappa \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$K = \{\epsilon, \beta, \delta\}$$

$$f : S_3 \underset{\{\epsilon,\beta,\delta\}}{\twoheadrightarrow} \mathbb{Z}_2$$

$$\mathbb{Z}_2 \cong S_3/\{\epsilon, \beta, \delta\}$$

1.4 Q4

From Chapter 3, part C (at the end):

$$P_D = \{A : A \subseteq D\}$$

If A and B are any two sets, their symmetric difference is the set $A + B$ defined as follows:

$$A + B = (A - B) \cup (B - A)$$

$A - B$ represents the set obtained by removing from A all the elements which are in B.

$$P_3 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Consider the function $f(C) = C \cap \{a, b\}$

$$P_2 = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$$

The kernel is $\{\varnothing, \{c\}\}$

Using the kernel we create the quotient cosets:

$$\begin{aligned} K &= \{\varnothing, \{c\}\} \\ &= K + \{c\} \\ K + \{a\} &= \{\{a\}, \{a, c\}\} \\ &= K + \{a, c\} \\ K + \{b\} &= \{\{b\}, \{b, c\}\} \\ &= K + \{b, c\} \\ K + \{a, b\} &= \{\{a, b\}, \{a, b, c\}\} \\ &= K + \{a, b, c\} \end{aligned}$$

Applying the function to the cosets, we get:

$$\begin{aligned} f(K) &= \{\varnothing\} \\ f(K \cap \{a\}) &= \{\{a\}\} \\ f(K \cap \{b\}) &= \{\{b\}\} \\ f(K \cap \{a, b\}) &= \{\{a, b\}\} \end{aligned}$$

Thus,

$$f : P_3 \underset{\{\varnothing,\{c\}\}}{\twoheadrightarrow} P_2$$

$$P_2 \cong P_3/\{\varnothing, \{c\}\}$$

1.5 Q5

$\mathbb{Z}_3$ and $(\mathbb{Z}_3 \times \mathbb{Z}_3)/K$ where $K = \{(0,0), (1,1), (2,2)\}$

Consider $f : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3$ by:

$$f(a, b) = a - b$$

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$$

$$\begin{aligned}
\overline{(0,0)} &= K + (0,0) = K + (1,1) = K + (2,2) \\
\overline{(0,1)} &= K + (0,1) = K + (1,2) = K + (2,0) \\
\overline{(0,2)} &= K + (0,2) = K + (1,0) = K + (2,1)
\end{aligned}$$

Applying the function to any element k from the cosets we get:

$$\begin{aligned}
f(0,0) &= f(1,1) = f(2,2) = 0 \\
f(0,1) &= f(1,2) = f(2,0) = 2 \\
f(0,2) &= f(1,0) = f(2,1) = 1
\end{aligned}$$

Thus,

$$f : \mathbb{Z}_3 \times \mathbb{Z}_3 \underset{K}{\twoheadrightarrow} \mathbb{Z}_3$$

$$\mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3/\{(0,0), (1,1), (2,2)\}$$