Tutorial 4

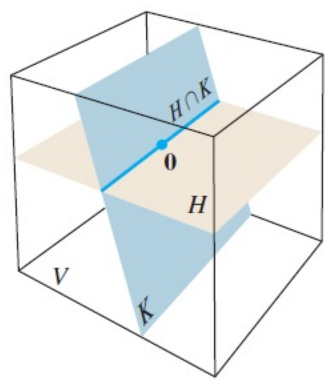
Vector Spaces

1. An $n \times n$ matrix A is said to be symmetric if $A^T = A$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times 3}$, the vector space of all 3×3 matrices.

2. (a) Let P be the plane in $\mathbb{R}^3$ with equation $x + y - 2z = 4$. Find two vectors in P and check that their sum is not in P.

   (b) Let P_0 be the plane through $(0, 0, 0)$ and parallel to P. Write the equation for P_0. Find two vectors in P_0 and check that their sum is in P_0.

3. Let H and K be subspaces of a vector space V. The **intersection** of H and K, written as $H \cap K$, is the set $\mathbf{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V.



4. Determine if the following set is a vector space:

$$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{lcl} a - 2b & = & 4c \\ 2a & = & c + 3d \end{array} \right\}$$

5. Find the matrix A if the following set is $\mathbf{C}(A)$:

$$\left\{ \begin{bmatrix} 2s + 3t \\ r + s - 2t \\ 4r + s \\ 3r - s - t \end{bmatrix} : r, s, t \ \text{ real} \right\}$$

6. For the matrix $D = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$, find a nonzero vector in $\mathbf{N}(D)$ and a nonzero vector in $\mathbf{C}(D)$.

7. Find the basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$.

8. Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$ and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It can be verified that $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$. Find a basis for H.

9. Consider the polynomials $\mathbf{p}_1(t) = 1 + t, \mathbf{p}_2(t) = 1 - t$ and $\mathbf{p}_3(t) = 2$ (for all t). By inspection, write a linear dependence relation among $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$. Then find a basis for $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

10. Use an inverse matrix to find the $\mathcal{B}-$coordinate of the vector $\mathbf{x}$, i.e., $[\mathbf{x}]_{\mathcal{B}}$, for $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.

11. Find the dimension of the subspace H of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

12. Determine the dimensions of $\mathbf{N}(A)$ and $\mathbf{C}(A)$ for $A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$.

13. If a 3×8 matrix A has rank 3, find dim $\mathbf{N}(A)$, dim $\mathbf{C}(A^T)$, and rank of A^T.

14. Suppose the solutions of a homogeneous system of 5 linear equations in 6 unknowns are all multiples of 1 nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations?

15. Verify that the rank of $\mathbf{u}\mathbf{v}^T \leq 1$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

16. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for V and suppose $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2, \mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.
    (a) Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.
    (b) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

Answers

1.

2. (a) e.g., $(4,0,0)$ and $(0,4,0)$ (b) e.g., $(2,0,1)$ and $(0,2,1)$

3.

4. Yes

5. $D = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$

6. e.g., $\mathbf{N}(A) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{C}(D)$ is either column of D.

7. $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

8. e.g., $\{\mathbf{v}_1, \mathbf{v}_2\}, \{\mathbf{v}_1, \mathbf{v}_3\}, \{\mathbf{v}_2, \mathbf{v}_3\}$.

9. $\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{0}$, $\{\mathbf{p}_1, \mathbf{p}_2\}$

10. $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$

11. 2

12. $\dim \mathbf{N}(A) = 2$, $\dim \mathbf{C}(A) = 2$.

13. $\dim \mathbf{N}(A) = 5$, $\dim \mathbf{C}(A^T) = 3$, rank of $A = 3$.

14. Yes

15.

16. (a) $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 4 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$

End