Q1.

(a) (i) $\int_{-1}^{1} C(1-x^2)\,dx = 1 \Longrightarrow \left[Cx - \frac{Cx^3}{3}\right]_{-1}^{1} = 2\left(C - \frac{C}{3}\right) = 1 \Longrightarrow C = \frac{3}{4}$.

(ii) $\mathbb{E}[X] = \int_{-1}^{1} x \cdot \frac{3}{4}(1-x^2)\,dx = \left[\frac{3x^2}{8} - \frac{3x^4}{16}\right]_{-1}^{1} = 0$.

$\mathbb{E}[X^2] = \int_{-1}^{1} x^2 \cdot \frac{3}{4}(1-x^2)\,dx = \left[\frac{x^3}{4} - \frac{3x^5}{20}\right]_{-1}^{1} = 2\left(\frac{1}{4} - \frac{3}{20}\right) = \frac{1}{5}$.

$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{5} - 0^2 = \frac{1}{5}$.

(iii) For $-1 \le x$, $F_X(x) = \int_{-1}^{x} \frac{3}{4}(1-t^2)\,dt = \left[\frac{3}{4}\left(t - \frac{t^3}{3}\right)\right]_{-1}^{x} = \frac{3}{4}(x - \frac{x^3}{3}) - \frac{3}{4}(-1+\frac{1}{3}) = \frac{3x}{4} - \frac{x^3}{4} + \frac{1}{2}$. Note that $F_X(x) = 0$ if $x < -1$, and $F_X(x) = 1$ if $x > 1$. Then $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \ln y) = F_X(\ln y) = \frac{3}{4}(\ln y) - \frac{(\ln y)^3}{4} + \frac{1}{2}$ for $-1 \le \ln y \le 1$.

The PDF of Y is given by $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{3}{4y} - \frac{3(\ln y)^2}{4y}$ for $-1 \le \ln y \le 1$; $f_Y(y) = 0$ elsewhere.

(b) $\mathbb{P}(|X-3| > 6) = 1 - \mathbb{P}(|X-3| < 6) = 1 - \mathbb{P}(-3 < X < 9) = 1 - \left(\Phi\left(\frac{9-3}{3}\right) - \Phi\left(\frac{-3-3}{3}\right)\right) = 1 - \Phi(2) + \Phi(-2)$.

(c) Note that the PDF is $f(x) = \frac{1}{2}$, $1 \le x \le 3$. So $M_X(t) = \mathbb{E}[e^{tX}] = \int_1^3 e^{tx}\frac{1}{2}\,dx = \frac{1}{2}\left[\frac{e^{tx}}{t}\right]_1^3 = \frac{e^{3t}-e^t}{2t}$, for $t \neq 0$. If $t = 0$, $M_X(0) = \mathbb{E}[e^{0\cdot X}] = \int_1^3 e^0 \frac{1}{2} = \frac{1}{2}(3-1) = 1$.

(d) Let X be the number of games played starting from the fifth game. Then X has Geometric distribution with $p = 1 - 0.6 = 0.4$. So the number of games played in total is

$$4 + \mathbb{E}[X] = 4 + \frac{1}{0.4} = 6.5.$$

Q2.

(a)(i) Test statistic: $Z = \frac{\overline{X}-\mu}{s/\sqrt{n}} \sim N(0,1)$.

p-value $= \mathbb{P}(Z < \frac{280-300}{60/\sqrt{30}}) = \mathbb{P}(Z < -1.826) \approx 1 - \mathbb{P}(Z < 1.83) = 1 - 0.9664 = 0.0336$

(a) (ii) Reject H_0 since p-value is less than $\alpha = 0.05$.

(b) (i) We shall reject if $S^2 \geq C$. We will find C. Test statistic $\chi = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$.

We want $\mathbb{P}\left(\chi \geq \frac{(n-1)C}{\sigma^2} | H_0\right) = \alpha = 0.05 \implies \mathbb{P}\left(\chi \geq \frac{11C}{10}\right) = 0.05 \implies \frac{11C}{10} = \chi^2_{0.05}(11) = 19.68$ (from table). So $C = \frac{10\times 19.68}{11} = 17.89$.

(b) (ii) $\beta = \mathbb{P}\left(S^2 < C | H_1\right) = \mathbb{P}(\chi < \frac{C(n-1)}{35}) = \mathbb{P}(\chi < \frac{17.89\times 11}{35}) = \mathbb{P}(\chi < 5.622) \approx 0.1$ from the table.

Q3.

(a) (i) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{x(1+3y^2)}{4}\,dy = \left[\frac{xy}{4} + \frac{xy^3}{4}\right]_0^1 = \frac{x}{2}$, $0 < x < 2$.

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{x(1+3y^2)}{4}\,dx = \left[\frac{x^2}{8} + \frac{3y^2x^2}{8}\right]_0^2 = \frac{1+3y^2}{2}$, $0 < y < 1$.

(a) (ii) Yes. Because $f(x,y) = f_X(x)f_Y(y)$.

(a) (iii) The conditional PDF of X given $Y = y$ is $g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x}{2}$, $0 < x < 2$. So

$$\mathbb{P}\left(\frac{1}{4} < X < \frac{1}{2}|Y = \frac{1}{3}\right) = \int_{1/4}^{1/2} \frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_{1/4}^{1/2} = \frac{1}{16} - \frac{1}{64} = \frac{3}{64}.$$

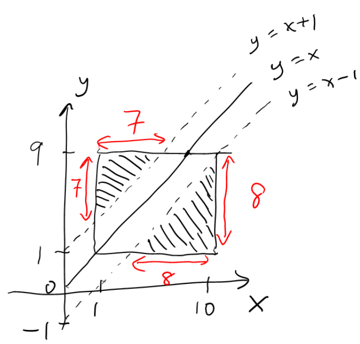
(a) (iv) $\mathbb{P}(X > Y) = \int_0^1 \int_y^2 f(x,y)\,dx\,dy = \int_0^1 \int_y^2 \frac{x}{4}(1+3y^2)\,dx\,dy = \int_0^1 \left[\frac{x^2}{8}(1+3y^2)\right]_y^2 dy$
$= \int_0^1 (1+3y^2)\left(\frac{1}{2} - \frac{y^2}{8}\right) dy = \int_0^1 \frac{1}{2} - \frac{y^2}{8} + \frac{3y^2}{2} - \frac{3y^4}{8}\,dy \approx 0.8833.$

(b) The shaded region below consists of points (x,y) for which $|x-y| \geq 1$. The area of the shaded region is

$$\frac{1}{2} \times 7 \times 7 + \frac{1}{2} \times 8 \times 8 = \frac{113}{2}.$$

Since $f(x,y)$ is uniform on the rectangle $[1,10] \times [1,9]$, we have

$$\mathbb{P}(\text{will be asked to guess again}) = 1 - \frac{1}{72} \times \text{area of the shaded region} = 1 - \frac{1}{72}\frac{113}{2} = \frac{31}{144}.$$



Q4.

(a) Maximum Likelihood function: $L(x_1, x_2, x_3, x_4|\lambda) = \prod_{i=1}^{4} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$. Then $\ln L = \sum_{i=1}^{4} \ln\left(\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right) = \sum_{i=1}^{4} (-\lambda + x_i \ln\lambda - \ln x_i!)$.

$\frac{d}{d\lambda}\ln L = 0 \Longrightarrow \sum_{i=1}^{4}(-1 + \frac{x_i}{\lambda}) = 0 \Longrightarrow \lambda = \frac{1}{4}\sum_{i=1}^{4} x_i = \frac{13+5+6+7}{4} = 7.75$.

(b) (i) Since X_i's are iid, $\mathbb{E}[\overline{X}] = \mathbb{E}[(\sum_{i=1}^{n} X_i)/n] = (1/n)\mathbb{E}[\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}n\mathbb{E}[X_1] = \mathbb{E}[X_1]$ since X_i's are iid. Similarly, $\mathrm{Var}[\overline{X}] = \mathrm{Var}[(\sum_{i=1}^{n} X_i)/n] = (1/n^2)\mathrm{Var}[\sum_{i=1}^{n} X_i] = (1/n^2)\sum_{i=1}^{n}\mathrm{Var}[X_i] = (1/n^2)n\mathrm{Var}[X_1] = (1/n)\mathrm{Var}[X_i]$.

$\mathbb{E}[\overline{X}] = \mathbb{E}[X_1] = 1\cdot\frac{\theta}{3} + 2\cdot\frac{2\theta}{3} + 3(1-\theta) = 3 - \frac{4}{3}\theta$.

$\mathbb{E}[X_1^2] = 1^2\cdot\frac{\theta}{3} + 2^2\cdot\frac{2\theta}{3} + 3^2\cdot(1-\theta) = 9 - 6\theta$.

$\mathrm{Bias}(\widehat{\theta}) = \mathbb{E}[\widehat{\theta}] - \theta = \mathbb{E}[X_1/3] - \theta = \frac{1}{3}\mathbb{E}[X_1] - \theta = \frac{1}{3}\left(3 - \frac{4\theta}{3}\right) - \theta = 1 - \frac{13\theta}{9}$.

$\mathrm{Var}[X_1] = \mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2 = (9-6\theta) - (3-4\theta/3)^2 = 2\theta - \frac{16}{9}\theta^2$.

$\mathrm{SE}(\widehat{\theta}) = \sqrt{\mathrm{Var}(\widehat{\theta})} = \sqrt{\mathrm{Var}[\frac{1}{3}\overline{X}]} = \frac{1}{3}\sqrt{\mathrm{Var}[\overline{X}]} = \frac{1}{3}\sqrt{\frac{1}{n}\mathrm{Var}[X_1]} = \frac{1}{3}\sqrt{\frac{1}{n}(2\theta - \frac{16}{9}\theta^2)}$

(ii) $\widehat{\theta} = \frac{1}{3}\left(\frac{2+2+1+3}{4}\right) = \frac{2}{3}$

(iii) Choose $\widehat{\theta} = \frac{9}{4} - \frac{3\overline{X}}{4}$.

Then $\mathbb{E}[\widehat{\theta}] = \frac{9}{4} - \frac{3}{4}\mathbb{E}[\overline{X}] = \frac{9}{4} - \frac{3}{4}\left(3 - \frac{4}{3}\theta\right) = \theta$, i.e. it is unbiased.