SC1004 Part 2

Lectured by Prof Guan Cuntai
(teaching materials by Prof Chng Eng Siong)

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Quiz 2 and Exam:

1. Quiz 2

- Coverage : Ch 6 ,7, 8

- Time/Date: Week 13, last lecture time (10:30-11.20am, 17th April 2024)

2. Final Exam

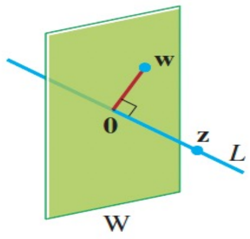
- **Coverage** : Ch 6, 7, 8 (Q3 & Q4)
- **Date/Time**: 2 May 2024 (Thursday), 1.00-3.00pm

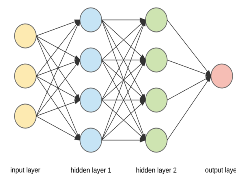
(Ch 9 will not be tested)

Syllabus for Part 2

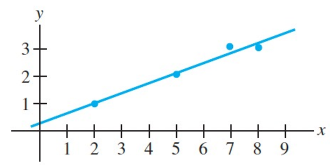
Chapter	Topics	Week (Lecture)	Week (Tut)
6	Orthogonality	8-9	9-10
7	Least Squares	9-10	10-11
8	EigenValue and Eigenvectors	11-12	12-13
9	Singular Value Decomposition (SVD)	13	

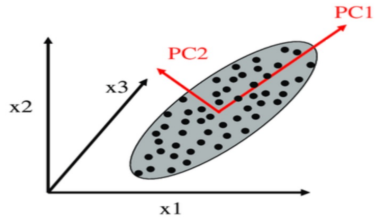
Table 1: schedule





$$\boldsymbol{A}\mathbf{x} = \mathbf{b}$$





Online Video learning Schedule

https://www.youtube.com/channel/UCBzG5jg3huxiPkCt-Serrjw

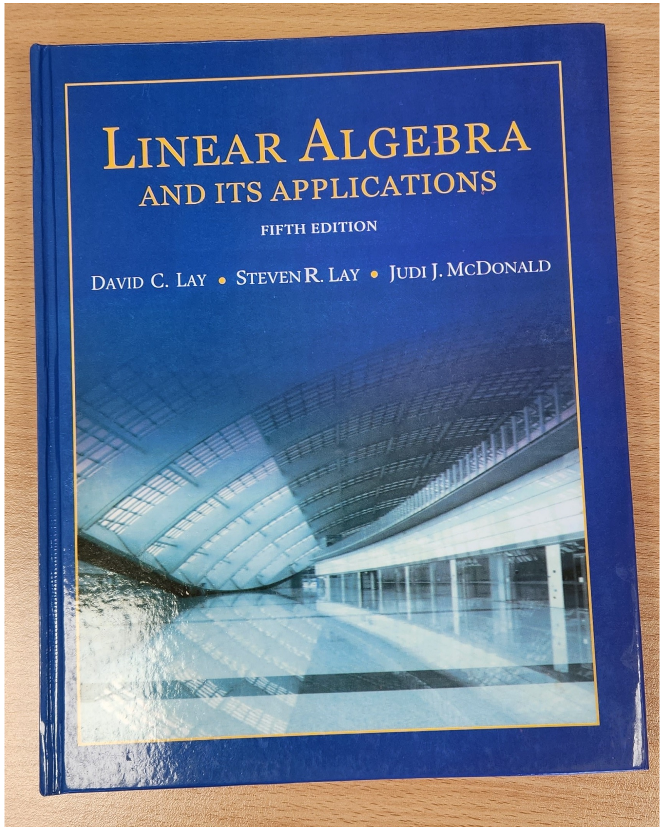
Week	Part	Topic	Notes
8	6.1.1-6.2.3	Orthogonality, Normalization, Dot-Product, Inequalities,	Lecture 1: **6.1.1 - 6.1.3** Lecture 2: **6.1.4 - 6.2.3**
9	6.2.4-6.3.2	Orthogonal/Orthonormal Sets, Basis, Gram Schmidt and QR Decomposition	Lecture 3: **6.2.4** Lecture 4: **6.2.5 – 6.3.2**
10	7.1.1-7.2.1	Least Squares and Normal Eqn, Projection Matrix, Applications	Lecture 5: **7.1.1 – 7.1.3** Lecture 6: **7.1.4 – 7.2.1**
11	8.1.1-8.1.2	Eigenvectors, Eigen-values, Characteristics Eqn	Lecture 7: **8.1.1** Lecture 8: **8.1.2**
12	8.1.3-8.1.5	Diagonalisation, Power of A, Change of basis	Lecture 9: **8.1.3** Lecture 10: **8.1.4 – 8.1.5**
13	9.1.1-9.2	Introduction to SVD and PCA (Not examined in quiz/exam)	Lecture 11: **9.1.1 – 9.2** Lecture 12: **Quiz 2**

How will we conduct the course?

1) Before the lectures, watch the videos according to the schedule in Table 1
    - You can watch past years zoom video recordings at
    https://www.youtube.com/@linearalgebra1884/playlists?view=50&sort=dd&shelf_id=2

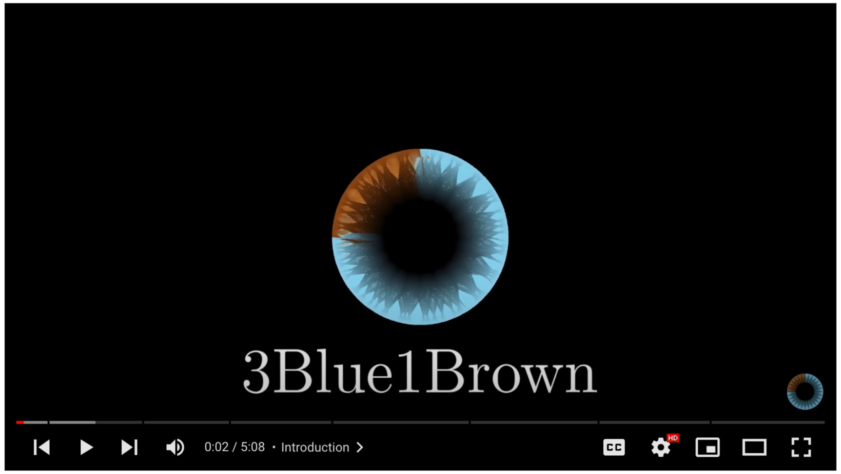
2) During lecture hours –
    - We will summarize the lectures and highlight the key points
    - Q&A.

References



Linear Algebra and Its Applications
by David Lay, Steven Lay, Judi McDonald

3Blue1Brown on YouTube



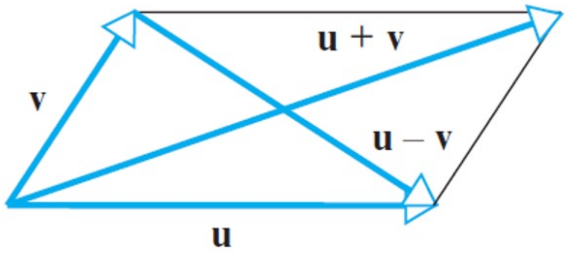
Essence of linear algebra preview

https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab

Lecture (Week 8)

(Chapter 6.1.1- 6.2.3)

Key points – 6.1.1 Geometric Vectors



- Vector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$
- Vector direction & length
  - $\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
- Vector addition & subtraction
  - $\boldsymbol{u} = \boldsymbol{v}_1 + \boldsymbol{v}_2$
  - $\boldsymbol{u} = \boldsymbol{v}_1 - \boldsymbol{v}_2$
- Euclidean space: R^n $- n$ dimensional real numbers

Key points – 6.1.2 Norm (Euclidean Norm)

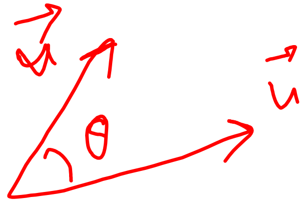
- Norm: $\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
  - $\|\boldsymbol{v}\| \geq \mathbf{0}$
  - $\|\boldsymbol{v}\| = \mathbf{0} \;\; iif \;\; \boldsymbol{v} = 0$
  - $\|k\boldsymbol{v}\| = |k|\|\boldsymbol{v}\|$

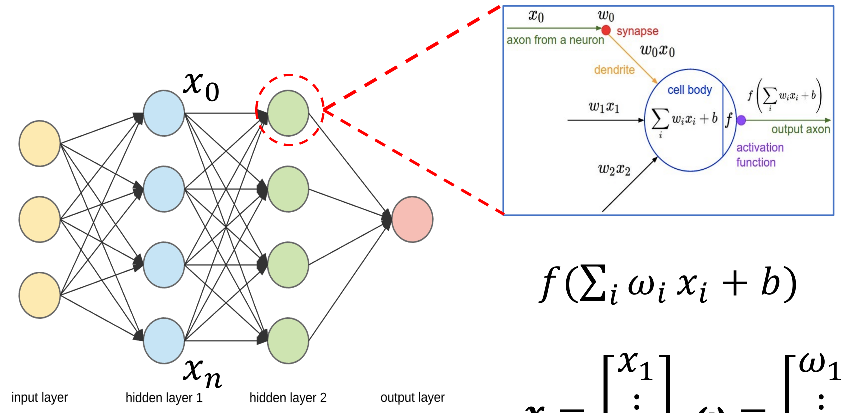
- Normalizing a vector (unit length vector)
  - $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$

- Vector distance
  - $\boldsymbol{dist}\,(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$

Key points – 6.1.3 Dot Product/Inner Product

- Definition
  - $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$
  - Geometric formula: $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \, cos\theta$
  - $cos\theta = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\| \, \|\boldsymbol{v}\|}$
  - $if \, \|\boldsymbol{u}\| = 1, \|\boldsymbol{v}\| = 1, cos\theta = \boldsymbol{u} \cdot \boldsymbol{v}$
  - $\|\boldsymbol{u}\|^2 = \boldsymbol{u} \cdot \boldsymbol{u}$, or $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$
  - Component formula: $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 \, v_1 + \cdots + u_n \, v_n$





$f(\sum_i \omega_i \, x_i + b)$

$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}$

- Explanation of dot product using the geometric formula
  - Projection: $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \, (\|\boldsymbol{v}\| \, cos\theta) = \|\boldsymbol{v}\| \, (\|\boldsymbol{u}\| \, cos\theta)$
  - Perpendicular: $\boldsymbol{u} \cdot \boldsymbol{v} = 0$

Key points – 6.1.3 Dot Product/Inner Product (2).

- Properties of dot product

Dot products have many of the same algebraic properties as products of real numbers.

$\vec{u} \cdot \vec{v} = \sum u_i v_i$

$= \underbrace{\underbrace{\vec{u}^T}_{1\times n} \underbrace{\vec{v}}_{n\times 1}}_{1\times 1}$

THEOREM 3.2.2 *If* **u**, **v**, *and* **w** *are vectors in* R^n, *and if* k *is a scalar, then:*

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ **[Symmetry property]**

(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$ **[Distributive property]**

(c) $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v}$ **[Homogeneity property]**

(d) $\mathbf{v}\cdot\mathbf{v} \geq 0$ *and* $\mathbf{v}\cdot\mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

- Transformation on dot product
  - $A\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{u}\cdot A^T\boldsymbol{v}$
  - $\boldsymbol{u}\cdot A\boldsymbol{v} = A^T\boldsymbol{u}\cdot\boldsymbol{v}$
    - Using $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{v}$, and $(AB)^T = B^TA^T$ to derive

$A\vec{u} \cdot \vec{v} = (A\vec{u})^T \vec{v}$

$\vec{u}^T \underbrace{(A^T\vec{v})}_{n\times 1} = \vec{u} \cdot (A^T\vec{v})$

Key points – 6.1.4 Inequalities

- Inequalities
  - $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$
  - Triangular inequality: $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$

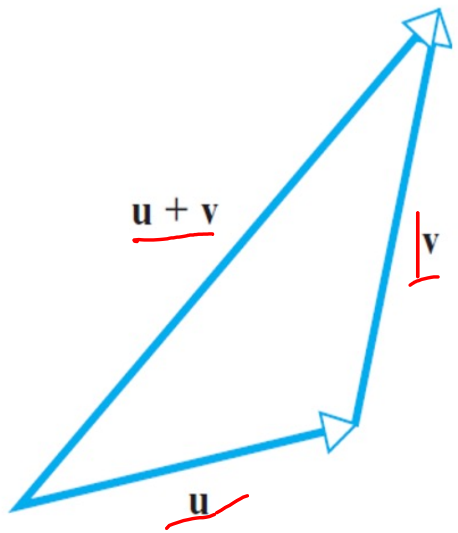
THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1 v_1 + u_2 v_2 + \cdots + u_n v_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2} (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$



  - Prove

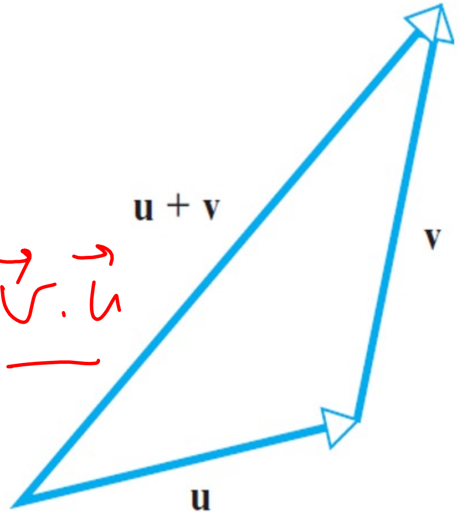
Explain

- $|\boldsymbol{u} \cdot \boldsymbol{v}| \le \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$
- $\|\boldsymbol{u} + \boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$

$$|\vec{u}\cdot\vec{u}| = u_1^2 + \cdots u_n^2$$
$$= \|\vec{u}\|^2$$

$$|\cos\theta| \le 1$$

$$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$$



$$|\vec{u}\cdot\vec{v}| = \|\vec{u}\|\,\|\vec{v}\|\,|\cos\theta|$$
$$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\,\|\vec{v}\|$$

$$\|\underbrace{\vec{u}+\vec{v}}_{w}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} + 2\,\vec{u}\cdot\vec{v}$$
$$\le \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2|\vec{u}\cdot\vec{v}|$$
$$\le \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|u\|\,\|v\| = (\|\vec{u}\| + \|\vec{v}\|)^2$$

Key points – 6.2.1 Orthogonality

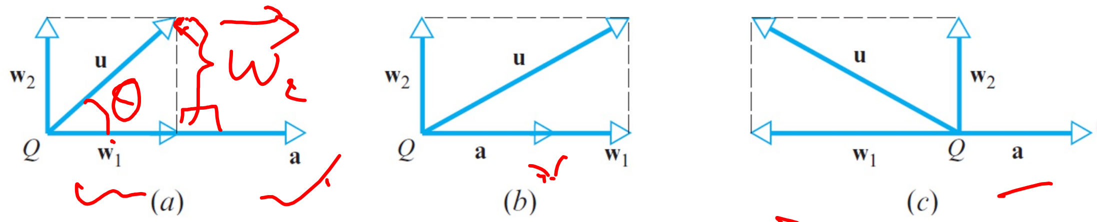
- Definition (vectors orthogonal to each other)
  - $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ $= \|\vec{u}\|\|\vec{v}\| \cos\theta$
  - $cos\theta = 0$ --> $\theta = 90^{\circ}$, or $\theta = \pi/2$

- Orthonormal
  - $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal with unit length ($\|\boldsymbol{u}\|$**=**1**,** $\|\boldsymbol{v}\| = 1$)

Key points – 6.2.2 Orthogonal Projection

- Decomposition of a vector
  - Standard basis in R^n

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$



- Projection theorem
  - $\boldsymbol{w}_1 = Proj_a\,\boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$ (projection) – prove & example
  - $\boldsymbol{w}_2 = \boldsymbol{u} - Proj_a\,\boldsymbol{u} = \boldsymbol{u} - \frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$ (residual)
  - $\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2$
  - Distance from $\boldsymbol{u}$ to $\boldsymbol{a}$: $\|\boldsymbol{u} - \boldsymbol{w}_1\|$ = $\|\boldsymbol{w}_2\|$

$\vec{w}_1 = k\vec{a}$, $\frac{\vec{a}}{\|\vec{a}\|}$

$= \|\vec{w}_1\|\frac{\vec{a}}{\|\vec{a}\|}$

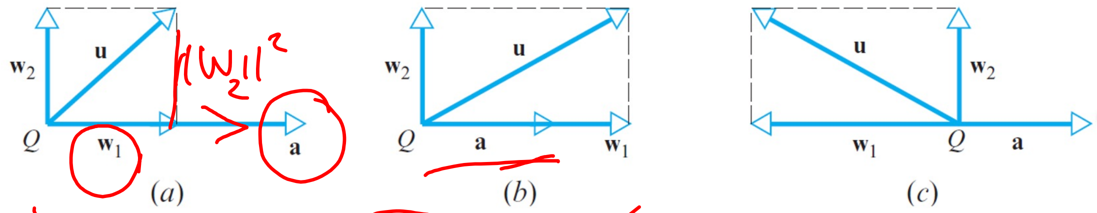
$= \|\vec{u}\|\cdot\cos\theta\cdot\frac{\vec{a}}{\|\vec{a}\|}$

$= \cancel{\|\vec{u}\|}\,\frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{u}\|}\,\|\vec{a}\|}\cdot\frac{\vec{a}}{\|\vec{a}\|}$

$= \frac{\vec{u}\cdot\vec{v}}{\vec{a}\cdot\vec{a}}\,\vec{a}$

Explain

- $\boldsymbol{w}_1 = Proj_a\, \boldsymbol{u} = \dfrac{\boldsymbol{u}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$



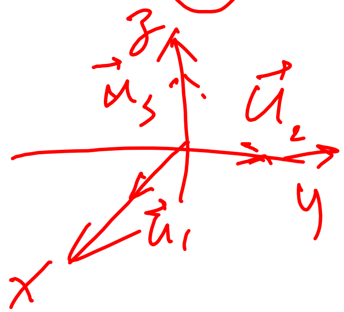
$$\vec{w}_1 = \frac{\vec{u}\cdot(\lambda\vec{a})}{(\lambda\vec{a})\cdot(\lambda\vec{a})}(\lambda\vec{a})$$

$\vec{w}_1 = Proj_{\vec{a}}\,\vec{u}$

$\bar{w}_1$? $c\cdot\vec{a}$

Key points – 6.2.3 Orthogonal Sets and Basis

- A set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ in R^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0$, whenever $i \neq j$.
  - If $p = n$, $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_n\}$ spans R^n
  - If $p < n$, $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ spans a subspace W in R^n
    - $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ are the basis of the subspace
    - Standard basis for Euclidian space of R^3 : $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



Key points – 6.2.3 Orthogonal Decomposition

- Project a vector $\boldsymbol{y}$ on to subpace spanned by $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ in R^n
  - Let W be a subspace of R^n. Then each $\boldsymbol{y}$ in R^n can be written **uniquely** in the form:

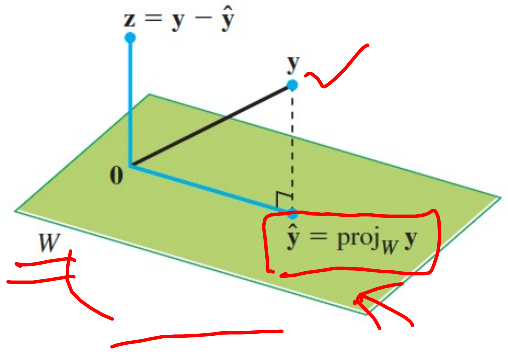
$$\boldsymbol{y} = \widehat{\boldsymbol{y}} + \boldsymbol{z}$$

Where $\widehat{\boldsymbol{y}}$ is in W and $\boldsymbol{z}$ is in $W^{\perp}$.

If $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ is any orthogonal basis of W, then

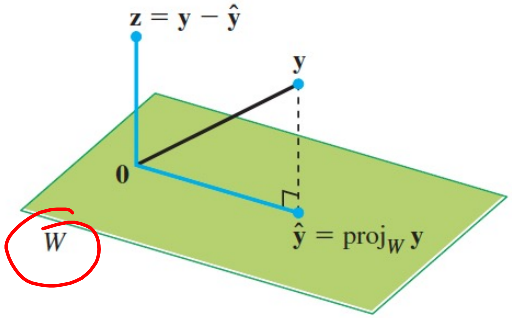
$$\widehat{\boldsymbol{y}} = Proj_W\boldsymbol{y} = \frac{\boldsymbol{y}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}\boldsymbol{u}_1 + \cdots + \frac{\boldsymbol{y}\cdot\boldsymbol{u}_p}{\boldsymbol{u}_p\cdot\boldsymbol{u}_p}\boldsymbol{u}_p$$

- Explain using: $\widehat{\boldsymbol{y}} = \boldsymbol{y} - \boldsymbol{z} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_p\boldsymbol{u}_p$



Explain

- $\hat{\boldsymbol{y}} = Proj_W \boldsymbol{y} = \boxed{\frac{\boldsymbol{y}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}}\boldsymbol{u}_1 + \cdots + \boxed{\frac{\boldsymbol{y}\cdot\boldsymbol{u}_p}{\boldsymbol{u}_p\cdot\boldsymbol{u}_p}}\boldsymbol{u}_p$ ✓



$$\hat{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots c_p \vec{u}_p$$

$$\vec{u}_1 \cdot \hat{y} = c_1 \vec{u}_1 \cdot \vec{u}_1 + \underbrace{c_2 \vec{u}_2 \cdot \vec{u}_1 + \cdots + c_p \vec{u}_p \cdot \vec{u}_1}_{0}$$

$$c_1 = \frac{\vec{u}_1 \cdot \vec{y}}{\vec{u}_1 \cdot \vec{u}_1} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}$$

Key points – for tutorial questions

$A^T A = I$ $\quad A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$

$AA^T = I$ $\quad n \times n$

- Orthogonal matrix A
  - If A is square with orthonormal columns (in fact, the row of an orthogonal matrix is also orthonormal)

$A^T = A^{-1}$

- Vector orthogonal to a subspace
  - If a vector $\boldsymbol{u}$ is orthogonal to every vector in a subspace W of R^n, then $\boldsymbol{u}$ is said to be orthogonal to W – all $\boldsymbol{u}$ called the orthogonal complement of W $(W^\perp)$

$A\underline{x} = 0$

Subspace. $\begin{cases} \vec{u} = 0 \\ c\vec{u} \\ \vec{u}_1 + \vec{u}_2 \end{cases}$ $\boxed{\begin{matrix} W^\perp \\ W^\perp \\ W^\perp \end{matrix}}$

End