

Sep 2020, Rev A

Q1) Lay5e/ch6.1/pg 338/Ex1

Compute the quantities in Exercises 1–8 using the vectors

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

1. $\mathbf{u} \cdot \mathbf{u}$, $\mathbf{v} \cdot \mathbf{u}$, and $\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$
2. $\mathbf{w} \cdot \mathbf{w}$, $\mathbf{x} \cdot \mathbf{w}$, and $\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}$
3. $\frac{1}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w}$
4. $\frac{1}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$
5. $\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$
6. $\left(\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{x} \cdot \mathbf{x}} \right) \mathbf{x}$
7. $\|\mathbf{w}\|$
8. $\|\mathbf{x}\|$

Q2) Lay5e/ch6.1/ pg 338/ Ex19

In Exercises 19 and 20, all vectors are in \mathbb{R}^n . Mark each statement True or False. Justify each answer.

19. a. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.
- b. For any scalar c , $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.
- c. If the distance from \mathbf{u} to \mathbf{v} equals the distance from \mathbf{u} to $-\mathbf{v}$, then \mathbf{u} and \mathbf{v} are orthogonal.
- d. For a square matrix A , vectors in $\text{Col } A$ are orthogonal to vectors in $\text{Nul } A$.
- e. If vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ span a subspace W and if \mathbf{x} is orthogonal to each \mathbf{v}_j for $j = 1, \dots, p$, then \mathbf{x} is in W^\perp .

Ans: T,T,T,F,T

30. Let W be a subspace of \mathbb{R}^n , and let W^\perp be the set of all vectors orthogonal to W . Show that W^\perp is a subspace of \mathbb{R}^n using the following steps.
- Take \mathbf{z} in W^\perp , and let \mathbf{u} represent any element of W . Then $\mathbf{z} \cdot \mathbf{u} = 0$. Take any scalar c and show that $c\mathbf{z}$ is orthogonal to \mathbf{u} . (Since \mathbf{u} was an arbitrary element of W , this will show that $c\mathbf{z}$ is in W^\perp .)
 - Take \mathbf{z}_1 and \mathbf{z}_2 in W^\perp , and let \mathbf{u} be any element of W . Show that $\mathbf{z}_1 + \mathbf{z}_2$ is orthogonal to \mathbf{u} . What can you conclude about $\mathbf{z}_1 + \mathbf{z}_2$? Why?
 - Finish the proof that W^\perp is a subspace of \mathbb{R}^n .

EXAMPLE 3 Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of \mathbf{y} onto \mathbf{u} . Then write \mathbf{y} as the sum of two orthogonal vectors, one in $\text{Span}\{\mathbf{u}\}$ and one orthogonal to \mathbf{u} .

In Exercises 17–22, determine which sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

17. $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$

18. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

In Exercises 23 and 24, all vectors are in \mathbb{R}^n . Mark each statement True or False. Justify each answer.

23. a. Not every linearly independent set in \mathbb{R}^n is an orthogonal set.
- b. If \mathbf{y} is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.
- c. If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.
- d. A matrix with orthonormal columns is an orthogonal matrix.
- e. If L is a line through $\mathbf{0}$ and if $\hat{\mathbf{y}}$ is the orthogonal projection of \mathbf{y} onto L , then $\|\hat{\mathbf{y}}\|$ gives the distance from \mathbf{y} to L .
24. a. Not every orthogonal set in \mathbb{R}^n is linearly independent.
- b. If a set $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.
- c. If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths.
- d. The orthogonal projection of \mathbf{y} onto \mathbf{v} is the same as the orthogonal projection of \mathbf{y} onto $c\mathbf{v}$ whenever $c \neq 0$.
- e. An orthogonal matrix is invertible.

31. Show that the orthogonal projection of a vector \mathbf{y} onto a line L through the origin in \mathbb{R}^2 does not depend on the choice of the nonzero \mathbf{u} in L used in the formula for $\hat{\mathbf{y}}$. To do this, suppose \mathbf{y} and \mathbf{u} are given and $\hat{\mathbf{y}}$ has been computed by formula (2) in this section. Replace \mathbf{u} in that formula by $c\mathbf{u}$, where c is an unspecified nonzero scalar. Show that the new formula gives the same $\hat{\mathbf{y}}$.

EXAMPLE 2 Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write \mathbf{y} as the sum of a vector in W and a vector orthogonal to W .

EXAMPLE 3 If $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, as in Example 2, then the closest point in W to \mathbf{y} is

EXAMPLE 4 The distance from a point \mathbf{y} in \mathbb{R}^n to a subspace W is defined as the distance from \mathbf{y} to the nearest point in W . Find the distance from \mathbf{y} to $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

THEOREM 10

If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of \mathbb{R}^n , then

$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p \quad (4)$$

If $U = [\mathbf{u}_1 \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_p]$, then

$$\text{proj}_W \mathbf{y} = UU^T \mathbf{y} \quad \text{for all } \mathbf{y} \text{ in } \mathbb{R}^n \quad (5)$$

In Exercises 21 and 22, all vectors and subspaces are in \mathbb{R}^n . Mark each statement True or False. Justify each answer.

21. a. If \mathbf{z} is orthogonal to \mathbf{u}_1 and to \mathbf{u}_2 and if $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then \mathbf{z} must be in W^\perp .
- b. For each \mathbf{y} and each subspace W , the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$ is orthogonal to W .
- c. The orthogonal projection $\hat{\mathbf{y}}$ of \mathbf{y} onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{\mathbf{y}}$.
- d. If \mathbf{y} is in a subspace W , then the orthogonal projection of \mathbf{y} onto W is \mathbf{y} itself.
- e. If the columns of an $n \times \tilde{p}$ matrix U are orthonormal, then $UU^T \mathbf{y}$ is the orthogonal projection of \mathbf{y} onto the column space of U .

19. Suppose $A = QR$, where Q is $m \times n$ and R is $n \times n$. Show that if the columns of A are linearly independent, then R must be invertible. [Hint: Study the equation $R\mathbf{x} = \mathbf{0}$ and use the fact that $A = QR$.]
20. Suppose $A = QR$, where R is an invertible matrix. Show that A and Q have the same column space. [Hint: Given \mathbf{y} in $\text{Col } A$, show that $\mathbf{y} = Q\mathbf{x}$ for some \mathbf{x} . Also, given \mathbf{y} in $\text{Col } Q$, show that $\mathbf{y} = A\mathbf{x}$ for some \mathbf{x} .]

EXAMPLE 4 Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

2. Suppose $A = QR$, where Q is an $m \times n$ matrix with orthogonal columns and R is an $n \times n$ matrix. Show that if the columns of A are linearly dependent, then R cannot be invertible.

===== End of Tut Ch 6 =====