SC1004 Part 2

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(teaching materials by Prof Chng Eng Siong)

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Quiz 2 and Exam:

1. Quiz 2
- **Coverage:** Ch 6 – 8.1.2
- **Time/Date**: Week 13, last lecture time (10:30-11.20am, 17th April 2024)
- **Venue**: LT1A

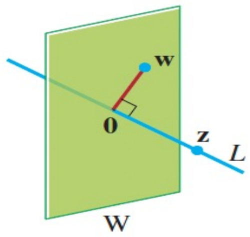
2. Final Exam
- **Coverage**: Ch 6, 7, 8 (Q3 & Q4)
- **Date/Time**: 2 May 2024 (Thursday), 1.00-3.00pm

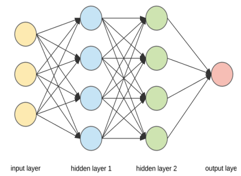
(Ch 9 will not be tested)

Syllabus for Part 2

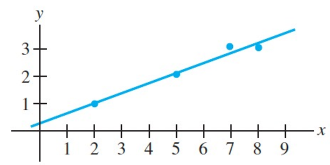
Chapter	Topics	Week (Lecture)	Week (Tut)
6	Orthogonality	8-9	9-10
7	Least Squares	9-10	10-11
8	EigenValue and Eigenvectors	11-12	12-13
9	Singular Value Decomposition (SVD)	13	

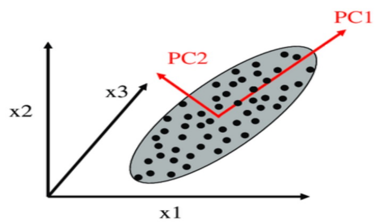
Table 1: schedule





$A\mathbf{x} = \mathbf{b}$





Online Video learning Schedule

https://www.youtube.com/channel/UCBzG5jg3huxiPkCt-Serrjw

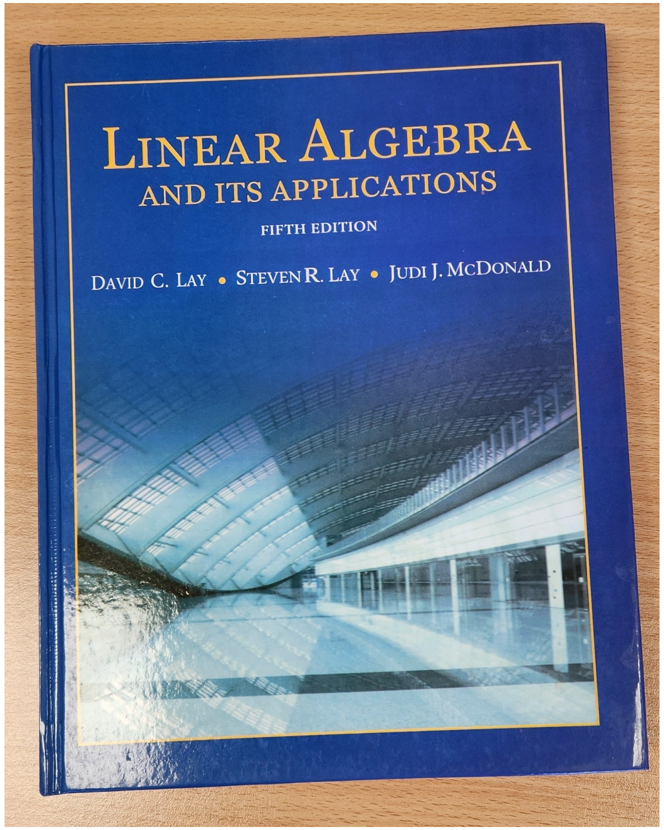
Week	Part	Topic	Notes
8	6.1.1-6.2.3	Orthogonality, Normalization, Dot-Product, Inequalities,	Lecture 1: **6.1.1 - 6.1.3** Lecture 2: **6.1.4 - 6.2.3**
9	6.2.4-6.3.2	Orthogonal/Orthonormal Sets, Basis, Gram Schmidt and QR Decomposition	Lecture 3: **6.2.4** Lecture 4: **6.2.5 – 6.3.2**
10	7.1.1-7.2.1	Least Squares and Normal Eqn, Projection Matrix, Applications	Lecture 5: **7.1.1 – 7.1.3** Lecture 6: **7.1.4 – 7.2.1**
11	8.1.1-8.1.2	Eigenvectors, Eigen-values, Characteristics Eqn	Lecture 7: **8.1.1** Lecture 8: **8.1.2**
12	8.1.3-8.1.5	Diagonalisation, Power of A, Change of basis	Lecture 9: **8.1.3** Lecture 10: **8.1.4 – 8.1.5**
13	9.1.1-9.2	Introduction to SVD and PCA (Not examined in quiz/exam)	Lecture 11: **9.1.1 – 9.2** Lecture 12: **Quiz 2**

How will we conduct the course?

1) Before the lectures, watch the videos according to the schedule in Table 1
   - You can watch past years zoom video recordings at https://www.youtube.com/@linearalgebra1884/playlists?view=50&sort=dd&shelf_id=2

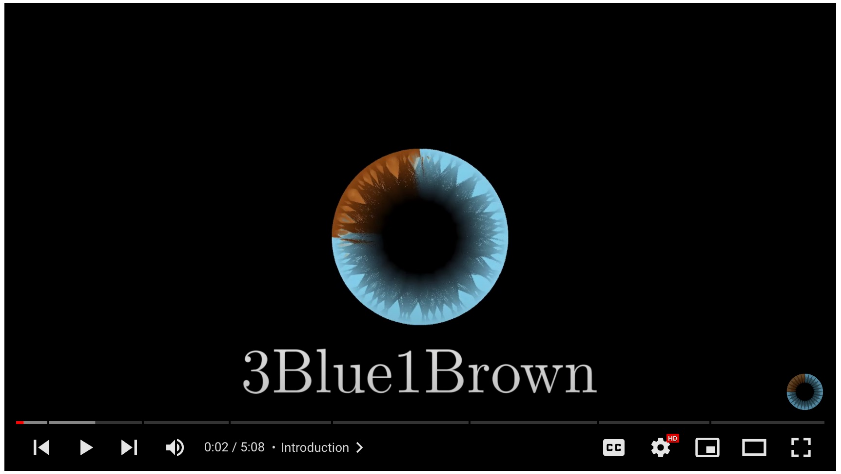
2) During lecture hours –
   - We will summarize the lectures and highlight the key points
   - Q&A.

References



Linear Algebra and Its Applications
by David Lay, Steven Lay, Judi McDonald

3Blue1Brown on YouTube



Essence of linear algebra preview

https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab

Lecture (Week 12)

(Chapter 8.1.3-8.1.5)

Revision

Key points – Eigenvalue and Eigenvector

- Steps to find the eigenvalue and eigenvector $(A\boldsymbol{x} = \lambda\boldsymbol{x})$

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

1) Find the eigenvalues
   - Using the Characteristic Equation: $\det(A - \lambda I) = 0$
   - Example: $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\det(A - \lambda I) = \det(\begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}) = 0$

     Characteristic polynomial: $\lambda^2 - 3\lambda - 28 = 0, (\lambda - 7)(\lambda + 4) = 0, \lambda = 7 \,\&\, \lambda = -4$

2) Find the eigenvectors
   - Solve the linear equation: $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$
   - Example: $(A - 7I)\boldsymbol{x} = \boldsymbol{0}$

     $\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\boldsymbol{x} = \boldsymbol{0} \Rightarrow \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0} \Rightarrow \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{0}$

     Using row reduction: $\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

     We get $x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow$ eigenvectors are $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (x_2 is a free variable)

$\{0, \vec{x}\}$

Revision

Key points – Eigenvector Intuition

$\vec{x} \to A\vec{x} = \lambda\vec{x}$

- $A_1 = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$
  - $\lambda_1 = -4$, eigenvectors $\boldsymbol{x} = t\begin{bmatrix} 6 \\ -5 \end{bmatrix}$
  - $\lambda_2 = 7$, eigenvectors $\boldsymbol{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

  (where t is a free variable)

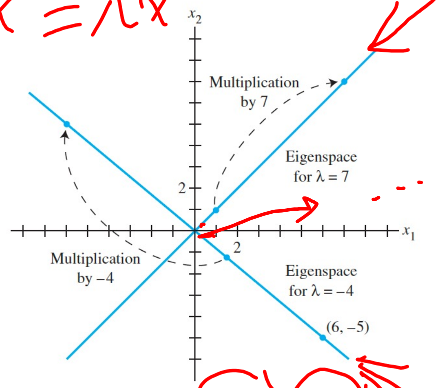
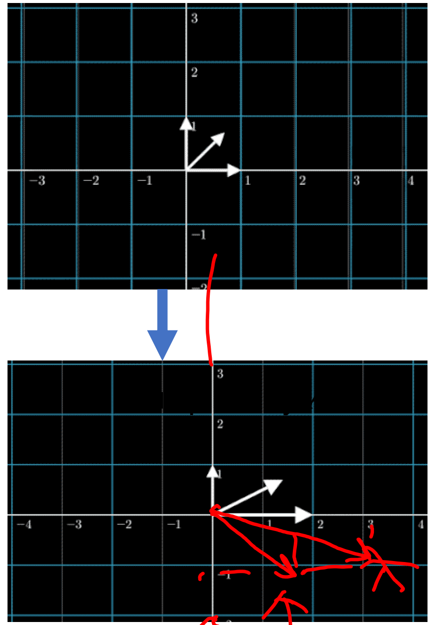


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

- $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $A_2 x$
  - $\lambda_1 = 2$, eigenvectors $\boldsymbol{x} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  - $\lambda_2 = 1$, eigenvectors $\boldsymbol{x} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$



×2

- Intuition
  - For a 2×2 matrix, we can find vectors in R^2, which remain the same directions after the linear transformation by A (a scalar multiplication).
  - In the case A_2, the eigenvectors happen to be on the directions of the standard basis:

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \; A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, only vectors on these two lines got dilated after the transformation. Any other vectors will “change” directions after the transformation.

Key points – 8.1.3 Similarity and Diagonalizability

- Definition
  - Similarity transformation:
    - Matrices A and B are called similar matrices, if there exits an invertible matrix P, such that. $A = PBP^{-1}$
    - We can find a similarity transformation: $B = P^{-1}AP$
    - The equivalent equation is: $\boxed{AP = PB}$
    - A and B have the same determinants, invertibility, rank, characteristic polynomial, eigenvalues, eigenspace dimensions (but eigenvectors are different).
  - Diagonalizability
    - If B is a diagonal matrix, we call A is diagonalizable

Explain: A and B have the same determinants, invertibility, rank, characteristic polynomial, eigenvalues, eigenspace dimensions (but eigenvectors are different)

- A and B are similar, implying that while A and B may look different, they represent the same linear transformation under different bases.
- **Same Determinants:** det(A)=det(PBP^{-1})=det($\cancel{P}$)det(B)det($\cancel{P^{-1}}$), det(P^{-1})=1/det(P), we have: det(A)= det(PBP^{-1})= det(P)det(B)(1/det(P)) = det(B)
- **Invertibility:** Since det(A)= det(B), if one of the matrices is invertible (nonzero determinant), the other must also be invertible.
- **Same Rank:** The rank of a matrix is the dimension of the vector space spanned by its columns. For similar matrices, the transformations they represent, albeit in different bases, span spaces of the same dimension. Thus, they have the same rank.
- **Characteristic Polynomial, Eigenvalues, and Eigenspace Dimensions:** The characteristic polynomial: $\det(A - \lambda I)$.
  - ✓ For similar matrices, $PBP^{-1} - \lambda I = PBP^{-1} - P\lambda P^{-1} = P(B - \lambda I)P^{-1}$, $\det(A - \lambda I)$= $\det(P(B - \lambda I)P^{-1}) = \cancel{\det}(P)\det(B - \lambda I)\det(\cancel{P^{-1}}) = \det(B - \lambda I)$ A

    $= \lambda P . P^{-1} = \lambda I$
  - ✓ So, they have the same characteristic polynomial, and consequently, the same eigenvalues.
  - ✓ The dimensions of the eigenspaces are determined by the eigenvalues and the algebraic multiplicity of these eigenvalues. Since A and B have the same eigenvalues with the same algebraic multiplicity, the dimensions of their eigenspaces (the geometric multiplicity) are the same.
- **Eigenvectors**
- While A and B have the same eigenvalues, their eigenvectors are not necessarily the same. This is because eigenvectors are directions that are invariant under the linear transformation represented by a matrix. Since A and B represent the same transformation under different bases, the "directions" in one base correspond to different "directions" in another. If $\boldsymbol{v}$ is an eigenvector of B, then $P\boldsymbol{v}$ is the corresponding eigenvector of A. Hence, the change-of-basis matrix P transforms eigenvectors of B into eigenvectors of A, explaining why the eigenvectors are different for similar matrices.

Key points – 8.1.3 Properties of Diagonal Matrices

- Summary of properties of diagonal matrices
  1) Eigenvalues: diagonal elements
  2) Determinant: product of diagonal entries
  3) Rank: number of non-zero entries on the diagonal
  4) Multiplication of A and a diagonal matrix D
     - DA: each row is multiplied by the corresponding entry in D
     - AD: each column is multiplied by the corresponding entry in D
  5) The matrix's inverse is reciprocal of diagonal elements
  6) Product of diagonal matrixes are easy to compute.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

$$DA = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$
$$= \begin{bmatrix} d_1 a_{11} & \cdots & d_1 a_{1n} \\ \vdots & \ddots & \vdots \\ d_n a_{n1} & \cdots & d_n a_{nn} \end{bmatrix}$$

$$AD = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$
$$= \begin{bmatrix} d_1 a_{11} & \cdots & d_n a_{1n} \\ \vdots & \ddots & \vdots \\ d_1 a_{n1} & \cdots & d_n a_{nn} \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & \ddots & \\ 0 & & 1/d_n \end{bmatrix}$$

Key points – 8.1.3 Diagonalize a Matrix

$A_{n \times n}$

- Definition (diagonalization theorem)
  - $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors
  - The diagonalization formula is: $\boxed{A = PDP^{-1}}$ ➔ $\boxed{AP = PD}$
    - P contains n linearly independent columns which are the eigenvectors of A
    - D is a diagonal matrix, whose elements are eigenvalues of A corresponding to the eigenvectors in P
    - The eigenvectors in P form a basis of R^n

$D = P^{-1}AP$

- Explain

$\vec{v}_1 \sim \lambda_1$

  - If P is an $n \times n$ matrix, $P = [\boldsymbol{v}_1\ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n]$, D is a diagonal matrix, $D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$
  - $AP = A[\boldsymbol{v}_1\ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n] = [A\boldsymbol{v}_1\ A\boldsymbol{v}_2 \cdots A\boldsymbol{v}_n]$
  - $PD = [\boldsymbol{v}_1\ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1\boldsymbol{v}_1\ \lambda_2\boldsymbol{v}_2 \cdots \lambda_n\boldsymbol{v}_n]$
  - With $AP = PD$, we have: $[A\boldsymbol{v}_1\ A\boldsymbol{v}_2 \cdots A\boldsymbol{v}_n] = [\lambda_1\boldsymbol{v}_1\ \lambda_2\ \boldsymbol{v}_2 \cdots \lambda_n\boldsymbol{v}_n]$ ➔ $A\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1, \cdots, A\boldsymbol{v}_n = \lambda_n\boldsymbol{v}_n$
  - Therefore, $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are the eigenvectors, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues!

Key points – 8.1.3 Diagonalize a Matrix (2)

- Example
  - $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, two eigenvectors $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
  - Given eigenvectors, we can find the eigenvalues
    - $A\boldsymbol{v}_1 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -9+12 \\ -6+7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\lambda_1 = 1$
    - $A\boldsymbol{v}_2 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+12 \\ -4+7 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 3$
  - Form matrices P and D:
    - $P = [\boldsymbol{v}_1\ \boldsymbol{v}_2] = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$
  - Verify:
    - $PDP^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix} \frac{1}{3\times 1 - 1\times 2} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} = A$

Key points – 8.1.3 Steps to Diagonalize a Matrix

For an $n \times n$ matrix A, following are the steps to diagonalize it.

1) Find the eigenvalues, $\lambda_1, \lambda_2, \cdots, \lambda_i, \cdots, \lambda_r$, using the characteristic equation

   $r \leq n$

2) For each eigenvalue λ_i, find the corresponding eigenvectors

3) If there are n independent eigenvectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$, then A can be represented as
   - $AP = PD$
   - $A = PDP^{-1}$
   - $P^{-1}AP = D$

   where D is a diagonal matrix with λ_i as it entries, P is a matrix whose columns are eigenvectors arranged in correspondence with λ_i.

Key points – 8.1.3 When is a Matrix is Diagonalizable?

- Sufficient condition to diagonalize a matrix:
  - For a $n \times n$ matrix A, if there are n distinct eigenvalues, then we can find n linearly independent eigenvectors $\{\boldsymbol{v}_1,\ \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ ➔ A is diagonalizable.
  - Explanations: see next slides
  - It is not a necessary condition
    - Some matrix A have less than n distinct eigenvalues, but we can still find n independent eigenvectors.
- Example
  - $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$: there are 3 distinct eigenvalues (5, 0, -2),

  so A is diagonalizable

Revision

Key points – Independence of Eigenvectors

- Definition:
  - For an $n \times n$ matrix A , if eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$ are corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$, then $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$ are linearly independent.

- Explain (proof of contradiction)
  - Assume $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r\}$ is linearly dependent.
  - Since $\boldsymbol{v}_i$ is nonzero, so, one of the vectors in the set is a linear combination of the preceding vectors which are independent.

$$\boldsymbol{v}_{p+1} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p$$

(multiply λ_{p+1} to both sides) ➔ $\lambda_{p+1}\boldsymbol{v}_{p+1} = c_1\lambda_{p+1}\boldsymbol{v}_1 + c_2\lambda_{p+1}\boldsymbol{v}_2 + \cdots + c_p\lambda_{p+1}\boldsymbol{v}_p$

(multiplying both sides by A) ➔ $A\boldsymbol{v}_{p+1} = c_1A\boldsymbol{v}_1 + c_2A\boldsymbol{v}_2 + \cdots + c_pA\boldsymbol{v}_p$

(use $A\,\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i$) ➔ $\lambda_{p+1}\boldsymbol{v}_{p+1} = c_1\lambda_1\boldsymbol{v}_1 + c_2\lambda_2\boldsymbol{v}_2 + \cdots + c_p\lambda_p\boldsymbol{v}_p$

(subtract above two equations) ➔ $c_1(\lambda_1 - \lambda_{p+1})\boldsymbol{v}_1 + c_2(\lambda_2 - \lambda_{p+1})\boldsymbol{v}_2 + \cdots + c_p(\lambda_p - \lambda_{p+1})\boldsymbol{v}_p = 0$

  - Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r\}$ is linearly independent, $\{c_1, c, \cdots, c_r\}$ has non-zero solution,
  - and $\lambda_i - \lambda_{p+1} \neq 0$, so $c_i = 0$ (for $i = 1, \cdots, p$) ➔ $\boldsymbol{v}_{p+1} = 0$, which contradicts with eigenvectors,

So, $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$ must be linearly independent.

Key points – 8.1.3 Is a Matrix with Repeated Eigenvalues Diagonalizable?

- Prerequisites
  - Algebraic multiplicity (n_{λ_i}): number of repetitions of a particular eigenvalue λ_i .
    - Matrix A: $\lambda_1 = 2\ (2)$, $\lambda_2 = 9(1)$
    - Matrix B: $\lambda_1 = 2(1)$, $\lambda_2 = 4(2)$, $\lambda_3 = 3(1)$
  - Geometric multiplicity (n_v): dimension of the eigenspace $E(\lambda_i)$ corresponding to eigenvalue λ_i.
    - Matrix B: for $\lambda_1 = 2(\dim = 1)$, for $\lambda_2 = 4(\dim = 2)$, for $\lambda_3 = 3(\dim = 1)$
    - Matrix A: for $\lambda_1 = 2(\dim = 2)$, for $\lambda_2 = 9(\dim = 1)$
  - Property of algebraic multiplicity and geometric multiplicity
    - algebraic multiplicity and geometric multiplicity may be different
    - geometric multiplicity is equal or less than algebraic multiplicity (the number of independent eigenvectors for λ_i is equal to or less than the number of repetitions of λ_i , ie $n_v \leq n_{\lambda_i}$)
    - $\sum n_{\lambda_i} = n$

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 6 & 1 & 3 \end{bmatrix}$$

We saw this example before. One of its eigenvalues is 2.

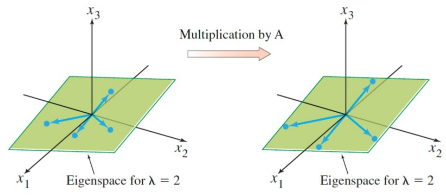


FIGURE 3 A acts as a dilation on the eigenspace.

Key points – 8.1.3 Is a Matrix with Repeated Eigenvalues Diagonalizable? (2)

- Definition
  - **For an $n \times n$ matrix A, for every eigenvalue, if the geometric multiplicity equals to the algebraic multiplicity, then A is diagonalizable.**
- Example
  - $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$
  - Step 1: find the eigenvalues using the characteristic equation is (λ-1)($\lambda + 2)^2 = 0$:

$$\lambda_1 = 1(1),\ \lambda_2 = -2(2)$$

  - Step 2: find the eigenvectors for each eigenvalue
    - $\lambda_1 = 1,\ \boldsymbol{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
    - $\lambda_2 = -2,\ \boldsymbol{x} = \boldsymbol{x}_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \boldsymbol{x}_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
    - Verify: $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a independent set
  - Step 3: Construct matrix $P = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3] = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
  - Step 4: Construct the diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$
  - Verification if $AP = PD$
    - $AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$
    - $PD = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

Key points – 8.1.3 Is a Matrix with Repeated Eigenvalues Diagonalizable? (3).

- Example
  - $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ 3 x 3
  - Step 1: find the eigenvalues using the characteristic equation is (λ-1) $(\lambda + 2)^2 = 0$:
$$\lambda_1 = 1(1), \lambda_2 = -2(2)$$
  - Step 2: find the eigenvectors for each eigenvalue
    - $\lambda_1 = 1, \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ 2
    - $\lambda_2 = -2$ (algebraic multiplicity is 2), $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (geometric multiplicity is 1)
    - Verify: only two linearly independent eigenvectors for $A \in R^{3\times 3}$. Any other eigenvectors will be a multiple of $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$. Therefore, A is not diagonalizable.

Continued to the previous example:

- $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, $\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}\right) = 0$, $(\lambda\text{-}1)\,(\lambda+2)^2 = 0,\ \lambda_1 = 1(1),\ \lambda_2 = -2(2)$
  - $(A - 1I)\boldsymbol{x} = \mathbf{0}$: $\begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix}\boldsymbol{x} = 0$

    $\Rightarrow \begin{bmatrix} 1 & 4 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$: $x_1 + 4x_2 + 3x_3 = 0, x_2 + x_3 = 0$

    $\Rightarrow\ x_1 = x_3,\ x_2 = -x_3, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
  - $(A + 2I)\boldsymbol{x} = \mathbf{0}$: $\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}\boldsymbol{x} = 0$

    $\Rightarrow \begin{bmatrix} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$: $x_1 + x_2 = 0, 3x_3 = 0$

    $\Rightarrow\ x_1 = -x_2,\ x_3 = 0, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Key points – 8.1.4 Compute the Powers of A

- Definition
  - For an $n \times n$ matrix A, if $A = PDP^{-1}$
  - Then, we can easily compute the powers of A: A^k
    - $A^2 = (PDP^{-1})\,(PDP^{-1}) = PD(P^{-1}P)\,DP^{-1} = PDD\,P^{-1} = PD^2\,P^{-1}$
    - …
    - $A^k = PD^k\,P^{-1}$

  - where
    - $D^2 = DD = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^2 \end{bmatrix}$
    - …
    - $D^k = \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix}$

Key points – 8.1.5A Coordinate System

- Definition of coordinate system
  - A vector space V is in R^n. A linear independent set $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_p\}$ is a basis of the subspace H (or $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_p$ span the subspace H).
    - Note 1: $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_p\}$ need not be an orthogonal set to be the basis of H
    - Note 2: think of the standard basis $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ for R^2

  - Coordinate system (Unique Representative Theorem)
    - In vector space V, any vector in V can be written as the linear combination of its basis

      $\boldsymbol{x} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 + \cdots + c_n\boldsymbol{b}_n$ (think of x, y, z coordinates as am example)

      c_i is called the coordinate for basis $\boldsymbol{b}_i$
    - Denote $[x]_{\mathcal{B}} = \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix}$ as the coordinate vector of $\boldsymbol{x}$ relative to $\mathcal{B}$
    - Notation $\boldsymbol{x} \longmapsto [x]_{\mathcal{B}}$ is called the coordinate mapping (determined by $\mathcal{B}$)
    - $\boldsymbol{x} \longmapsto [\boldsymbol{x}]_{\mathcal{B}}$ is a one-to-one mapping!

Key points – 8.1.5A Coordinate System (2)

- Example 1
  - Under the standard basis, vector $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ can be represented as

    $\boldsymbol{x} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - So the coordinates of $\boldsymbol{x}$ is $c_1 = 1$, $c_2 = 2$. $[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  - If we change the coordinates:
  - $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $[x]_{\mathcal{B}} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$ for $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
  - $\boldsymbol{x} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 = (3/2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - (1/2)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Key points – 8.1.5A Coordinate System (3)

- Example 2 (from coordinates to vector)
  - $\boldsymbol{b}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}$, $[x]_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$
  - $\boldsymbol{x} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 = (-2)\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$.

    Note: $\begin{bmatrix}1\\6\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 6\begin{bmatrix}0\\1\end{bmatrix}$ under the standard basis

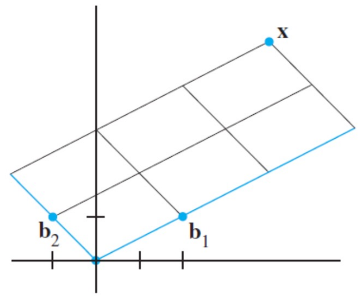


FIGURE 4
The $\mathcal{B}$-coordinate vector of **x** is (3, 2).

- Example 3 (from vector to coordinates): Figure 4
  - $\boldsymbol{b}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix}4\\5\end{bmatrix}$ (this is $\boldsymbol{x}$ under the standard basis)
  - $\boldsymbol{x} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$, we can find $\begin{bmatrix}c_1\\c_2\end{bmatrix}$ from the system equation: $\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$
  - Using row operation, we have $\begin{bmatrix}2 & -1 & 4\\1 & 1 & 5\end{bmatrix} \sim \begin{bmatrix}2 & 0 & 6\\0 & 1 & 2\end{bmatrix}$ ➔ $\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$ (this is $\boldsymbol{x}$ under the new basis)

Key points – 8.1.5B Change of Basis

- Definition
  - For a linear system, the linear transformation is defined as follows:
    - Input vector: $\boldsymbol{x} \in R^n$
    - Output vector: $\boldsymbol{y} \in R^m$
    - $\boldsymbol{y} = T(\boldsymbol{x}) = A\boldsymbol{x}$
    - A consists of the coordinate basis $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots \boldsymbol{a}_n]$
    - We can write as $A = [T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \cdots T(\boldsymbol{e}_n)]$
    - $\boldsymbol{a}_i = T(\boldsymbol{e}_i)$

Key points – 8.1.5B Change of Basis (2)

- Change the basis from standard basis to the new basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots \boldsymbol{a}_n\}$
  - For an $n \times n$ matrix A, $A = PDP^{-1}$ via eigenvalues and eigenvectors
  - We get $\boldsymbol{y} = T(\boldsymbol{x}) = A\boldsymbol{x} = PDP^{-1}\boldsymbol{x}$

- Explain
  - $\boldsymbol{y} = PDP^{-1}\boldsymbol{x} = PD(P^{-1}\boldsymbol{x}) = PD\ [\boldsymbol{x}]_{\mathcal{B}} = P[\boldsymbol{y}]_{\mathcal{B}}$
    - $\boldsymbol{x}$ can be interpreted as $[\boldsymbol{x}]_{standard}$
    - Since $\boldsymbol{x} = P[\boldsymbol{x}]_{\mathcal{B}}$ (linear combination of the basis), pre- multiply P^{-1}: $P^{-1}\boldsymbol{x} = P^{-1}P[\boldsymbol{x}]_{\mathcal{B}}$, we get $[\boldsymbol{x}]_{\mathcal{B}} = P^{-1}\boldsymbol{x}$.
  - So the coordinate for $\boldsymbol{y}$ is $[\boldsymbol{y}]_{\mathcal{B}} = DP^{-1}\boldsymbol{x}$

End