

Discrete Mathematics MH1812

Topic 3 - Predicate Logic Summary

SINGAPORE

Quantification: Order of Nesting Matters

Is
$$\forall x \in D, \exists y \in D, P(x,y) \equiv \exists y \in D, \forall x \in D, P(x,y) \text{ in general?}$$

7 - for 1/1

LHS

 $\forall x \in D, \exists y \in D, P(x,y)$

y can vary with x

RHS

 $\exists y \in D, \forall x \in D, P(x,y)$

y is fixed, but x varies

Let P(x,y) ="x admires y"

"Every person admires someone"

"Some people are admired by everyone"

Example

Consider the domains $X = \{2,3\}$ and $Y = \{2,4,6\}$, and the predicate P(x,y) = "x divides y".

for
$$y=2$$
, have x/y
for $y=4$, have x/y
for $y=6$ have x/y

Example

Consider the domains $X = \{2,3\}$ and $Y = \{2,6,9\}$, and the predicate P(x,y) = "x divides y".

What are the truth values of

- 1. $\forall y \in Y, \exists x \in X, P(x,y). \vdash For y = 2, take a = 2 then x ly$
- 2. $\exists x \in X, \forall y \in Y, P(x,y)$. For y = 6, take x = 2 then $x \mid y$ For y = 9, take x = 3 then $x \mid y$
- 2. Try x=2: for y=2, have x/yfor y=6, have x/yfor y=9, then x/y

Try z=3: for y=2 then x fy X

Quantification: Order of Nesting Matters

Consider (arbitrary) domains X and Y with \underline{m} and \overline{n} members respectively.

Then
$$\exists x \in X, \exists y \in Y, P(x,y) \equiv \exists y \in Y, \exists x \in X, P(x,y)$$
 $X = \{ x_1, x_2, ..., x_m \}$ and $\forall x \in X, \forall y \in Y, P(x,y) \equiv \forall y \in Y, \forall x \in X, P(x,y)$ $Y = \{ y_1, y_2, ..., y_n \}$ $\exists x \in X \{ \exists y \in Y, P(x,y) \} \equiv \{ \exists y \in Y, P(x_1,y) \} \}$ $\exists x \in X \{ \exists y \in Y, P(x,y) \} \equiv \{ \exists y \in Y, P(x_1,y) \} \}$ $\exists x \in X \{ \exists y \in Y, P(x,y) \} \equiv \{ \exists y \in Y, P(x_1,y_1) \lor ... \lor P(x_n,y_1) \} \lor ... \lor P(x_m,y_1) \}$ $\exists x \in X, P(x,y_1) \} \lor ... \lor P(x_1,y_1) \lor ... \lor P(x_1,y_1) \lor ... \lor P(x_n,y_n) \}$ $\exists x \in X, P(x,y_1) \} \lor ... \lor \{ \exists x \in X, P(x,y_n) \}$ $\exists x \in X, P(x,y_n) \}$ $\exists x \in X, P(x,y_n) \}$

Determining Truth Values: Method of Case

Positive Example to Prove Existential Quantification

Let \mathbb{Z} denote all integers.

Is $\exists x \in \mathbb{Z}$, $x^2 = x$ true or false?

Take x = 0 or 1 and we have it.

Positive Example

It is **not** a proof of universal quantification.

one example suffices

Counterexample to Disprove Universal Quantification

Let \mathbb{R} denote all reals.

Is $\forall x \in \mathbb{R}$, $x^2 > x$ true or false?

Take x = 0.3 as a counterexample.



Negative Example

It is **not** disproof of existential quantification.

Conditional Quantification: Definitions

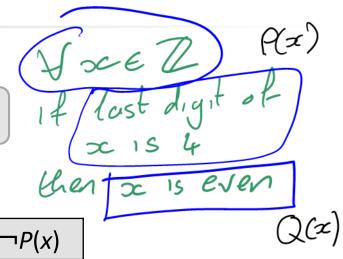
Given a conditional quantification such as...

$$\forall x \in A \ (P(x) \to Q(x))$$

Then, we define...

Contrapositive	$\forall x \in A, \neg Q(x) \rightarrow \neg P(x)$
Converse	$\forall x \in A, Q(x) \rightarrow P(x)$
Inverse	$\forall x \in A, \neg P(x) \rightarrow \neg Q(x)$

Note: a conditional proposition is logically equivalent to its contrapositive.



Conditional Quantification: Negation

What is
$$\neg (\forall x \in X, P(x) \rightarrow Q(x))$$
?

$$\neg (\forall x \in X, P(x) \rightarrow Q(x))$$

$$\equiv \exists x \in X, \neg (P(x) \rightarrow Q(x))$$

$$\equiv \exists x \in X, \neg (\neg P(x) \lor Q(x))$$

$$\equiv \exists x \in X, P(x) \land \neg Q(x)$$

Negation of Quantified Statements

Conversion of Conditionals

De Morgan

Example

Consider the domains $X = \{2,3\}$ and $Y = \{2,4,6\}$, and the predicate P(x,y) = "x divides y".

What are the truth values of

1.
$$\neg (\exists x \in X, (\exists y \in Y P(x,y)))$$
. $\vdash take x = 2 & y = 2$

$$= \forall x \in X, \ 7Q(x) = \forall x \in X, \ 7(\exists y \in X, \ R(x,y))$$

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$$= \forall x \in X, \forall y \in Y, \exists x \in X, P(x,y)) = f$$
2. $\neg (\forall y \in Y, \exists x \in X, P(x,y)) = f$

$$= \forall x \in X, \forall y \in Y, \exists x \in X, P(x,y) = f$$

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$$= \forall x \in X, P(x,y) =$$

$$= \exists y \in Y, 7(\exists x \in X, \ell(x,y))$$

 $= \exists y \in Y, \forall x \in X, 7\ell(x,y)$

Basic Inference Rules:



P(c) for any arbitrary c from the domain D.

 $\therefore \forall x \in D, P(x)$



P(c)

 $\therefore \exists x \in D, P(x)$

for c some specific element of the domain D.



 $\forall x \in D, P(x)$

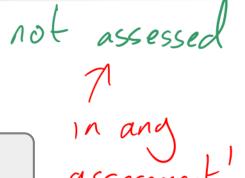
 $\therefore P(c)$

where c is any element of the domain D.



 $\exists x \in D, P(x)$

 \therefore P(c) for some c in the domain D.





A: conjunction (or)

V: disjunction (or)

¬: negation (not, alternatively \sim)

p \rightarrow q: conditional (if then)

p \leftrightarrow q: biconditional (if and only

proposition

equivalence laws (e.g. De Morgan, Conversion Theorem, Distributivity) valid argument (premises and conclusion) inference rules, e.g. Modus ponens/tollens

predicate

Quantification:

- Universal ∀
- Existential 3
- Nested
- Negation
- Conditional
- Negation of conditional

Inference rules:

- Universal generalization/instantiation
- Existential generalization/instantiation