NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2022–2023

MH1812 – Discrete Mathematics

May 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

QUESTION 1. **(16 marks)**

The last digit of 12345 is 5.

(a) What is the last digit of 2023^{1812}? **(4 marks)**

(b) Let $S \subset \mathbb{N}$ such that $\forall x \in S$ the last digit of x^{1812} is 6.

(i) Is S closed under addition? Justify your answer. **(6 marks)**

(ii) Is S closed under multiplication? Justify your answer. **(6 marks)**

Solution:

(a) 1

(b) (i) No. $2 \in S$ and $8 \in S$ but 10 is not in S.

(ii) Yes. Each element of S is even and not divisible by 10. The same is true of any product of elements of S.

QUESTION 2. **(8 marks)**

Use De Morgan's Law and mathematical induction to prove that, for each positive integer n,

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n,$$

where $A_1, A_2, \ldots, A_n$ are sets and $\overline{A}$ denotes the complement of the set A.

Solution: Let $P(n)$ be the hypothesis that

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n.$$

Base case: $P(1)$ is true. Assume that $P(n)$ is true for some $n \in \mathbb{N}$. Now consider $P(n+1)$. Using the hypothesis $P(n)$ we see that the LHS of $P(n+1)$ is

$$\begin{aligned}
\overline{A_1 \cap A_2 \cap \cdots \cap A_{n+1}} &= \overline{(A_1 \cap A_2 \cap \ldots A_n) \cap A_{n+1}} \\
&= \overline{(A_1 \cap A_2 \cap \ldots A_n)} \cup \overline{A_{n+1}} \\
&= \overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n \cup \overline{A_{n+1}} \\
&= \overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A_{n+1}}
\end{aligned}$$

as required.

QUESTION 3. **(18 marks)**

In an experiment, a 6-sided die is rolled six times. The outcome of each roll is recorded in order to obtain a six-digit number. (One possible outcome for the resulting six-digit number is 512256.)

(a) What is the total number of possible six-digit numbers that can be obtained from this experiment? **(4 marks)**

(b) How many of the possible six-digit numbers contain exactly three digits equal to 3? **(4 marks)**

(c) How many of the possible six-digit numbers are divisible by 3? **(5 marks)**

(d) How many of the possible six-digit numbers are less than 123456? **(5 marks)**

For each part, provide your answer as an explicit number (not an expression). No justification is required.

Solution:

(a) $6^6 = 46656$

(b) $\binom{6}{3} \times 5^3 = 2500$

(c) $2 \times 6^5 = 15552$

(d) $6^4 + 2 \times 6^3 + 3 \times 6^2 + 4 \times 6 + 5 = 1865$

QUESTION 4. **(28 marks)**

(a) Define the relation R on the set of rational numbers $\mathbb{Q}$ as

$$R = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} \mid a - b \in \mathbb{Z}\}$$

(i) Is R reflexive? **(4 marks)**

(ii) Is R symmetric? **(4 marks)**

(iii) Is R anti-symmetric? **(4 marks)**

(iv) Find the transitive closure of R. **(4 marks)**

Justify your answers.

(b) Define the relation S on the set of natural numbers $\mathbb{N}$ such that $(a, b) \in S$ if and only if $\sqrt{2}\sqrt{a^2 + b^2} \in \mathbb{N}$.

(i) Is S reflexive? **(4 marks)**

(ii) Is S symmetric? **(4 marks)**

(iii) Is S anti-symmetric? **(4 marks)**

Justify your answers.

Solution:

(a) (i) Yes: $0 \in \mathbb{Z}$

(ii) Yes: if $a - b \in \mathbb{Z}$ then $b - a = -(a - b) \in \mathbb{Z}$

(ii) No: $(1, 2) \in R$ and $(2, 1) \in R$ but $1 \neq 2$.

(ii) $R^t = R$, since R is transitive: if $a - b = z_1$ and $b - c = z_2$ then $a - c = a - b + b - c = z_1 + z_2$.

(b) (i) Yes: $\sqrt{2}\sqrt{2a^2} = 2a \in \mathbb{N}$.

(ii) Yes: if $\sqrt{2}\sqrt{a^2 + b^2} \in \mathbb{N}$ then $\sqrt{2}\sqrt{b^2 + a^2} \in \mathbb{N}$ (commutative addition)

(iii) No: $(1, 7) \in S$ and $(7, 1) \in S$ but $1 \neq 7$.

QUESTION 5. **(20 marks)**

(a) Define the function $F : \mathbb{N} \to \mathbb{N}$ by the formula $F(x) = \lceil \sqrt{x} \rceil$.

  (i) Is $F(x)$ one-to-one? If so then prove it, if not then give a counterexample. **(5 marks)**

  (ii) Is $F(x)$ onto? If so then prove it, if not then give a counterexample. **(5 marks)**

(b) Define the function $G : \mathbb{N} \to \mathbb{N}$ by the formula $G(x) = F(4x^2 + 4x + 1)$.

  (i) Is $G(x)$ one-to-one? If so then prove it, if not then give a counterexample. **(5 marks)**

  (ii) Is $G(x)$ onto? If so then prove it, if not then give a counterexample. **(5 marks)**

Solution:

(a) (i) No: $F(2) = F(3) = 2$.

  (ii) Yes: $\forall y \in \mathbb{N}$, we have $F(y^2) = y$.

(b) (i) Yes: $G(x) = x + 1$ so $G(x) = G(y)$ implies $x = y$.

  (ii) No: There is no preimage for $y = 1 \in \mathbb{N}$.

QUESTION 6. **(10 marks)**

Let $A \subset \{1, 2, 3, \ldots, 88\}$ such that $|A| = 45$. Must there exist a and b in A such that $a \neq b$ and a divides b? Justify your answer.

Solution: Define the pigeonholes as the sets $H_k = \{2^i(2k-1) \mid i \in \mathbb{N}\}$. Then each of the 45 "pigeons" of A belongs to one of the 44 pigeonholes. By the pigeonhole principle, there must be at least two elements of A in on of the pigeonholes. The smaller of these two elements divides the larger (the quotient is a power of two).

END OF PAPER