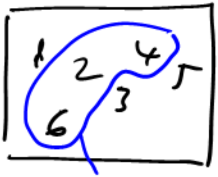
1. Properties of Probability
2. Conditional Probability → filtering / factoring in new information
3. Independent Events
4. Bayes' Theorem

Recap :

① method of counting.

② roll a dice

Ω



A event.

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

Properties of probability

Properties of probability

$A \cup \overline{A} = \Omega$

$\mathbb{P}(A) + \mathbb{P}(\overline{A}) = 1$

The following properties can be deduced from the definition of probability.

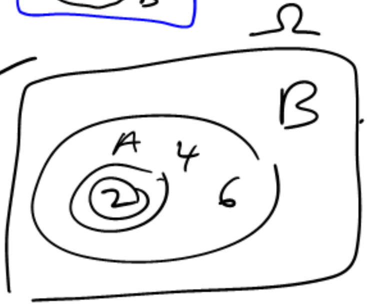
(a) $\mathbb{P}(\emptyset) = 0$. ✓

(b) $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A)$. ✓

(c) If $A \subseteq B$, then $\mathbb{P}(A) \le \mathbb{P}(B)$. ✓

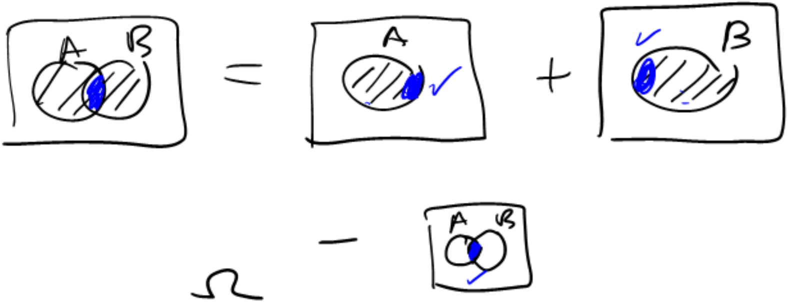
(d) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. ✓

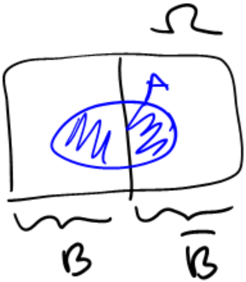
(e) $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \overline{B})$. ✓



$\mathbb{P}(A) = \frac{1}{6}$

$\mathbb{P}(B) : \frac{3}{6}$


A
B
A
B
A
B


Ω
A
B
B̄

Example 1

Find the probability that a randomly chosen card from a 52-card poker deck is of hearts or from one of the ranks ace, king, queen. Assume all outcomes in Ω have the same probability.

- Ω: set of all 52 cards, $|\Omega| = 52$
- A: set of all cards that are hearts, $|A| = 13$
- B: set of all cards that are aces, kings or queens, $|B| = 12$
- $A \cap B = \{$ace of heart, king of heart, queen of heart$\}$, $|A \cap B| = 3$

(4) (4) (4)

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26}.$$

Example 2

Of a group of small-business owners, 30% consult both an accountant and a planner and 10% consult neither. The probability of consulting an accountant exceeds the probability of consulting a planner by 20%.

What is the probability of a randomly selected person X from this group consulting an accountant?

A : consult accountant.

B : consult planner.

$\mathbb{P}(A \cap B) = 0.3$

$\mathbb{P}(\bar{A} \cap \bar{B}) = 0.10$

$\mathbb{P}(A) = \mathbb{P}(B) + 0.2$

$\mathbb{P}(A) = ??$

De morgan

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$$\mathbb{P}(A \cup B) \overset{1}{=} \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$= \mathbb{P}(A) + \mathbb{P}(A) - 0.2 - 0.3$$

$$\mathbb{P}(A \cup B) = 2\,\mathbb{P}(A) - 0.5 .$$

$$\overset{2}{=}$$

$$1 - \mathbb{P}(\overline{A \cup B})$$

$$\overset{3}{=} \text{ De morgen}$$

$$1 - [\mathbb{P}(\bar{A} \cap \bar{B})]$$

$$=$$

$$1 - 0.1 = 0.9$$

$$0.9 = 2\mathbb{P}(A) - 0.5$$

$$\mathbb{P}(A) = 0.7$$

#.

Conditional Probability

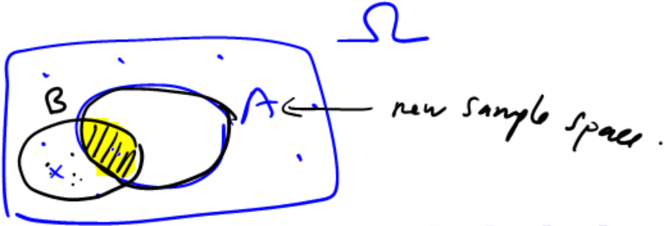
Motivation: We are interested in the probability of the event B given that another event A has already occured.

Under this condition, A becomes the new sample space and event $B \subseteq \Omega$ is represented by $A \cap B$ in the new sample space

We use the notation $\boxed{\mathbb{P}(B|A)}$ for the probability that an event B occurred under the condition that A occured.

$$\mathbb{P}(B) = \frac{|B|}{|\Omega|}$$

$$\mathbb{P}(B|A) = \frac{|A \cap B|}{|A|}$$



For illustration, if we assume that all outcomes have the same probability in the finite sample space Ω, then

$$\mathbb{P}(B|A) = \frac{|A \cap B|}{|A|}.$$

But we can write this as

$$\begin{aligned}
\mathbb{P}(B|A) &= \frac{|A \cap B|}{|A|} \\
&= \frac{|A \cap B|/|\Omega|}{|A|/|\Omega|} \\
\mathbb{P}(B|A) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.
\end{aligned}$$

This motivates the followng definition of conditional probability.

Let $A, B \subseteq \Omega$ be events with $\mathbb{P}(A) > 0$. The **conditional probability** of B given A is

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

Note that $\boxed{\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A)}$. This formula often can be used to compute $\mathbb{P}(A \cap B)$.

$$\Omega = \{ \text{5-card combinations} \}$$
$$|\Omega| = \binom{52}{5}$$

Example 3

Consider a poker deck of 52 cards. What is the probability that 5 randomly chosen cards form a four of a kind under the assumption that they do not contain any ace, king, or queen?

$$A = \{\text{no } \underset{(4)}{\text{ace}}, \underset{(4)}{\text{king}}, \underset{(4)}{\text{queen}}\}$$

unordered

$$B = \{\text{four of a kind}\} = \{AAAA\square, 2222\square, \ldots\}$$

want $\mathbb{P}(B|A)$?

$$P(B|A) = \frac{|A \cap B|}{|A|}.$$

$$|A| = \binom{52 - (4+4+4)}{5}$$

$$= \binom{40}{5} \checkmark$$

these are aces, kings, queens to be avoided

$A \cap B = \{ 2222\square,\ 3333\square,\ \ldots\ldots,\ JJJJ\square \}$

10 ways. ✓ $52-16 = 36$

$\binom{52-16}{1}$

choose a rank for 4 cards

choose the last card.

~~2~~, 3, 4, ..., J. ~~K~~, ~~Q~~

cannot be ace, king, queen, 1st card just now

4 4 4

4

So $|A \cap B| = 10 \times 36$

$$= 360.$$

$$\mathbb{P}(B|A) = \frac{|A \cap B|}{|A|} = \frac{360}{\binom{40}{5}}$$

#.

Example 4

An organ transplant operation succeeds with probability 0.65. Given that the operation succeeded, the probability that the body rejects the organ is 0.2. What is the probability that a randomly selected patient is treated successfully?

- A: event that a transplant succeds
- B: event that body does not reject organ
- $A \cap B$: event that a patient is treated successfully

$$\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A) = 0.8 \cdot 0.65 = 0.52.$$

Given $\mathbb{P}(\bar{B}|A) = 0.2$

Independent Events

Usually steps of an experiment are conducted independently.

For instance, if a dice is rolled 3 times, then the result of the one of the rolls does not influence the results of the other rolls.

In other words: If we know the result of one roll, this does not give us any information on the results of the other rolls. Events with this property are called **independent**.

Formally, events A and B are **independent** if

$$\boxed{\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).}$$

In this case,

- $\mathbb{P}(B|A) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)} = \frac{\cancel{\mathbb{P}(A)}\mathbb{P}(B)}{\cancel{\mathbb{P}(A)}} = \mathbb{P}(B).$ → A has no influence on B.
- $\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$ → B has no influence on A.

That is, information that A occurs does not change probability that B occurs and vice versa.

$|\Omega| = 6 \times 6 = 36$

Example 5

A dice is rolled two times

- A: first roll is a 1
- B: second roll is a 6

$A = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\}$

$B = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6)\}$

A and B are independent:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

$A \cap B = \{(1,6)\}$

$\mathbb{P}(A \cap B) = \frac{1}{36}$

$\mathbb{P}(A) = \frac{6}{36} = \frac{1}{6}$

$\mathbb{P}(B) = \frac{6}{36} = \frac{1}{6}$

Example 6

A red die and a white die are rolled. Assume the dice are fair. Consider the events

- $C = \{5 \text{ on red die}\}$
- $D = \{\text{sum of dice is } 11\}$. $= \{(5,6),(6,5)\}$

Are the event C and D independent?

Note: $C = \{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$, $|C| = 6$, and $D = \{(5,6),(6,5)\}$, $|D| = 2$. So $C \cap D = \{(5,6)\}$, $|C \cap D| = 1$.

$$\mathbb{P}(C \cap D) = \frac{1}{36} \neq \mathbb{P}(C)\mathbb{P}(D) = \frac{6}{36}\frac{2}{36}.$$

Hence, C and D are dependent events.

Independence for several events

Events A_1, …, A_n are **mutually independent** if

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{1_i}) \cdots \mathbb{P}(A_{i_k})$$

for every nonempty subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ with $k \geq 2$. $2 \leq k \leq n.$

Intuitive interpretation: Knowledge of any particular event A_i does not give information whether the other event A_j, where $i \neq j$.

$n=3.$

Example: Events A_1, A_2, A_3 are mutually independent if all of the following hold:

- $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_3)\mathbb{P}(A_3)$ ✓
- $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$ ✓
- $\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_3)$ ✓
- $\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2)\mathbb{P}(A_3)$ ✓

Example 7

A fair six-sided die is rolled 6 independent times. Let A_i be the event that side i is observed on the ith roll, called a match on the ith trial, $i = 1, \ldots, 6$. What is the probability that at least one match occurs?

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

Eg: $(2, 3, 4, 5, 6, 1) \longrightarrow$ no match.

$(2, 2, 4, 1, 6, 6) \longrightarrow$ 2 matches.

$$A_i = \{ \; i \text{ on the } i\text{-th roll} \}$$

Want to calculate

$$\mathbb{P}(A_1 \cup A_2 \dots \cup A_6).$$

$$= 1 - \mathbb{P}(\overline{A_1 \cup \dots \cup A_6})$$

$$= 1 - \mathbb{P}(\bar{A}_1 \cap \bar{A}_2 \dots \cap \bar{A}_6)$$

$$= 1 - \mathbb{P}(\bar{A}_1) \dots \mathbb{P}(\bar{A}_6) = 1 - \left(\frac{5}{6}\right)^{6}$$

$$A_1 = \{ \underset{\substack{\uparrow \\ 2,3,4,5,6}}{x} \ \overline{\_\ \_\ \_\ \_\ \_} \}$$

$$\mathbb{P}(A_1) = \frac{5}{6}$$

$$\mathbb{P}(A_i) = \frac{5}{6}.$$

Let B: event that at least one match occurs. Then

$$\mathbb{P}(B) = 1 - \mathbb{P}(\overline{B}) = 1 - \mathbb{P}(\overline{A_1} \cap \cdots \cap \overline{A_6})$$

Events $\overline{A_i}$ are mutually independent. So

$$\mathbb{P}(B) = 1 - \mathbb{P}(\overline{A_1}) \cdots \mathbb{P}(\overline{A_6}) = 1 - \frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6} = 1 - \left(\frac{5}{6}\right)^6 .$$

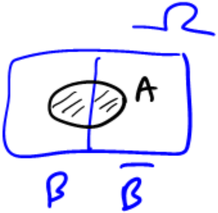
□

Bayes' Theorem

flip $\mathbb{P}(B \mid A)$ to $\mathbb{P}(A \mid B)$

& vice versa.

Law of Total Probability



Suppose the sample space Ω is partitioned as

$$\Omega = B_1 \cup \cdots \cup B_n$$

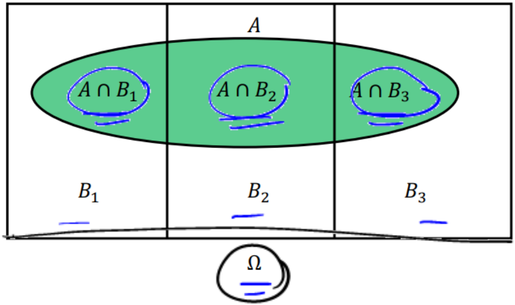
where the B_i are disjoint events. Then

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(A \cap B_1) + \cdots + \mathbb{P}(A \cap B_n) \\
&= \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \cdots + \mathbb{P}(A|B_n)\mathbb{P}(B_n)
\end{aligned}$$

for every event A.

$$\mathbb{P}(A|B_1) = \frac{\mathbb{P}(A \cap B_1)}{\mathbb{P}(B_1)}$$

Intuition behind law of total probability:



Theorem 8 (Bayes' Theorem)

Let A, B *be events with* $\mathbb{P}(A), \mathbb{P}(B) > 0$. *Then*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Proof.

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \implies \mathbb{P}(A \cap B) = \boxed{\mathbb{P}(B|A)\mathbb{P}(A).}$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\boxed{\mathbb{P}(B|A)\mathbb{P}(A)}}{\mathbb{P}(B)}.$$

□

Bayes' Theorem allows us to flip what is given.

Calculate : $\mathbb{P}(D \mid T) = ?$

Example 9

Consider a lab test for a disease:

- It is 95% effective at detecting the disease $\mathbb{P}(T \mid D) = 0.95$
- It has a false positive rate of 1% $\mathbb{P}(T \mid \bar{D}) = 0.01$ ✓
- The rate of occurence of the disease in the general population is 0.5%

I take the screening test and get a positive result. What is the likelihood I have the disease?

T = tested positive

D = disease.

$\mathbb{P}(D) = 0.005$

$$\mathbb{P}(D \mid T) = \frac{\mathbb{P}(T \mid D) \cdot \mathbb{P}(D)}{\mathbb{P}(T)}$$

$$= \frac{0.95 \times 0.005}{\underline{\underline{\mathbb{P}(T \mid D)}}\,\mathbb{P}\underline{\underline{(D)}} + \mathbb{P}(T \mid \bar{D})\,\mathbb{P}(\bar{D})}$$

$$= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times (1 - 0.005)}$$

$$= 0.323 \ \#$$

There are 3 doors, behind which are two goats and a car.
You pick a door (call it Door 1). You're hoping for the car of course.
Monty Hall, the game show host, examines the other doors (Door 2 & 3) and opens one with a goat. (If both doors have goats, he picks randomly.)

Which of the following is a better strategy?

1. I will stick to Door 1 — 28% — 9
2. I will switch to the unopened door. — 47% — 15
3. It does not matter. — 25% — 8

Monty Hall :

Choose Door 1. Host opened Door 3
with a goat.

H = car behind Door 1.

E = Host open door 3 to show
goat.

$\mathbb{P}(H \mid E)$ = chance to win Car if
stick to Door 1.

$\mathbb{P}(\bar{H} \mid E)$ = chance of win car if switch.