MH1820 Introduction to Probability and Statistical Methods

Tutorial 5 (Week 6) Solution

Note: You may use the tables provided in NTULearn > Content > TABLES.pdf for calculations using the standard normal and chi-square distribution. No interpolation is required for calculating $\Phi(z)$, i.e. you may round the number to 2 decimal places before using the table. E.g. $\Phi(0.7897)$ can be approximated by just $\Phi(0.79)$ etc.

Problem 1 (Normal distribution)

(a) If X is normally distributed with a mean of 6 and a variance of 25, find

(i) $\mathbb{P}(6 \leq X \leq 12)$

(ii) $\mathbb{P}(|X-6| < 15)$

(iii) $\mathbb{P}(X > 21)$

(b) If $X \sim N(650, 625)$, find the constant c such that $\mathbb{P}(|X-650| \leq c) = 0.9544$.

Solution (a)

(i) $\mathbb{P}(6 \leq X \leq 12) = \Phi\left(\frac{12-6}{5}\right) - \Phi\left(\frac{6-6}{5}\right) = \Phi(1.2) - \Phi(0) = 0.8849 - 0.5 = 0.3849.$

(ii) Expanding $|X-6| < 15$, we have $-15 < X-6 < 15$, and so $-9 < X < 21$. Thus,

$$\begin{aligned}
\mathbb{P}(|X-6| < 15) &= \mathbb{P}(-9 < X < 21) \\
&= \Phi((21-6)/5) - \Phi((-9-6)/5) \\
&= \Phi(3) - \Phi(-3) \\
&= \Phi(3) - (1 - \Phi(3)) \\
&= 2\Phi(3) - 1 = 2(0.9987) - 1 = 0.9974.
\end{aligned}$$

(ii) $\mathbb{P}(X > 21) = 1 - \mathbb{P}(X \leq 21) = 1 - \Phi((21-6)/5) = 1 - \Phi(3) = 1 - 0.9987 = 0.0013.$

(b)

$$\begin{aligned}
\mathbb{P}(|X-650|\le c) &= 0.9544\\
\mathbb{P}(650-c\le X\le 650+c) &= 0.9544\\
\mathbb{P}\left(\frac{650-c-650}{25}\le Z\le \frac{650+c-650}{25}\right) &= 0.9544\\
\mathbb{P}\left(\frac{-c}{25}\le Z\le \frac{c}{25}\right) &= 0.9544\\
\Phi\left(\frac{c}{25}\right)-\Phi\left(-\frac{c}{25}\right) &= 0.9544\\
\Phi\left(\frac{c}{25}\right)-1+\Phi\left(\frac{c}{25}\right) &= 0.9544\\
2\Phi\left(\frac{c}{25}\right) &= 1.9544\\
\Phi\left(\frac{c}{25}\right) &= 0.9772
\end{aligned}$$

From the table, we have $\Phi(2) = 0.9772$. Therefore, $\frac{c}{25} = 2$, whence $c = 50$. $\square$

Problem 2 (Normal distribution) Find the distribution of $W = X^2$ when

(i) X is $N(0,4)$

(ii) X is $N(0,\sigma^2)$

Solution (i) The CDF of W is

$$\begin{aligned}
F(w) &= \mathbb{P}(W\le w)\\
&= \mathbb{P}(X^2\le w)\\
&= \mathbb{P}(-\sqrt{w}\le X\le \sqrt{w})\\
&= \Phi\left(\frac{\sqrt{w}}{2}\right)-\Phi\left(-\frac{\sqrt{w}}{2}\right)\\
&= \Phi\left(\frac{\sqrt{w}}{2}\right)-1+\Phi\left(\frac{\sqrt{w}}{2}\right)\\
&= 2\Phi\left(\frac{\sqrt{w}}{2}\right)-1
\end{aligned}$$

The PDF is given by

$$\begin{aligned}
f(w) &= F'(w)\\
&= 2\frac{d}{du}\Phi(u)\cdot\frac{du}{dw} \quad \text{where } u=\sqrt{w}/2\\
&= 2\frac{1}{\sqrt{2\pi}}e^{-u^2/2}\frac{1}{4\sqrt{w}}\\
&= \frac{1}{2\sqrt{2\pi}}w^{-1/2}e^{-w/8}.
\end{aligned}$$

(ii) Repeating the above with σ, the CDF of W is

$$\begin{aligned} F(w) &= \mathbb{P}(W \leq w) \\ &= \mathbb{P}(X^2 \leq w) \\ &= \mathbb{P}(-\sqrt{w} \leq X \leq \sqrt{w}) \\ &= \Phi\left(\frac{\sqrt{w}}{\sigma}\right) - \Phi\left(-\frac{\sqrt{w}}{\sigma}\right) \\ &= \Phi\left(\frac{\sqrt{w}}{\sigma}\right) - 1 + \Phi\left(\frac{\sqrt{w}}{\sigma}\right) \\ &= 2\Phi\left(\frac{\sqrt{w}}{\sigma}\right) - 1 \end{aligned}$$

The PDF is given by

$$\begin{aligned} f(w) &= F'(w) \\ &= 2\frac{d}{du}\Phi(u) \cdot \frac{du}{dw} \quad \text{where } u = \sqrt{w}/\sigma \\ &= 2\frac{1}{\sqrt{2\pi}}e^{-u^2/2}\frac{1}{2\sigma\sqrt{w}} \\ &= \frac{1}{\sigma\sqrt{2\pi}}w^{-1/2}e^{-\frac{w}{2\sigma^2}}. \end{aligned}$$

□

Problem 3 (Normal distribution) A candy maker produces mints that have a label weight of 20.4 grams. Assume that the distribution of the weights of these mints is $N(21.37, 0.16)$. Suppose that 15 mints are selected independently and weighted. Let Y equal the number of these mints that weigh less than 20.857 grams. Find $\mathbb{P}(Y \leq 2)$.

$\boxed{\textbf{Solution}}$ Let X be the weight of a mint. Let p be the probability that the weight of a mint is less than 20.857. Then $p = \mathbb{P}(X < 20.857) = \Phi\left(\frac{20.857-21.37}{0.4}\right) = \Phi(-1.2825) \approx 1 - \Phi(1.28) = 1 - 0.8997 = 0.1003$.
Y is binomially distributed, i.e. $Y \sim \mathit{Binomial}(15, p)$. So

$$\mathbb{P}(Y \leq 2) = \binom{15}{0}p^0(1-p)^{15} + \binom{15}{1}p^1(1-p)^{14} + \binom{15}{2}p^2(1-p)^{13}$$

$$= (0.8997)^{15} + 15(0.1003)(0.8997)^{14} + 105(0.1003)^2(0.8997)^{13} \approx 0.815.$$

□

Problem 4 (Normal distribution) The price of an asset is such that its distribution is found by $Y = e^X$, where X is $N(10, 1)$. Find the CDF and PDF of X, and compute $\mathbb{P}(10,000 < Y < 20,000)$.

Note: $F(y) = \mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \ln y)$. For CDF, you can leave your answer in terms of $\Phi(\cdot)$.

$\boxed{\textbf{Solution}}$ The CDF of Y is

$$\begin{aligned} F(y) &= \mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \ln y) \\ &= \mathbb{P}\left(Z \le \frac{\ln y - 10}{1}\right), \quad \text{where } Z \sim N(0,1) \\ &= \Phi\left(\ln y - 10\right) \end{aligned}$$

The PDF of Y is

$$\begin{aligned} f(y) &= \frac{d}{dy}\Phi(\ln y - 10) \\ &= \frac{d}{du}\Phi(u) \cdot \frac{du}{dy}, \quad \text{where } u = \ln y - 10 \\ &= \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \cdot \frac{1}{y} \\ &= \frac{1}{y\sqrt{2\pi}} e^{-(\ln y - 10)^2/2} \end{aligned}$$

$$\begin{aligned} \mathbb{P}(10,000 < Y < 20,000) &= \mathbb{P}(\ln 10,000 < X < \ln 20,000) \\ &= \Phi\left(\frac{\ln 20,000 - 10}{1}\right) - \Phi\left(\frac{\ln 10,000 - 10}{1}\right) \\ &= \Phi(-0.0965) - \Phi(-0.7897) \\ &= 1 - \Phi(0.0965) - 1 + \Phi(0.7897) \\ &\approx \Phi(0.79) - \Phi(0.10) \\ &\approx 0.7852 - 0.5398 \approx 0.25. \end{aligned}$$

$\square$

Problem 5 (Chi-square distribution) If X is $\chi^2(17)$, find

(i) $\mathbb{P}(X < 7.564)$

(ii) $\mathbb{P}(X > 27.59)$

(iii) $\mathbb{P}(6.408 < X < 27.59)$

(iv) $\chi^2_{0.95}(17)$

(v) $\chi^2_{0.025}(17)$

$\boxed{\textbf{Solution}}$

(i) $\mathbb{P}(X < 7.564) = \mathbb{P}(X < \chi^2_{0.975}(17)) = 0.025.$

(ii) $\mathbb{P}(X > 27.59) = \mathbb{P}(X < \chi^2_{0.05}(17)) = 0.05.$

(iii)

$$\begin{aligned} \mathbb{P}(6.408 < X < 27.59) &= \mathbb{P}(X < 27.59) - \mathbb{P}(6.408) \\ &= \mathbb{P}(X < \chi^2_{0.05}(17)) - \mathbb{P}(X < \chi^2_{0.99}(17)) \\ &= 0.95 - 0.01 = 0.94. \end{aligned}$$

(iv) $\chi^2_{0.95}(17) = 8.672.$

(v) $\chi^2_{0.025}(17) = 30.19.$

$\square$

Problem 6 (Chi-square distribution)
Cars arrive at a tollbooth at a mean rate of 5 cars every 10 minutes according to a Poisson distribution. Let X be the time (in minutes) that the toll collector will have to wait before collecting the **eighth** toll.

(i) Find $\mathbb{E}[X]$ and the standard deviation of X.

(ii) Find the probability that the toll collector will have to wait longer than 26.30 minutes before collecting the eighth toll.

$\boxed{\textbf{Solution}}$ The cars arrive at the tollbooth on an average of $\lambda = \frac{5}{10} = \frac{1}{2}$ cars per minutes. Let X be the waiting time (in minutes) until the eighth toll. Then $X \sim \mathit{Gamma}(\alpha = 8, \theta = \frac{1}{\lambda} = 2) = \chi^2(16)$.

(i)

$$\mathbb{E}[X] = \alpha\theta = 8(2) = 16.$$

$$\mathrm{Var}[X] = \alpha\theta^2 = 8(2)^2 = 32 \Longrightarrow \text{standard deviation} = \sqrt{32} \approx 5.66.$$

(ii) From the chi-square table, $\chi^2_{0.05}(16) = 26.30$. Thus, $\mathbb{P}(X > 26.30) = 0.05$.

$\square$

Answer Keys. **1(a)(i).** 0.3849 **(ii).** 0.9974 **(iii).** 0.0013 **1(b).** 50 **2(i).** CDF: $F(w) = 2\Phi\left(\frac{\sqrt{w}}{2}\right) - 1$, PDF: $f(w) = \frac{1}{2\sqrt{2\pi}} w^{-1/2} e^{-w/8}$ **2(ii).** CDF: $F(w) = 2\Phi\left(\frac{\sqrt{w}}{\sigma}\right) - 1$, PDF: $f(w) = \frac{1}{\sigma\sqrt{2\pi}} w^{-1/2} e^{-\frac{w}{2\sigma^2}}$ **3.** 0.815 **4.** 0.25 **5(i).** 0.025 **(ii).** 0.05 **(iii).** 0.94 **(iv).** 8.672 **(v).** 30.19 **6(i).** 16, 5.66 **(ii).** 0.05