Discrete Mathematics

MH1812

PYP

Question 1 (from 2016-2017 Sem 1 PYP):

$a+b \leftarrow$ remainder $/4$

$3+2=5\equiv 1$

(a) Let A be the set of integers modulo 4. Compute the cardinality of the set

$$S=\{f : A \to A,\ f(x) = ax + b,\ \text{for some } a, b \in A,\ f \text{ is injective (one-to-one)}\}.$$

$A=\{\overline{0},\overline{1},\overline{2},\overline{3}\}$

4, 8, 12   5, 9

$f(x) = \overset{a}{0}\cdot x+\overset{b}{0}$

$1\cdot x+3$

check $a=0$ ✗

$f(x)=0\cdot x+b$

$f(0)=0\cdot 0+b=b$

$f(1)=0\cdot 1+b=b$

$a = 1$

$f(x) = 1 \cdot x + b$

$f(0) = 1 \cdot 0 + b = b$

$f(1) = 1 \cdot 1 + b = 1 + b$

$f(2) = 1 \cdot 2 + b = 2 + b$

$f(3) = 1 \cdot 3 + b = 3 + b$

$a = 2$

$S = \{ x+0, x+1, x+2, x+3,$

$\ldots$

$3x+0, 3x+1, 3x+2, 3x+3 \}$

$|S| = 8$

$f(x) = 2 \cdot x + b$

$f(0) = 2 \cdot 0 + b = b$

$f(1) = 2 \cdot 1 + b = 2 + b$

$f(2) = 2 \cdot 2 + b = 4 + b \equiv 0 + b \pmod 4$

$\underline{a = 3}$ ✓

$f(x) = 3 \cdot x + b$

$f(0) = 3 \cdot 0 + b = \underline{b}$;

$f(1) = 3 \cdot 1 + b = \underline{3 + b}$

$f(2) = 3 \cdot 2 + b = \underline{2 + b}$

$f(3) = 3 \cdot 3 + b = \underline{1 + b}$

Question 2 (from 2015-2016 Sem 2 PYP):

(c) How many solutions are there for the following equation

$$x_1 + x_2 + \cdots + x_r = n$$

with r, n, x_i positive integers for $i = 1, 2, \ldots, r$ and $n \geq r$. **(10 marks)**

$n = 6$; $r = 3$

$x_1 + x_2 + x_3 = 6$ $x_i \in \mathbb{N}$

$4+1+1=6$

$1+4+1=6$

$1+1+4=6$

$3+2+1=6$

$3+1+2=6$

$2+1+3=6$

$2+3+1=6$

$1+2+3=6$

$1+3+2=6$

$2+2+2=6$

1 1 1 1 / 1 / 1

1 / 1 1 1 1 / 1

1 , 1 , 1 1 1 1

1 1 1 , 1 1 , 1

To count # solutions

= # ways of placing 2 commas between the 6 1s.

$$\underbrace{1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1}_{\text{5 places to choose from}}$$

So total number of ways to choose

$$= {}^5C_2 = \binom{5}{2} = \frac{5 \cdot 4}{2} = 10$$

In general, have n 1s.

$\Rightarrow$ have $n-1$ places

need to choose $r-1$ places

$$\therefore \binom{n-1}{r-1}$$

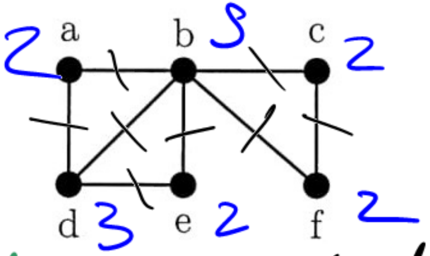
General technique: "stars & bars"

QUESTION 3.

(a) Let A, B, and C be sets, show $(B - A) \cup (C - A) = (B \cup C) - A$. **(10 marks)**

(b) Refer to the graph below, find Euler Path, Euler Circuit and Hamilton Circuit if any, justify your answer if it does not exist. **(8 marks)**

last week.



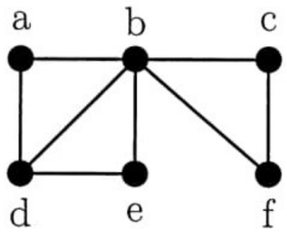
b) Euler path: debdabcfb

Euler circuit: DNE

QUESTION 3.

(a) Let A, B, and C be sets, show $(B - A) \cup (C - A) = (B \cup C) - A$. **(10 marks)**

(b) Refer to the graph below, find Euler Path, Euler Circuit and Hamilton Circuit if any, justify your answer if it does not exist. **(8 marks)**



QUESTION 2. $f : A \to B$ **(30 marks)**

$S, T \subseteq A$

(a) Let f be a function from the set A to the set B. Let S and T be subsets of A.

(i) Show that

$$f(S \cup T) = f(S) \cup f(T).$$

$f(U) = \{f(u) \mid u \in U\}$

(ii) Prove or $\boxed{\text{disprove}}$ that

$$f(S \cap T) = f(S) \cap f(T).$$

i) LHS $\subseteq$ RHS

Take $x \in$ LHS $= f(S \cup T)$

$\Rightarrow x = f(u)$ for some $u \in S \cup T$

$\Rightarrow u \in S$ or $u \in T$

$\Rightarrow x \in f(S)$ | $\Rightarrow x \in f(T)$

$\Rightarrow x \in f(S) \cup f(T)$ | $\Rightarrow x \in f(S) \cup f(T)$

RHS $\subseteq$ LHS

Take $x \in \text{RHS} = f(S) \cup f(T)$

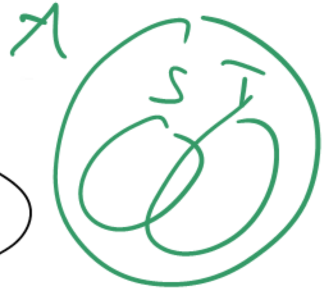
$\Rightarrow \quad x \in f(S) \text{ or } x \in f(T)$

$\Rightarrow x = f(u),\ u \in S$ | $\Rightarrow x = f(u),\ u \in T$

$\Rightarrow u \in S \cup T$ | $\Rightarrow u \in S \cup T$

$\Rightarrow x \in f(S \cup T)$ | $\Rightarrow x \in f(S \cup T)$

QUESTION 2. $S, T \subseteq A$ **(30 marks)**



(a) Let f be a function from the set A to the set B. Let S and T be subsets of A.

(i) Show that

$$f(S \cup T) = f(S) \cup f(T).$$

(ii) Prove or disprove that

$$f(\underbrace{S \cap T}_{= \emptyset}) = f(S) \cap f(T).$$

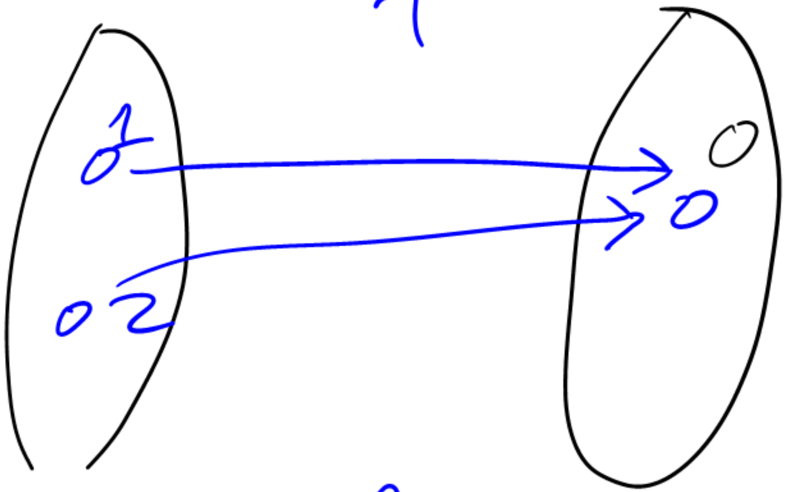
want this to have ≥ 1 element

$f(\emptyset) = \emptyset$

Idea: choose S, T so that $S \cap T = \emptyset$

Try $S = \{1\}$; $T = \{2\}$; set $A = \{1, 2\}$

$A \xrightarrow{f} B = \mathbb{R}$



$$f(S) = \{f(u) \mid u \in S\} = \{f(1)\} = \{0\}$$

& $$f(T) = \{f(u) \mid u \in T\} = \{f(2)\} = \{0\}$$

So $f(S) \cap f(T) = \{0\}$

But $f(S \cap T) = f(\emptyset) = \emptyset$

QUESTION 3. $P(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \ldots$ **(20 marks)**

Consider the set $S = \{1, \ldots, n\}$, for some integer $n \geq 2$, and its power set $P(S)$. For $A, B \in P(S)$, define a relation R by $ARB \iff A \cap B = \emptyset$.

(a) Is R reflexive? $\forall A \in P(S),\ ARA$? No

(b) Is R symmetric? $\forall A, B \in P(S),\ ARB \to BRA$

(c) Is R antisymmetric? $\forall A, B \in P(S),\ ARB \wedge BRA \to A = B$

(d) Is R transitive? $\forall A, B, C \in P(S),\ ARB \wedge BRC \to ARC$

Justify your answers.

a) No, $\{1\} \cap \{1\} = \{1\} \neq \emptyset$

b) Yes, Take $A, B \in P(S)$, suppose ARB

$$\iff A \cap B = \emptyset$$

$$\Rightarrow B \cap A = \emptyset \Rightarrow BRA$$

c) No, Take $A = \{1\}$, $B = \{2\}$

ARB ✓ but $A \neq B$

BRA ✓

$n \geq 2$, we lose generality if we assume $n \geq 4$

Following doesn't work: $A = \{1\}$, $B = \{\underline{4}\}$

d) No. $A = \{1\}$ ✓ $ARB \land BRC$ ✓

$B = \{2\}$ but $(A, C) \notin R$

$C = \{1\}$

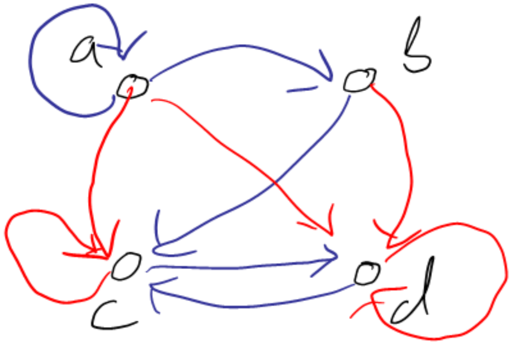
QUESTION 4.

(a) Let set $A = \{a, b, c, d\}$ and relation $R = \{(a,a), (a,b), (b,c), (c,d), (d,c)\}$. **(12 marks)**

No No No

- Is R reflexive, symmetric, transitive ?
- Find R^t, i.e., the transitive closure of R.

$R^t = \{(a,a), (a,b), (b,c), (c,d), (d,c), (a,c), (b,d), (a,d), (c,c), (d,d)\}$



QUESTION 2. **(20 marks)**

Define for a finite set A

$$T(A) = \{ S \subseteq A \mid |S| \cdot |A - S| = 2|A| \}.$$

(a) Find $T(\emptyset)$. **(4 marks)**

(b) Find $|T(\{1, 2, \ldots, 8\})|$. Your answer should be an explicit number. **(4 marks)**

(c) Find $|T(\{1, 2, \ldots, 9\})|$. Your answer should be an explicit number. **(4 marks)**

(d) Find all $n \geq 1$ such that $|T(\{1, 2, \ldots, n\})| \geq n$. **(8 marks)**

You need not justify your answers for parts (a) to (c), but you must justify your answer for part (d).

a) $T(\emptyset) = \{\emptyset\}$

b) $|T(\{1,2,\dots,8\})| = \binom{8}{4} = 70$

c) $|T(\{1,2,\dots,9\}| = \binom{9}{3} + \binom{9}{6} = 168$

d) Let $A = \{1,\dots,n\}$. Suppose $S \subseteq A$ with $|S| = x$ such that

$$|S| \cdot |A - S| = 2|A|$$

$$\iff x(n-x) = 2n$$

$$\iff x^2 - nx + 2n = 0$$

$$\Rightarrow \quad x = \frac{n \pm \sqrt{n^2 - 8n}}{2}$$

Since x must be an integer, we must have that $n^2 - 8n$ is a perfect square i.e., $n^2 - 8n = m^2 \quad \exists\, m \in \mathbb{Z}$

$$\Rightarrow (n-4)^2 - 4^2 = m^2$$

$$\Rightarrow (n-4)^2 - m^2 = 4^2$$

$$\Rightarrow (n-4-m)(n-4+m) = 16$$

16 can be factored as $16 = a \cdot b$ where

$$(a,b) = (1,16), (2,8), (4,4), (8,2), (16,1),$$
$$(-1,-16), (-2,-8), (-4,-4), (-8,-2), (-16,-1)$$

Now we solve the simultaneous equations

$$\left.\begin{array}{l} n-4-m=a \\ n-4+m=b \end{array}\right\} \text{ for each pair } (a,b)$$

Summing the above equations gives $2(n-4)=a+b$.

This implies $a+b$ is even. Thus, $(a,b)\in\left\{\begin{array}{l}(2,8),(4,4),(8,2) \\ (-2,-8),(-4,-4),(-8,-2)\end{array}\right\}$

Hence $a+b=\pm 8$ or ± 10

$a+b$	$n-4$	n
8	4	8
-8	-4	0
10	5	9
-10	-5	-1

These are the only possible values for n.

QUESTION 3. **(15 marks)**

The Fibonacci sequence $\{f_n\}$ is defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 3,$$
$$f_1 = f_2 = 1.$$

(a) Prove by mathematical induction that

$$f_n > \left(\frac{3}{2}\right)^{n-1}$$

for all $n \geq 6$. **(10 marks)**

(b) What is the largest β such that $f_n \geq \beta^{n-1}$ for all $n \geq 6$? Justify your answer without using a calculator. **(5 marks)**

a) $f_1, f_2, f_3, f_4, f_5, f_6, f_7$

$1, 1, 2, 3, 5, 8, 13$

Base case(s)

$$f_6 = 8 > \left(\frac{3}{2}\right)^5 = 7.59375$$ ✓

$$f_7 = 13 > \left(\frac{3}{2}\right)^6 = 11.390625$$ ✓

Inductive hypothesis

Suppose $f_k > \left(\frac{3}{2}\right)^{k-1} \quad \forall \; 6 \leq k \leq n$

inductive step

Want to show $f_{n+1} > \left(\frac{3}{2}\right)^n$

LHS: $f_{n+1} = f_n + f_{n-1}$

$$> \left(\frac{3}{2}\right)^{n-1} + \left(\frac{3}{2}\right)^{n-2} = \left(\frac{3}{2}\right)^{n-2}\left(\overset{2.5}{\frac{3}{2}+1}\right)$$

$$> \left(\frac{3}{2}\right)^{n-2} \cdot \underbrace{\left(\frac{3}{2}\right)^{2}}_{2.25} = \left(\frac{3}{2}\right)^{n} = \text{RHS}$$

b) We need $f_6 \geq \beta^5$

thus, $\beta \leq \sqrt[5]{8}$.