Type I Errors and Size of a Test

There are two types of errors in hypothesis testing:

H_0	H_0 True	H_0 False
Reject H_0	Type I Error	
Do not Reject H_0		Type II Error

- In the p-value approach, we reject H_0 when p-value is less than the significance level α.

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- Instead of using p-value, we can also formulate rejection criteria using **rejection region**, where we reject H_0 if the test statistic statisfies certain inequalities.

E.g. Reject $H_0 \iff \underbrace{\sum_{i=1}^{n} X_i}_{T} > c$, $\underbrace{\left|\sum_{i=1}^{n} X_i\right| > c}_{|T| > c}$ etc.

$T > c$

$T > c$ or $T < -c$

Size of a Test

- If the null hypothesis H_0 is true, but **rejected**, then a **Type I Error** occurs.
- The probability of a Type I Error is

H_0 true.

$$\mathbb{P}(H_0 \text{ rejected} | H_0).$$

- $\mathbb{P}(H_0 \text{ rejected}|H_0)$ is also called the **size** of the test.
- The smaller the size, the more conclusive is the test – the size measures how conclusive a test is.

$\sigma = 1$

Example 1

- $X_1, \ldots, X_9$ i.i.d $\sim N(\mu, 1)$
- Null hypothesis $H_0 : \mu = 0$
- Rejection criteria: Reject $H_0 \iff |\sum_{i=1}^{9} X_i| > 5.88$.

Compute the size of the test.

$$\text{size} = \mathbb{P}(H_0 \text{ rejected} \mid H_0).$$

$$= \mathbb{P}\left(\left| \sum_{i=1}^{9} X_i \right| > 5.88 \;\middle|\; \underline{\underline{\mu = 0}} \right)$$

$$\frac{\frac{1}{9}\sum_{i=1}^{9} X_i - \cancel{\mu}^{\,0}}{\sigma/\sqrt{9}} \sim N(0,1).$$

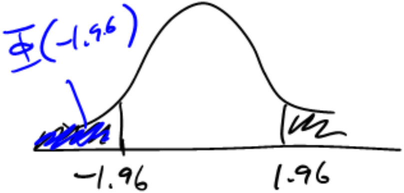
$\sigma = 1$

$$\boxed{\frac{\sum_{i=1}^{9} X_i}{3}} \sim N(0,1)$$

$$\text{size} = \mathbb{P}\left(\sum_{i=1}^{9} X_i > 5.88 \text{ or } \sum_{i=1}^{9} X_i < -5.88 \;\middle|\; \underline{\underline{\mu = 0}} \right)$$

$$= \mathbb{P}\left(\boxed{\frac{\sum_{i=1}^{9} X_i}{3}} > \frac{5.88}{3} \text{ or } \boxed{\frac{\sum_{i=1}^{9} X_i}{3}} < -\frac{5.88}{3} \right)$$

$$= \mathbb{P}\left(\phi > 1.96 \text{ or } \phi < -1.96 \right)$$



$$\text{Size} = 2 \times \Phi(-1.96)$$
$$= 2 \times (1 - \Phi(1.96))$$
$$= 2 \times (1 - 0.975)$$
$$= 2 \times 0.025 = 0.05 \ \#.$$

Example 2

- $X_1, \ldots, X_{100}$ i.i.d $\sim Bernoulli(p)$, $0 \leq p \leq 1$
- Null hypothesis $H_0 : p = 0.5$
- Test statistic $T = \underline{X_1 + \cdots + X_{100}}$
- Rejection criteria: Reject $H_0 \iff T - 50 > 8.$ $\boxed{T > 58}$

Compute the size of the test.

$X_i \sim Bernoulli(p)$

$\mathbb{E}[X_i] = p$ $\quad$ s.d $[X_i] = \sqrt{p(1-p)}$.

$p = 0.5$ $\quad$ $\mathbb{E}(X_i) = \underline{0.5}$ $\quad$ s.d $[X_i] = \sqrt{0.5(1-0.5)} = \underline{0.5}$

$$\text{size} = \mathbb{P}(H_0 \text{ rejected} \mid H_0)$$

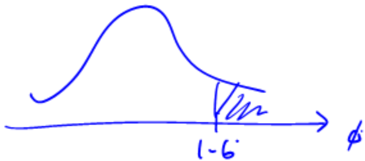
$$= \mathbb{P}(T > 58 \mid p = 0.5)$$

$$= \mathbb{P}\Bigg(\underbrace{\frac{\frac{T}{100} - 0.5}{0.5/\sqrt{100}}}_{\phi} > \frac{\frac{58}{100} - 0.5}{0.5/\sqrt{100}}\Bigg)$$

by CLT

$$\approx \mathbb{P}\left(\phi > \frac{0.58 - 0.5}{0.5/10} = \frac{0.08}{0.05} = \frac{8}{5}\right).$$

$$= \mathbb{P}(\phi > 1.6)$$



$$= 1 - \Phi(1.6)$$

$$= 1 - 0.9452$$

$$= 0.0548$$

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Example 3

- $X_1, \ldots, X_4$ i.i.d $\sim$ *Bernoulli*(p), $0 \leq p \leq 1$
- Null hypothesis H_0 : $p = 0.5$
- Test statistic $T = X_1 + X_2 + X_3 + X_4$
- Rejection criteria: Reject $H_0 \iff \boxed{|T - 2| \geq 2.}$

Compute the size of the test.

$$T = x_1 + x_2 + x_3 + x_4 \sim \text{Binomial}(4, p)$$

$$\underline{\underline{T = 0, 1, 2, 3, 4}}$$

$$
\begin{aligned}
\text{size} &= \mathbb{P}(H_0 \text{ rejected} \mid H_0) \\
&= \mathbb{P}(|T-2| \geq 2 \mid p=0.5) \\
&= \mathbb{P}(T-2 \geq 2 \text{ or } T-2 \leq -2 \mid p=0.5) \\
&= \mathbb{P}(\underline{\underline{T \geq 4}} \text{ or } T \leq 0 \mid p=0.5) \\
&= \mathbb{P}(T=4 \text{ or } \underline{\underline{T=0}} \mid \boxed{p=0.5})
\end{aligned}
$$

$$\boxed{\begin{gathered}\mathbb{P}(Y=k)=\binom{n}{k}p^k(1-p)^{n-k} \\ Y \sim \text{Binomial}(n,\, p).\end{gathered}}$$

$$\text{Site} = \binom{4}{4}p^4(1-p)^{4-4} + \binom{4}{0}p^0(1-p)^{4-0}$$

$$= \binom{4}{4}0.5^4\,0.5^0 + \binom{4}{0}0.5^0\,0.5^4$$

$$= 0.125 \quad \#.$$