MH1820 Introduction to Probability and Statistical Methods

Tutorial 7 (Week 8) Solution

Problem 1 (Joint PMF, Marginal PMF, Conditional PMF)
Let W equal the weight of laundry soap in a 1-kilogram box that is distributed in Southeast Asia. Suppose that $\mathbb{P}(W < 1) = 0.02$ and $\mathbb{P}(W > 1.072) = 0.08$. Call a box of soap light, good, or heavy depending on whether $W < 1$, $1 \le W \le 1.072$, or $W > 1.072$, respectively. In $n = 50$ independent observations of these boxes, let X equal the number of light boxes and Y the number of good boxes.

(a) What is the joint PMF of X and Y?

(b) Give the name of the distribution of Y along with the values of the parameters of this distribution.

(c) Given that $X = 3$, how is Y distributed conditionally?

(d) Determine $\mathbb{E}[Y|X = 3]$

Solution

(a) Let x, and y be fixed. Then there are $\binom{50}{x}$ ways to choose the boxes to be light, followed by $\binom{50-x}{y}$ ways to select the boxes to be good, and $\binom{50-x-y}{50-x-y} = 1$ way for the rest of the boxes to be heavy. By multiplication principle, the joint PMF $p(x, y)$ is

$$p(x,y) = \binom{50}{x}\binom{50-x}{y}(0.02)^x(0.9)^y(0.08)^{50-x-y} = \frac{50!}{x!y!(50-x-y)!}(0.02)^x(0.9)^y(0.08)^{50-x-y}.$$

(b) For a fixed y, there are $\binom{50}{y}$ ways to choose the boxes to be good, followed by $\binom{50-y}{50-y} = 1$ way for the rest of the boxes to be light or heavy. By multiplication principle, the (marginal) PMF of Y is

$$p_Y(y) = \binom{50}{y}(0.9)^y(0.1)^{50-y}.$$

That is, $Y \sim Binomial(50, 0.9)$.

(c) The marginal PMF of X is

$$p_X(x) = \binom{50}{x}(0.02)^x(0.98)^{50-x}.$$

The conditional PMF of Y given $X = x$ is

$$h(y|x) = \frac{p(x,y)}{p_X(x)} = \frac{\binom{50}{x}\binom{50-x}{y}(0.02)^x(0.9)^y(0.08)^{50-x-y}}{\binom{50}{x}(0.02)^x(0.98)^{50-x}}.$$

Substituting $x = 3$, we have

$$\begin{aligned} h(y|3) &= \frac{\binom{47}{y}(0.9)^y(0.08)^{47-y}}{0.98^{47}} \\ &= \binom{47}{y}\frac{(0.9)^y(0.08)^{47-y}}{(0.98)^y(0.98)^{47-y}} \\ &= \binom{47}{y}\left(\frac{0.9}{0.98}\right)^y\left(\frac{0.08}{0.98}\right)^{47-y} \\ &= \binom{47}{y}\left(\frac{0.9}{0.98}\right)^y\left(1-\frac{0.9}{0.98}\right)^{47-y}. \end{aligned}$$

Thus, given that $X = 3$, the conditional distribution of Y is given by $Binomial(47, 0.9/0.98)$.

(d) By part (c),

$$\begin{aligned} \mu_{Y|3} &= \mathbb{E}[Y|X=3] \\ &= 47 \times 0.9/0.98 \\ &\approx 43.1633. \end{aligned}$$

□

Problem 2 (Joint PMF, Marginal PMF, Conditional PMF)
An insurance company sells both homeowners' insurance and automobile deductible insurance. Let X be the deductible on the homeowners' insurance and Y the deductible on automobile insurance. Among those who take both types of insurance with this company, we find the following probabilities:

	$x = 100$	$x = 500$	$x = 1000$
$y = 1000$	0.05	0.10	0.15
$y = 500$	0.10	0.20	0.05
$y = 100$	0.20	0.10	0.05

(a) Compute the probabilities $\mathbb{P}(Y = 500|X = 500)$, $\mathbb{P}(Y = 100|X = 500)$.

(b) Compute the conditional means $\mathbb{E}[X|Y = 100]$, $\mathbb{E}[Y|X = 500]$.

Solution The marginal PMF $p(x, y)$ of X and Y are given in the last row and last column of the table below:

	$x = 100$	$x = 500$	$x = 1000$	$p_Y(y)$
$y = 1000$	0.05	0.10	0.15	0.30
$y = 500$	0.10	0.20	0.05	0.35
$y = 100$	0.20	0.10	0.05	0.35
$p_X(x)$	0.35	0.40	0.25	

(a)

$$\begin{aligned}
\mathbb{P}(Y = 500|X = 500) &= \frac{\mathbb{P}(Y = 500 \text{ and } X = 500)}{\mathbb{P}(X = 500)} \\
&= \frac{p(500, 500)}{p_X(500)} = \frac{0.20}{0.40} = 0.5.
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(Y = 100|X = 500) &= \frac{\mathbb{P}(Y = 100 \text{ and } X = 500)}{\mathbb{P}(X = 500)} \\
&= \frac{p(500, 100)}{p_X(500)} = \frac{0.10}{0.40} = 0.25.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbb{E}[X|Y = 100] &= 100 \cdot \frac{p(100, 100)}{p_Y(100)} + 500 \cdot \frac{p(500, 100)}{p_Y(100)} + 1000 \cdot \frac{p(1000, 100)}{p_Y(100)} \\
&= 100 \cdot \frac{0.20}{0.35} + 500 \cdot \frac{0.10}{0.35} + 1000 \cdot \frac{0.05}{0.35} \\
&= \frac{2400}{7}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[Y|X = 500] &= 100 \cdot \frac{p(500, 100)}{p_X(500)} + 500 \cdot \frac{p(500, 500)}{p_X(500)} + 1000 \cdot \frac{p(500, 1000)}{p_X(500)} \\
&= 100 \cdot \frac{0.10}{0.40} + 500 \cdot \frac{0.20}{0.40} + 1000 \cdot \frac{0.10}{0.40} \\
&= 525.
\end{aligned}$$

□

Problem 3 (Joint PMF, Marginal PMF, Conditional PMF)
Let X and Y have a uniform distribution on the set of points with **integer** coordinates in $S = \{(x, y) : 0 \leq x \leq 7, x \leq y \leq x + 2\}$. That is, $p(x, y) = 1/24$, $(x, y) \in S$, and both x and y are integers. Find

(a) the marginal PMF $p_X(x)$ and $p_Y(y)$.

(b) the conditional PMF $h(y|x)$ of Y given $X = x$.

(c) $\mathbb{E}[Y|X = x]$.

(d) $\sigma^2_{Y|x}$.

$\boxed{\textbf{Solution}}$

(a)

$$p_X(x) = \sum_y p(x,y) = \sum_{y=x,x+1,x+2} p(x,y) = \sum_{y=x,x+1,x+2} 1/24 = \frac{3}{24} = \frac{1}{8},$$

for $x = 0, 1, 2, 3, 4, 5, 6, 7$.

Let y be fixed. We have the following possible values of x:

- If $2 \leq y \leq 7$, then x can be y, $y-1$ or $y-2$;
- If $y = 1$, then x can be y or $y-1$;
- If $y = 8$, then x can be $y-1$ or $y-2$;
- If $y = 0$, then $x = 0$;
- If $y = 9$, then $x = y - 2$.

Hence,

$$p_Y(y) = \begin{cases} \frac{3}{24} = \frac{1}{8} & \text{if } y = 2,3,4,5,6,7 \\ \frac{2}{24} = \frac{1}{12} & \text{if } y = 1,8 \\ \frac{1}{24} & \text{if } y = 0,9. \end{cases}$$

(b)

$$h(y|x) = \frac{p(x,y)}{p_X(x)} = \frac{1/24}{1/8} = \frac{1}{3},$$

for $x = 0, 1, 2, 3, 4, 5, 6, 7$, $y = x, x+1, x+2$.

(c)

$$\begin{aligned} \mu_{Y|x} = \mathbb{E}[Y|X=x] &= \sum_y y h(y|x) \\ &= x h(x|x) + (x+1)h(x+1|x) + (x+2)h(x+2|x) \\ &= \frac{x}{3} + \frac{x+1}{3} + \frac{x+2}{3} \\ &= \frac{3x+3}{3} \\ &= 1 + x, \end{aligned}$$

for $x = 0, 1, 2, 3, 4, 5, 6, 7$,

(d)

$$\begin{aligned} \sigma^2_{Y|x} &= \mathbb{E}[Y^2|X=x] - \mu^2_{Y|x} \\ &= x^2 h(x|x) + (x+1)^2 h(x+1|x) + (x+2)^2 h(x+2|x) - (1+x)^2 \\ &= \frac{x^2}{3} + \frac{(x+1)^2}{3} + \frac{(x+2)^2}{3} - (1+x)^2 \\ &= \frac{2}{3}, \end{aligned}$$

for $x = 0, 1, 2, 3, 4, 5, 6, 7$.

$\square$

Problem 4 (Joint PMF, Marginal PMF, Conditional PMF)
Let $p_X(x) = 1/10$, $x = 0, 1, 2, \ldots, 9$, and let the conditional PMF of Y given $X = x$ be $h(y|x) = 1/(10-x)$, $y = x, x+1, \ldots, 9$. Find

(a) $p(x, y)$.

(b) $p_Y(3)$.

(c) $\mathbb{E}[Y|X = 7]$.

Solution

(a) Since $h(y|x) = \dfrac{p(x,y)}{p_X(x)}$, we have

$$p(x, y) = h(y|x) \cdot p_X(x) = \frac{1}{10(10-x)},$$

for $x = 0, 1, \ldots, 9$, $y = x, x+1, \ldots, 9$.

(b)

	$y=0$	$y=1$	$y=2$	$y=3$	$y=4$	$y=5$	$y=6$	$y=7$	$y=8$	$y=9$
$x=0$	1/100	1/100	1/100	1/100	1/100	1/100	1/100	1/100	1/100	1/100
$x=1$		1/90	1/90	1/90	1/90	1/90	1/90	1/90	1/90	1/90
$x=2$			1/80	1/80	1/80	1/80	1/80	1/80	1/80	1/80
$x=3$				1/70	1/70	1/70	1/70	1/70	1/70	1/70
$x=4$					1/60	1/60	1/60	1/60	1/60	1/60
$x=5$						1/50	1/50	1/50	1/50	1/50
$x=6$							1/40	1/40	1/40	1/40
$x=7$								1/30	1/30	1/30
$x=8$									1/20	1/20
$x=9$										1/10

$$\begin{aligned}
p_Y(3) &= \sum_{x=0}^{3} p(x, 3) \\
&= p(0,3) + p(1,3) + p(2,3) + p(3,3) \\
&= \frac{1}{100} + \frac{1}{90} + \frac{1}{80} + \frac{1}{70} \approx 0.0479.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbb{E}[Y|X=7] &= \sum_{y=7}^{9} y h(y|7) \\
&= 7h(7|7) + 8h(8|7) + 9h(9|7) \\
&= 7(1/3) + 8(1/3) + 9(1/3) \\
&= 8.
\end{aligned}$$

$\square$

Problem 5 (Joint PMF, Marginal PMF, Conditional PMF)
From a standard poker deck of 52 cards, 3 cards are drawn. Let X be number of clubs among the 3 cards and let Y be the number of hearts among the 3.

(a) Find the joint PMF of X and Y.

(b) Find the marginal PMFs of X and Y.

(c) Compute $P(X = 1|Y = 1)$.

(d) Let F be the joint CDF of X and Y. Compute $F(1, 1)$.

Solution

(a) Note that $X = x$ and $Y = y$ mean that exactly x of the 3 cards are clubs and exactly y of the 3 cards are hearts, and $0 \le x \le 3$, $0 \le y \le 3$, $0 \le x + y \le 3$. There are $\binom{13}{x}$ ways to choose x cards from the 13 clubs and $\binom{13}{y}$ ways to choose y cards of hearts. There are $3 - x - y$ remaining that have to be chosen from the 26 cards that are not clubs and not hearts. Hence there are $\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}$ ways to choose the 3 cards so that x of them are clubs and y of them are hearts. We conclude that the joint PMF of X and Y is given by

$$p(x,y) = P(X = x, Y = y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}}{\binom{52}{3}} \text{ for } 0 \le x \le 3, 0 \le y \le 3, 0 \le x + y \le 3.$$

and $p(x, y) = 0$ otherwise. The following table shows the values of the joint PMF.

$x \setminus y$	0	1	2	3
0	$\frac{200}{1700}$	$\frac{325}{1700}$	$\frac{156}{1700}$	$\frac{22}{1700}$
1	$\frac{325}{1700}$	$\frac{338}{1700}$	$\frac{78}{1700}$	0
2	$\frac{156}{1700}$	$\frac{78}{1700}$	0	0
3	$\frac{22}{1700}$	0	0	0

(b) We could compute the marginal PMFs of X and Y by summing over the rows and columns of the above table, respectively. However, we alternatively can find the marginal PMFs directly as follows.

$$p_X(x) = P(X = x) = \frac{\binom{13}{x}\binom{39}{3-x}}{\binom{52}{3}},$$
$$p_Y(y) = P(Y = y) = \frac{\binom{13}{y}\binom{39}{3-y}}{\binom{52}{3}}$$

for $0 \le x \le 3$ and $0 \le y \le 3$ and $p_X(x) = 0$ and $p_Y(y) = 0$ otherwise. For example,

$$p_X(0) = \frac{\binom{13}{0}\binom{39}{3}}{\binom{52}{3}} = \frac{703}{1700},$$

which coincides with the sum of the PMF values in row of the table corresponding to $x = 0$.

(c) We compute

$$\begin{aligned} P(X = 1|Y = 1) &= \frac{P(X = 1, Y = 1)}{P(Y = 1)} \\ &= \frac{p(1,1)}{p_Y(1)} = \frac{\frac{338}{1700}}{\frac{741}{1700}} = \frac{338}{741}. \end{aligned}$$

(d) We have

$$\begin{aligned} F(1,1) &= P(X \le 1, Y \le 1) \\ &= p(0,0) + p(0,1) + p(1,0) + p(1,1) \\ &= \frac{200}{1700} + \frac{325}{1700} + \frac{325}{1700} + \frac{338}{1700} = \frac{1188}{1700}. \end{aligned}$$

□

Answer Keys. **1(a).** $\frac{50!}{x!y!(50-x-y)!}(0.02)^x(0.9)^y(0.08)^{50-x-y}$ **1(b).** $Binomial(50, 0.9)$ **1(c).** $Binomial(47, 0.9/0.98)$ **1(d).** 43.1633 **2(a).** 0.5, 0.25 **2(b).** $\frac{2400}{7}$, 525 **3(a).** $p_X(x) = \frac{1}{8}$ **3(b).** $h(y|x) = \frac{1}{3}$ **3(c).** $1 + x$ **3(d).** $\frac{2}{3}$ **4(a).** $\frac{1}{10(10-x)}$ **4(b).** 0.0479 **4(c).** 8 **5(a).** $\frac{\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}}{\binom{52}{3}}$ **5(b).** $p_X(x) = \frac{\binom{13}{x}\binom{39}{3-x}}{\binom{52}{3}}$, $p_Y(y) = \frac{\binom{13}{y}\binom{39}{3-y}}{\binom{52}{3}}$ **5(c).** $\frac{338}{741}$ **5(d).** $\frac{1188}{1700}$