MH1820 Introduction to Probability and Statistical Methods
Tutorial 4 (Week 5)

Problem 1 (Poisson distribution)

(a) In the past, computer tape was used to store files, and flaws occured on these tapes. Suppose that flaws occur on the average of one flaw per 1200 feet. Let X be the number of flaws in a 4800-foot tape.

   (i) What is the PMF of X?

   (ii) What is the probability that $2 \leq X \leq 4$?

(b) Let X have a Poisson distribution such that $3\mathbb{P}(X = 1) = \mathbb{P}(X = 2)$. Find $\mathbb{P}(X = 4)$.

Problem 2 (Continuous random variables, PDF and CDF)
Let X be a random variable with PDF given by

$$f(x) = \frac{x^2}{9} \text{ for } 0 \leq x \leq 3 \text{ and } f(x) = 0 \text{ otherwise.}$$

(a) Draw a graph of f.

(b) Compute the CDF F of X.

(c) Compute $\mathbb{E}[X]$, $\mathbb{E}[X^2]$, and $\mathrm{Var}[X]$.

(d) Find $\mathbb{P}(X \geq 1)$, $\mathbb{P}(2 \leq X \leq 3)$, and $\mathbb{P}(X^2 \leq 2)$.

Problem 3 (Continuous random variables, PDF and CDF)
Suppose X is a random variable whose PDF is defined by

$$f(x) = \begin{cases} x, & 0 \leq x < 1, \\ \dfrac{c}{x^3}, & 1 \leq x < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Find (i) the value of the constant c so that $f(x)$ is a PDF and (ii) the mean of X.

Problem 4 (Continuous random variables, expected value) A trader receives a bonus if the total losses X (in \$100,000) of his investment is less than 1.5. The bonus equals $0.05 - \frac{X}{30}$ if $X < 1.5$, and equals 0 otherwise. Suppose X has PDF $f(x) = \frac{3}{x^4}$ for $x > 1$ and $f(x) = 0$ for $x \leq 1$. Find the expected value of the bonus.

Hint: $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$, where $f(x)$ is the PDF of X.

Problem 5 (Exponential distribution)

(a) A certain type of aluminium screen has, on the average, 3 flaws in a 100-foot roll.

  (i) What is the probability that the first 40 feet in a roll contain no flaws?

  (ii) What assumption did you make to solve part (i)?

(b) Cars arrive at a tollbooth at a mean rate of 5 cars every 10 minutes according to a Poisson distribution. Let X be the time in minutes that the toll collector will have to wait before collecting the first toll. Find $\mathbb{E}[X]$ and the standard deviation of X.

Problem 6 (Uniform distribution)

(a) Let X be uniformly distributed over the interval $[a, b]$, i.e. $X \sim U(a, b)$. Find $\mathbb{E}[X]$.

(b) Let X be uniformly distributed over the interval $[0, 1]$. Let $Y = X^2$. Find the CDF and hence the PDF of Y.

Hint: Differentiating CDF will give the PDF.

Problem 7 (Poisson distribution) A store selling newspapers orders only $n = 4$ of a certain newspaper because the manager does not get many requests for that publication. If the number of requests per day follows a Poisson distribution with mean 3,

(i) What is the expected value of the number sold per day?

(ii) What is the minimum number that the manager should order so that the chance of having more requests than available newspapers is less than 0.05?

You may use the CDF table for Poisson distribution attached at the end of this worksheet for calculations.

Hint: Let X be the number of requests per day, and Y be the number of newspapers sold per day. For part (i), we are interested in finding $\mathbb{E}[Y]$. How can we get the PMF for Y based on what we know about X? Remember that the store only has 4 newspapers.

Problem 8 (Monty Hall Problem - Revisited)
There are 3 doors, behind which are two goats and a car. A contestant picked a door (call it Door 1). The contestant is hoping for the car, of course. Monty Hall, the game show host, examines the other doors (Door 2 & 3) and opens one with a goat. (If both doors have goats, he picks randomly.)
Let H be the event that the car is behind Door 1. Let E be the event that the game host opens Door 3 with a goat after Door 1 has been picked.

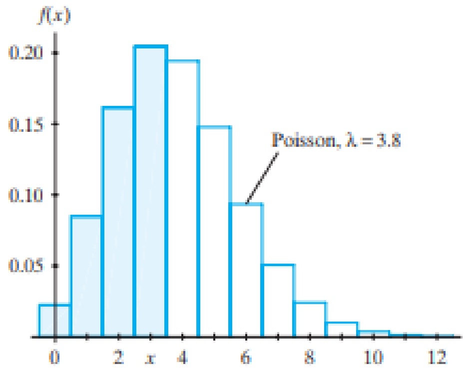
(i) Calculate the probability $\mathbb{P}(H)$, and conditional probabilities $\mathbb{P}(E|H)$ and $\mathbb{P}(E|\overline{H})$.

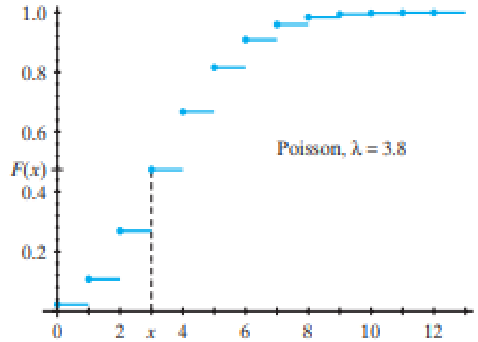
(ii) Calculate the conditional probabilities $\mathbb{P}(H|E)$ and $\mathbb{P}(\overline{H}|E)$.

(iii) Suppose that the game host opens Door 3 with a goat. Explain why the contestant would have a higher chance of winning the car if s/he switches to Door 2 from Door 1.

Answer Keys. 1(a)(ii). 0.5373 1(b). 0.1339 2(c). 9/4, 27/5, 27/80 2(d). 26/27, 19/27, $2\sqrt{2}/27$ 3(i). 1 3(ii). 4/3 4. \$740.74 5(a)(i). 0.3012 5(b). 2, 2 6(a). $(a+b)/2$ 7(i). 2.681 7(ii). 6

Table III The Poisson Distribution





$$F(x) = P(X \le x) = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}$$

	$\lambda = E(X)$									
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0.905	0.819	0.741	0.670	0.607	0.549	0.497	0.449	0.407	0.368
1	0.995	0.982	0.963	0.938	0.910	0.878	0.844	0.809	0.772	0.736
2	1.000	0.999	0.996	0.992	0.986	0.977	0.966	0.953	0.937	0.920
3	1.000	1.000	1.000	0.999	0.998	0.997	0.994	0.991	0.987	0.981
4	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.998	0.996
5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999
6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	0.333	0.301	0.273	0.247	0.223	0.202	0.183	0.165	0.150	0.135
1	0.699	0.663	0.627	0.592	0.558	0.525	0.493	0.463	0.434	0.406
2	0.900	0.879	0.857	0.833	0.809	0.783	0.757	0.731	0.704	0.677
3	0.974	0.966	0.957	0.946	0.934	0.921	0.907	0.891	0.875	0.857
4	0.995	0.992	0.989	0.986	0.981	0.976	0.970	0.964	0.956	0.947
5	0.999	0.998	0.998	0.997	0.996	0.994	0.992	0.990	0.987	0.983
6	1.000	1.000	1.000	0.999	0.999	0.999	0.998	0.997	0.997	0.995
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.999
8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

x	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
0	0.111	0.091	0.074	0.061	0.050	0.041	0.033	0.027	0.022	0.018
1	0.355	0.308	0.267	0.231	0.199	0.171	0.147	0.126	0.107	0.092
2	0.623	0.570	0.518	0.469	0.423	0.380	0.340	0.303	0.269	0.238
3	0.819	0.779	0.736	0.692	0.647	0.603	0.558	0.515	0.473	0.433
4	0.928	0.904	0.877	0.848	0.815	0.781	0.744	0.706	0.668	0.629
5	0.975	0.964	0.951	0.935	0.916	0.895	0.871	0.844	0.816	0.785
6	0.993	0.988	0.983	0.976	0.966	0.955	0.942	0.927	0.909	0.889
7	0.998	0.997	0.995	0.992	0.988	0.983	0.977	0.969	0.960	0.949
8	1.000	0.999	0.999	0.998	0.996	0.994	0.992	0.988	0.984	0.979
9	1.000	1.000	1.000	0.999	0.999	0.998	0.997	0.996	0.994	0.992
10	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.998	0.997
11	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000