MH1820 Week 13 (Review)

QUESTION 1. **(30 Marks)**

(a) Let X be a continuous random variable with PDF given by

$$f(x) = \begin{cases} C(1-x^2), & \text{for } -1 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) What is the value of C?

(ii) Compute $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.

(iii) Find the PDF of $Y = e^X$.

(b) If X has a normal distribution with mean $\mu = 3$ and variance $\sigma^2 = 9$, find $\mathbb{P}(|X-3| > 6)$ in terms of $\Phi(z)$, the CDF of the standard normal random variable Z.

(c) Suppose X has the uniform distribution $U(1, 3)$ on the interval $[1, 3]$. Using the definition of moment generating function (MGF), find the MGF $M_X(t)$ of X.

(d) Each game you play is a win with probability 0.6. You plan to play 5 games, but if you win the fifth game, then you will keep on playing until you lose. Find the expected number of games that you will play.

QUESTION 2. **(20 Marks)**

(a) The weight X (in grams) of a randomly selected chocolate bar produced by a company is normally distributed with mean μ and variance σ^2 which is unknown. Due to a potential fault in a machine, the company suspects that the mean weight is less than 300 grams. We shall test the null hypothesis H_0: $\mu = 300$ against the alternative hypothesis H_1: $\mu < 300$, with a significance level of $\alpha = 0.05$. A random sample of $n = 30$ yielded a mean of $\overline{x} = 280$ and standard deviation $s = 60$.

(i) What is the p-value of the test?

(ii) What is the conclusion of the test?

(b) Let X_1, X_2, ..., X_{12} be a random sample of size $n = 12$ from the normal distribution $N(\mu, \sigma^2)$. We shall test the null hypothesis H_0: $\sigma^2 = 10$ against the alternative hypothesis H_1: $\sigma^2 = 35$.

(i) Find a rejection criteria for the test, where the size of the test is $\alpha = 0.05$.

(ii) Estimate the probability of a Type II Error with the rejection criteria in (i).

QUESTION 3. **(25 Marks)**

(a) The joint PDF of two random variables X and Y is given by

$$f(x, y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1; \\ 0, & \text{elsewhere.} \end{cases}$$

(i) Find the marginal PDFs of X and Y.

(ii) Are X and Y independent? Justify your answer.

(iii) Compute $\mathbb{P}\left(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{3}\right)$.

(iv) Compute $\mathbb{P}\left(X > Y\right)$.

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(b) In a number game, two participants make guesses of X and Y respectively. The joint PDF of X and Y is **uniform** (i.e. constant) on the region $1 \leq x \leq 10$, $1 \leq y \leq 9$. If $|X - Y| < 1$, then the two participants will be asked to guess again. What is the probability that they will be asked to guess again?

QUESTION 4. **(25 Marks)**

(a) Let $X_1, \ldots, X_n$ be i.i.d from $Poisson(\lambda)$, where λ is unknown. Find the maximum likelihood estimator for λ based on the observations $x_1 = 13$, $x_2 = 5$, $x_3 = 6$, $x_4 = 7$ (here $n = 4$). (Recall that if $X \sim Poisson(\lambda)$, then $\mathbb{P}(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}$.)

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(b) Let D_θ, $0 < \theta < 1$, be the discrete distribution with the following PMF:

x	1	2	3
$p(x)$	$\theta/3$	$2\theta/3$	$1-\theta$

Let $X_1, \ldots, X_n$ be i.i.d drawn from D_θ and let $\overline{X}$ be the sample mean. Consider an estimator for θ given by $\widehat{\theta} = \frac{1}{3}\overline{X}$.

(i) Compute the bias and standard error for $\widehat{\theta}$.

(ii) Find $\widehat{\theta}$ using the observations $x_1 = 2$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$ (here $n = 4$).

(iii) Find an estimator of θ which is unbiased, i.e. it has zero bias.