NANYANG TECHNOLOGICAL UNIVERSITY

MIDTERM I (CA1)

MH1812 – Discrete Mathematics

February 2020 TIME ALLOWED: 50 minutes

Name:

Matric. no.: Tutor group:

INSTRUCTIONS TO CANDIDATES

1. **DO NOT TURN OVER PAPER UNTIL INSTRUCTED.**
2. This midterm paper contains **THREE (3)** questions.
3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. This **IS NOT** an **OPEN BOOK** exam.
6. Candidates should clearly explain their reasoning when answering each question.

QUESTION 1. **(30 marks)**

(a) Which integer $a \in \{0, 1, \ldots, 14\}$ is congruent to $2020+1010+550+225$ modulo 15? (10 marks)

(b) Write down each integer $a \in \{0, 1, 2\}$ for which there exists an integer n such that $a \equiv n^2 + n - 1 \pmod 3$. (10 marks)

(c) Let $S = \{\text{integers congruent to 1 modulo 5}\}$ and Δ be multiplication. Is S closed under Δ? Justify your answer.

Solution:

(a) We have

$$\begin{aligned} 2020 + 1010 + 550 + 225 &= 3030 + (2 \cdot 225 + 100) + 225 \\ &= 2 \cdot 15 \cdot 101 + 3 \cdot 225 + 100 \\ &= 2 \cdot 15 \cdot 101 + 15 \cdot 45 + 90 + 10 \\ &= 15 \cdot (2 \cdot 101 + 45 + 6) + 10 \end{aligned}$$

Hence $2020 + 1010 + 550 + 225 \equiv 10 \pmod{15}$.

(b) Modulo 3, we have 3 possibilities for n.

- For $n \equiv 0 \pmod 3$ we have $n^2 + n - 1 \equiv 2 \pmod 3$.
- For $n \equiv 1 \pmod 3$ we have $n^2 + n - 1 \equiv 1 \pmod 3$.
- For $n \equiv 2 \pmod 3$ we have $n^2 + n - 1 \equiv 4 + 2 - 1 \equiv 2 \pmod 3$.

So $a = 1$ or $a = 2$.

(c) Here S is closed under Δ. Indeed, for generic elements $x \in S$ and $y \in S$, we can write $x = 5p + 1$ and $y = 5q + 1$ for some integers p and q. Then

$$x \cdot y = (5p + 1)(5q + 1) = 25pq + 5p + 5q + 1 = 5(5pq + p + q) + 1,$$

which is congruent to 1 modulo 5.

QUESTION 2. **(40 marks)**

(a) Prove or disprove the following logical equivalences:

(i) (10 marks)

$$p \wedge (T \to p) \equiv p$$

(ii) (10 marks)

$$(p \wedge q \wedge r) \to (p \vee s) \equiv (p \to s) \vee (q \to s) \vee (r \to s)$$

(b) Decide whether or not the following argument is valid (20 marks):

$$\begin{array}{l} \neg q \vee p; \\ \neg q \to F; \\ p \to (\neg r \to s); \\ q \to \neg r \\ \therefore s \end{array}$$

Briefly justify your answers.

Solution:

(a) (i)

p	$T \to p$	$p \wedge (T \to p)$
T	T	T
F	F	F

This proves the logical equivalence.

(ii) For $p = T$, $q = T$, $r = T$, $s = F$ the LHS is true and the RHS is false. This disproves the logical equivalence.

(b) The argument is valid.

(1) $\neg q \vee p$

(2) $\neg q \to F$

(3) $p \to (\neg r \to s)$

(4) $q \to \neg r$

(5) $\therefore q$ from (2)

(6) $\therefore p$ from (5) and (1)

(7) $\therefore \neg r \to s$ from (6) and (3)

(8) $\therefore \neg r$ from (5) and (4)

(9) $\therefore s$ from (8) and (7)

Alternatively, one can show that the argument is valid using a truth table.

QUESTION 3. **(30 marks)**

(a) Let X and Y be domains, and let $P(x)$ and $Q(y)$ be predicates. Which of the following statements is the *negation* of the statement $\forall x \in X,\ \exists y \in Y,\ P(x) \vee \neg Q(y)$? (10 marks)

(i) $\forall y \in Y,\ \exists x \in X,\ \neg P(x) \wedge Q(y)$;
(ii) $\exists x \in X,\ \forall y \in Y,\ P(x) \to \neg Q(y)$;
(iii) $\exists y \in Y,\ \forall x \in X,\ \neg P(x) \to \neg Q(y)$;
(iv) $\exists x \in X,\ \forall y \in Y,\ \neg P(x) \wedge Q(y)$;
(v) none of the above.

Consider the domains $A = \{3, 4\}$ and $B = \{0, 3, 6\}$ and the predicate $P(x, y) =$ "$x^2 - y \geqslant 9$".

Determine the truth value of the following statements:

(i) $\forall x \in A,\ \exists y \in B,\ P(x, y)$. (10 marks);

(ii) $\exists x \in A,\ \forall y \in B,\ P(x, y)$. (10 marks).

Briefly justify your answers.

Solution:

(a) We can write

$$\forall x \in X,\ \exists y \in Y,\ P(x) \vee \neg Q(y) \equiv \exists \forall x \in X,\ R(x),$$

where $R(x)$ is the predicate $R(y) = \exists y \in Y,\ P(x) \vee \neg Q(y)$. The negation of "$\forall x \in X,\ R(x'$" is "$\exists x \in X,\ \neg R(y)$". Next we see that the negation of $R(x)$ is just $\forall y \in Y,\ \neg(P(x) \vee \neg Q(y))$. Then

$$\begin{aligned} \neg(P(x) \vee \neg Q(y)) &\equiv \neg P(x) \wedge \neg\neg Q(y) && \text{(De Morgan's law)} \\ &\equiv \neg P(x) \wedge Q(y) && \text{(double negation)} \end{aligned}$$

Hence the answer is (iv).

(b) (i) True. For $x = 3$ take $y = 0$. For $x = 4$ take $y = 0$.

(ii) True. For $x = 4$ the predicate is true for each $y \in B$.