QUESTION 1. (16 marks)

(a) [4 marks] Tick the corresponding box to indicate which of the following are linear homogeneous recurrence relations? No justification is required.

(i) ☐ $a_n = 7a_{n-2} + 6a_{n-4}$

(iv) ☐ $a_n = -a_{n-1} + a_{n-2} - a_{n-3}$

(ii) ☐ $a_n = a_{n-1} + 7$

(v) ☐ $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$

(iii) ☐ $a_n = 5a_{n-3}^2$

(vi) ☐ $a_n = a_{n-1}a_{n-2} + a_{n-2}a_{n-3}$

(b) [6 marks] Solve the following linear recurrence, that is, write a_n in terms of n:

$a_n = 3a_{n-1} + 10a_{n-2}$ for each $n \geqslant 2$, with initial conditions $a_0 = 2$, $a_1 = 10$.

(c) [6 marks] Use induction to show that, for each $n \in \mathbb{N} - \{0\}$,

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}.$$

QUESTION 2.   (17 marks)

In this question **no justification is required**. For each part, give **an explicit number** as your answer, not an expression.

(a) A coin is tossed five times. In each case, the outcome H (for heads) or T (for tails) is recorded. (One possible outcome for the five tosses is denoted THHTT.)

  (i) [2 marks] What is the total number of possible outcomes of the coin-tossing experiment?

  (ii) [2 marks] In how many of the possible outcomes are exactly two tails obtained?

(b) Recall that a standard deck of cards has 52 cards. The cards can be classified according to suits or denominations. There are 4 suits, hearts, diamonds, spades and clubs. There are 13 cards in each suit; one for each of the 13 denominations: Aces, Kings, Queens, ..., Twos. A *poker hand* consists of five cards drawn from a standard deck. Note that the order in which the cards are drawn does not matter.

  (i) [2 marks] How many poker hands consist of 2 Aces and 3 Kings?

  (ii) [3 marks] How many poker hands consist of 2 Aces, 2 Kings, and a card whose denomination is neither an Ace nor a King?

  (iii) [3 marks] How many poker hands have three cards from one denomination and two from another?

  (iv) [3 marks] How many poker hands consist of five cards all from the same suit?

(c) [2 marks] How many distinguishable permutations of the word BOOKKEEPER are there?

QUESTION 3. (17 marks)

(a) Let $A = \{1, 2\}$, $B = \{a, b\}$, and $C = \{a, c\}$. Find each of the following sets. No justification is required.

    (i) [2 marks] $A \times (B - C)$: _____

    (ii) [3 marks] $(A \times B) \cup (A \times C)$: _____

    (iii) [3 marks] The power set $P(B \cap C)$: _____

    (iv) [3 marks] The power set $P(P(\emptyset)) - \{\emptyset\}$: _____

(b) Let A and B be sets.

    (i) [3 marks] Show that $(A \times B) \cup (B \times A) \subseteq (A \cup B) \times (A \cup B)$.

    (ii) [3 marks] Must $(A \times B) \cup (B \times A) = (A \cup B) \times (A \cup B)$? If so, prove it, otherwise give a counterexample.