AY 22/23 MH1820 Midterm Test Solution

Q1. There are 8 choices $(1,2,3,4,5,6,8,9)$ for the first digit, and 9 choices $(0,1,2,3,4,5,6,8,9)$ for the other 4 digits. By the multiplication principle, the total number of choices is $8 \cdot 9^4$.

Q2. There 4! ways to arrange the people with the 5 men together in a block. There are 5! ways to arrange the people in the 5-men block who must sit next to each other. By the multiplication principle, the total number of ways is $5! \cdot 4!$.

Q3. Let $X =$ number of choices with exactkly three kings, $Y =$ number of cjhoices with at least four spades, $Z =$ number of choices with three spades and two hearts. Note that $X = \binom{4}{3}\binom{48}{2} = 4512$, $Y = \binom{13}{4}\binom{39}{1} + \binom{13}{5} = 29,172$. $Z = \binom{13}{3}\binom{13}{2} = 22,308$. So the event with at least four spades has the highest probability.

Q4. There are 10 outcomes whose total rolled is 15: $(3,6,6)$, $(6,3,6)$, $6,6,3)$, $(4,5,6)$, $(4,6,5)$, $(5,4,6)$, $(6,4,5)$, $(5,6,4)$, $(6,5,4)$, $(5,5,5)$. Exactly three of them have at least (and hence exactly) one rolls of a 3. So the probability is $\frac{3}{10}$.

Q5. Let X be the number of cars arriving in the first hour (60 minutes). The $X \sim Poisson(\lambda = 4 \times 10 = 40)$. So $\mathbb{P}(X = 10) = e^{-40}\frac{40^{10}}{10!}$.

Q6. Let S, M, W denote the event of a strong, moderate and weak recommendation, and J be the event that there is a job offer. It is given that

$$\mathbb{P}(J|S) = 0.8,\ \mathbb{P}(J|M) = 0.5,\ \mathbb{P}(J|W) = 0.05,\ \mathbb{P}(S) = 0.6,\ \mathbb{P}(M) = 0.3,\ \mathbb{P}(W) = 0.1.$$

(a) [3 marks] $\mathbb{P}(J) = \mathbb{P}(J|S)\mathbb{P}(S) + \mathbb{P}(J|M)\mathbb{P}(M) + \mathbb{P}(J|W)\mathbb{P}(W) = 0.8(0.6) + 0.5(0.3) + 0.05(0.1) = 0.635$.
(b) [3 marks]

$$\mathbb{P}(S|\overline{J}) = \frac{\mathbb{P}(\overline{J}|S)\mathbb{P}(S)}{\mathbb{P}(\overline{J})} = \frac{(1 - \mathbb{P}(J|S))\mathbb{P}(S)}{1 - \mathbb{P}(J)} = \frac{(1 - 0.8)(0.6)}{1 - 0.635} = 0.3288.$$

Q7.
(a) [2 marks] $\mathbb{P}(X > 30) = \int_{30}^{\infty} \frac{10}{x^2}\, dx = \left[-\frac{10}{x}\right]_{30}^{\infty} = \frac{10}{30} = \frac{1}{3}$.

(b) [3 marks] For $x \le 10$, $F(x) = 0$. For $x > 10$, $F(x) = \int_{10}^{x} \frac{10}{t^2}\, dt = \left[-\frac{10}{t}\right]_{10}^{x} = 1 - \frac{10}{x}$.

(c) [3 marks] For $y \le 10^3$, $0 \le \mathbb{P}(Y \le y) = \mathbb{P}(X^3 \le y) \le \mathbb{P}(X^3 \le 10^3) = \mathbb{P}(X \le 10) = 0$, so $F_Y(y) = 0$ if $y \le 10^3$. For $y > 10^3$, $F_Y(y) = \mathbb{P}(X^3 \le y) = \mathbb{P}(X \le y^{1/3}) = F(y^{1/3}) = 1 - \frac{10}{y^{1/3}}$. Differentiating, we get the PDF of Y: $f_Y(y) = \frac{10}{3}y^{-4/3}$, for $y > 10^3$; and $f_Y(y) = 0$ for $y \le 10^3$.

Q8.
(a) [3 marks] $\mathbb{P}(X > 300) = \int_{300}^{\infty} \frac{1}{500}e^{-x/500}\, dx = \left[-e^{-x/500}\right]_{300}^{\infty} = e^{-3/5} = 0.5488$.

(b) [3 marks] $\mathbb{P}(\text{operate more than 300 hours}) = \mathbb{P}(\text{not more than two radio tube failed}) = \sum_{i=0}^{2}\binom{5}{i}(1 - 0.5488)^i(0.5488)^{5-i} = \binom{5}{0}(0.4512)^0(0.5488)^5 + \binom{5}{1}(0.4512)^1(0.5488)^4 + \binom{5}{2}(0.4512)^2(0.5488)^3) = 0.5906$.