

MH1820 Introduction to Probability and Statistical Methods

Tutorial 6 (Week 7) Solution

Problem 1 (MGF) (a) Let X be a discrete random variable with PMF p given by

$$\begin{array}{c|c|c|c} x & -1 & 0 & 1 \\ \hline p(x) & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array} \text{ and } p(x) = 0 \text{ for } x \notin \{-1, 0, 1\}.$$

Compute the MGF of X and use the properties of MGFs from the lecture to compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

(b) Let X be a continuous random variable with PDF f given as follows.

$$f(x) = 2x \text{ for } 0 \leq x \leq 1 \text{ and } f(x) = 0 \text{ otherwise.}$$

Compute the MGF $\mathbb{E}[e^{tX}]$ of X for $t \neq 0$.

(c) Let X be a random variable with $M_X(t) = \frac{e^t}{2 - e^t}$ for all $t < \ln 2$. What is the distribution of X ? Hint: Check the table for MGF of common distributions.

Solution (a) We get the following.

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}] \\ &= \sum_{x=-1}^1 e^{tx} p(x) \\ &= e^{-t} p(-1) + e^0 p(0) + e^t p(1) \\ &= \frac{1}{2} e^{-t} + \frac{1}{4} + \frac{1}{4} e^t \\ M_X^{(1)}(t) &= \frac{dM_X(t)}{dt} = -\frac{1}{2} e^{-t} + \frac{1}{4} e^t \\ M_X^{(2)}(t) &= \frac{d^2 M_X(t)}{dt^2} = \frac{1}{2} e^{-t} + \frac{1}{4} e^t \\ \mathbb{E}[X] &= M_X^{(1)}(0) = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4} \\ \mathbb{E}[X^2] &= M_X^{(2)}(0) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \end{aligned}$$

(b) Using integration by parts at one step, we get the following.

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^1 2xe^{tx} dx \\ &= \left[\frac{2xe^{tx}}{t} \right]_0^1 - \int_0^1 \frac{2e^{tx}}{t} dx = \left[\frac{2xe^{tx}}{t} \right]_0^1 - \left[\frac{2e^{tx}}{t^2} \right]_0^1 \\ &= \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2}, \text{ provided } t \neq 0. \end{aligned}$$

(c) Recall that if $X \sim \text{Geom}(p)$, then its MGF is given by

$$\frac{pe^t}{1 - (1-p)e^t}, \text{ for } t < -\ln(1-p).$$

Note that $M_X(t)$ is the MGF of $\text{Geom}(\frac{1}{2})$. Hence $X \sim \text{Geom}(\frac{1}{2})$. □

Problem 2 (MGF) Let $X \sim \text{Poisson}(\lambda)$. Using the definition of MGF, verify that the MGF of X is given by

$$e^{\lambda(e^t-1)}.$$

Hence, find the mean and variance of X .

Hint: You may assume that $e^z = \sum_{x=0}^{\infty} \frac{z^x}{x!}$. This is the Maclaurin series for e^z .

Solution

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}] \\ &= \sum_x e^{tx} \cdot \frac{e^{-\lambda} \lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda(e^t-1)}. \end{aligned}$$

Differentiating the MGF yields (by Chain Rule)

$$M_X^{(1)}(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t.$$

Differentiating again (via product rule),

$$\begin{aligned} M_X^{(2)}(t) &= e^{\lambda(e^t-1)} \cdot \frac{d}{dt} \lambda e^t + \lambda e^t \frac{d}{dt} e^{\lambda(e^t-1)} \\ &= e^{\lambda(e^t-1)} \cdot \lambda e^t + \lambda e^t (e^{\lambda(e^t-1)} \cdot \lambda e^t) \\ &= e^{\lambda(e^t-1)} \cdot \lambda e^t + (\lambda e^t)^2 \cdot e^{\lambda(e^t-1)} \\ &= e^{\lambda(e^t-1)} (\lambda e^t) (\lambda e^t + 1) \end{aligned}$$

Hence,

$$\mathbb{E}[X] = M_X^{(1)}(0) = \lambda, \quad \mathbb{E}[X^2] = \lambda(\lambda + 1).$$

So the mean is $\mathbb{E}[X] = \lambda$, and the variance is

$$\begin{aligned}\sigma^2 &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \lambda(\lambda + 1) - \lambda^2 \\ &= \lambda.\end{aligned}$$

□

Problem 3 (MGF) Suppose X and Y are independent Poisson random variables with means λ_1 and λ_2 respectively. What is the distribution of $X + Y$?

Solution

$$\begin{aligned}M_{X+Y}(t) &= M_{2X}(t)M_Y(t) \\ &= M_X(2t)M_Y(t) \\ &= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} \\ &= e^{(\lambda_1+\lambda_2)(e^t-1)}\end{aligned}$$

Hence, $X + Y$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$.

□

Problem 4 (Joint PMF, CDF, Marginal PMF) Let the joint PMF of X and Y be defined by

$$p(x, y) = \frac{x + y}{32},$$

for $x = 1, 2, y = 1, 2, 3, 4$.

- (a) Find $p_X(x)$, the marginal PMF of X .
- (b) Find $p_Y(y)$, the marginal PMF of Y .
- (c) Find $\mathbb{P}(X > Y)$
- (d) Find $\mathbb{P}(Y = 2X)$
- (e) Find $\mathbb{P}(X + Y = 3)$.
- (f) Find $\mathbb{P}(X \leq 3 - Y)$.
- (g) Are X and Y independent or dependent? Why or why not?

Solution (a)

$$p_X(x) = \sum_{y=1}^4 p(x, y) = \frac{x+1}{32} + \frac{x+2}{32} + \frac{x+3}{32} + \frac{x+4}{32} = \frac{4x+10}{32},$$

for $x = 1, 2$.

(b)

$$p_Y(y) = \sum_{x=1}^2 p(x, y) = \frac{1+y}{32} + \frac{2+y}{32} = \frac{3+2y}{32},$$

for $y = 1, 2, 3, 4$.

(c) The event $X > Y$ occurs only when $x = 2$ and $y = 1$. So

$$\mathbb{P}(X > Y) = p(2, 1) = \frac{3}{32}.$$

(d) The event $Y = 2X$ is given by $\{(1, 2), (2, 4)\}$. So

$$\mathbb{P}(Y = 2X) = p(1, 2) + p(2, 4) = \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32}.$$

(e) The event $X + Y = 3$ is given by $\{(1, 2), (2, 1)\}$. So

$$\mathbb{P}(X + Y = 3) = p(1, 2) + p(2, 1) = \frac{1+2}{32} + \frac{2+1}{32} = \frac{6}{32}.$$

(f) The event $X + Y \leq 3$ is given by $\{(1, 1), (1, 2), (2, 1)\}$. So

$$\mathbb{P}(X \leq 3 - Y) = \mathbb{P}(X + Y \leq 3) = p(1, 1) + p(1, 2) + p(2, 1) = \frac{1+1}{32} + \frac{1+2}{32} + \frac{2+1}{32} = \frac{8}{32}.$$

(g) Note that $p(1, 1) = \frac{1+1}{32} = \frac{2}{32}$, $p_X(1) = \frac{14}{32}$, $p_Y(1) = \frac{5}{32}$. Since $p(1, 1) \neq p_X(1)p_Y(1)$, X and Y are not independent.

Problem 5 (Joint PMF, CDF, Marginal PMF) From a standard poker deck of 52 cards, 3 cards are chosen randomly. Let X be number of clubs among the 3 cards and let Y be the number of hearts among the 3.

(a) Find the joint PMF of X and Y .

(b) Find the marginal PMFs of X and Y .

(c) Are X and Y independent?

Solution (a) Note that $X = x$ and $Y = y$ mean that exactly x of the 3 cards are clubs and exactly y of the 3 cards are hearts. There are $\binom{13}{x}$ ways to choose x cards from the 13 hearts and $\binom{13}{y}$ ways to choose y cards of hearts. There are $3 - x - y$ remaining that have to be chosen

from the 26 cards that not clubs and not hearts. Hence there are $\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}$ ways to choose the 3 cards so that x of them are clubs and y of them are hearts. We conclude that the joint PMF of X and Y is given by

$$p(x, y) = \mathbb{P}(X = x, Y = y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}}{\binom{52}{3}} \text{ for } x, y = 0, 1, 2, 3,$$

and $p(x, y) = 0$ otherwise. The following table shows the values of the joint PMF.

$x \setminus y$	0	1	2	3
0	$\frac{200}{1700}$	$\frac{325}{1700}$	$\frac{156}{1700}$	$\frac{22}{1700}$
1	$\frac{325}{1700}$	$\frac{338}{1700}$	$\frac{78}{1700}$	0
2	$\frac{156}{1700}$	$\frac{78}{1700}$	0	0
3	$\frac{22}{1700}$	0	0	0

(b) We could compute the marginal PMFs of X and Y by summing over the rows and columns of the above table, respectively. However, we alternatively can find the marginal PMFs directly as follows.

$$p_X(x) = P(X = x) = \frac{\binom{13}{x}\binom{39}{3-x}}{\binom{52}{3}},$$

$$p_Y(y) = P(Y = y) = \frac{\binom{13}{y}\binom{39}{3-y}}{\binom{52}{3}}$$

for $0 \leq x \leq 3$ and $0 \leq y \leq 3$ and $p_X(x) = 0$ and $p_Y(y) = 0$ otherwise. For example,

$$p_X(0) = \frac{\binom{13}{0}\binom{39}{3}}{\binom{52}{3}} = \frac{703}{1700},$$

which coincides with the sum of the PMF values in the second row of the table.

(c) Intuitively, X and Y cannot be independent, since information on X also gives information on Y . For instance, if $X = 3$, then all 3 cards are clubs and thus $Y = 0$. For a rigorous argument, we compute, for instance,

$$p(3, 1) = 0,$$

$$p_X(3) = \frac{22}{1700},$$

$$p_Y(1) = 741/1700.$$

Hence $p(3, 1) \neq p_X(3)p_Y(1)$ which proves that X and Y are not independent. □

Problem 6 (Joint PMF, CDF, Marginal PMF) There are eight similar chips in a bowl: three marked $(0, 0)$, two marked $(1, 0)$, two marked $(0, 1)$, and one marked $(1, 1)$. A player selects

a chip at random and is given the sum of the two coordinates in dollars. What is the expected payoff?

Solution Let X and Y represent the x - and y -coordinate of the marking on the chips. The joint PMF is given by

$x \setminus y$	0	1
0	$\frac{3}{8}$	$\frac{2}{8}$
1	$\frac{2}{8}$	$\frac{1}{8}$

The payoff is $X + Y$. So the expected payoff is

$$\begin{aligned}
 \mathbb{E}[X + Y] &= \sum_{x=0}^1 \sum_{y=0}^1 (x + y)p(x, y) \\
 &= (0 + 0)\frac{3}{8} + (0 + 1)\frac{2}{8} + (1 + 0)\frac{2}{8} + (1 + 1)\frac{1}{8} \\
 &= \frac{10}{8} = 0.75.
 \end{aligned}$$

The expected payoff is \$0.75. □

Answer Keys. 1(a) $M_X(t) = \frac{1}{2}e^{-t} + \frac{1}{4} + \frac{1}{4}e^t$, $-1/4$, $3/4$. 1(b) $M_X(t) = \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2}$ 1(c) $X \sim \text{Geom}(1/2)$. 3 $X + Y$ is Poisson with mean $\lambda_1 + \lambda + 2$. 4(a) $\frac{4x+10}{32}$ 4(b) $\frac{3+2y}{32}$ 4(c) $3/32$ 4(d) $9/32$ 4(e) $6/32$ 4(f) $8/32$ 4(g) Dependent. 5(a) $p(x, y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}}{\binom{52}{3}}$ 5(b) $p_X(x) = \frac{\binom{13}{x}\binom{39}{3-x}}{\binom{52}{3}}$, $p_Y(y) = \frac{\binom{13}{y}\binom{39}{3-y}}{\binom{52}{3}}$ 5(c) Dependent 6 \$0.75.