Name:

Matric. no.: Tutor group:

March 2022 **CA2** TIME ALLOWED: 50 minutes

QUESTION 1. **(16 marks)**

(a) [8 marks] Solve the following linear recurrence, that is, write a_n in terms of n:
$a_n = 9a_{n-2}$ for each $n \geqslant 2$, with initial conditions $a_0 = 0$, $a_1 = 2$.

(b) [8 marks] A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 3$ and $b_k = (b_{k-1})^2$ for every integer $k \geqslant 1$. Using induction, show that $b_n = 3^{2^n}$ for every integer $n \geqslant 0$.

Solution

(a) [8 marks] The characteristic equation is

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0. \end{aligned}$$

This equation has roots $s_1 = 3$ and $s_2 = -3$. Hence $a_n = u3^n + v(-3)^n$ for some u and v. Using the initial conditions, we find that $u = -v = \frac{1}{3}$. Thus $a_n = 3^{n-1} + (-3)^{n-1}$ for all $n \in \mathbb{N}$.

[Distribution: 4 marks for correct expression for a_n and 4 marks for the justification]

(b) [8 marks] Let $P(k)$ denote the predicate $b_k = 3^{2^k}$. First we check the base case $P(0)$. Here the LHS, $b_0 = 3$ is equal to the RHS $3^{2^0} = 3$.

Now we want to prove the proposition $\forall k \in \mathbb{N}, P(k) \to P(k+1)$. For our inductive hypothesis, assume $P(k)$ is true for some $k \in \mathbb{N} \cup \{0\}$. The LHS of $P(k+1)$, b_{k+1}, is equal to the square of the LHS of $P(k)$. Hence, using the inductive hypothesis, we have

$$\begin{aligned} b_{k+1} &= b_k^2 \\ &= (3^{2^k})^2 \\ &= 3^{2^k + 2^k} \\ &= 3^{2^{k+1}}. \end{aligned}$$

Thus, we have shown that $P(k+1)$ follows from $P(k)$, as required.

[Distribution: 2 marks for correct predicate, 2 marks for base case, 2 marks for inductive hypothesis, 2 marks for correctly using induction.]

For graders only:	Question	1(a)	1(b)	2(a)	2(b)	2(c)	2(d)	3(a)	3(b)	Total
	Marks									

QUESTION 2. **(17 marks)**

In this question **no justification is required**.
A computer programming team has 13 members.

(a) [2 marks] How many ways can a group of seven be chosen to work on a project?

(b) Suppose seven team members are women and six are men.

(i) [3 marks] How many groups of seven can be chosen that contain four women and three men?

(ii) [3 marks] How many groups of seven can be chosen that contain at least one man?

(iii) [3 marks] How many groups of seven can be chosen that contain at most three women?

(c) [3 marks] Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project?

(d) [3 marks] Suppose two team members insist on either working together or not at all on projects. How many groups of seven can be chosen to work on a project?

Solution

(a) [2 marks] $\binom{13}{7} = 1716$.

[Distribution: 2 marks for correct answer (either expression or number)]

(b)

(i) [3 marks] $\binom{7}{4} \cdot \binom{6}{3} = 700$.

(ii) [3 marks] Total minus number of groups with 0 men: $\binom{13}{7} \cdot \binom{7}{7} = 1715$.

(iii) [3 marks] Groups can have 1, 2, or 3 women: $\binom{7}{3} \cdot \binom{6}{4} + \binom{7}{2} \cdot \binom{6}{5} + \binom{7}{1} \cdot \binom{6}{6} = 658$.

[Distribution: 3 marks for each correct answer (either expression or number)]

(c) [3 marks] Let X and Y be the two team members in question. We sum the number of groups without both X and Y and the number of groups that contain X (and not Y) and the number of groups that contain Y (and not X): $\binom{11}{7} + 2\binom{11}{6} = 1254$.

[Distribution: 3 marks for correct answer (either expression or number)]

(d) [3 marks] Let X and Y be the two team members in question. We sum the number of groups without both X and Y and the number of groups that contain both X and Y: $\binom{11}{7} + \binom{11}{5} = 792$.

[Distribution: 3 marks for correct answer (either expression or number)]

QUESTION 3. **(17 marks)**

(a) Let $A = \{a, b\}$, $B = \{1, 2\}$, and $C = \{2, 3\}$. Find each of the following sets.

(i) [3 marks] $A \times (B \cup C)$

(ii) [3 marks] $(A \times B) \cap (A \times C)$

(iii) [3 marks] The power set $P(B - C)$

(iv) [3 marks] The power set $P(P(\emptyset))$

(b) [5 marks] For all sets A and B, is the power set $P(A \times B)$ equal to $P(A) \times P(B)$? If so then prove it, if not then give a counterexample.

Solution

(a)

(i) [3 marks] $\{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

(ii) [3 marks] $\{(a, 2), (b, 2)\}$

(iii) [3 marks] $\{\emptyset, \{1\}\}$

(iv) [3 marks] $\{\emptyset, \{\emptyset\}\}$

[Distribution: 1-2 marks for partially correct answer, 3 marks for perfect answer of each part.]

(b) [5 marks] No. Let's check the cardinalities of $P(A \times B)$ and $P(A) \times P(B)$. The cardinality of $P(A \times B)$ is equal to $2^{|A| \cdot |B|}$ and the cardinality of $P(A) \times P(B)$ is equal to $2^{|A|} \cdot 2^{|B|} = 2^{|A|+|B|}$. Therefore, a counterexample could be $A =$ and $B = \{1\}$. Then $P(A \times B)$ and $P(A) \times P(B)$ must be not equal since they have different cardinalities.

[Distribution: 2 marks for correct answer, 2 marks for counterexample, 1 mark for correct justification of the counterexample.]