MH1812 Discrete Mathematics: Quiz (CA) 2

Name: Tutorial Group:

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There are 3 (THREE) questions, please try all of them, and justify all your answers! Best of luck!

Question 1 (35 points)

a) Let A, B, C be sets. Prove or disprove the following set equality (20 points):

$$A \times (B - C) = (A \times B) - (A \times C).$$

Solution. The first important thing to notice here is that we have a cartesian product of sets. Take $x \in A \times (B - C)$. Then $x = (x_1, x_2)$ with $x_1 \in A$ and $x_2 \in B - C$ (or equivalently $x_2 \in B$ and $x_2 \notin C$). Thus $(x_1, x_2) \in A \times B$ and $(x_1, x_2) \notin A \times C$, which shows that

$$A \times (B - C) \subseteq (A \times B) - (A \times C).$$

Conversely, take $x = (x_1, x_2) \in A \times B$ but not in $A \times C$. Since $x_1 \in A$, it must be that $x_2 \in B$ and also $x_2 \notin C$ for x not to be in $A \times C$. Thus $x_1 \in A$ and $x_2 \in B - C$ which shows that $x \in A \times (B - C)$ and we have the reverse inclusion:

$$(A \times B) - (A \times C) \subseteq A \times (B - C).$$

Note that it is also possible to do a membership table, but then the membership table needs to reflect the cartesian product.

b) Prove the following set equality (15 points):

$$\{12a + 25b, a, b \in \mathbb{Z}\} = \mathbb{Z}.$$

Solution. We prove double inclusion. First we note that since $a, b \in \mathbb{Z}$, $12a + 25b \in \mathbb{Z}$ (closure of addition and multiplication of integers), thus

$$\{12a + 25b, a, b \in \mathbb{Z}\} \subseteq \mathbb{Z}.$$

Next we need to prove the reverse inclusion. Take any $x \in \mathbb{Z}$. We need to prove that x can be written of the form $x = 12a + 25b$ for $a, b \in \mathbb{Z}$. One way of doing this is to pick $a = -2x$, and $b = x$ then $x = 12(-2x) + 25x$. This shows that every element $x \in \mathbb{Z}$ is of the form $12a + 25b$ for some $a, b \in \mathbb{Z}$ therefore

$$\mathbb{Z} \subseteq \{12a + 25b, a, b \in \mathbb{Z}\}.$$

and we have equality.

Question 2 (40 points)

a) Prove by mathematical induction that

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Solution. We have $P(n) =$"$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$". The basis step is $P(1)$ which is true, since $1^2 = 1 = \frac{1}{6}1(1+1)(2+1)$. Suppose true for $P(k)$, that is $1^2+2^2+\ldots+k^2 = \frac{1}{6}k(k+1)(2k+1)$ holds. We need to prove that $P(k+1)$ is true, namely we need to prove that

$$1^2 + 2^2 + \ldots + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2(k+1)+1) = \frac{1}{6}(k+1)(k+2)(2k+3)$$

is true. So let us start by computing the left hand side:

$$\begin{aligned} 1^2 + 2^2 + \ldots + (k+1)^2 &= 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \end{aligned}$$

using the induction hypothesis. We continue to compute

$$\begin{aligned} 1^2 + 2^2 + \ldots + (k+1)^2 &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\ &= \frac{1}{6}(k+1)\left(2k^2 + k + 6k + 6\right) \end{aligned}$$

Since

$$\frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)\left(2k^2 + k + 6k + 6\right)$$

this concludes the proof.

b) How many subsets of $\{1, \ldots, n\}$ are there with an even number of elements? Justify your answer.
Solution. One way to solve this is to say that the total number of subsets of $\{1, \ldots, n\}$ is 2^n. Now this total number counts subsets of odd and even numbers of elements. The way we proved that the total number is 2^n is by noting that counting all subsets is adding the choice of k elements out of n, that is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = (1+1)^n$$

by using the binomial theorem:

$$(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k}.$$

Now use again the binomial theorem but this time with $x = -1$ and $y = 1$:

$$0 = \sum_{k=1}^{n} \binom{n}{k}(-1)^k = \sum_{k\ odd}^{n} \binom{n}{k}(-1) + \sum_{k\ even}^{n} \binom{n}{k}$$

This shows that

$$\sum_{k\ odd}^{n} \binom{n}{k} = \sum_{k\ even}^{n} \binom{n}{k}$$

or in other words the number of subsets with even number of elements is the the same as the number of subsets with odd number of elements, therefore we have

$$2^n/2 = 2^{n-1}$$

subsets of $\{1, \ldots, n\}$ with an even number of elements.

Question 3 (25 points)

Solve the following linear recurrence relation:

$$b_n = 4b_{n-1} - b_{n-2},\ b_0 = 2,\ b_1 = 4.$$

Solution. Since

$$x^n = 4x^{n-1} - x^{n-2} \iff x^{n-2}(x^2 - 4x + 1) = 0$$

The characteristic equation is

$$x^2 - 4x + 1 = 0.$$

The roots are

$$\frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

The general solution is

$$b_n = u(2+\sqrt{3})^n + v(2-\sqrt{3})^n.$$

The initial conditions tell us that

$$b_0 = u + v = 2,\ b_1 = u(2+\sqrt{3}) + v(2-\sqrt{3}) = 4.$$

Thus $u = 2 - v$ and

$$4 = (2-v)(2+\sqrt{3}) + v(2-\sqrt{3}) = 4 + 2\sqrt{3} - 2v - v\sqrt{3} + 2v - v\sqrt{3} = 4 + 2\sqrt{3} - 2v\sqrt{3}$$

showing that $2\sqrt{3} = 2v\sqrt{3}$ that is $v = 1$ and thus $u = 1$. The final solution is then

$$b_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n.$$