MH1820 Introduction to Probability and Statistical Methods

Tutorial 12 (Week 13) $\boxed{\text{Solution}}$

Problem 1 Let $X_1, \ldots, X_{10} \sim N(\mu, \sigma^2)$ be i.i.d, where μ and σ are both unknown. Consider a test for $H_0 : \mu = 10$ against $H_1 : \mu \neq 10$ based on the test statistic $T = \frac{\overline{X}-\mu}{s/\sqrt{n}}$, where s^2 is the sample variance. Suppose we reject H_0 if and only if $|T| \geq t_0$.

(a) Find t_0 so that the size of the test is 0.05.

(b) Using the t_0 from part (a), is H_0 rejected for the following observations?

$$23.3,\ 3.5, -1.0,\ 40.3,\ 34.5,\ 9.6,\ 23.4,\ 18.5,\ 0.7,\ 9.0.$$

Solution

(a) Here, we have $n = 10$, a small sample size. Recall that $T = \frac{\overline{X}-\mu}{s/\sqrt{n}} \sim t(n-1)$, where $t(n-1)$ is the t-distribution with $n-1$ degree of freedom.

We need to find t_0 so that

$$\begin{aligned}
\mathbb{P}(|T| \geq t_0|H_0) &= 0.05 \\
\iff \mathbb{P}(T \geq t_0|H_0) + \mathbb{P}(T \leq -t_0|H_0) &= 0.05 \\
\iff 2\mathbb{P}(T \geq t_0|H_0) &= 0.05 \\
\iff \mathbb{P}(T \geq t_0|H_0) &= 0.025 \\
\iff t_0 = t_{0.025}(n-1) = t_{0.025}(9) &= 2.262.
\end{aligned}$$

(b) The sample mean and sample variance for the observations are

$$\overline{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = 16.18, \quad s = \sqrt{\frac{1}{9}\sum_{i=1}^{10}(x_i - \overline{x})^2} \approx 14.2.$$

Thus, $t = \frac{16.18-10}{14.2/\sqrt{10}} = 1.376$. Since $|t| \not\geq t_0 = 2.262$, we will not reject H_0. $\square$

3.5, -1,

Problem 2 Let $X_1, \ldots, X_{10}$ be an i.i.d sample drawn from $\text{Exp}(\theta)$ where $\theta \in (0, \infty)$ is an unknown parameter. Consider a test for $H_0 : \theta = 1$ against $H_1 : \theta = \frac{1}{2}$ based on the test statistic $T = \sum_{i=1}^{n} X_i$.

(a) Find the observed value t_0 of T such that the p-value $\mathbb{P}(T \leq t_0|H_0)$ is equal to 0.05.

(b) Using the t_0 from part (a), consider that test that rejects H_0 if and only if $T \leq t_0$. What is the size and the power of the test?

(c) Using the test from part (b), is H_0 rejected for the following observations?

$$0.1, 0.2, 0.1, 0.3, 0.5, 0.01, 1.2, 0.05, 0.001, 0.1$$

You may use the following online calculator for Gamma distribution (Notation: the shape parameter β on the website is our θ for Gamma distribution). `https://homepage.divms.uiowa.edu/~mbognar/applets/gamma.html`

Solution (a) We know that the Exp(θ) distribution is the same as Gamma$(1, \theta)$. The MGF of $X \sim \Gamma(\alpha, \theta)$ is $M_X(t) = (1-\theta t)^{\alpha}$. Since $X_i \sim \Gamma(1, \theta)$ are i.i.d, we have $M_T(t) = M_{X_1+\cdots+X_{10}}(t) = \prod_{i=1}^{10}(1-\theta t)^{-1} = (1-\theta t)^{-10}$, and so $T = \sum_{i=1}^{10} X_i \sim \Gamma(10, \theta)$.

Thus, under H_0, we have $T \sim$ Gamma$(10, 1)$. By the requirement in the problem, we want $\mathbb{P}(T \leq t_0|H_0) = 0.05$ and hence $F(t_0) = 0.05$ where F is the CDF of Gamma$(10, 1)$. Using a calculator or software, we find $t_0 \approx 5.425$.

(b) The size of the test is

$$\mathbb{P}(H_0 \text{ is rejected} \mid H_0) = \mathbb{P}(T \leq t_0|H_0) = 0.05.$$

Under H_1 we have $T \sim$ Gamma$(10, 1/2)$ and hence the power is

$$P(H_0 \text{ is rejected}|H_1) \quad = P(T \leq 5.425|H_1) = G(5.424),$$

where G is the CDF of Gamma$(10, 1/2)$. Using a calculator or software, we see that the power is ≈ 0.643.

(c) For the given observations,

$$t = \sum_{i=1}^{10} x_i = 2.561 < 5.425.$$

Therefore, H_0 is rejected. $\square$

Problem 3 Let $X_1, \ldots, X_5$ be an i.i.d sample drawn from Bernoulli(p) where $p \in [0, 1]$ is an unknown parameter. Consider the test for $H_0 : p = 0.2$ against $H_1 : p = 0.5$ which rejects H_0 if and only if $\sum_{i=1}^{5} X_i > 2$.

(a) Compute the probabilities for Type-I and Type-II Errors

(b) Find the size and the power of the test.

Solution Set $Y = \sum_{i=1}^{5} X_i$. We have $Y \sim$ Binomial$(5, p)$.

(a) First, we calculate the probability for a Type-I Error:

$$\begin{aligned}
\alpha &= \mathbb{P}(H_0 \text{ is rejected} \mid H_0) \\
&= \mathbb{P}(Y > 2 \mid p = 0.2) \\
&= 1 - \mathbb{P}(Y = 0|p = 0.2) - \mathbb{P}(Y = 1|p = 0.2) - \mathbb{P}(Y = 2|p = 0.2) \\
&= 1 - 0.8^5 - 5 \times 0.2 \times 0.8^4 - \binom{5}{2} \times 0.2^2 \times 0.8^3 \\
&= 0.05792.
\end{aligned}$$

The probability for a Type-II Error is

$$\begin{aligned}
\beta &= \mathbb{P}(H_0 \text{ is not rejected} \mid H_1) \\
&= \mathbb{P}(Y \leq 2 \mid p = 0.5) \\
&= \mathbb{P}(Y = 0|p = 0.5) + \mathbb{P}(Y = 1|p = 0.5) + \mathbb{P}(Y = 2|p = 0.5) \\
&= 0.5^5 + 5 \times 0.5 \times 0.5^4 + \binom{5}{2} \times 0.5^2 \times 0.5^3 \\
&= 0.5.
\end{aligned}$$

(b) The size of the test is $\alpha = 0.05792$. The power of the test is $1 - \beta = 0.5$. □

Answer Keys. **Q1(a)** $t_0 = 2.262$ **Q1(b)** Do not reject H_0 **Q2(a)** $t_0 \approx 5.425$ **Q2(b)** 0.643 **Q2(c)** Reject H_0 **Q3(a)** 0.05792, 0.5 **Q3(b)** 0.05792, 0.5