Name:

Matric. no.: Tutor group:

February 2023 **CA1** TIME ALLOWED: 50 minutes

QUESTION 1. **(15 marks)**

(a) [5 marks] For each element $x \in \{0, \ldots, 4\}$ find which element of $\{0, \ldots, 4\}$ is congruent to x^2 modulo 5. Fill in the following table accordingly.

$x \pmod 5$	0	1	2	3	4
$x^2 \pmod 5$	0	1	4	4	1

[mark distribution: 1 mark for each correct table entry]

(b) [5 marks] For each $a \in \{0, 1, \ldots, 4\}$ evaluate the truth value of the following statement

$$\exists x \in \{0, 1, \ldots, 4\} \text{ such that } 2x - 1 \equiv a \pmod 5.$$

Fill in the following table accordingly.

a	0	1	2	3	4
T/F	T	T	T	T	T

[mark distribution: 1 mark for each correct table entry]

(c) [5 marks] Let S be the set of integers that are congruent to 2 modulo 4. Is S closed under multiplication? Justify your answer.

Solution: Not closed. Indeed $2 \in S$ but $2 \times 2 = 4 \equiv 0 \pmod 4$.

[mark distribution: 2 marks for correct answer, 3 marks for justification]

For graders only:	Question	1(a)	1(b)	1(c)	2(a)	2(b)	2(c)	3(a)	3(b)	3(c)	Total
	Marks										

QUESTION 2. **(15 marks)**

Let $\mathbb{Q}$ denote the set of rational numbers and S denote the set of odd integers. Determine the truth value of the following statements. Justify your answers.

(a) [5 marks] $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ \exists z \in S,\ xyz = 2023$;

Solution: False. Counterexample: $x = 0$ then $xyz = 0$ for all y and z.

[mark distribution: 2 marks for correct answer, 3 marks for justification]

(b) [5 marks] $\exists x \in S,\ \exists y \in \mathbb{Q},\ \forall z \in S,\ x + yz = 2023$;

Solution: True. Example: $x = 2023$ and $y = 0$ then $x + yz = 2023$ for all z.

[mark distribution: 2 marks for correct answer, 3 marks for justification]

(c) [5 marks] $\neg(\forall x \in S,\ \forall y \in \mathbb{Q},\ \exists z \in S,\ xy + z = 2023)$.

Solution: True.

We have

$\neg(\forall x \in S,\ \forall y \in \mathbb{Q},\ \exists z \in S,\ xy + z = 2023) \equiv \exists x \in S,\ \exists y \in \mathbb{Q},\ \forall z \in S,\ xy + z \neq 2023.$

Example: take $x = 1$ and $y = 1$ then since z is odd, we have $xy + z \equiv 0 \pmod 2$. Since $2023 \equiv 1 \pmod 2$, we know that $xy + z \neq 2023$ for all $z \in S$.

[mark distribution: 2 marks for correct answer, 3 marks for justification]

QUESTION 3. **(20 marks)**

(a) [5 marks] Write out the truth table for the compound propositions $(\neg p \to q) \vee \neg q$ and $\neg(q \wedge p) \vee q$. Is $(\neg p \to q) \vee \neg q \equiv \neg(q \wedge p) \vee q$?

Solution:

p	q	$(\neg p \to q) \vee \neg q$	$\neg(q \wedge p) \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	T	T

Therefore, $(\neg p \to q) \vee \neg q \equiv \neg(q \wedge p) \vee q$.

[mark distribution: 1/2 mark for each correct entry in the last two columns of the table, 1 mark for deducing equivalence.]

(b) [5 marks] Identify all the critical rows for the argument:

$$\begin{aligned} &\neg p \to q; \\ &r \to q; \\ &s \to \neg p; \\ &\neg s; \\ \therefore\ &\neg r. \end{aligned}$$

Solution:

p	q	r	s
T	F	F	F
T	T	T	F
T	T	F	F
F	T	T	F
F	T	F	F

[mark distribution: 1 mark for each critical row]

(c) For each of the following arguments, decide whether or not it is valid. If it is invalid give a counterexample, if it is valid then demonstrate how the conclusion follows from the premises, pointing out which inference rule you are using at each step. You may need the following inference rules: modus ponens, modus tollens, and disjunctive syllogism.

(i) [5 marks]

$$\begin{aligned} & p \wedge q; \\ & \neg r \rightarrow s; \\ & q \vee r; \\ & p \vee s; \\ \therefore\ & r. \end{aligned}$$

Solution: Invalid. Counterexample:

p	q	r	s
T	T	F	T

[mark distribution: 2 marks for correct validity, 3 marks for justification]

(ii) [5 marks]

$$\begin{aligned} & \neg p \rightarrow r; \\ & r \rightarrow s; \\ & \neg p \vee q; \\ & \neg q; \\ \therefore\ & s. \end{aligned}$$

Solution: Valid. Justification:

1.	$\neg p \rightarrow r;$	
2.	$r \rightarrow s;$	
3.	$\neg p \vee q;$	
4.	$\neg q;$	
$\therefore$ 5.	$\neg p$	disjunctive syllogism on 3 and 4
$\therefore$ 6.	r	modus ponens on 1 and 5
$\therefore$	s	modus ponens on 2 and 6

[mark distribution: 2 marks for correct validity, 3 marks for justification]