NANYANG TECHNOLOGICAL UNIVERSITY

MIDTERM I (CA1)

MH1812 - Discrete Mathematics

February 2017		TIME ALLOWED: 40 minutes		
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Name:				
Matric. no.	:		Tutor group:	

INSTRUCTIONS TO CANDIDATES

- 1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
- 2. This midterm paper contains **THREE** (3) questions.
- 3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
- 4. Candidates can write anywhere on this midterm paper.
- 5. This **IS NOT** an **OPEN BOOK** exam.
- 6. Candidates should clearly explain their reasoning when answering each question.

MH1812

QUESTION 1.

(40 marks)

- (a) Which integer $a \in \{0, 1, 2\}$ is congruent to 2017 + 2020 + 2023 modulo 3? (10 marks)
- (b) Write down each integer $a \in \{0, 1, 2, 3\}$ for which there exists an integer n such that $a \equiv n^2 \pmod{4}$? (10 marks)
- (c) Decide whether the set S is closed under the operation Δ when
 - $S = \{\text{even integers}\}\ \text{and}\ \Delta \text{ is subtraction.}\ (10 \text{ marks})$
 - $S = \{\text{irrational numbers}\}\$ and Δ is multiplication. (10 marks)

Briefly justify your answers.

Solution:

(a) We have

$$2017 + 2020 + 2023 = 2017 + (2017 + 3) + (2017 + 2 \cdot 3)$$
$$= 3 \cdot 2017 + 3 \cdot 3.$$

So 2017 + 2020 + 2023 is divisible by 3, hence $2017 + 2020 + 2023 \equiv 0 \pmod{3}$.

- (b) Let n be an integer. Then n is congruent to either 0, 1, 2, or 3 modulo 4.
 - If $n \equiv 0 \pmod{4}$ then $n^2 \equiv 0^2 \equiv 0 \pmod{4}$;
 - if $n \equiv 1 \pmod{4}$ then $n^2 \equiv 1^2 \equiv 1 \pmod{4}$;
 - if $n \equiv 2 \pmod{4}$ then $n^2 \equiv 2^2 \equiv 4 \equiv 0 \pmod{4}$;
 - and if $n \equiv 3 \pmod{4}$ then $n^2 \equiv 3^2 \equiv 9 \equiv 1 \pmod{4}$.

So the only $a \in \{0, 1, 2, 3\}$ for which there exists an integer n such that $a \equiv n^2 \pmod{4}$ are 0 and 1.

(c) \bullet $S = \{\text{even integers}\}$ and Δ is subtraction. Here S is closed under Δ : Take two elements $a, b \in S$; both a and b are even integers so a = 2a' and b = 2b' for some integers a' and b'. Now a - b = 2a' - 2b' = 2(a' - b') is also an even integer.

• $S = \{\text{irrational numbers}\}\$ and Δ is multiplication. Here S is not closed under Δ . Indeed, $\sqrt{2}$ is irrational and $\sqrt{2} \cdot \sqrt{2} = 2$ is not irrational. In other words, $\sqrt{2} \in S$ and $\sqrt{2} \cdot \sqrt{2} \not \in S$.

QUESTION 2.

(40 marks)

(a) Prove or disprove the following statement (20 marks):

$$(p \land \neg q) \to r \equiv (p \land \neg r) \to q.$$

(b) Decide whether or not the following argument is valid (20 marks):

$$p \rightarrow q;$$

 $\neg p \rightarrow r;$
 $\neg (r \land q);$
 $\therefore \neg p$

Briefly justify your answer.

Solution:

(a)

$(p \land \neg q) \to r \equiv \neg (p \land \neg q) \lor r$	conversion theorem	
$\equiv (\neg p \vee \neg (\neg q)) \vee r$	De Morgan	
$\equiv (\neg p \vee q) \vee r$	double negation	
$\equiv \neg p \lor (q \lor r)$	associativity	
$\equiv \neg p \lor (r \lor q)$	commutativity	
$\equiv (\neg p \lor r) \lor q$	associativity	
$\equiv \neg(p \land \neg r) \lor q$	De Morgan	
$\equiv (p \land \neg r) \to q$	conversion theorem	

Alternatively, one can prove the statement using a truth table.

- (b) The argument is invalid. We want to find when the conclusion is false and all premises are true. The conclusion being false implies p is true. Now with p true, we want to make all the premises true.
 - (i) With p true and $p \to q$ true we must have that q is true (modus ponens).
 - (ii) Since $\neg p$ is false $\neg p \rightarrow r$ is true regardless of the truth value of r.
 - (iii) To make $\neg(r \land q)$ true, we need r false, since q is true.

So we have a counterexample: p true, q true, and r false.

QUESTION 3.

(20 marks)

Consider the domains $X=\mathbb{Z}=\{\text{integers}\}$ and $Y=\{0,1,2\}$, and the predicate P(x,y)= "3 divides x-y".

Determine the truth values of the following statements:

- (a) $\forall x \in X, \exists y \in Y, P(x, y); (10 \text{ marks})$
- (b) $\neg (\forall y \in Y, \exists x \in X, \neg P(x, y))$. (10 marks)

Briefly justify your answers.

Solution:

- (a) For all $x \in X$ we can write x = 3k + 0, x = 3k + 1, or x = 3k + 2 for some integers k.
 - If x = 3k + 0 then 3 divides x y for y = 0;
 - if x = 3k + 1 then 3 divides x y for y = 1;
 - if x = 3k + 2 then 3 divides x y for y = 2.

Hence the statement is true.

- (b) Let us check the statement $\forall y \in Y, \exists x \in X, \neg P(x, y).$
 - If y = 0 then 3 does not divide x y for x = 1;
 - if y = 1 then 3 does not divide x y for x = 2;
 - if y = 2 then 3 does not divide x y for x = 1.

Therefore the statement $\forall y \in Y, \exists x \in X, \neg P(x, y)$ is true and its negation is false. So $\neg (\forall y \in Y, \exists x \in X, \neg P(x, y))$ is false.