1. Bivariate Distribution (Joint PDF, CDF and Marginal PDF)

2. Conditional Distributions

Bivariate Distribution (Joint PDF, CDF and Marginal PDF)

For continuous bivariate distributions, the definitions are really the same as the those in the discrete case except that integrals replace summations.

The **joint probability density function (joint PDF)** of two continuous-type random variables is an integrable function $f(x, y)$ with the following properties:

(a) $f(x, y) \geq 0$.

(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$

(c) $\mathbb{P}((X, Y) \in A) = \iint_A f(x, y)\, dx\, dy$, where A is an event defined by a region on the xy-plane.

The **joint cumulative density function (joint CDF)** of X and Y is given by

$$F(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\, dt\, ds,$$

where $f(x,y)$ is the joint PDF of X and Y.

The respective **marginal PDF** of continuous-type random variables X and Y are given by

- $$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy,$$
- $$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

Example 1

Let X and Y have the joint PDF

$$f(x,y) = \frac{4}{3}(1 - xy), \quad 0 \le x \le 1, 0 \le y \le 1.$$

Find

(a) the marginal PDFs $f_X(x)$ and $f_Y(y)$;

(b) $\mathbb{P}(Y \le X/2)$;

(c) the mean and variance of X.

Solution.
(a)

$$f_X(x) = \int_0^1 \frac{4}{3}(1 - xy)\, dy = \frac{4}{3}\left[y - \frac{xy^2}{2}\right]_0^1 = \frac{4}{3}\left(1 - \frac{x}{2}\right).$$

$$f_Y(y) = \int_0^1 \frac{4}{3}(1 - xy)\, dx = \frac{4}{3}\left[x - \frac{x^2 y}{2}\right]_0^1 = \frac{4}{3}\left(1 - \frac{y}{2}\right).$$

(b)

$$
\begin{aligned}
\mathbb{P}(Y \leq X/2) &= \int_0^1 \int_0^{x/2} \frac{4}{3}(1 - xy)\, dy\, dx \\
&= \frac{4}{3} \int_0^1 \left[y - \frac{xy^2}{2} \right]_0^{x/2} dx \\
&= \frac{4}{3} \int_0^1 \frac{x}{2} - \frac{x^3}{8}\, dx \\
&= \frac{7}{24}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\text{Mean of } X = \mu_X &= \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \\
&= \int_0^1 x \frac{4}{3}\left(1 - \frac{x}{2}\right)\, dx \\
&= \frac{4}{3}\left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^1 \\
&= \frac{4}{9}.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[X^2] - \mu_X^2 \\
&= \int_0^1 x^2 \frac{4}{3}\left(1 - \frac{x}{2}\right)\, dx - \left(\frac{4}{9}\right)^2 \\
&= \frac{4}{3}\left[\frac{x^3}{3} - \frac{x^4}{8}\right]_0^1 - \left(\frac{4}{9}\right)^2 \\
&= \frac{13}{162}.
\end{aligned}
$$

□

Example 2

Let X and Y have the joint PDF

$$f(x, y) = 2, \quad \text{for } 0 \leq x \leq y \leq 1.$$

(a) Find $\mathbb{P}(X \leq \frac{1}{2}, Y \leq \frac{1}{2})$.

(b) Find the marginal PDFs $f_X(x)$, $f_Y(y)$.

Solution. The condition that $0 \leq x \leq y \leq 1$ means that $f(x, y) = 2$ whenever (x, y) comes from the triangular region bounded by x-axis, the line $y = x$ and vertical line $x = 1$; and $f(x, y) = 0$ otherwise.

(a)

$$
\begin{aligned}
\mathbb{P}\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right) &= \mathbb{P}\left(0 \leq X \leq \frac{1}{2}, X \leq Y \leq \frac{1}{2}\right) \\
&= \int_0^{1/2} \int_x^{1/2} 2 \, dy \, dx \\
&= \int_0^{1/2} [2y]_x^{1/2} \, dx \\
&= \int_0^{1/2} 1 - 2x \, dx \\
&= \left[x - x^2\right]_0^{1/2} \\
&= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.
\end{aligned}
$$

□

(b)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_x^1 2\, dy = 2(1-x), \quad 0 \le x \le 1.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^y 2\, dx = 2y, \quad 0 \le y \le 1.$$

$\square$

Conditional Distributions

Conditional Distributions

Suppose $f(x, y)$ is the joint PMF/PDF of X and Y, and $f_X(x)$ and $f_Y(y)$ are the marginal PMFs/PDFs.

- The **conditional PMF/PDF** of X, given that $Y = y$, is defined by

$$g(x|y) = \frac{f(x, y)}{f_Y(y)}.$$

- The **conditional PMF/PDF** of Y, given that $X = x$, is defined by

$$h(y|x) = \frac{f(x, y)}{f_X(x)}.$$

We can use conditional PDF/PMF to compute conditional probabilities.

Discrete case:

$$\mathbb{P}(a \le X \le b | Y = y) = \sum_{x: a \le x \le b} g(x|y).$$

$$\mathbb{P}(a \le Y \le b | X = x) = \sum_{y: a \le y \le b} h(y|x).$$

Continuous case:

$$\mathbb{P}(a \le X \le b | Y = y) = \int_a^b g(x|y)\, dx.$$

$$\mathbb{P}(a \le Y \le b | X = x) = \int_a^b h(y|x)\, dy.$$

- **Conditional mean** of Y given $X = x$:

$$\mu_{Y|x} = \mathbb{E}[Y|X = x]$$

- **Conditional variance** of Y given $X = x$:

$$\sigma^2_{Y|x} = \mathbb{E}[Y^2|X = x] - \left(\mu_{Y|x}\right)^2$$

Remark: $\mu_{X|y}$ and $\sigma^2_{X|y}$ are defined similarly.

Example 3

Suppose X and Y have the joint PMF

$$p(x, y) = \frac{x + y}{21}, \quad x = 1, 2, 3, \quad y = 1, 2.$$

(a) Find the conditional PMF $g(x|y)$ of X given $Y = y$, and $h(y|x)$ of Y given $X = x$.

(b) Find $\mu_{Y|x}$ and $\sigma^2_{Y|x}$ when $x = 3$.

Solution. (a) Note that

$$p_Y(y) = \sum_x p(x, y) = \frac{1 + y}{21} + \frac{2 + y}{21} + \frac{3 + y}{21} = \frac{2 + y}{7}.$$

$$p_X(x) = \sum_y p(x, y) = \frac{x + 1}{21} + \frac{x + 2}{21} = \frac{2x + 3}{21}.$$

Hence,

$$g(x|y) = \frac{p(x, y)}{p_Y(y)} = \frac{(x + y)/21}{(2 + y)/7} = \frac{x + y}{6 + 3y}.$$

$$h(y|x) = \frac{p(x, y)}{p_X(x)} = \frac{(x + y)/21}{(2x + 3)/21} = \frac{x + y}{2x + 3}.$$

(b)

$$\begin{aligned}
\mu_{Y|x} &= \mathbb{E}[Y|X=x] \\
&= \sum_y yh(y|x) \\
&= 1 \cdot h(1|x) + 2 \cdot h(2|x) \\
&= 1 \cdot \frac{x+1}{2x+3} + 2 \cdot \frac{x+2}{2x+3} = \frac{3x+5}{2x+3}.
\end{aligned}$$

Hence,

$$\mu_{Y|3} = \frac{3(3)+5}{2(3)+3} = \frac{14}{9}.$$

$$
\begin{aligned}
\sigma^2_{Y|3} &= \mathbb{E}[Y^2|X=3] - \mu^2_{Y|3} \\
&= \sum_y y^2 h(y|3) - (14/9)^2 \\
&= 1^2 \cdot h(1|3) + 2^2 \cdot h(2|3) - (14/9)^2 \\
&= 1 \cdot \frac{3+1}{2(3)+3} + 2^2 \cdot \frac{3+2}{2(3)+3} - (14/9)^2 = \frac{20}{81}.
\end{aligned}
$$

$\square$

Example 4

Let X and Y have the joint PDF

$$f(x, y) = 2, \quad \text{for } 0 \leq x \leq y \leq 1.$$

Find

(a) the conditional mean of Y given $X = x$.

(b) the conditional variance of Y given $X = x$.

(c) $\mathbb{P}\left(\frac{3}{4} \leq Y \leq \frac{7}{8} \ |X = \frac{1}{4}\right)$

Solution. (a) By Example 2, the marginal PDF of X is $f_X(x) = 2(1-x)$, $0 \le x \le 1$. So the conditional PDF of Y given $X = x$ is

$$h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}, \quad 0 \le x \le 1, x \le y \le 1.$$

The conditonal mean of Y given $X = x$ is

$$\begin{aligned}
\mu_{Y|x} &= \mathbb{E}[Y|X = x] \\
&= \int_x^1 y \frac{1}{1-x}\, dy \\
&= \frac{1}{1-x}\left[\frac{y^2}{2}\right]_x^1 \\
&= \frac{1}{1-x}\left(\frac{1}{2} - \frac{x^2}{2}\right) \\
&= \frac{1+x}{2},
\end{aligned}$$

for $0 \leq x \leq 1$.

(b) The conditional variance is

$$\begin{aligned}
\sigma_{Y|x}^2 &= \mathbb{E}[Y^2|X=x] - \mu_{Y|x}^2 \\
&= \int_x^1 y^2 \frac{1}{1-x}\,dy - \left(\frac{1+x}{2}\right)^2 \\
&= \frac{1}{1-x}\left(\frac{1}{3} - \frac{x^3}{3}\right) - \frac{(1+x)^2}{4} \\
&= \frac{(1-x)^2}{12},
\end{aligned}$$

for $0 \le x \le 1$.

(c)

$$\begin{aligned}
\mathbb{P}\left(\frac{3}{4} \leq Y \leq \frac{7}{8} \ \middle| X = \frac{1}{4}\right) &= \int_{3/4}^{7/8} h(y|1/4)\, dy \\
&= \int_{3/4}^{7/8} \frac{1}{1-(1/4)}\, dy \\
&= \frac{4}{3}\int_{3/4}^{7/8} 1\, dy \\
&= \frac{4}{3}\cdot\frac{1}{8} = \frac{1}{6}.
\end{aligned}$$

□

Example 5

Let X have a uniform distribution $U(0,2)$, i.e. $f_X(x) = 1/2$ if $0 < x < 2$ and 0 otherwise. Let the conditional distribution of Y, given that $X = x$, be $U(0, x^2)$.

(a) Find the joint PDF $f(x,y)$ of X and Y. Sketch the region where $f(x,y) > 0$.

(b) Find the marginal PDF $f_Y(y)$ of Y.

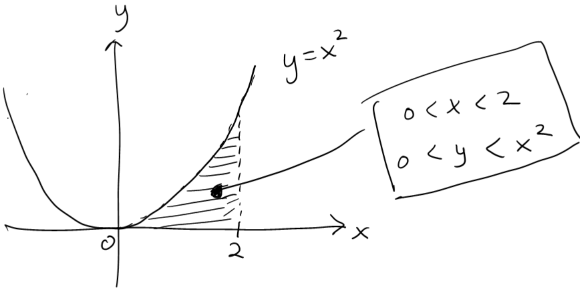
Solution. (a) Given $0 < x < 2$, we have

$$\frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{1}{x^2} & 0 < y < x^2 \\ 0 & \text{otherwise.} \end{cases}$$

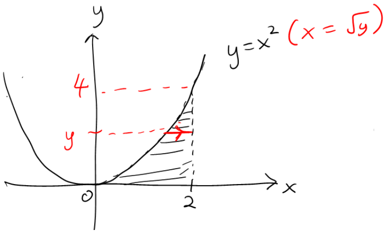
Since $f_X(x) = 1/2$ if $0 < x < 2$, and 0 otherwise, we have

$$f(x,y) = \begin{cases} \frac{1}{2x^2} & 0 < y < x^2 \\ 0 & \text{otherwise,} \end{cases}$$

for $0 < x < 2$.



y
$y=x^2$
$0 < x < 2$
$0 < y < x^2$
0
2
x



(b)

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{\sqrt{y}}^{2} \frac{1}{2x^2}\,dx \\
&= \left[-\frac{1}{2}x^{-1}\right]_{\sqrt{y}}^{2} \\
&= -\frac{1}{4} + \frac{1}{2\sqrt{y}} = \frac{2-\sqrt{y}}{4\sqrt{y}},
\end{aligned}
$$

for $0 < y < 4$.