MH1820 Week 6

1. Moment generating functions
2. Bivariate distribution (Joint PMF, CDF and Marginal PMF)

(discrete case) for today.

e.g $\mathbb{P}(X=x, Y=y)$.

Moment generating functions

applies to both discrete
& continuous,

- $\mathbb{E}[X]$ mean <u>1st</u> moment
- $\mathbb{E}[X^2]$ 2nd moment.

e.g $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

- $\mathbb{E}[X^n]$ n-th moment.

fully characterize distribution of X.

Recall: school:

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + \cdots$$

$$\mathbb{E}[e^{tX}] = 1 + t\,\underbrace{\mathbb{E}[X]} + \frac{t^2}{2!}\underbrace{\mathbb{E}[X^2]} + \cdots$$

moment generating function.

Moment generating functions

- A distribution of a random variable X is determined by its CDF or by its PDF/PMF.
- If a random variable is defined by some expression (e.g. $X = \frac{1}{10}(X_1 + \cdots + X_{10})$, then it may be tedious to compute the CDF/PDF/PMF directly.
- Moment generating functions sometimes can be used in these cases to identify the distribution of X indirectly in a much quicker way.

Recall: $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

$\uparrow$ PDF.

Let X be a random variable. Its **moment generating function (MGF)** is defined by

$$M_X(t) = \mathbb{E}[e^{tX}], \quad t \in \mathbb{R}.$$

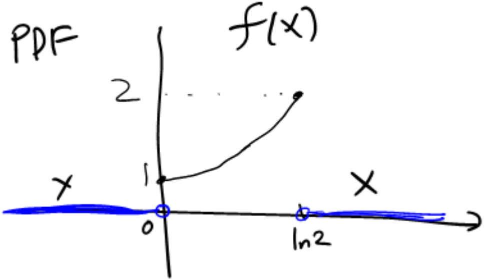
- Continuous case: $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$, where $f(x)$ is the PDF of X.
- Discrete case: $M_X(t) = \sum_x e^{tx} p(x)$, where $p(x)$ is the PMF of X.

$\mathbb{E}[g(X)] = \sum_x g(x) p(x)$

Example 1

Let X be a random variable with PDF $f(x) = e^x$ for $0 \leq x \leq \ln 2$, and $f(x) = 0$ otherwise. Compute the moment generating function of X.

Solution.



MGF $\quad M_X(t) = \mathbb{E}[e^{tX}]$

$$= \int_{-\infty}^{\infty} e^{tx} \underline{\underline{f(x)}}\, dx$$

$$= \int_0^{\ln 2} e^{tx} e^{x}\, dx$$

$$= \int_0^{\ln 2} e^{(t+1)x}\, dx$$

$$= \left[\frac{e^{(t+1)x}}{t+1} \right]_0^{\ln 2} = \frac{e^{(t+1)\ln 2}}{t+1} - \frac{1}{t+1}$$

$$= \frac{e^{\ln\left(2^{t+1}\right)} - 1}{t+1}$$

$$= \frac{2^{t+1} - 1}{t+1} \qquad \underline{\underline{(t \neq -1)}}$$

#.

Recall $e^{\ln a} = a$.

Example 2

Let X be a discrete random variable with PMF $p(x)$ given as follows:

x	0	1	2	3
$p(x)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

average = $\mathbb{E}[X]$

$= \cancel{0 \cdot \frac{3}{8}} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8}$.

$= 1$

We compute the moment generating function of X:

$$
\begin{aligned}
M_X(t) &= \sum_{x=0}^{3} e^{tx} p(x) \\
&= \frac{3}{8} e^{t \cdot 0} + \frac{3}{8} \cdot e^{t \cdot 1} + \frac{1}{8} \cdot e^{t \cdot 2} + \frac{1}{8} \cdot e^{t \cdot 3} \\
&= \frac{1}{8}(3 + 3e^t + e^{2t} + e^{3t}).
\end{aligned}
$$

$$M_X(t) = \frac{3}{8} + \frac{3}{8}e^t + \frac{e^{2t}}{8} + \frac{e^{3t}}{8}.$$

$$\frac{d\,M_X(t)}{dt} = \frac{3}{8}e^t + \frac{2}{8}e^{2t} + \frac{3e^{3t}}{8}$$

$$\frac{dM_X}{dt}(0) = \frac{3}{8} + \frac{2}{8} + \frac{3}{8} = 1$$

$\uparrow$

$t=0$

Theorem 3 (Properties of MGF – Part I)

Let X, Y be random variables with $M_X(t) < \infty$, $M_Y(t) < \infty$ for $-h < t < h$. Then

(a) $\mathbb{E}[X^n] = M_X^{(n)}(0)$, *where* $M_X^{(n)}(t) = \frac{d^n}{dt^n} M_X(t)$, *the n-th derivative of* $M_X(t)$.

(b) *(Inversion Theorem) If* $M_X(t) = M_Y(t)$ *for all* t, *then* X *and* Y *have the same distribution, i.e. they have the same CDF/PDF.*

$\mathbb{E}[X^n]$ is called the n-**th moment** of X. E.g. the first moment is the same as the expected value (or mean).

Theorem 4 (Properties of MGF – Part II)

(c) *If $Y = aX + b$, where $a, b \in \mathbb{R}$, then*

$$M_Y(t) = e^{tb} M_X(at)$$

(d) *If X and Y are independent, then*

$$M_{X+Y}(t) = M_X(t) M_Y(t).$$

$$\frac{X_1 + X_2 + \cdots + X_n}{n}.$$

MGFs of common distributions.

Distribution	MGF
$Bernoulli(p)$	$pe^t + 1 - p$
$Geom(p)$	$\dfrac{pe^t}{1-(1-p)e^t}$ for $t < -\ln(1-p)$
$Binomial(n, p)$	$(pe^t + 1 - p)^n$
$Poisson(\lambda)$	$e^{\lambda(e^t - 1)}$
$U(a, b)$	$\dfrac{e^{tb} - e^{ta}}{t(b-a)}$ for $t \neq 0$, 1 for $t = 0$
$N(\mu, \sigma^2)$	θ $e^{\mu t + \sigma^2 t^2/2}$ θ
$Gamma(\alpha, \theta)$	$(1-\theta t)^{-\alpha}$ for $t < \frac{1}{\theta}$
$Exp(\theta)$	$(1-\theta t)^{-1}$ for $t < \frac{1}{\theta}$

$\dfrac{1}{1-\theta t}$

Example 5

Let $X \sim Binomial(n, p)$. Show that the MGF of X is $(pe^t + 1 - p)^n$.

Solution. Let $Y \sim$ Bernoulli(p)

y	0	1
$p(y)$	$1-p$	p

$$M_Y(t) = \sum_{y=0}^{1} e^{ty} \cdot p(y) = e^0 \cdot p(0) + e^t p(1)$$
$$= 1 - p + pe^t$$
$$= \underline{pe^t + 1 - p}. \checkmark$$

$X \sim \text{Binomial}(n, p)$

$$X = \sum_{i=1}^{n} Y_i \qquad Y_i \sim \text{Bernoulli}(p)$$

(independent)

By property (d):

$$M_X(t) = M_{Y_1+Y_2+\cdots+Y_n}(t)$$

$$= M_{Y_1}(t) \cdot M_{Y_2}(t) \cdots M_{Y_n}(t)$$

$$= (pe^t+1-p)(pe^t+1-p)\cdots(pe^t+1-p)$$

$$= (pe^t+1-p)^n \; \#.$$

$X \sim \text{Binomial}(n, p)$

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\mathbb{E}[X^3] = \sum x^3 \underline{p(x)}$$

$$= \ldots\ldots\ldots = ??$$

Example 6

Let $X \sim Exp(\theta)$. Derive the MGF of X and use it to find the mean and variance of X.

Solution. Recall PDF $f(x) = \frac{1}{\theta} e^{-x/\theta}$ $\quad x \geq 0$

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} \frac{1}{\theta} e^{-x/\theta}\,dx$$

$$= \frac{1}{\theta}\int_0^{\infty} e^{(t-\frac{1}{\theta})x}\,dx$$

$$= \frac{1}{\theta}\left[\frac{e^{(t-\frac{1}{\theta})x}}{t-\frac{1}{\theta}}\right]_0^{\infty}$$

$$\left(t-\frac{1}{\theta}\neq 0.\ \text{i.e } t\neq\frac{1}{\theta}\right)$$

$$= \frac{1}{\theta}\left[0-\frac{1}{t-\frac{1}{\theta}}\right]$$

provided $t-\frac{1}{\theta}\neq 0$.

$$= \frac{1}{\cancel{\theta}}\left(-\frac{\cancel{\theta}}{t\theta-1}\right) = \frac{1}{1-t\theta}$$

provided $\boxed{t-\frac{1}{\theta}<0}$

$$e^{\left(t-\frac{1}{\theta}\right)\infty}.$$

$$e^{-0.001\cdot\infty} = e^{-\infty}$$

e.g

$$e^{-2\cdot\infty} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0.$$

$$e^{+2\infty} = +\infty \quad X$$

$$M_X(t) = (1-\theta t)^{-1}$$

Differentiating the MGF, we have $\frac{dM_X(t)}{dt} = -(1-\theta t)^{-2}(-\theta)$

$$M_X^{(1)}(t) = \frac{d}{dt}M_X(t) = \theta(1-\theta t)^{-2}$$

$$M_X^{(2)}(t) = \frac{d^2}{dt^2}M_X(t) = 2\theta^2(1-\theta t)^{-3}$$

By Property (a) of MGF, we have

$$\mathbb{E}[X] = M_X^{(1)}(0) = \theta, \quad \mathbb{E}[X^2] = M_X^{(2)}(\overset{\uparrow}{\underset{t=0}{0}}) = 2\theta^2.$$

The mean of X is $\mathbb{E}[X] = \theta$.

The variance of X is

$$\begin{aligned} \mathrm{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= 2\theta^2 - \theta^2 \\ &= \theta^2. \end{aligned}$$

Example 7

Suppose $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent. Use MGF to find the distribution of $X_1 + X_2$.

$\mathbb{P}(X_1 + X_2 \le \omega)$

Solution. From the Table of MGF,

$$M_{X_1}(t) = e^{\mu_1 t + \sigma_1^2 t^2/2}, \quad M_{X_2} = e^{\mu_2 t + \sigma_2^2 t^2/2}.$$

By Property (c) of MGF, $X = X_1 \quad Y = X_2$.

$$M_{X+Y}(t) = M_X(t) M_Y(t) = e^{(\mu_1+\mu_2)t + \frac{(\sigma_1^2+\sigma_2^2)}{2} t^2}$$

From the Table of MGF and Property (d) of MGF, we deduce that

$$X + Y \sim N\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right),$$

i.e. $X + Y$ is normally distributed with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. $\square$

$N(\mu, \sigma^2)$ MGF $e^{\mu t + \frac{\sigma^2}{2} t}$.

Bivariate distribution (Joint PMF, CDF and Marginal PMF)

PPF/PMF

$X \longrightarrow f_X(x) / P_X(x)$

$Y \longrightarrow f_Y(x) / P_Y(y)$

PDF/PMF

$X+Y$??

Motivating Example: 2 balls are drawn from a box which contains 2 blue, 3 red, and 4 yellow balls.

- X = number of blue balls drawn
- Y = number of red balls drawn

For each possible pair of values of (x, y), we are interested in the probability that $X = x$, $Y = y$ occur simultaneously, i.e.

$$\mathbb{P}(X = x, Y = y).$$

Here, we require $0 \leq x \leq 2$, $0 \leq y \leq 2$ and $0 \leq x + y \leq 2$.

$x = 0, 1, 2.$ $y = 0, 1, 2$

Given x blue, y red.

$$\mathbb{P}(X=x, Y=y) = \frac{\binom{2}{x}\binom{3}{y}\binom{4}{2-x-y}}{\binom{9}{2}}$$

The **joint PMF** of X and Y is given by

$$p(x, y) = \mathbb{P}(X = x, Y = y) = \frac{\binom{2}{x}\binom{3}{y}\binom{4}{2-x-y}}{\binom{9}{2}}$$

$$p(0,0) = \frac{\binom{2}{0}\binom{3}{0}\binom{4}{2}}{\binom{9}{2}} = \frac{1}{6}.$$

$x \backslash y$	0	1	2
0	$\boxed{\frac{1}{6}}$	$\frac{1}{3}$	$\frac{1}{12}$
1	$\frac{2}{9}$	$\frac{1}{6}$	0
2	$\frac{1}{36}$	0	0

$x+y=3$ X

$0 \le x+y \le 2$

The distribution given by the joint PMF is called the **joint distribution** of X and Y.

Let X, Y be discrete random variables.

- The **joint probability mass distribution (joint PMF)** of X and Y is defined by

$$p(x, y) = \mathbb{P}(X = x, Y = y).$$

- The **joint cumulative density function (joint CDF)** of X and Y is defined by

$$F(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \sum_{s \leq x} \sum_{t \leq y} p(s, t).$$

$$F(x) = \mathbb{P}(X \leq x) = \sum_{s \leq x} p(s).$$

$$\Omega = \left\{ (a,b) : \begin{array}{l} a = 1, 2, \ldots, 6 \\ b = 1, 2, \ldots 6 \end{array} \right\}$$

Example 8

Roll a pair of fair dice. For each of the 36 sample points with probability $1/36$, let X denote the smaller and Y the larger outcome on the dice. If both numbers of the dice are the same, then X and Y take on the same value.

Find the joint PMF of X and Y.

Joint PMF $p(x,y) = \mathbb{P}(X=x, Y=y)$.

Given (x, y),

Event smaller number is x & larger number is y. $= \{ (x,y), (y,x) \}$.

$\underline{\underline{x<y}}$.

$$\mathbb{P}(X=x, Y=y) = \begin{cases} \frac{2}{36} & \text{if } x<y. \\ 0 & \text{if } x>y \end{cases}$$

Event that $x=y$ $= \{(x,x)\}$

$$\mathbb{P}(X=x, Y=y) = \frac{1}{36}. \quad \text{if } x=y.$$

$x \backslash y$	1	2	3	4	5	6	$P_X(x)$
1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$11/36$
2	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$9/36$
3	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$7/36$
4	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$5/36$
5	0	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$3/36$
6	0	0	0	0	0	$\frac{1}{36}$	$1/36$
$P_Y(y)$							

Let X and Y have the joint probability mass function $f(x, y)$.

- The probability mass function of X alone, which is called the **marginal probability mass function** of X, is defined by

$$p_X(x) = \sum_y p(x, y) = \mathbb{P}(X = x).$$

$p_X(x)$, $p_Y(y)$ are the called the marginal PMF of X and Y.

- If $u(X, Y)$ is a function of X and Y, then

$$\mathbb{E}[u(X, Y)] = \sum_x \sum_y u(x, y) p(x, y)$$

is the **expected value** of $u(X, Y)$.

$$\mathbb{E}[u(x)] = \sum_x u(x) p(x)$$

Theorem 9 (Independence via marginals)

The random variables X and Y are **independent** *if and only if*

$$p(x, y) = p_X(x)p_Y(y) \quad \text{for all } x,\ y.$$

Otherwise, X and Y are said to be dependent.

	y	$p_X(x)$
x	$p(x,y)$	$p_X(x)$
$p_Y(y)$	$p_Y(y)$	

Example 10

A dice is rolled 2 times. Let

- X = number of rolls that are 1
- Y = number of rolls that are 2

$x = 0, 1, 2$

$y = 0, 1, 2$

(i) Find $F(2, 1)$ where $F(x, y)$ is the joint CDF of X and Y.

(ii) Find the marginal PMF $p_X(x)$, where $p(x, y)$ is the joint PMF of X and Y.

(iii) Are X and Y independent?

PMF

$x \backslash y$	0	1	2
0	$\frac{16}{36}$.	$\frac{8}{36}$.	$\frac{1}{36}$.
1	$\frac{8}{36}$.	$\frac{2}{36}$	0
2	$\frac{1}{36}$.	0	0

$$p(0,0) = \mathbb{P}\mathrm{rob}\begin{pmatrix} \text{no roll is } 1 \\ \& \text{ no roll is } 2 \end{pmatrix}.$$

$$= \frac{4 \times 4}{36}$$

$$= \frac{16}{36}.$$

$$p(0,1) = \mathbb{P}\mathrm{rob}\begin{pmatrix} \text{no roll is } 1 \\ \& \ \underline{\underline{1}} \text{ roll is } \underline{\underline{2}} \end{pmatrix}$$

$$= \mathbb{P}\mathrm{rob}\left\{ (2, \underset{3,4,5,6}{\uparrow *}), (\underset{3,4,5,6}{\uparrow *}, 2) \right\} = \frac{8}{36}.$$

Solution. Joint PMF:

$x \setminus y$	0	1	2	$P_X(x)$
0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$	$\frac{25}{36}$.
1	$\frac{8}{36}$	$\boxed{\frac{2}{36}}$	0	$\frac{10}{36}$
2	$\frac{1}{36}$	0	0	$\frac{1}{36}$.
$P_Y(y)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$.	

(i)

$$\underline{\underline{F(2,1)}} = \mathbb{P}(\underline{\underline{X \le 2}}, \underline{\underline{Y \le 1}}) = \frac{16}{36} + \frac{8}{36} + \frac{8}{36} + \frac{2}{36} + \frac{1}{36} = \frac{35}{36}.$$

(ii) The marginal PMF of X is given by

$$p_X(x) = \sum_y p(x, y).$$

So

- $p_X(0) = \sum_y p(0, y) = \frac{16}{36} + \frac{8}{36} + \frac{1}{36} = \frac{25}{36}$.
- $p_X(1) = \sum_y p(1, y) = \frac{8}{36} + \frac{2}{36} + 0 = \frac{10}{36}$.
- $p_X(2) = \sum_y p(2, y) = \frac{1}{36} + 0 + 0 = \frac{1}{36}$.

$x = 2$
$y = 2$

(iii) Note that

$$p(2,2) = 0, \quad p_X(2) = \frac{1}{36}, \quad p_Y(2) = \frac{1}{36}.$$

Since

$$p(2,2) \neq p_X(2)p_Y(2),$$

the random variables X and Y are dependent.

$p(1,1) = \frac{2}{36}$ $\quad$ $p_X(1) = \frac{10}{36} = p_Y(1).$

$p(1,1) \neq p_X(1)\,p_Y(1)$