Examples of Hypothesis Testing

Procedue for Hypothesis Testing:

- Given are observations $x_1, \ldots, x_n$.
- Formulate **null hypothesis** H_0 describing the population distribution from which observations were drawn.
- Choose significance level α (often $\alpha = 0.05$)
- Choose test statistic $T(X_1, \ldots, X_n)$ that contains information on the parameters involved in H_0 and whose distribution is known under H_0.
- Assuming H_0, compute probability (p-value) to observe $t = T(x_1, \ldots, x_n)$ or something "**at least as extreme as t" (in the direction of rejection of H_0)**. depends on Alternative Hypothesis H_1.
- If the p-value is smaller than α, reject null hypothesis.

p-value

- p-value is the probability to observe t or something "at least as extreme as t" assuming H_0 is true.
- If p-value is small, it means that chances of observing what we have observed (assuming H_0 is true) is small.

$$\Longrightarrow \text{ the smaller the } p\text{-value, the less we should believe in } H_0.$$

- The significance level α is the minimum value of this probability that we are willing to accept before perfoming the test.

$$\Longrightarrow \begin{cases} \text{Reject } H_0 & \text{if } p\text{-value} < \alpha \\ \text{Do not reject } H_0 & \text{otherwise.} \end{cases}$$

Example 1

A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilograms.

Test the hypothesis that $\mu = 8$ kilograms against the alternative hypothesis that $\mu \neq 8$ kilograms if a sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms.

Use a 0.01 level of significance.

$\alpha = 0.01$.

The Setup:

- $X_1, X_2, \ldots, X_n \longrightarrow$ $\bar{X} = 7.8$ kg.

  $n = 50$ (large).

  know: $\sigma = 0.5$ kg

- $H_0: \mu = 8$ kg ✓
- $H_1: \mu \neq 8$ kg.
- $\alpha = 0.01$.
- Choose $T = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(\boxed{0}, 1)$ (CLT)

  $\mathbb{E}[T]$

Compute p-value based on data:

$$p\text{-value} = \mathbb{P}\left(|T - \cancelto{0}{\mathbb{E}[T]}| \geq \underline{|t - \cancelto{0}{\mathbb{E}[T]}|} \right).$$

$$= \mathbb{P}\left(|T| \geq |t| \right).$$

$$t = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{7.8 - 8}{0.5/\sqrt{50}} = -2.83.$$

$$p\text{-value} = \mathbb{P}(|T| \geq |-2.83|)$$

$$= \mathbb{P}(|T| \geq 2.83).$$

$$= \mathbb{P}(T \geq 2.83 \text{ or } T \leq -2.83)$$

$$= 2 \times \Phi(-2.83) = 2(1-\Phi(2.83))$$

$$= 2(1-0.9977)) = 0.0046.$$

Decision:

Reject H_0

because p-value $= 0.0046 < \alpha = 0.01$.

Example 2

Suppose that the distribution of X is *Bernoulli*(p). We shall test the null hypothesis $H_0 : p = 0.5$ against the alternative hypothesis $H_1 : p \neq 0.5$.

Suppose a random sample of $n = 100$ observations yielded $\sum_{i=1}^{100} x_i = 65$. Define a test statistic, calculate the p-value and state your conclusion using a significance level of $\alpha = 0.05$.

The Setup:

- $X_1, \ldots, X_n \longrightarrow \boxed{\sum_{i=1}^{100} X_i = 65}$ ✓ $\leftarrow t$
- $H_0 : p = 0.5$ ✓ $\quad p = \mathbb{E}[X_i]$
- $H_1 : p \neq 0.5$. $\quad p(1-p) = Var[X_i]$
  $\sigma = \sqrt{p(1-p)} = s.d(X_i)$ ✓
- $\alpha = 0.05$
- Choose $\boxed{T = \sum_{i=1}^{n} X_i} \sim \underbrace{Binomial(100, p)}_{\mathbb{E}[T] = 100p}$

Also: $\left(\frac{\sum_{i=1}^{100} X_i}{100} - p\right) \Big/ \sqrt{p(1-p)}/\sqrt{n} \sim N(0,1)$ (by CLT)

(assuming H_0 true)

Compute p-value based on data:

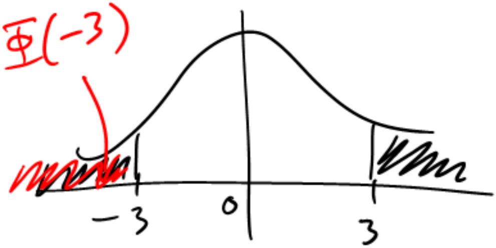
$$
\begin{aligned}
p\text{-value} &= \mathbb{P}\left(|T - \mathbb{E}[T]| \geq |t - \mathbb{E}[T]|\right) \\
&= \mathbb{P}\left(|T - 100(0.5)| \geq |t - 100(0.5)|\right). \\
&= \mathbb{P}\left(|T - 50| \geq |t - 50|\right). \\
&= \mathbb{P}\left(|T - 50| \geq |65 - 50|\right). \\
&= \mathbb{P}\left(|T - 50| \geq 15\right) \\
&= \mathbb{P}\left(T - 50 \geq 15 \text{ or } T - 50 \leq -15\right).
\end{aligned}
$$

$$= \mathbb{P}\left(T \geq 65 \text{ or } T \leq 35 \right)$$

$$= \mathbb{P}\left(\frac{\frac{T}{100} - p}{\sigma/\sqrt{n}} \geq \frac{\frac{65}{100} - p}{\sigma/\sqrt{n}} \text{ or } \frac{\frac{T}{100} - p}{\sigma/\sqrt{n}} \leq \frac{\frac{35}{100} - p}{\sigma/\sqrt{n}} \right)$$

$$= \mathbb{P}\left(\phi \geq \frac{0.65 - 0.5}{\sqrt{0.5(1-0.5)}/\sqrt{100}} \text{ or } \phi \leq \frac{0.35 - 0.5}{\sqrt{0.5(1-0.5)}/\sqrt{100}} \right)$$

$$= \mathbb{P}\left(\phi \geq 3, \text{ or } \phi \leq -3 \right).$$



$$
\begin{aligned}
p\text{-value} &= 2 \times \Phi(-3) \\
&\doteq 2 \times (1 - \Phi(3)) \\
&= 2 \times (1 - 0.9987) \\
&= 2 \times 0.0013 = 0.0026
\end{aligned}
$$

Decision:

Reject H_0

because p-value $= 0.0026 < \alpha = 0.05$