Example Class 3

Combinatorics & Linear Recurrences

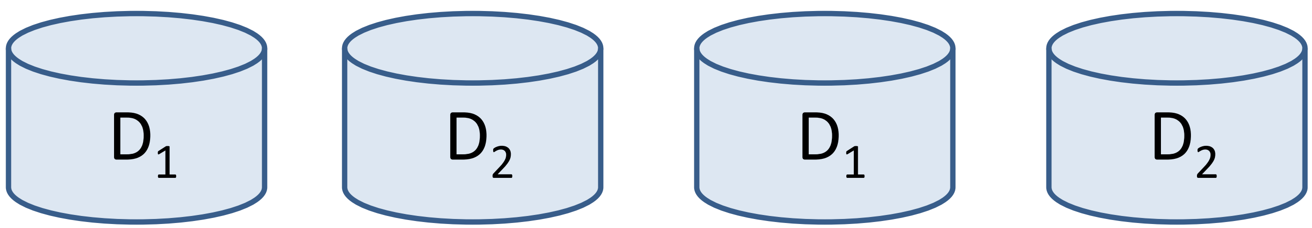
Outline

- Probability & data storage
- The Hat Puzzle
- Hanoi Tower

Example (I)

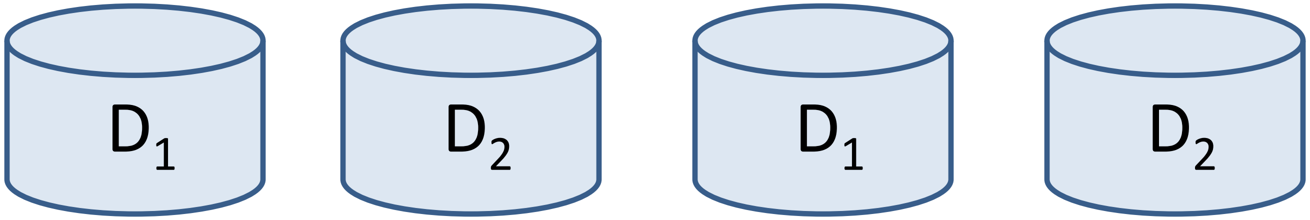
- Suppose you want to store 200GB of (binary) data
- Option 1: buy 4 disks of 100 GB each, store 2 copies of your data.

$D = (D_1, D_2)$



Example (II)

$D = (D_1, D_2)$



- If one hard disk fails, your data is safe.
- What is the probability of losing your data in case two hard disks fail?

Example (II)

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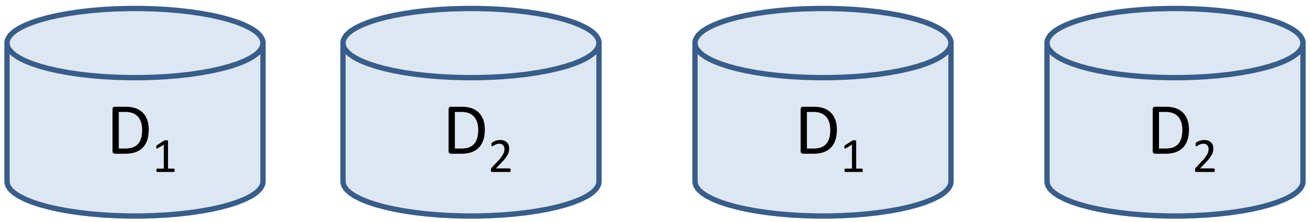
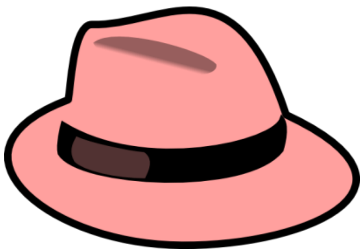


- If one hard disk fails, your data is safe.
- What is the probability of losing your data in case two hard disks fail?

$$\frac{2}{C(4,2)} = \frac{2}{6} = \frac{1}{3}$$

The Hat Problem



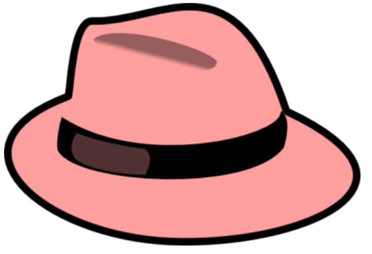
N players enter a room

- A red or blue hat is placed on each person's head.
  - P(red)=P(blue)=1/2, independently.
- Each player sees the other hats but not his own.
  - The players must simultaneously guess the color of their own hats or pass.
- Win *if at least one player guesses correctly and no players guess incorrectly*.
  - No communication is allowed, except for any initial strategy session before the game begins.

The Hat Problem (N=1)

N =1

- A red or blue hat is placed on the player 's head.

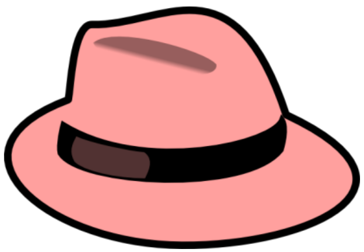


- Win:

The Hat Problem (N=1)

N =1

- A red or blue hat is placed on the player 's head.

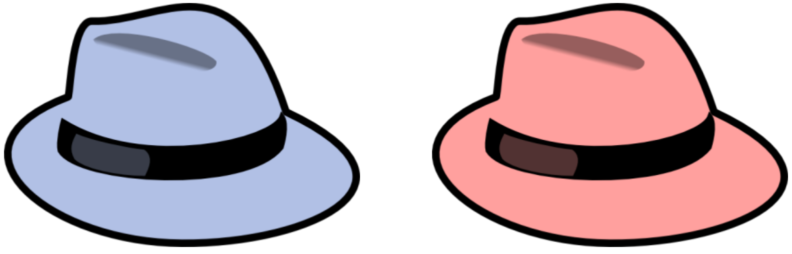


- Win: with probability ½.

The Hat Problem (N=2)

N =2

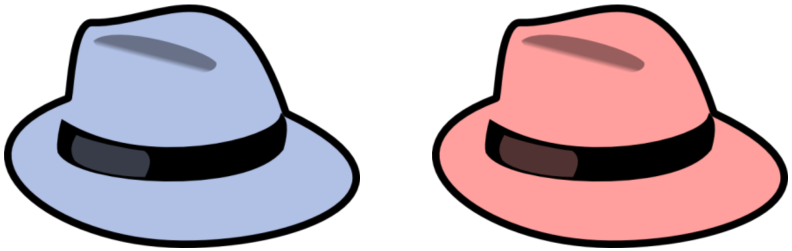
- A red or blue hat is placed on each player 's head.



The Hat Problem (N=2)

N =2

- A red or blue hat is placed on each player 's head.

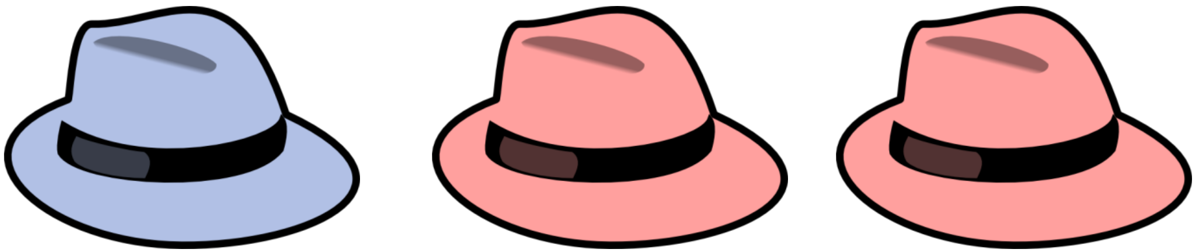


- Both players make a guess: win with probability 1/4.
- One player makes a guess: win with probability ½.

The Hat Problem (N=3)

N =3

- A red or blue hat is placed on each player 's head.



- One player makes a guess: win with probability ½.
  - Can a strategy do better?

http://www.nytimes.com/2001/04/10/science/why-mathematicians-now-care-about-their-hat-color.html

The Hat Problem

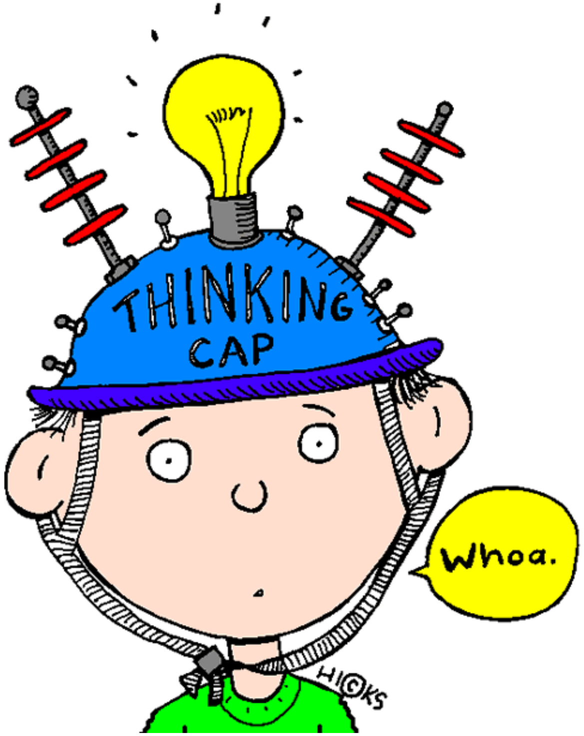
000 100 010 110 001 101 011 111

Strategy: If the other two guys have the same hat color, "guess the opposite", if they have different colors, stay silent!

- *Chance of winning with this strategy*: 3/8+3/8=0.75

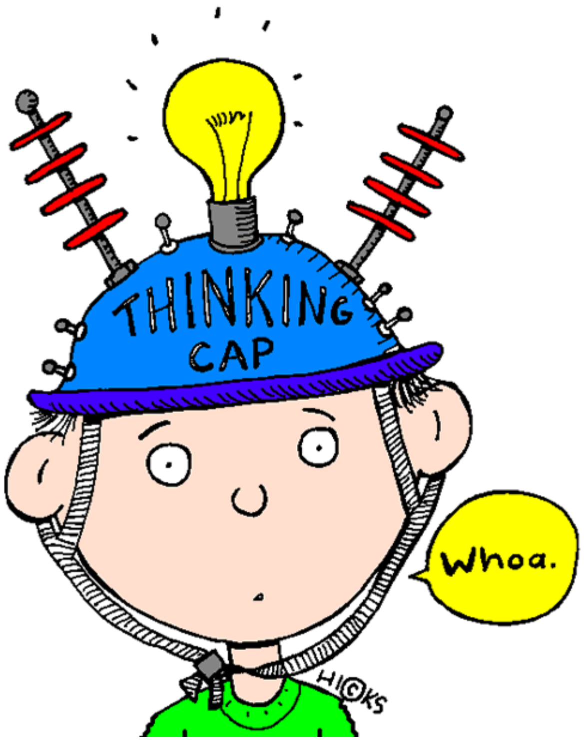
The Hat Problem

- Optimal strategy?



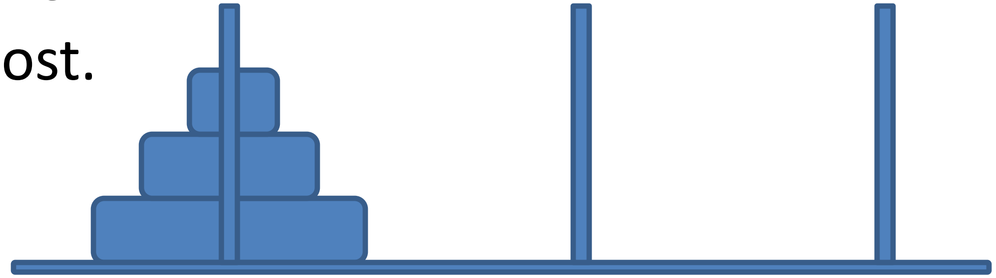
The Hat Problem

- Optimal strategy?
- Number of correct guesses = number of incorrect guesses
- Better strategy: 7 wins & 1 loss
- At least 7 correct guesses, impossible to have 7 incorrect guesses in one loss and 3 players



Hanoi Tower

- **Goal:** move all n disks in the same order, but on a different post.
- **Only permitted action:** remove the top disk from a post and drop it onto another post.
- **Rule:** a larger disk can never lie above a smaller disk on any post.



Hanoi Tower (n=3)

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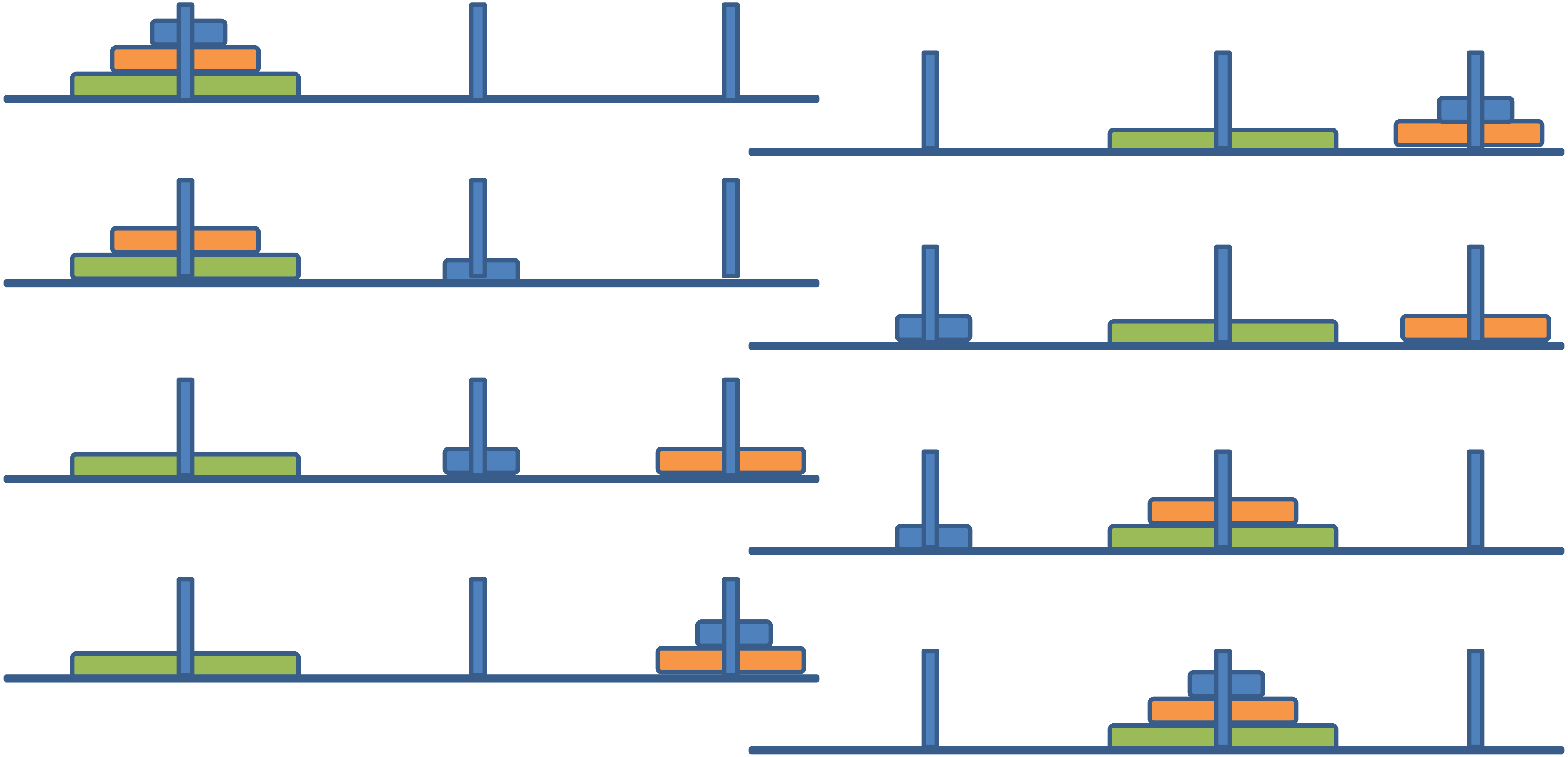
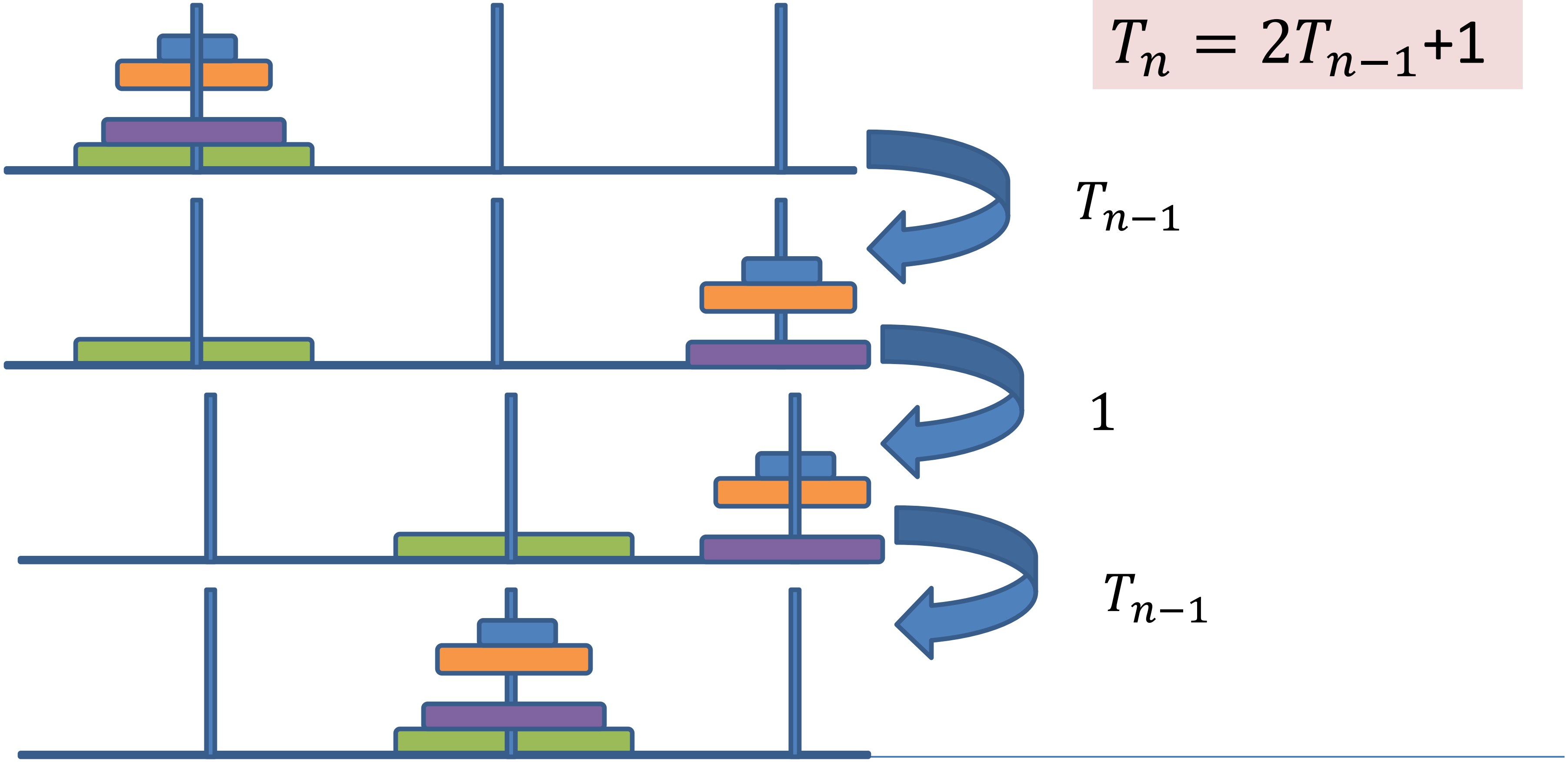


Find a Recurrence

Find a Recurrence

- T_n = minimum number of steps needed to move an n-disk tower from one post to another

$$T_n = 2T_{n-1}+1$$



Backtracking

$T_1 = 1$ $\boxed{T_n = 2T_{n-1} + 1}$

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$$
\begin{aligned}
&T_n \\
&= 2T_{n-1} + 1 = 2(2T_{n-2} + 1) + 1 \quad \Leftarrow \quad 3 \\
&= 4T_{n-2} + 3 = 4(2T_{n-3} + 1) + 3 \quad \Leftarrow \quad 7 \\
&= 8T_{n-3} + 7 \quad \Leftarrow \quad 15
\end{aligned}
$$

$T_n = 2^n - 1$

Induction

- P(n)=``$T_n = 2^n - 1$''

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- P(n)=``$T_n = 2^n - 1$''
- Basis step: P(1)=``$T_1 = 1$''
- Inductive step: suppose P(n) is true.
- To show, P(n+1).
- $T_{n+1} = 2T_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1$