MH1820 Week 5

1. Normal distribution

2. Chi-square distribution

$\chi^2(r)$ degree of freedom,

Chi-square distribution

- Gamma distribution.
- how table for percentiles of χ^2

$$Z^2 \qquad Z \sim N(0,1).$$

white noise

$$\mathrm{Var}(Z) = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$$

$$= \mathbb{E}[Z^2].$$

Distribution of Z^2

$$Z^2 = \chi^2(r=1).$$

Chi-square distribution

The preceding example is a **chi-square distribution** with 1 degree of freedom, denoted by $\chi^2(1)$. In general, we have the following:

Theorem 8 (Chi-square distribution)

Suppose $X_i = Z^2$ where $Z \sim N(0,1)$, for $i = 1, \ldots, r$, are independently and identically distributed (i.i.d).

Then $X = \sum_{i=1}^{r} X_i$ has a **chi-square distribution with r degree of freedom**, *denoted by $X \sim \chi^2(r)$.*

$r = 2\alpha$

In fact,

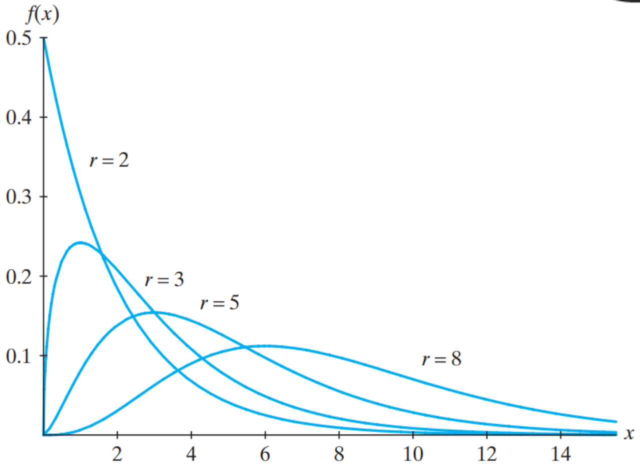
$$\chi^2(r) = Gamma\left(\alpha = \frac{r}{2}, \theta = 2\right).$$

$\text{Gamma}(\alpha = 9, \theta = 2) =$ waiting time until αth (9th) arrival of Poisson $(\lambda = \frac{1}{\theta} = \frac{1}{2})$

$\chi^2(r = 2\alpha)$

$= \chi^2(r = 18)$. ✓

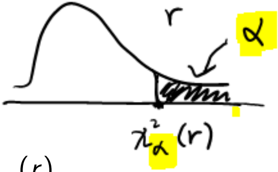
Chi-square distribution $\chi^2(r)$ for different degrees of freedom $r = 2, 3, 5, 8$.



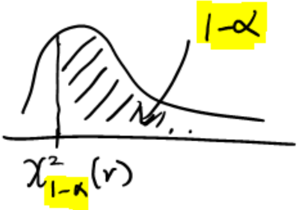
Let $X \sim \chi^2(r)$. Similar to standard normal, we define

- the $100(1-\alpha)$**th percentile** (or **upper** 100α**th percentage point**) $\chi^2_\alpha(r)$ to be the number such that

$$\mathbb{P}(X \geq \chi^2_\alpha(r)) = \alpha.$$



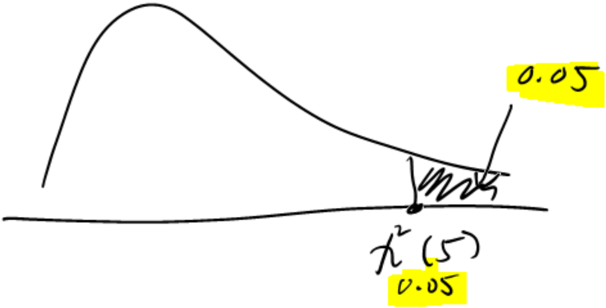
- the 100α**th percentile** to be the number $\chi^2_{1-\alpha}(r)$.



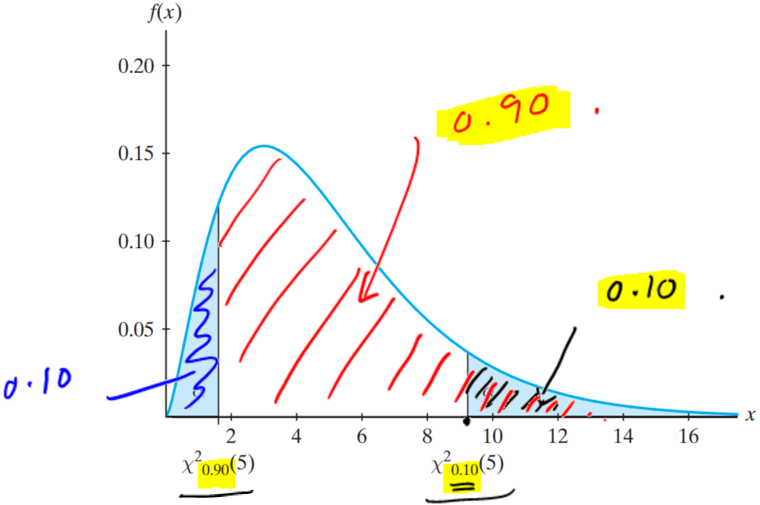
$r = 5$

$\chi^2_{0.05}(5)$

$\chi^2(5)$



Chi-square tails, $r = 5$, $\alpha = 0.10$:



For a table of $\chi^2_\alpha(r)$, see NTULearn $->$ Content $->$ TABLES.pdf

Example 9

Let X have a chi-square distribution with $r = 5$ degree of freedom. Find the probabilty that X is between 1.145 and 12.83.

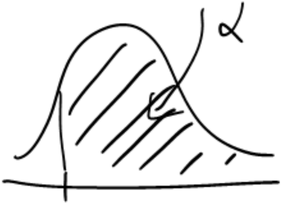
Solution. Use the table for $\chi^2_\alpha(5)$.

$$\mathbb{P}(1.145 \leq X \leq 12.83)$$

$$= \mathbb{P}(X \geq 1.145) - \mathbb{P}(X \geq 12.83)$$

$$= \mathbb{P}(X \geq \chi^2_{0.95}(5)) - \mathbb{P}(X \geq \chi^2_{0.025}(5)).$$

$$= 0.95 - 0.025 = 0.925 \ \#.$$



$\chi^2_{\alpha}(r) = 1.145$

$\chi^2_{\alpha}(r) = 12.83$.

Example 10

If customers arrive at a shop on the average of 30 per hour in accordance with a Poisson process, what is the probability that the shopkeeper will have to wait longer than 9.390 minutes for the first nine customers to arrive?

X = waiting time (minutes) until 9th arrival.

$X \sim \text{Gamma}\left(\alpha = 9,\ \theta = \frac{1}{\lambda} = \frac{1}{1/2} = 2\right)$.

average of 30 customers per 60 minutes.

" " $\frac{30}{60} = \frac{1}{2}$ " " minute.

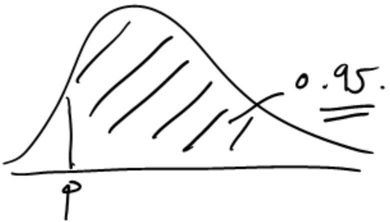
Poisson$\left(\lambda = \frac{1}{2}\right)$

$$X \sim \text{Gamma}(\alpha = 9, \theta = 2) = \chi^2(r = 2\alpha = 18)$$

$$\mathbb{P}(X \geq 9.390) = \mathbb{P}\left(X \geq \chi^2_{0.95}(18)\right)$$

$$= \underline{\underline{0.95}}.$$

#.



$$\chi^2_{0.95}(18) = 9.390$$