**NANYANG TECHNOLOGICAL UNIVERSITY**

SEMESTER II EXAMINATION 2020–2021

**MH1812 – Discrete Mathematics**

May 2021 TIME ALLOWED: 2 HOURS

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INSTRUCTIONS TO CANDIDATES

**QUESTION 1.** **(10 marks)**

Prove or disprove the following statements.

(a) For all sets $A$, $B$, and $C$, if $B \cup C \subseteq A$ then

$$(A - B) \cap (A - C) = \emptyset.$$

(b) For all sets $A$ and $B$,

$$P(A) \cup P(B) \subseteq P(A \cup B),$$

where $P(A)$ and $P(B)$ denotes the power set of $A$ and $B$, respectively.

**Solution:**

(a) False. Counterexample: $A = \{1\}$, $B = C = \emptyset$.

(b) True. We prove it. Let $X \in P(A) \cup P(B)$. This implies that either $X \in P(A)$ or $X \in P(B)$.

Case $X \in P(A)$. This implies $X \subset A \subset A \cup B$. Hence $X \in P(A \cup B)$.

Case $X \in P(B)$. This implies $X \subset B \subset A \cup B$. Hence $X \in P(A \cup B)$.

**QUESTION 2.** **(20 marks)**

(a) Suppose that $C_1 = 1$ and for each integer $k \geqslant 2$ we have $C_k = \frac{4k-2}{k+1} C_{k-1}$. Using induction, prove that

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad \text{for each positive integer } n,$$

where $\binom{2n}{n}$ is the number of subsets of $\{1, \ldots, 2n\}$ that have cardinality $n$.

(b) Let $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3, \ldots$ be the Fibonacci sequence, i.e., $F_k = F_{k-1} + F_{k-2}$ for each integer $k \geqslant 2$. Prove by induction that, for each integer $n \geqslant 3$,

$$F_n < 2^{n-1}.$$

**Solution:**

(a) Let $P(n)$ be the hypothesis that

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Base case: $P(1)$ is true. Assume that $P(n)$ is true for some $n \in \mathbb{N}$. Now consider $P(n+1)$. Using the hypothesis $P(n)$ we see that the LHS of $P(n+1)$ is

$$\begin{aligned}
C_{n+1} &= \frac{4n+2}{n+2} C_n \\
&= \frac{4n+2}{n+2}\frac{1}{n+1}\binom{2n}{n} \\
&= \frac{4n+2}{n+2}\frac{1}{n+1}\frac{(2n)!}{n!n!} \\
&= \frac{4n+2}{n+2}\frac{n+1}{1}\frac{(2n)!}{(n+1)!(n+1)!} \\
&= \frac{2n+1}{n+2}\frac{2n+2}{1}\frac{(2n)!}{(n+1)!(n+1)!} \\
&= \frac{1}{n+2}\binom{2(n+1)}{n+1},
\end{aligned}$$

as required.

(b) Let $P(n)$ be the hypothesis that

$$F_n < 2^{n-1}.$$

Basis case: $n = 3$ we have the LHS is 3 and the RHS is 4. So $P(3)$ is true. $n = 4$ we have the LHS is 5 and the RHS is 8. So $P(4)$ is true. Assume that $P(n)$ is true for some integer $n \geqslant 4$. Now consider $P(n+1)$. Using the hypothesis $P(n)$ we see that the LHS of $P(n+1)$ is

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ &< 2^{n-2} + 2^{n-3} \\ &= 2^{n-3}(2+1) \\ &< 2^{n-1}, \end{aligned}$$

as required.

**QUESTION 3.** **(10 marks)**

A coin is tossed ten times. In each case, the outcome $H$ (for heads) or $T$ (for tails) is recorded. (One possible outcome for the ten tosses is denoted $THHTTTHTTH$.)

(a) What is the total number of possible outcomes of the coin-tossing experiment?

(b) In how many of the possible outcomes are exactly five heads obtained?

(c) In how many of the possible outcomes are at least eight heads obtained?

**Solution:**

(a) $2^{10} = 1024$

(b) $10!/(5!5!) = 252$

(c) $1 + 10!/9! + 10!/(2!8!) = 1 + 10 + 45 = 56$

**QUESTION 4.** **(15 marks)**

(a) Let $A = \{a, b\}$.

(i) How many relations on $A$ are equivalence relations?

(ii) How many relations on $A$ are partial orders?

(b) Define a relation $R$ on $\mathbb{Z}$, the set of all integers as follows. For each $m, n \in \mathbb{Z}$, $(m, n) \in R$ if and only if $m + n$ is even.

(i) Is $R$ reflexive?

(ii) Is $R$ symmetric?

(iii) Is $R$ anti-symmetric?

(iv) Is $R$ transitive?

(v) Is $R$ an equivalence relation?

(vi) Is $R$ a partial order?

Justify your answers.

**Solution:**

(a) (i) 2: $\{(a, a), (b, b)\}$, $\{(a, a), (b, b), (a, b), (b, a)\}$

(ii) 3: $\{(a, a), (b, b)\}$, $\{(a, a), (b, b), (a, b)\}$, $\{(a, a), (b, b), (b, a)\}$

(b) (i) $R$ is reflexive: $m + m = 2m$ is even for all $m \in \mathbb{Z}$

(ii) $R$ is symmetric: if $m + n$ is even then so is $n + m$.

(iii) $R$ is not anti-symmetric: $1 + 3$ is even and is $3 + 1$ but $1 \neq 3$.

(iv) $R$ is transitive: if $m + n = 2k$ and $n + p = 2l$ then $m + p = 2k + 2l - 2n$ is also even.

(v) $R$ is an equivalence relation: it is reflexive, symmetric, and transitive

(vi) $R$ is not a partial order: it is not anti-symmetric

**QUESTION 5.** **(20 marks)**

(a) Define the function $F : \mathbb{R} \to \mathbb{Z}$ by the formula $F(x) = \lfloor x \rfloor$ for each $x \in \mathbb{R}$.

(i) Is $F(x)$ one-to-one? If so then prove it, if not then give a counterexample.

(ii) Is $F(x)$ onto? If so then prove it, if not then give a counterexample.

(b) Define the function $G : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by the formula $G(x, y) = x + \sqrt{2}y$ for each $(x, y) \in \mathbb{Q} \times \mathbb{Q}$. Is $G(x)$ one-to-one? If so then prove it, if not then give a counterexample.

(c) True or False? Given any set $X$ and given any functions $f : X \to X$, $g : X \to X$, and $h : X \to X$, if $h$ is one-to-one and $h \circ f = h \circ g$ then $f = g$. Justify your answer.

**Solution:**

(a) (i) No: $F(1) = F(1.5) = 1$.

(ii) Yes: $\forall y \in \mathbb{Z}$, we have $F(y) = y$.

(b) Yes: if $G(a, b) = G(x, y)$ then $a + \sqrt{2}b = x + \sqrt{2}y$, which implies $a - x = (y - b)\sqrt{2}$. We must have $y = b$ since $\sqrt{2}$ is irrational. Whence $x = a$, as required.

(c) True. Suppose $h$ is one-to-one and $h \circ f = h \circ g$. We want to show that $\forall x in X$, $f(x) = g(x)$. Let $x \in X$. Then $h(f(x)) = h(g(x))$. Since $h$ is one-to-one, we must have $f(x) = g(x)$.

**QUESTION 6.** **(10 marks)**

Each of (a)-(c) describes a graph. In each case answer, *yes*, *no*, or *not necessarily* to this question: Does the graph have an Euler circuit? No justification is required.

(a) $G = (V, E)$ is a connected graph where $|V| = 5$ and the five vertices of $G$ have the five degrees 2, 2, 3, 3, 4.

(b) $G = (V, E)$ is a connected graph where $|V| = 5$ and the five vertices of $G$ have the five degrees 2, 2, 4, 4, 6.

(c) $G = (V, E)$ is a graph where $|V| = 5$ and the five vertices of $G$ have the five degrees 2, 2, 4, 4, 6.

**Solution:**

(a) no

(b) yes

(c) not necessarily

**QUESTION 7.** **(15 marks)**

Let $A$ be a set of six positive integers, each of which is less than 15. Show that there exists two distinct subsets $S \subset A$ and $T \subset A$ such that the sum of the elements in $S$ is equal to the sum of the elements in $T$. For example, if $A = \{3, 4, 5, 10, 11, 12\}$ then there exists subsets $S = \{3, 12\}$ and $T = \{5, 10\}$ whose elements both add up to $15 = 3 + 12 = 5 + 10$.

**Solution:** The number of nonempty subsets is $2^6 - 1 = 64 - 1 = 63$. Let $S = \{a_1, a_2, \ldots, a_6\}$ where $a_1 < a_2 < \cdots < a_6$. For any nonempty subset $S \subset A$ the sum of the elements in $S$ is at most $a_1 + a_2 + \cdots + a_6$ and at least $a_1$. Therefore there are at most $1 + a_2 + \cdots + a_6 \leqslant 1 + 10 + 11 + 12 + 13 + 14 = 61$ possible values for the sum of elements of $S$. The statement then follows from the pigeonhole principle.

**END OF PAPER**