For graders only:	Question	1(a)	1(b)	1(c)	2(a)	2(b)	2(c)	3(a)	3(b)	3(c)	Total
	Marks										

MIDTERM I (CA1)

MH1812 – Discrete Mathematics

March 2024 TIME ALLOWED: 50 minutes

Name:

Matric. no.: Tutor group:

INSTRUCTIONS TO CANDIDATES

1. **DO NOT TURN OVER PAPER UNTIL INSTRUCTED.**
2. This midterm paper contains **THREE (3)** questions.
3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
4. Read the question carefully to see how to write your answers.
5. Clearly indicate your answers. Unclear or ambiguous answers will receive **zero marks**.
6. For questions that require you to **circle** to indicate your answer, the choice that you circle will be interpreted as your answer.
7. This **IS NOT** an **OPEN BOOK** exam.
8. Calculators are allowed.

QUESTION 1. **(10 marks)**

(a) [1 mark] Find the remainder r of 7^{1812} after division by 8.

$$r = \boxed{}$$

(b) Decide whether the set S is closed under the operation Δ when

(i) [1 mark] $S_1 = \{\text{negative integers}\}$ and Δ_1 is multiplication.

S_1 is $\boxed{\text{closed/not closed}}$ under the operation Δ_1

(**Circle** "closed" or "not closed" to indicate your answer.)

(ii) [1 mark] $S_2 = \{\text{non-zero rational numbers}\}$ and Δ_2 is addition.

S_2 is $\boxed{\text{closed/not closed}}$ under the operation Δ_2

(**Circle** "closed" or "not closed" to indicate your answer.)

No justification is required.

(c) [7 marks] In the table below, mark with a 'Y' each integer $a \in \{0, 1, 2, 3, 4, 5, 6\}$ that satisfies the congruence $(5436)^a \equiv 3^{2024} \pmod 7$ and an 'N' for those that do not.

a	0	1	2	3	4	5	6
Y/N							

Solution:

(a) We have

$$7 \equiv -1 \pmod 8$$

Hence $7^{1812} \equiv (-1)^{1812} \equiv 1 \pmod 8$.

(b) Decide whether the set S is closed under the operation Δ when

(i) not closed since $(-1) \times (-1) = 1 \notin S_1$.

(ii) not closed since $-1 + 1 = 0 \notin S_2$.

(c) [7 marks] In the table below, mark with a 'Y' each integer $a \in \{0, 1, 2, 3, 4, 5, 6\}$ that satisfies the congruence $(5436)^a \equiv 3^{2024} \pmod 7$ and an 'N' for those that do not.

First, $3^6 \equiv 1 \pmod 7$. And $2024 \equiv 2 \pmod 6$. Hence $3^{2024} \equiv 3^2 \equiv 2 \pmod 7$. Now we need to find which $a \in \{0, 1, 2, 3, 4, 5, 6\}$ satisfies $(5436)^a \equiv 2 \pmod 7$. We have $5436 = 3 * 1812 \equiv 3 * 6 \equiv 4 \pmod 7$. Hence $(5436)^a \equiv 4^a \pmod 7$. Now we make a table to find which a satisfy that congruence.

a	0	1	2	3	4	5	6
$4^a \pmod 7$	1	4	2	1	4	2	1

QUESTION 2. **(10 marks)**

(a) [3 marks] Show that the compound propositions

$$(q \vee r) \to (q \wedge p) \text{ and } (q \to p) \wedge (r \to p)$$

are not equivalent by finding a row where their truth tables differ.

p	q	r

(b) [5 marks] Show that the following argument is valid by completing the table below. You may need the following inference rules: Modus Ponens, Modus Tollens, Conjunctive Simplification, Conjunctive Addition, Disjunctive Addition, and Disjunctive Syllogism.

$$\begin{aligned} & u \to r \wedge \neg s; \\ & \neg w; \\ & t \to s; \\ & u \vee w; \\ & \therefore t \to F. \end{aligned}$$

(1)	$u \to r \wedge \neg s$	
(2)	$\neg w$	
(3)	$t \to s$	
(4)	$u \vee w$	
$\therefore$ (5)	u	Disjunctive Syllogism on (2) and (4)
$\therefore$ (6)	$r \wedge \neg s$	
$\therefore$ (7)		Conjunctive Simplification on (6)
$\therefore$ (8)	$\neg t$	
$\therefore$ (9)	$t \to F$	Rule of Contradiction on (8)

(c) [2 marks] Find the number of *critical rows* and *counter-examples* of the following argument.

$$\begin{aligned} & p; \\ & p \vee q; \\ & q \to (r \to s); \\ & t \to r; \\ & \therefore \neg s \to \neg t. \end{aligned}$$

Number of critical rows: $\boxed{}$

Number of counter-examples: $\boxed{}$

Solution:

(a)

p	q	r
T	F	T

(b)

(1)	$u \to r \wedge \neg s$	
(2)	$\neg w$	
(3)	$t \to s$	
(4)	$u \vee w$	
$\therefore$ (5)	u	Disjunctive Syllogism on (5) and (2)
$\therefore$ (6)	$r \wedge \neg s$	$\boxed{\text{Modus Ponens on (5) and (1)}}$
$\therefore$ (7)	$\boxed{\neg s}$	Conjunctive Simplification on (6)
$\therefore$ (8)	$\neg t$	$\boxed{\text{Modus Tollens on (7) and (3)}}$
$\therefore$ (9)	$t \to F$	Rule of Contradiction on (8)

(c) Note that $p = T$ in all critical rows. If $q = T$ then $r \to s$ must be true.

q	r	s	t	$t \to r$	$\neg s \to \neg t$
T	T	T	T	T	T
T	T	T	F	T	T
T	F	T	T	F	
T	F	T	F	T	T
T	F	F	T	F	
T	F	F	F	T	T

This gives us four critical rows when $q = T$.

Now assume that $q = F$.

q	r	s	t	$t \to r$	$\neg s \to \neg t$
F	T	T	T	T	T
F	T	T	F	T	T
F	T	F	T	T	F
F	T	F	F	T	T
F	F	T	T	F	
F	F	T	F	T	T
F	F	F	T	F	
F	F	F	F	T	T

This yields six more critical rows. Hence the total number of critical rows is 10. And there is just 1 counter-example.

QUESTION 3. **(7 marks)**

(a) [3 marks] Consider the domains

$\mathbb{Q} = \{\text{rational numbers}\}$, $\mathbb{Z} = \{\text{integers}\}$, and $\mathbb{N} = \{\text{positive integers}\}$.

Determine the truth value of the following statements.

(**Circle** "T" or "F" to indicate your answer.)

(i) $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Z},\ \exists z \in \mathbb{N},\ xz - y \in \mathbb{N}$. | T | F |

(ii) $\forall x \in \mathbb{Q},\ \forall y \in \mathbb{Z},\ \exists z \in \mathbb{N},\ y + xz \in \mathbb{N}$. | T | F |

(iii) $\exists x \in \mathbb{Q},\ \forall y \in \mathbb{Z},\ \forall z \in \mathbb{N},\ xy + z \in \mathbb{N}$. | T | F |

No justification is required.

(b) [3 marks] Consider the domains

$X = \{1, 2, 3, 4, 5\}$, $Y = \{-2, -1, 0, 1, 2\}$, and $Z = \{-5, -4, -3, -2, -1\}$.

Determine the truth value of the following statements.

(**Circle** "T" or "F" to indicate your answer.)

(i) $\forall x \in X,\ \exists y \in Y,\ \exists z \in Z,\ xy = z$. | T | F |

(ii) $\forall x \in X,\ \forall y \in Y,\ \exists z \in Z,\ xy > z$. | T | F |

(iii) $\forall x \in X,\ \exists y \in Y,\ \forall z \in Z,\ xyz < 0$. | T | F |

No justification is required.

(c) [1 mark] Consider the domains

$P = \{\text{prime numbers}\}$ and $Q = \{\text{integers congruent to 7 modulo 11}\}$.

Determine the truth value of the following statement.

(**Circle** "T" or "F" to indicate your answer.)

$\neg\left(\forall x \in P,\ \exists y \in Q,\ xy \in Q\right)$. | T | F |

No justification is required.

Solution:

(a)

(i) T. For any $x \in \mathbb{Q}$, we can write $x = a/b$ where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Take $z = b \in \mathbb{N}$ and $y = a - 1 \in \mathbb{Z}$.

(ii) F. For $x = 0$ and $y = 0$ we have $y + xz = 0$, which is not in $\mathbb{N}$.

(iii) T. Take $x = 0$. Then $xy + z = z \in \mathbb{N}$.

(b)

(i) T. Take $y = -1$ and $z = -x$.

(ii) F. For $x = 5$ and $y = -2$, we have $xy < z$ for all $z \in Z$.

(iii) T. Take $y = 1$ then $xyz < 0$.

(c) T. First we distribute the negation

$$\neg \left(\forall x \in P,\ \exists y \in Q,\ xy \in Q \right) \equiv \exists x \in P,\ \forall y \in Q,\ xy \notin Q.$$

Take $x = 11$. Then $xy \equiv 0 \pmod{11}$ for all $y \in Q$. Hence, $xy \notin Q$ and the statement is true.