Discrete Mathematics

MH1812

Topic 3 - Predicate Logic
Summary

Quantification: Order of Nesting Matters

Is $\forall x \in D, \exists y \in D, P(x,y) \equiv \exists y \in D, \forall x \in D, P(x,y)$ in general?

LHS	RHS
$\forall x \in D, \exists y \in D, P(x,y)$	$\exists y \in D, \forall x \in D, P(x,y)$
y can **vary** with x	y is **fixed**, but x **varies**
"Every person admires someone"	"Some people are admired by everyone"

Let $P(x,y)$ = "x admires y"

Example

Consider the domains $X = \{2,3\}$ **and** $Y = \{2,4,6\}$, **and the predicate** $P(x,y)$ = “x **divides** y”.

What are the truth values of

1. $\forall y \in Y, \exists x \in X, P(x,y).$
2. $\exists x \in X, \forall y \in Y, P(x,y).$

Example

Consider the domains $X = \{2,3\}$ **and** $Y = \{2,6,9\}$, **and the predicate** $P(x,y)$ = "x **divides** y".

What are the truth values of

1. $\forall y \in Y, \exists x \in X, P(x,y)$.
2. $\exists x \in X, \forall y \in Y, P(x,y)$.

Quantification: Order of Nesting Matters

Consider (arbitrary) domains *X* and *Y* with *m* and *n* members respectively.

Then $\exists x \in X, \exists y \in Y, P(x,y) \equiv \exists y \in Y, \exists x \in X, P(x,y)$

and $\forall x \in X, \forall y \in Y, P(x,y) \equiv \forall y \in Y, \forall x \in X, P(x,y)$

$$\exists x \in X, \exists y \in Y, P(x,y) \equiv [\exists y \in Y, P(x_1,y)] \vee \ldots \vee [\exists y \in Y, P(x_m,y)]$$

$$\equiv [P(x_1, y_1) \vee \ldots \vee P(x_1, y_n)] \vee \ldots \vee [P(x_m, y_1) \vee \ldots \vee P(x_m, y_n)]$$

$$\equiv [P(x_1, y_1) \vee \ldots \vee P(x_m, y_1)] \vee \ldots \vee [P(x_1, y_n) \vee \ldots \vee P(x_m, y_n)]$$

$$\equiv [\exists x \in X, P(x, y_1)] \vee \ldots \vee [\exists x \in X, P(x, y_n)]$$

$$\equiv \exists y \in Y, \exists x \in X, P(x,y)$$

Determining Truth Values: Method of Case

Positive Example to Prove Existential Quantification

Let $\mathbb{Z}$ denote all integers.

Is $\exists x \in \mathbb{Z}, x^2 = x$ true or false?

Take $x = 0$ or 1 and we have it.

Positive Example

It is not a proof of universal quantification.

Counterexample to Disprove Universal Quantification

Let $\mathbb{R}$ denote all reals.

Is $\forall x \in \mathbb{R}, x^2 > x$ true or false?

Take $x = 0.3$ as a counterexample.

Negative Example

It is not disproof of existential quantification.

Conditional Quantification: Definitions

Given a conditional quantification such as…

$$\forall x \in A\ (P(x) \rightarrow Q(x))$$

Then, we define…

Contrapositive	$\forall x \in A,\ \neg Q(x) \rightarrow \neg P(x)$
Converse	$\forall x \in A,\ Q(x) \rightarrow P(x)$
Inverse	$\forall x \in A,\ \neg P(x) \rightarrow \neg Q(x)$

Note: a conditional proposition is logically equivalent to its contrapositive.

Conditional Quantification: Negation

What is $\neg\,(\forall x \in X,\ P(x) \rightarrow Q(x))$**?**

$$\neg\,(\forall x \in X,\ P(x) \rightarrow Q(x))$$

$$\equiv \exists x \in X,\ \neg\,(P(x) \rightarrow Q(x))$$

Negation of Quantified Statements

$$\equiv \exists x \in X,\ \neg\,(\neg\, P(x) \vee Q(x))$$

Conversion of Conditionals

$$\equiv \exists x \in X,\ P(x) \wedge \neg\, Q(x)$$

De Morgan

Example

Consider the domains $X = \{2,3\}$ **and** $Y = \{2,4,6\}$, **and the predicate** $P(x,y)$ = "x **divides** y".

What are the truth values of

1. $\neg(\exists x \in X, \exists\, y \in Y\ P(x,y))$.

2. $\neg(\forall y \in Y, \exists x \in X, P(x,y))$.

Basic Inference Rules:

$P(c)$ for **any arbitrary** c **from the domain** D.
$\therefore \forall x \in D, P(x)$

$P(c)$
$\therefore \exists x \in D, P(x)$
for c **some specific element of the domain** D.

$\forall x \in D, P(x)$
$\therefore P(c)$
where c **is any element of the domain** D.

$\exists x \in D, \mathrm{P}(x)$
$\therefore P(c)$ **for some** c **in the domain** D.

Logic

proposition

$\wedge$: conjunction (and)

$\vee$: disjunction (or)

$\neg$: negation (not, alternatively ~)

$p \to q$: conditional (if then)

$p \leftrightarrow q$: biconditional (if and only if)

equivalence laws (e.g. De Morgan, Conversion Theorem, Distributivity)

valid argument (premises and conclusion)

inference rules, e.g. Modus ponens/tollens

$$\begin{array}{r} p \to q; \\ p; \\ \therefore q \end{array} \qquad \begin{array}{r} p \to q; \\ \neg q; \\ \therefore \neg p \end{array}$$

predicate

Quantification:

- Universal $\forall$
- Existential $\exists$
- Nested
- Negation
- Conditional
- Negation of conditional

Inference rules:

- Universal generalization/instantiation
- Existential generalization/instantiation