Type II Errors and Power of a Test

There are two types of errors in hypothesis testing: H_1 true.

H_0	H_0 True	H_0 False
Reject H_0	Type I Error	
Do not Reject H_0		Type II Error

Type II Errors and Power of a Test

Type II Error

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- It is usually not possible to make both Type I and Type II errors arbitrarily small.

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- It is usually not possible to make both Type I and Type II errors arbitrarily small.
- Realistic goal: Find test with a prescribed probability of a Type I Error that minimizes the probability of a Type II Error.
- Type II Error can be controlled using the Alternative Hypothesis.

Example 1

Coin is tossed 100 times to test if there is a bias towards heads.

- $X_1, \ldots, X_{100}$ i.i.d $\sim$ *Bernoulli*(p)
- $H_0 : p = 0.5, \quad H_1 : p > 0.5$

Type I Error: Coin is fair, but H_0 is rejected.

Type II Error: Coin is biased towards heads, but H_0 is not rejected.

Power of a Test

- The probability of a Type II Error is denoted by β:

$$\beta = \mathbb{P}(H_0 \text{ not rejected} | H_1)$$

- The probability that H_0 is rejected if it is wrong is the **power** of the test, i.e.

$$\textbf{Power} = \mathbb{P}(H_0 \text{ rejected} | H_1) = 1 - \beta.$$

Example 2

- $X_1, \ldots, X_{10}$ i.i.d $\sim Poisson(\lambda)$
- Test $H_0 : \lambda = \frac{1}{10}$ against $H_1 : \lambda = 1$
- H_0 is rejected $\iff \sum_{i=1}^{10} X_i \geq 2$.

Find the size and power of this test.

Recall: Tutorial 6 Q3:

$$Y = \sum_{i=1}^{10} X_i \sim \text{Poisson}(10\lambda)$$

① $\text{Size} = \mathbb{P}(H_0 \text{ is rejected} \mid H_0)$

$$= \mathbb{P}(Y \geq 2 \mid \lambda = 0.1)$$

$$Y \sim \text{Poisson}(10 \times 0.1) = \underline{\underline{\text{Poisson}(1)}}$$

$$= 1 - \mathbb{P}(Y \leq 1)$$

$$= 1 - \left(\mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) \right)$$

$$= 1 - \left(\frac{e^{-1} 1^{0}}{0!} + \frac{e^{-1} 1^{1}}{1!} \right) \approx 0.26$$

#.

② Power $= 1 - \beta$

$$\beta = \mathbb{P}(H_0 \text{ is } \underline{\text{not}} \text{ rejected} \mid H_1)$$

$$= \mathbb{P}(\underline{Y \le 1} \mid \lambda = 1)$$

$$Y \sim \text{Poisson}(10\lambda) = \text{Poisson}(10)$$

$$= \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1)$$

$$= \frac{e^{-10}\,10^{0}}{0!} + \frac{e^{-10}\,10^{1}}{1!} \approx 0.0005.$$

$$\text{Power} = 1 - \beta$$
$$\approx 1 - 0.0005$$
$$\approx 0.9995$$
#

Example 3

- $X_1, \ldots, X_{10}$ i.i.d $\sim Poisson(\lambda)$
- Test $H_0 : \lambda = \frac{1}{10}$ against $H_1 : \lambda = 1$
- Test statistic: $\sum_{i=1}^{10} X_i \sim Poisson(10\lambda)$
- Rejection criteria: Reject $H_0 \iff (\sum_{i=1}^{10} X_i > c)$.

(unknown)

Suppose we require the size of the test to be at most 0.05? What is the maximum power we can achieve?

$$Y = \sum_{i=1}^{10} X_i \sim \text{Poisson}(10\lambda).$$

$$\text{Size} = \mathbb{P}(H_0 \text{ rejected} \mid H_0)$$

$$= \mathbb{P}(Y > c \mid \lambda = 0.1)$$

$$Y \sim \text{Poisson}(10\lambda) = \text{Poisson}(1)$$

$$= 1 - \underbrace{\mathbb{P}(Y \leq c)}_{\text{CDF of } Y \sim \text{Poisson}(1)}$$

Our requirement says that

size ≤ 0.05

$$1 - \mathbb{P}(Y \leq c) \leq 0.05$$

$$\Rightarrow \quad \mathbb{P}(Y \leq c) \geq 1 - 0.05 = 0.95$$

$$\boxed{\mathbb{P}(Y \leq c) \geq 0.95}$$

$Y \sim \text{Poisson}(1)$

From the table, $\mathbb{P}(Y \leq 2) \approx 0.92$

$\mathbb{P}(Y \leq 3) \approx 0.981$

If $c = 2$, then $\mathbb{P}(Y \leq 2) \approx 0.92$, which is impossible

$c = 3$ is fine.

$c = 4$ is fine

$\vdots$

Conclusion:

$$\boxed{c \geqslant 3}$$

$$\text{Power} = 1 - \beta$$

$$= \mathbb{P}(H_0 \text{ rejected} \mid H_1)$$

$$= \mathbb{P}(Y > c \mid \boxed{\lambda = 1})$$

$$Y \sim \text{Poisson}(10\lambda) = \text{Poisson}(10)$$

$$= 1 - \mathbb{P}(Y \leq c).$$

$$\leq 1 - \mathbb{P}(Y \leq 3) \quad (\text{since } c \geqslant 3).$$

Remark: $3 \le c \Rightarrow \mathbb{P}(Y \le 3) \le \mathbb{P}(Y \le c)$

$$\Rightarrow -\mathbb{P}(Y \le c) \le -\mathbb{P}(Y \le 3)$$

$$\Rightarrow 1 - \mathbb{P}(Y \le c) \le 1 - \mathbb{P}(Y \le 3)$$

So power $\le 1 - \mathbb{P}(Y \le 3) = 1 - 0.01$

$= 0.99$ #

In fact power $= 0.99$ if we choose $c = 3$.

With size of the test ≤ 0.05, the maximum power the test can achieve based on given rejection criteria is 0.99 (is attained if we set $c = 3$). $\square$.

Example 4

- $X_1, \ldots, X_{25}$ i.i.d $\sim N(\mu, 100)$.
- Test $H_0 : \mu = 60$ against $H_1 : \mu > 60$
- Reject $H_0 \iff \overline{X} \geq c$.

$\sigma^2 = 100$

$\sigma = 10$.

Compute the size of the test for $c = 62$ and $c = 63.29$. For each of the c above, what is the power of the test if $\mu = 65$?

Example 4

- $X_1, \ldots, X_{25}$ i.i.d $\sim N(\mu, 100)$.
- Test $H_0 : \mu = 60$ against $H_1 : \mu > 60$
- Reject $H_0 \iff \overline{X} \geq c$.

$\sigma = 10$

Compute the size of the test for $c = 62$ and $c = 63.29$. For each of the c above, what is the power of the test if $\mu = 65$?

Solution.

Assuming H_0 is true, i.e $\mu = 60$. Note: the standardized sample mean is standard normal:

$$\boxed{\frac{\overline{X} - 60}{10/\sqrt{25}} = \frac{\overline{X} - 60}{2} \sim N(0, 1).}$$

$$\boxed{c = 62}$$

$$\begin{aligned}
\text{size} &= \mathbb{P}(H_0 \text{ rejected} \mid H_0) \\
&= \mathbb{P}(\bar{X} \geq 62 \mid \mu = 60) \\
&= \mathbb{P}\left(\frac{\bar{X} - 60}{2} \geq \frac{62 - 60}{2}\right) \\
&= \mathbb{P}(\phi \geq 1) = 1 - \mathbb{P}(\phi \leq 1) \\
&= 1 - \Phi(1) = 1 - 0.8413 = \underline{\underline{0.1587}}
\end{aligned}$$

$$\text{Power} = 1 - \beta$$

$$= 1 - \mathbb{P}(H_0 \text{ is } \underline{\underline{\text{not}}} \text{ rejected} \mid H_1)$$

$$= 1 - \mathbb{P}(\bar{X} < c \mid \underline{\underline{\mu = 65}})$$

$$= 1 - \mathbb{P}\left(\frac{\bar{X} - 65}{2} < \frac{c - 65}{2}\right)$$

$$= 1 - \mathbb{P}\left(\phi < \frac{62 - 65}{2} = -1.5\right)$$

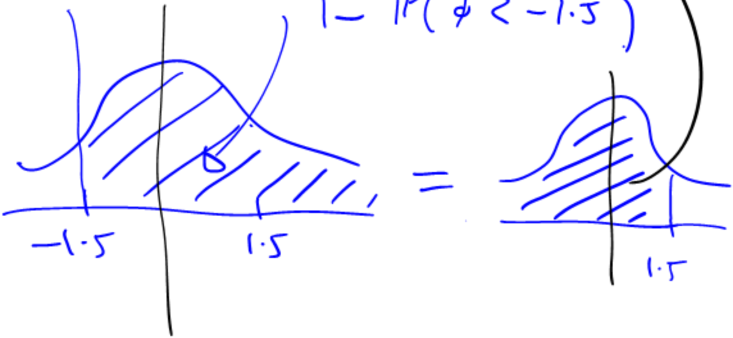
$$= 1 - \mathbb{P}(\phi < -1.5)$$

$$= 1 - (1 - \mathbb{P}(\phi < 1.5))$$

$\doteq \mathbb{P}(\phi < 1.5).$

$= 0.9332$ #.

$1 - \mathbb{P}(\phi < -1.5)$



$$\boxed{c = 63.29}$$

$$\text{Size} = \mathbb{P}\left(\phi > \frac{63.29 - 60}{2}\right)$$

$$= \mathbb{P}\left(\phi > \frac{3.29}{2}\right)$$

$$= \mathbb{P}(\phi > 1.645)$$

$$= 1 - \Phi(1.645)$$

$$= 1 - 0.9495 = 0.0505$$

$$\text{Power} = 1 - \mathbb{P}\left(\phi < \frac{63.29 - 65}{2}\right)$$

$$= 1 - \mathbb{P}(\phi < -0.855)$$

$$= 0.8051$$

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