Name:

Matric. no.: Tutor group:

March 2023 **CA2** TIME ALLOWED: 50 minutes

QUESTION 1. **(16 marks)**

(a) [4 marks] Which of the following are linear homogeneous recurrence relations? No justification is required.

- (i) ☐ $a_n = 7a_{n-2} + 6a_{n-4}$
- (ii) ☐ $a_n = a_{n-1} + 7$
- (iii) ☐ $a_n = 5a_{n-3}^2$
- (iv) ☐ $a_n = -a_{n-1} + a_{n-2} - a_{n-3}$
- (v) ☐ $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$
- (vi) ☐ $a_n = a_{n-1}a_{n-2} + a_{n-2}a_{n-3}$

(b) [6 marks] Solve the following linear recurrence, that is, write a_n in terms of n:

$a_n = 3a_{n-1} + 10a_{n-2}$ for each $n \geqslant 2$, with initial conditions $a_0 = 2$, $a_1 = 10$.

(c) [6 marks] Use induction to show that, for each $n \in \mathbb{N} - \{0\}$,

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}.$$

Solution

(a) [4 marks] (i), (iv), (v)

[Distribution: 1 mark for each item. 1 bonus mark for identifying all items correctly.]

(b) [6 marks] The characteristic equation is

$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0.$$

This equation has roots $s_1 = 5$ and $s_2 = -2$. Hence $a_n = u5^n + v(-2)^n$ for some u and v. Using the initial conditions, we find that $u = 2$ and $v = 0$. Thus $a_n = 2 \times 5^n$ for all $n \in \mathbb{N}$.

[Distribution: 3 marks for correct expression for a_n and 3 marks for the justification]

(c) [6 marks] Let $P(k)$ denote the predicate $1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{4k^3-k}{3}$. First we check the base case $P(1)$. Here the LHS, $1^2 = 1$ is equal to the RHS $\frac{4\times1^3-1}{3} = 1$.

Now we want to prove the proposition $\forall k \in \mathbb{N}, P(k) \rightarrow P(k+1)$. For our inductive hypothesis, assume $P(k)$ is true for some $k \in \mathbb{N} - \{0\}$. The LHS of $P(k+1)$ is equal to the LHS of $P(k)$ plus $(2(k+1)-1)^2$. Hence, using the inductive hypothesis, we have

$$\begin{aligned}
1^2 + 3^2 + 5^2 + \cdots + (2(k+1)-1)^2 &= 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2(k+1)-1)^2 \\
&= \frac{4k^3 - k}{3} + (2(k+1)-1)^2 \\
&= \frac{4k^3 - k}{3} + \frac{12k^2 + 12k + 3}{3} \\
&= \frac{4k^3 + 12k^2 + 11k + 3}{3} \\
&= \frac{4k^3 + 12k^2 + 12k + 4 - k - 1}{3} \\
&= \frac{4(k+1)^3 - (k+1)}{3}.
\end{aligned}$$

Thus, we have shown that $P(k+1)$ follows from $P(k)$, as required.

[Distribution: 2 marks for correct predicate, 1 mark for base case, 2 marks for inductive hypothesis, 1 mark for correctly using induction.]

For graders only:	Question	1(a)	1(b)	1(c)	2(a)	2(b)	2(c)	3(a)	3(b)	Total
	Marks									

QUESTION 2. **(17 marks)**

In this question **no justification is required**. For each part, give **an explicit number** as your answer, not an expression.

(a) A coin is tossed five times. In each case, the outcome H (for heads) or T (for tails) is recorded. (One possible outcome for the five tosses is denoted THHTT.)

(i) [2 marks] What is the total number of possible outcomes of the coin-tossing experiment?

(ii) [2 marks] In how many of the possible outcomes are exactly two tails obtained?

(b) Recall that a standard deck of cards has 52 cards. The cards can be classified according to suits or denominations. There are 4 suits, hearts, diamonds, spades and clubs. There are 13 cards in each suit; one for each of the 13 denominations: Aces, Kings, Queens, ..., ,Twos. A *poker hand* consists of five cards drawn from a standard deck. Note that the order in which the cards are drawn does not matter.

(i) [2 marks] How many poker hands consist of 2 Aces and 3 Kings?

(ii) [3 marks] How many poker hands consist of 2 Aces, 2 Kings, and a card whose denomination is neither an Ace nor a King?

(iii) [3 marks] How many poker hands have three cards from one denomination and two from another?

(iv) [3 marks] How many poker hands consist of five cards all from the same suit?

(c) [2 marks] How many distinguishable permutations of the word BOOKKEEPER are there?

Solution

(a) (i) [2 marks] $2^5 = 32$

(ii) [2 marks] $C(5,2) = 10$

(b) (i) [2 marks] $C(4,2) \times C(4,3) = 6 \times 4 = 24$

(ii) [3 marks] $C(4,2) \times C(4,2) \times (52-8) = 6 \times 6 \times 44 = 1584$

(iii) [3 marks] $13 \times C(4,3) \times 12 \times C(4,2) = 13 \times 4s \times 12 \times 6 = 3744$

(iv) [3 marks] $4 \times C(13,5) = 4 \times 1287 = 5148$

(c) [2 marks] 151200

QUESTION 3. **(17 marks)**

(a) Let $A = \{1, 2\}$, $B = \{a, b\}$, and $C = \{a, c\}$. Find each of the following sets. No justification is required.

(i) [2 marks] $A \times (B - C)$

(ii) [3 marks] $(A \times B) \cup (A \times C)$

(iii) [3 marks] The power set $P(B \cap C)$

(iv) [3 marks] The power set $P(P(\emptyset)) - \{\emptyset\}$

(b) Let A and B be sets.

(i) [3 marks] Show that $(A \times B) \cup (B \times A) \subseteq (A \cup B) \times (A \cup B)$.

(ii) [3 marks] Must $(A \times B) \cup (B \times A) = (A \cup B) \times (A \cup B)$? If so, prove it, otherwise give a counterexample.

Solution

(a)

(i) [2 marks] $\{(1, b), (2, b)\}$
[Distribution: 1 mark for each element]

(ii) [3 marks] $\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
[Distribution: 1/2 mark for each element]

(iii) [3 marks] $\{\emptyset, \{a\}\}$
[Distribution: 1 mark for each element, 1 bonus mark for everything correct]

(iv) [3 marks] $\{\{\emptyset\}\}$
[Distribution: 1 mark for $\emptyset$, 1/2 for each bracket]

(b) (i) [3 marks] Let $x \in (A \times B) \cup (B \times A)$. Then $x \in (A \times B)$ or $x \in (B \times A)$. First, assume $x \in (A \times B)$. Then $x = (x_1, x_2)$, where $x_1 \in A$ and $x_2 \in B$. Hence $x_1 \in A \cup B$ and $x_2 \in A \cup B$. Thus, $x \in (A \cup B) \times (A \cup B)$, as required. Lastly, assume $x \in (B \times A)$. Then $x = (x_1, x_2)$, where $x_1 \in B$ and $x_2 \in A$. Hence $x_1 \in A \cup B$ and $x_2 \in A \cup B$. Thus, $x \in (A \cup B) \times (A \cup B)$, as required.
[Distribution: 1 mark for $x = (x_1, x_2)$, 2 marks for the rest of the justification]

(ii) [3 marks] No. Counterexample: $A = \{1\}$ and $B = \{2\}$.
[Distribution: 1 mark for "No", 2 marks for the counterexample]