SC1004 Part 2

Lectured by Prof Guan Cuntai
(teaching materials by Prof Chng Eng Siong)

Email: ctguan@ntu.edu.sg

Quiz 2 and Exam:

1. **Quiz 2**

- Coverage : Ch 6 ,7, 8

- Time/Date: Week 13, last lecture time (10:30-11.20am, 17th April 2024)

2. Final Exam

- **Coverage** : Ch 6, 7, 8 (Q3 & Q4)

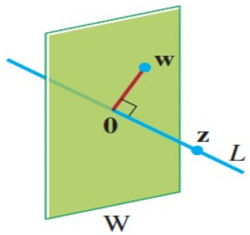
- **Date/Time**: 2 May 2024 (Thursday), 1.00-3.00pm

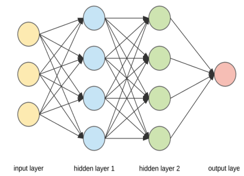
(Ch 9 will not be tested)

Syllabus for Part 2

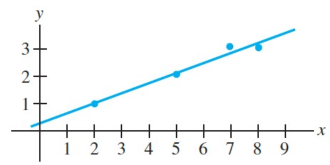
Chapter	Topics	Week (Lecture)	Week (Tut)
6	Orthogonality	8-9	9-10
7	Least Squares	9-10	10-11
8	EigenValue and Eigenvectors	11-12	12-13
9	Singular Value Decomposition (SVD)	13	

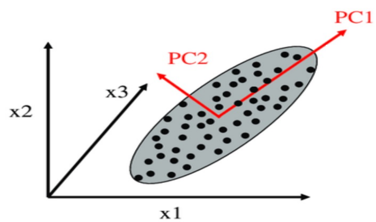
Table 1: schedule





$A\mathbf{x} = \mathbf{b}$





Online Video learning Schedule

https://www.youtube.com/channel/UCBzG5jg3huxiPkCt-Serrjw

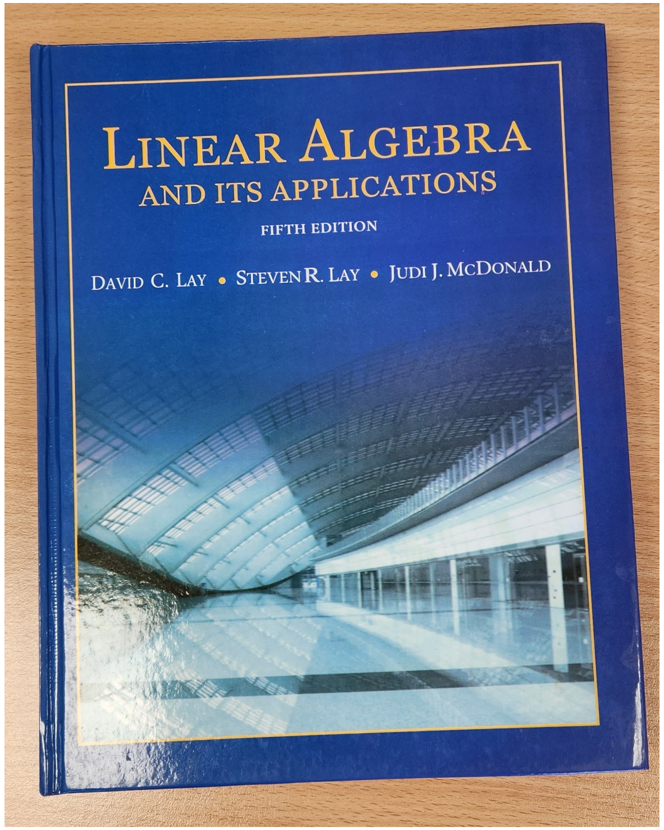
Week	Part	Topic	Notes
8	6.1.1-6.2.3	Orthogonality, Normalization, Dot-Product, Inequalities,	Lecture 1: **6.1.1 - 6.1.3** Lecture 2: **6.1.4 - 6.2.3**
9	6.2.4-6.3.2	Orthogonal/Orthonormal Sets, Basis, Gram Schmidt and QR Decomposition	Lecture 3: **6.2.4** Lecture 4: **6.2.5 – 6.3.2**
10	7.1.1-7.2.1	Least Squares and Normal Eqn, Projection Matrix, Applications	Lecture 5: **7.1.1 – 7.1.3** Lecture 6: **7.1.4 – 7.2.1**
11	8.1.1-8.1.2	Eigenvectors, Eigen-values, Characteristics Eqn	Lecture 7: **8.1.1** Lecture 8: **8.1.2**
12	8.1.3-8.1.5	Diagonalisation, Power of A, Change of basis	Lecture 9: **8.1.3** Lecture 10: **8.1.4 – 8.1.5**
13	9.1.1-9.2	Introduction to SVD and PCA (Not examined in quiz/exam)	Lecture 11: **9.1.1 – 9.2** Lecture 12: **Quiz 2**

How will we conduct the course?

1) Before the lectures, watch the videos according to the schedule in Table 1
   - You can watch past years zoom video recordings at
   https://www.youtube.com/@linearalgebra1884/playlists?view=50&sort=dd&shelf_id=2

2) During lecture hours –
   - We will summarize the lectures and highlight the key points
   - Q&A.

References



Linear Algebra and Its Applications
by David Lay, Steven Lay, Judi McDonald

3Blue1Brown on YouTube

Essence of linear algebra preview

https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab

Lecture (Week 11)

(Chapter 8.1.1-8.1.2)

Key points – Overview of Chapter 8

- Week 11
  - Eigenvalues and eigenvectors
    - Definition and explanations
    - Find eigenvectors given an eigenvalue
    - Eigenspace
    - Find eigenvalues

- Week 12
  - Diagonalization
    - Motivation of diagonalization
    - Using eigenvalues and eigenvectors to diagonalize a matrix
    - Calculation of the power of a matrix
  - Coordinate system and change of basis
    - Understanding the concept of changing basis

Key points – 8.1.1 Eigenvalue & Eigenvector

- Definition
  - For a $n \times n$ square matrix A: if $A\boldsymbol{x} = \lambda\boldsymbol{x}$, then
    - λ is an eigenvalue of matrix A
    - $\boldsymbol{x}$ is the eigenvector corresponding to λ ($\boldsymbol{x}$ is non-zero)
    - Each A has up to n eigenvalues

A

- Example:
  - $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, if $\boldsymbol{u}$ and $\boldsymbol{v}$ are the eigenvectors?
  - $A\boldsymbol{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda\boldsymbol{u}$. So, $\boldsymbol{u}$ is not an eigenvector
  - $A\boldsymbol{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\,\boldsymbol{v}$, $\lambda = 2$. So, $\boldsymbol{v}$ is an eigenvector
  - Geometric interpretation of eigenvector and eigenvalue: transformed vector by A is the scaling of the vector – scaled by eigenvalue λ.
  - In linear algebra, knowing which vectors have their directions unchanged by a given linear transformation is important. The eigenvectors and eigenvalues of a transformation serve to characterize it. They play important roles in all the areas where linear algebra is applied, from geology to quantum mechanics.

The word eigenvalue comes from the German Eigenwert which means "proper or characteristic value."

- Note: eigenvalue/eigenvector is one of the most important concept in linear algebra, with many applications. We will learn two applications later: diagonalize a matrix, Principal Component Analysis (PCA).

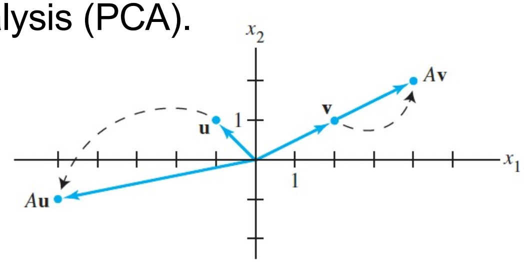
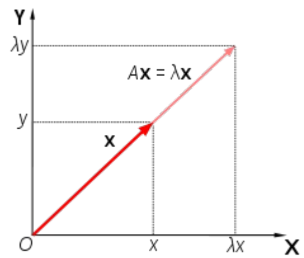


FIGURE 1 Effects of multiplication by A.



Key points – 8.1.1 Find Eigenvectors

- How to find the eigenvectors given an eigenvalue (we will learn how to find eigenvalues later)
  - General formula: $A\boldsymbol{x} = \lambda\boldsymbol{x}$ ➔ $A\boldsymbol{x} - \lambda\boldsymbol{x} = \mathbf{0}$ ➔ $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$

$(A - \lambda I)$

  - So, the eigenvector is the non-zero solution of above equation.

$Ax = 0$

- Example: $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ has an eigenvalue of 7.
  - $(A - 7I)\boldsymbol{x} = \mathbf{0}$
  - $\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\boldsymbol{x} = \mathbf{0}$ ➔ $\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\boldsymbol{x} = \mathbf{0}$ ➔ $\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$
  - Using row reduction: $\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
  - We get $x_1 - x_2 = 0$ ➔ $x_1 = x_2$ ➔ solution is $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
  - where x_2 is a free variable.
  - There are infinite eigenvectors corresponding to $\lambda = 7$.

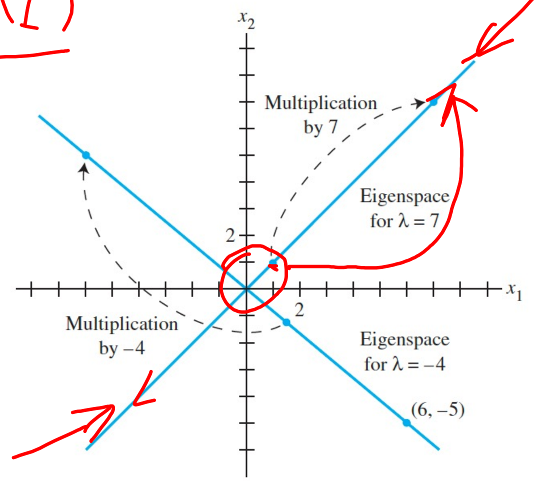


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

Therefore, eigenvector corresponding to $\lambda = 7$ is not a single vector. The entire line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors!

Key points – 8.1.1 Find Eigenvectors (2)

- Example: $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ has another eigenvalue of -4.
  - $(A + 4I)\boldsymbol{x} = \boldsymbol{0}$
  - $\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\boldsymbol{x} = \boldsymbol{0}$ ➔ $\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}$ ➔ $\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{0}$
  - Using row reduction: $\begin{bmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
  - We get $5x_1 + 6x_2 = 0$ ➔ $x_1 = -\frac{6}{5}x_2$
  - solution is $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{6}{5}x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix} = x_2\begin{bmatrix} 6 \\ -5 \end{bmatrix}$
  - where x_2 is a free variable.
  - There are infinite eigenvectors corresponding to $\lambda = -4$.

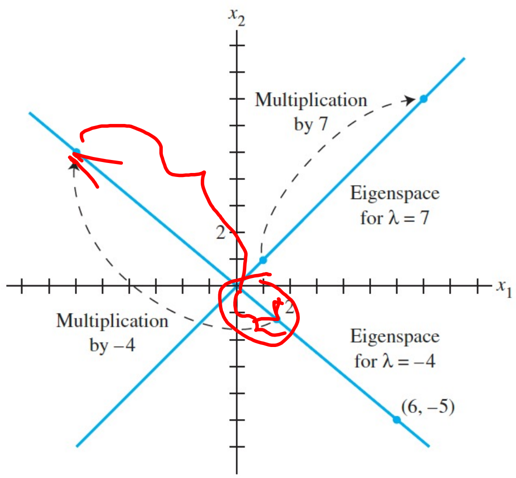


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

Key points – 8.1.1 Eigenspace

- Definition: for an $n{\times}n$ square matrix A
  - The set of all solutions of $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ is the null space of matrix $A - \lambda I$: $\{\mathbf{0}, \boldsymbol{x}\}$
  - This set is a subspace in R^n, called an eigenspace of A corresponding to λ (Note: $\boldsymbol{x}$ is in R^n).

- Recall the eigenvectors for $\lambda = -4$ and $\lambda = 7$

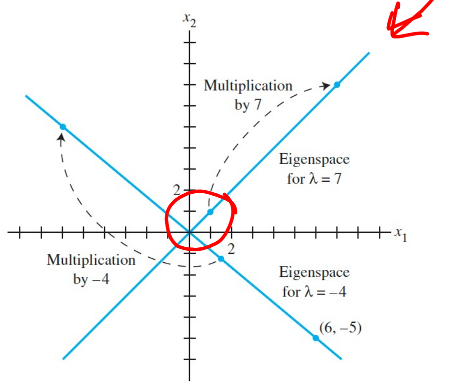


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

Key points – 8.1.1 Eigenspace (2).

$Ax = \lambda x$

- Example: $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, $\lambda = 2$
- From $A - \lambda I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$
- Row deduction: $\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- We find: $2x_1 - x_2 + 6x_3 = 0$, $x_1 = \frac{1}{2}x_2 - 3x_3$

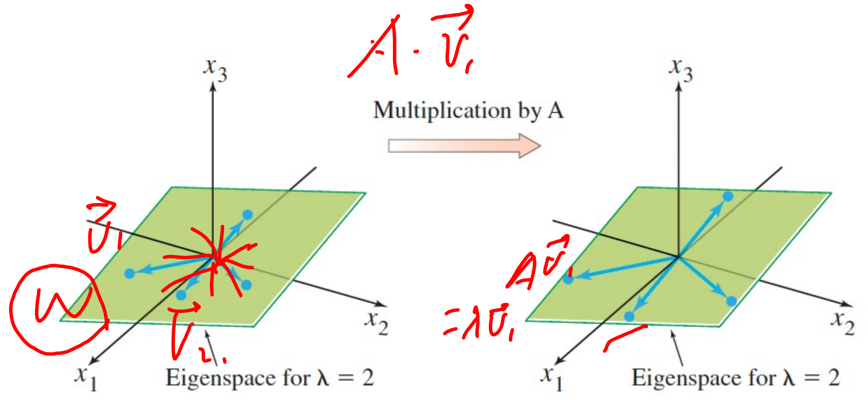


FIGURE 3 *A* acts as a dilation on the eigenspace.

- Eigenvectors are: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = x_2 \boldsymbol{a}_1 + x_3 \boldsymbol{a}_2$, where $\boldsymbol{a}_1 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are independent!
- Here, we have infinite eigenvectors corresponding to $\lambda = 2$.
- The eigenvectors are, in fact, the linear combinations of two independent vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, which span the subspace (it is called an eigenspace).
- Geometric interpretation: eigenvectors are all the vectors in the eigenspace spanned by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. In the eigenspace, each eigenvector will be dilated by λ after applying the transformation A to it.

Key points – 8.1.2 Find Eigenvalues

- Definition: for an $n \times n$ square matrix A
  - Eigenvalues can be found using the "characteristic equation" by solving a polynomial.
  - From the definition of eigenvectors: $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$
  - It has non-zero solutions, so $A - \lambda I$ has dependent columns
    - So, $A - \lambda I$ does not have full rank (not invertible)
    - which is equivalent to $\det(A - \lambda I) = 0$
  - From $\det(A - \lambda I) = |A - \lambda I| = 0$ we can find eigenvalues.
  - $\det(A - \lambda I) = 0$ is called "characteristic equation" which is in polynomial form.

$A - \lambda I$

A

- Examples: $A_1 = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Key points – 8.1.2 Find Eigenvalues: examples

- Examples: $A_1 = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

$A_{n \times n}$ max: n

$n_\lambda < n$

(1) $\det(A_1 - \lambda I) = 0 \rightarrow \det(\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}) = \det(\begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}) = 0$

$(3-\lambda)(-\lambda) - (-2) = 0$, $\lambda^2 - 3\lambda + 2 = 0$, $(\lambda - 2)(\lambda - 1) = 0$,

So, we found the eigenvalues: $\lambda = 1$ & $\lambda = 2$

Note:
$\lambda^2 - 3\lambda + 2 = 0$
is called characteristic polynomial

(2) $\det(A_2 - \lambda I) = 0 \rightarrow \det(\begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}) = 0$

$(1-\lambda)(2-\lambda) - 30 = 0$, $\lambda^2 - 3\lambda - 28 = 0$, $(\lambda - 7)(\lambda + 4) = 0$,

So, we found the eigenvalues: $\lambda = 7$ & $\lambda = -4$

Note:
For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its determinant
$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \boxed{ad - bc}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(3) $\det(A_3 - \lambda I) = 0 \rightarrow \det(\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}) = 0$

$(2-\lambda)(-6-\lambda) - 9 = 0$, $\lambda^2 + 4\lambda - 21 = 0$, $(\lambda - 3)(\lambda + 7) = 0$,

So, we found the eigenvalues: $\lambda = 3$ & $\lambda = -7$

① repeat
② eigenvalue $= 0$? ✓

Key points – 8.1.2 Find Eigenvalues: Triangular Matrix

- Definition:
  - For any triangular matrix (upper or lower triangle):

    $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ or $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

  - Its characteristic equation $\det(A - \lambda I) = \det\left(\begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}\right) = 0,$

    Or $\det\left(\begin{bmatrix} a_{11} - \lambda & 0 & 0 \\ a_{21} & a_{22} - \lambda & 0 \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix}\right) = 0$

  - Becomes: $\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0$

    So, the eigenvalues are: $\lambda = a_{11}, \lambda = a_{22}, \lambda = a_{33}$, which are the values of the diagonal entries.

Key points – 8.1.2 Eigenvalues: More Examples

- $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$
- Eigenvalues for A: $3, 0, 2$

x non-zero

- Explain:
  - What does an eigenvalue 0 mean?
    - By definition: $A\boldsymbol{x} = \lambda\boldsymbol{x}$, since $\lambda = 0$, we have $A\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$
    - It means A has dependent columns, so we can get non-zero solution for $A\boldsymbol{x} = 0$
    - In this case, A is not invertible. $\Leftrightarrow$ A has an eigenvalue of 0.

Key points – 8.1.2 Eigenvalues: More Examples (2)

- $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 11 & 0 \\ 5 & 3 & 4 \end{bmatrix}$ 3×3

$n \times n$

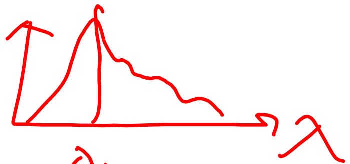
- Eigenvalues for B: $11, 4$ (4 repeated twice)

$\det(A - \lambda I) = (4-\lambda)(11-\lambda)(4-\lambda)$

$= (4-\lambda)^2 (11-\lambda)$

- Explain:
  - ○ $\lambda = 4$ repeated twice, we denote the number of repetitions as algebraic multiplicity.
  - ○ algebraic multiplicity will be discussed in 8.1.3 to determine if a matrix can be diagonalized.

Key points – 8.1.2 Spectrum of a matrix



$N_\lambda < n$

- Definition:
  - For an $n \times n$ square matrix A
  - The set of eigenvalues $(\lambda_1, \lambda_2, \cdots, \lambda_{N_\lambda})$ is called a spectrum of A.
  - The characteristic equation is:

$$P(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots \left(\lambda - \lambda_{N_\lambda}\right)^{n_{N_\lambda}} = 0.$$

$$\sum_{i=1}^{N_\lambda} n_i = n$$

(PCA)

n_i

  - For each eigenvalue λ_i, there is a corresponding EigenSpace $E(\lambda_i)$
  - n_i is the number of repetitions of the i^{th} eigenvalues λ_i, also called algebraic multiplicity.

① $\lambda_i \leq n$ distinct

② each $\lambda_i \sim$ eigenspace

③ $\lambda = 0 \sim A$ is not invertible

④ none of $\lambda = 0$, A is invertible

⑤ $\underbrace{\lambda_1 \cdots \lambda_{N_\lambda}}$ $\sim$ $\underbrace{\vec{v}_1 \cdots \vec{v}_{N_\lambda}}_{\text{indep}}$

Key points – Independence of Eigenvectors Corresponding to Eigenvalues

- Definition:
  - If $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ of an $n \times n$ matrix A, then $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$ are linearly independent.
- Explain
  - Assume $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r\}$ is linearly dependent.
  - Since $\boldsymbol{v}_i$ is nonzero, so, one of the vectors in the set is a linear combination of the preceding vectors which are independent.

$$\boldsymbol{v}_{p+1} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p$$

  - Multiplying both sides by A, we obtain

$$A\boldsymbol{v}_{p+1} = c_1 A\boldsymbol{v}_1 + c_2 A\boldsymbol{v}_2 + \cdots + c_p A\boldsymbol{v}_p \quad (\text{use } A\,\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i) \;\Rightarrow\; \lambda_{p+1}\boldsymbol{v}_{p+1} = c_1\lambda_1\boldsymbol{v}_1 + c_2\lambda_2\boldsymbol{v}_2 + \cdots + c_p\lambda_p\boldsymbol{v}_p$$

  - Multiply λ_{p+1} to both sides of $\boldsymbol{v}_{p+1} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p \Rightarrow \lambda_{p+1}\boldsymbol{v}_{p+1} = c_1\lambda_{p+1}\boldsymbol{v}_1 + c_2\lambda_{p+1}\boldsymbol{v}_2 + \cdots + c_p\lambda_{p+1}\boldsymbol{v}_p$
  - Subtract above two equations, we get $c_1(\lambda_1 - \lambda_{p+1})\boldsymbol{v}_1 + c_2(\lambda_2 - \lambda_{p+1})\boldsymbol{v}_2 + \cdots + c_p(\lambda_p - \lambda_{p+1})\boldsymbol{v}_p = 0$
    - Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r\}$ is linearly independent, the weights must be zero.
    - But $\lambda_i - \lambda_{p+1} \neq 0$ as the eigenvalues are distinct
    - Hence $c_i = 0$ (for $i = 1, \cdots, p$) $\Rightarrow \boldsymbol{v}_{p+1} = 0$, which contradicts with non-zero eigenvectors.
  - So, $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$ must be linearly independent.

End