SC1004 Part 2

Lectured by Prof Guan Cuntai
(teaching materials by Prof Chng Eng Siong)

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Quiz 2 and Exam:

1. Quiz 2

- Coverage : Ch 6 ,7, 8

- Time/Date: Week 13, last lecture time (10:30-11.20am, 17th April 2024)

2. Final Exam

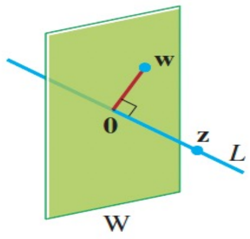
- **Coverage** : Ch 6, 7, 8 (Q3 & Q4)
- **Date/Time**: 2 May 2024 (Thursday), 1.00-3.00pm

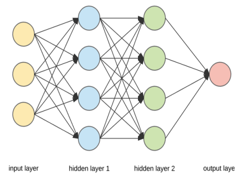
(Ch 9 will not be tested)

Syllabus for Part 2

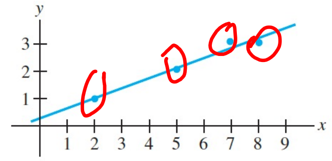
Chapter	Topics	Week (Lecture)	Week (Tut)
6	Orthogonality dot prod u v	8-9	9-10
7	Least Squares	9-10	10-11
8	EigenValue and Eigenvectors	11-12	12-13
9	Singular Value Decomposition (SVD)	13	

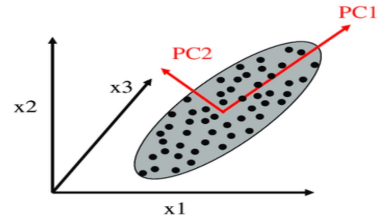
Table 1: schedule





$A\mathbf{x} = \mathbf{b}$





Online Video learning Schedule (2022/S2)

Table 1: schedule (2022/S2)

Week	Part	Topic	Notes
8	6.0-6.2.2	Orthogonality, Normalization, Dot-Product, Inequalities,	
9	6.2.3-6.3.3	Orthogonal/Orthonormal Sets, Basis, Gram Schmidt and QR Decomposition	
10	7.1.1-7.2.1	Least Squares and Normal Eqn, Projection Matrix, Applications	
11	8.1.0-8.1.2	Eigenvectors, Eigen-values, Characteristics Eqn	
12	8.1.3-8.1.5,	Diagonalisation, Power of A, Change of basis	
13	9.1.1-9.1.4	Introduction to SVD and PCA	Not examined in quiz/exam

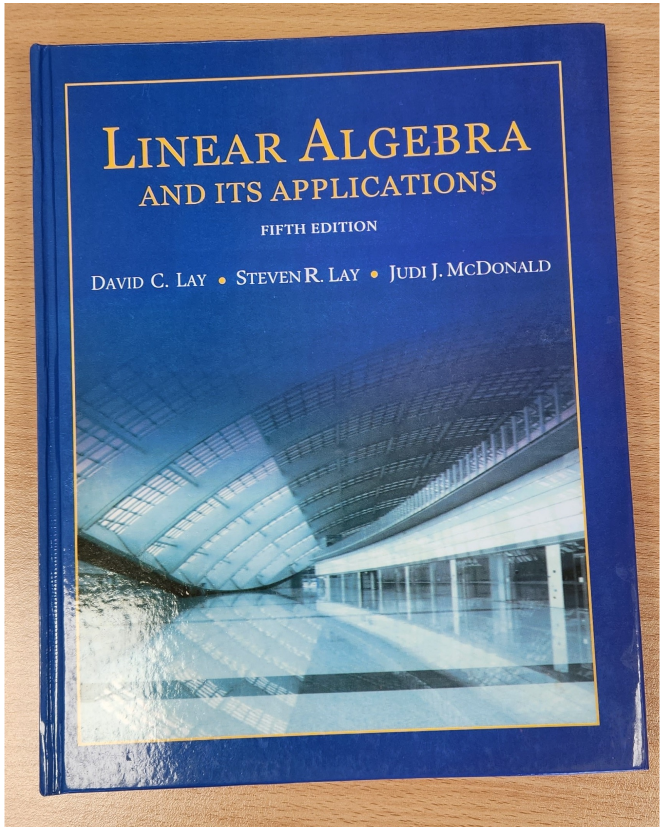
https://www.youtube.com/channel/UCBzG5jg3huxiPkCt-Serrjw

How will we conduct the course?

1) Before the lectures, watch the videos according to the schedule in Table 1
   - You can watch past years zoom video recordings at https://www.youtube.com/@linearalgebra1884/playlists?view=50&sort=dd&shelf_id=2

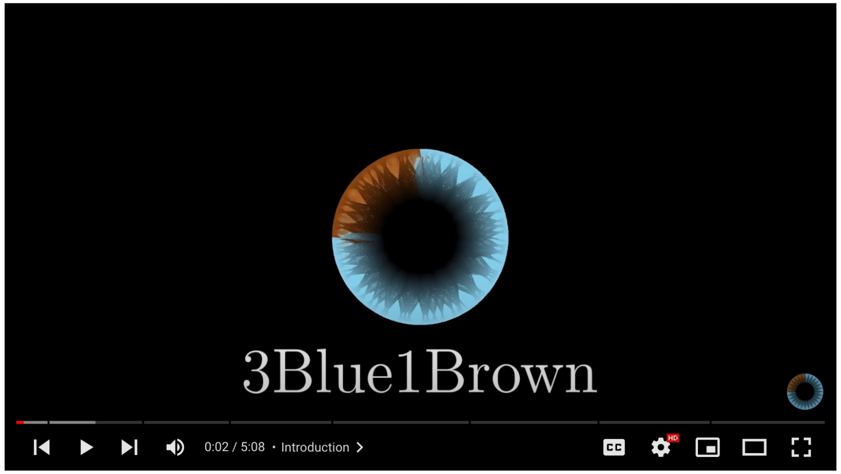
2) During lecture hours –
   - We will summarize the lectures and highlight the key points
   - Q&A.

References



Linear Algebra and Its Applications
by David Lay, Steven Lay, Judi McDonald

3Blue1Brown on YouTube



Essence of linear algebra preview

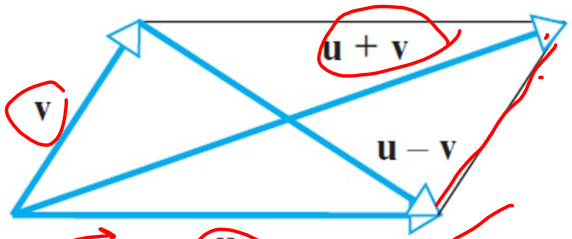
https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab

Lecture (Week 8)

(Chapter 6.1.1- 6.2.2)

Key points – 6.1.1 Geometric Vectors

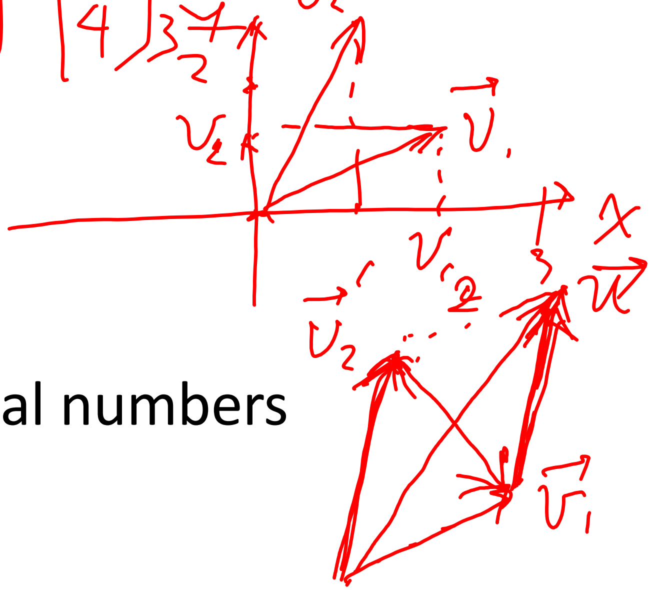
$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$



- Vector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ v_i

$u = \begin{bmatrix} 2+1 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

- Vector direction & length
  - $\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ ✓
- Vector addition & subtraction
  - $\boldsymbol{u} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ ✓
  - $\boldsymbol{u} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ $\vec{v}_1 = \vec{u} + \vec{v}_2$
- Euclidean space: R^n $- n$ dimensional real numbers



Key points – 6.1.2 Norm (Euclidean Norm)

- Norm: $\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
  - $\|\boldsymbol{v}\| \geq \mathbf{0}$
  - $\|\boldsymbol{v}\| = \mathbf{0} \;\; iif \;\; \boldsymbol{v} = 0$
  - $\|k\boldsymbol{v}\| = |k|\|\boldsymbol{v}\|$

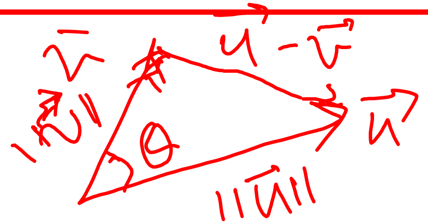
$$\vec{y} = \vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ \vdots \\ u_n - v_n \end{bmatrix}$$

- Normalizing a vector (unit length vector)
  - $\boldsymbol{u} = \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$

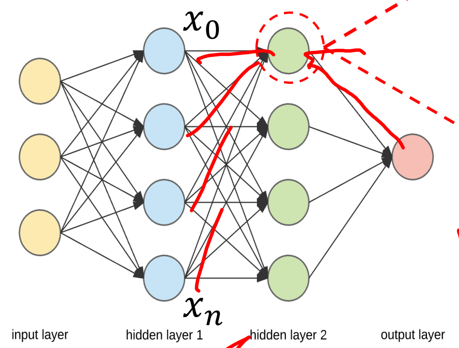
$$\|\vec{u}\| = \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \frac{\|\vec{v}\|}{\|\vec{v}\|} = 1$$

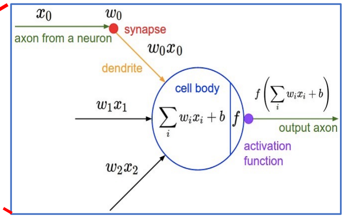
- Vector distance
  - $\boldsymbol{dist}\,(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$

Key points – 6.1.3 Dot Product/Inner Product



- Definition
  - $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$
  - Geometric formula: $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\ \|\boldsymbol{v}\|\ cos\theta$
  - $cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}$
  - $if\ \|\boldsymbol{u}\| = 1, \|\boldsymbol{v}\| = 1, cos\theta = \boldsymbol{u} \cdot \boldsymbol{v}$
  - $\|\boldsymbol{u}\|^2 = \boldsymbol{u} \cdot \boldsymbol{u}$, or $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$
  - Component formula: $\boxed{\boldsymbol{u} \cdot \boldsymbol{v}} = u_1\ v_1 + \cdots + u_n\ v_n$





$f(\sum_i \omega_i\, x_i + b)$

$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}$

Law of cos:

$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$

$\|\vec{u}-\vec{v}\|^2 = (u_1-v_1)^2 + (u_2-v_2)^2$

$= u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2(u_1v_1 + u_2v_2)$

$= \|u\|^2 + \|\vec{v}\|^2 - 2(u_1v_1 + u_2v_2)$

- Explanation of dot product using the geometric formula
  - Projection: $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\ (\|\boldsymbol{v}\|\ cos\theta) = \|\boldsymbol{v}\|\ (\|\boldsymbol{u}\|\ cos\theta)$
  - Perpendicular: $\boldsymbol{u} \cdot \boldsymbol{v} = 0$

$\cos\theta = 0$, $\theta = \frac{\pi}{2}$

Key points – 6.1.3 Dot Product/Inner Product (2).

scalar $\vec{u} \cdot \vec{v} = u_1 v_1 + \dots + u_n v_n$

- Properties of dot product

Dot products have many of the same algebraic properties as products of real numbers.

THEOREM 3.2.2 *If* **u**, **v**, *and* **w** *are vectors in* R^n, *and if* k *is a scalar, then:*

(a)	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	[Symmetry property]
(b)	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$	[Distributive property]
(c)	$k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$	[Homogeneity property]
(d)	$\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$	[Positivity property]

$\vec{u} \cdot \vec{v} = \boxed{\vec{u}^T \vec{v}}$

- Transformation on dot product
  - $\boxed{A\boldsymbol{u}} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot A^T \boldsymbol{v}$
  - $\boldsymbol{u} \cdot A\boldsymbol{v} = A^T \boldsymbol{u} \cdot \boldsymbol{v}$
    - ➢ Using $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$, and $(AB)^T = B^T A^T$ to derive

$A\vec{u} \cdot \vec{v}$

$\vec{u} = \begin{bmatrix} u_1 \\ u_n \end{bmatrix}$

$\vec{u}^T = [u_1 \quad u_n]$

$\vec{u}^T \cdot \vec{v} = [u_1 \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

Key points – 6.1.4 Inequalities

- Inequalities
  - $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$
  - Triangular inequality: $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$

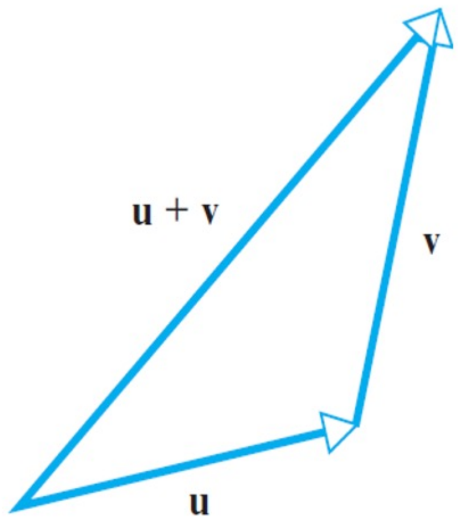
THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1 v_1 + u_2 v_2 + \cdots + u_n v_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2} (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$



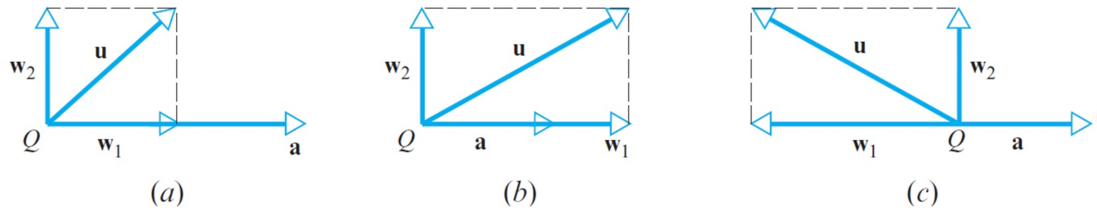
Key points – 6.2.1 Orthogonality

- Definition (vectors orthogonal to each other)
  - $\boldsymbol{u} \cdot \boldsymbol{v} = 0$
  - $cos\theta = 0$ --> $\theta = 90^{\circ}$, or $\theta = \pi/2$

- Orthonormal
  - $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal with unit length ($\|\boldsymbol{u}\|\boldsymbol{=}1\boldsymbol{,}$ $\|\boldsymbol{v}\| = 1$)

Key points – 6.2.2 Orthogonal Projection

- Decomposition of a vector
  - Standard basis in R^n



- Projection theorem
  - $\boldsymbol{w}_1 = Proj_a\, \boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$ (projection)
  - $\boldsymbol{w}_2 = \boldsymbol{u} - Proj_a\, \boldsymbol{u} = \boldsymbol{u} - \frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$ (residual)
  - $\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2$
  - Distance from $\boldsymbol{u}$ to $\boldsymbol{a}$: $\|\boldsymbol{u} - \boldsymbol{w}_1\| = \|\boldsymbol{w}_2\|$

Key points – 6.2.3 Orthogonal Sets and Basis

- A set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ in R^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0$, whenever $i \neq j$.
  - If $p = n$, $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_n\}$ spans R^n
  - If $p < n$, $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ spans a subspace W in R^n
    - $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ are the basis of the subspace
    - Standard basis for Euclidian space of R^3 : $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Key points – 6.2.3 Orthogonal Decomposition

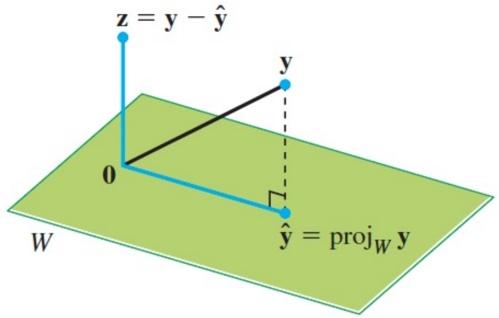
- Project a vector $\boldsymbol{y}$ on to subpace spanned by $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ in R^n
  - Let W be a subspace of R^n. Then each $\boldsymbol{y}$ in R^n can be written **uniquely** in the form:

$$\boldsymbol{y} = \widehat{\boldsymbol{y}} + \boldsymbol{z}$$

Where $\widehat{\boldsymbol{y}}$ is in W and $\boldsymbol{z}$ is in $W^{\perp}$.

If $\{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_p\}$ is any orthogonal basis of W, then

$$\widehat{\boldsymbol{y}} = Proj_W \boldsymbol{y} = \frac{\boldsymbol{y}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}\boldsymbol{u}_1 + \cdots + \frac{\boldsymbol{y}\cdot\boldsymbol{u}_p}{\boldsymbol{u}_p\cdot\boldsymbol{u}_p}\boldsymbol{u}_p$$



Key points – for tutorial questions

- Orthogonal matrix A
  - If A is square with orthonormal columns (in fact, the row of an orthogonal matrix is also orthonormal)
- Vector orthogonal to a subspace
  - If a vector $\boldsymbol{u}$ is orthogonal to every vector in a subspace W of R^n, then $\boldsymbol{u}$ is said to be orthogonal to W – all $\boldsymbol{u}$ called the orthogonal complement of W ($W^{\perp}$)

End