QUESTION 1. **(15 marks)**

(a) [5 marks] Which integer $a \in \{0, 1, \ldots, 4\}$ is congruent to 2021 modulo 5?

(b) [5 marks] Which integer $a \in \{0, 1, \ldots, 9\}$ is congruent to 1812^{56} modulo 10? Justify your answer.

(c) [5 marks] Let $S = \{\text{integers congruent to 7 modulo 6}\}$ and Δ be multiplication. Is S closed under Δ? Justify your answer.

Solution:

(a) $a = 1$.

(b) $a = 6$.

Indeed, $1812 \equiv 2 \pmod{10}$. And $2^5 \equiv 2 \pmod{10}$.

$$\begin{aligned} 1812^{56} &\equiv 2^{56} \pmod{10} \\ &\equiv 2 \cdot (2^5)^{11} \pmod{10} \\ &\equiv 2 \cdot 2^{11} \pmod{10} \\ &\equiv 2^2 \cdot (2^5)^2 \pmod{10} \\ &\equiv 2^4 \equiv 6 \pmod{10}. \end{aligned}$$

[Distribution: 2 marks for $a = 2$ and 3 marks for the justification]

(c) Here S is closed under Δ. Indeed, for generic elements $x \in S$ and $y \in S$, we can write $x = 6p + 7 = 6(p + 1) + 1$ and $y = 6q + 7 = 6(q + 1) + 1$ for some integers p and q. Then

$$\begin{aligned} x \cdot y &= (6(p + 1) + 1)(6(q + 1) + 1) \\ &= 6^2(p + 1)(q + 1) + 6(p + 1) + 6(q + 1) + 1 \\ &= 6(6(p + 1)(q + 1) + p + q + 2) + 1, \end{aligned}$$

which is congruent to 1 modulo 6. Therefore

$$xy \equiv 1 \equiv 7 \pmod{6}$$

[Distribution: 2 marks for correctly identifying that S is closed under Δ and 3 marks for the justification]

QUESTION 2. **(15 marks)**

Let $\mathbb{Q}$ denote the set of rational numbers. Consider the predicate $P(x, y, z) =$ "$x(y+z) = 2021$". Determine the truth value of the following statements. Justify your answers.

(i) [5 marks] $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ \exists z \in \mathbb{Q},\ P(x, y, z)$;

(ii) [5 marks] $\exists x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ \exists z \in \mathbb{Q},\ P(x, y, z)$;

(iii) [5 marks] $\exists x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ \forall z \in \mathbb{Q},\ P(x, y, z)$.

Solution:

(i) False: counterexample when $x = 0$

(ii) True: Take $x = 1$ and for any fixed y, take $z = 2021 - y$

(iii) False. Consider the negation:

$$\forall x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ \exists z \in \mathbb{Q},\ \neg P(x, y, z).$$

If $x = 0$ then $\neg P(x, y, z)$ is true. Otherwise, for nonzero $x \in \mathbb{Q}$, and fixed $y \in \mathbb{Q}$, take $z = -y$, whence $\neg P(x, y, z)$ is true.

[Distribution: for each part, 2 marks for correctly identifying the truth value and 3 marks for the justification]

QUESTION 3. **(20 marks)**

(a) [5 marks] Use a truth table to prove or disprove the following equivalence.

$$(p \lor (p \to F)) \land q \equiv p \to q$$

(b) [5 marks] Prove the following equivalence using conversion theorem, De Morgan's law, double negation, and distributivity (noting where each is used)

$$p \lor (\neg(q \to r)) \equiv (p \lor q) \land (\neg p \to \neg r)$$

(c) [10 marks] Decide whether or not the following argument is valid:

$$\begin{aligned} &q \land r \to p; \\ &T \to p \land r; \\ &p \to (\neg r \to s); \\ &r \to \neg s; \\ \therefore\, &s \lor q. \end{aligned}$$

Briefly justify your answers.

Solution:

(a)

p	q	$p \to F$	$p \lor (p \to F)$	$(p \lor (p \to F)) \land q$	$p \to q$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	T	T
F	F	T	T	F	T

The truth table disproves the equivalence!

[Distribution: 2 marks for correctly identifying nonequivalence of the statements and 3 marks for the justification]

(b)

$$\begin{aligned} p \lor (\neg(q \to r)) &\equiv p \lor (\neg(\neg q \lor r)) && \text{conversion theorem} \\ &\equiv p \lor (\neg\neg q \land \neg r)) && \text{De Morgan} \\ &\equiv p \lor (q \land \neg r)) && \text{double negation} \\ &\equiv (p \lor q) \land (p \lor \neg r) && \text{distributivity} \\ &\equiv (p \lor q) \land (\neg p \to \neg r) && \text{conversion theorem} \end{aligned}$$

[Distribution: 1 mark for each line]

(c) The argument is invalid. Counterexample:

p	q	r	s
T	F	T	F

[Distribution: 5 marks for correctly identifying the argument is invalid and 5 marks for the justification]