NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 2 - Propositional Logic Summary

Logical Operators: Three Basic Operators

- $\wedge$ Conjunction (and)
- $\vee$ Disjunction (or)
- $\neg$ Negation (not, alternatively ~)

Propositions

p, q

T F

Equivalent Expressions: The Statements

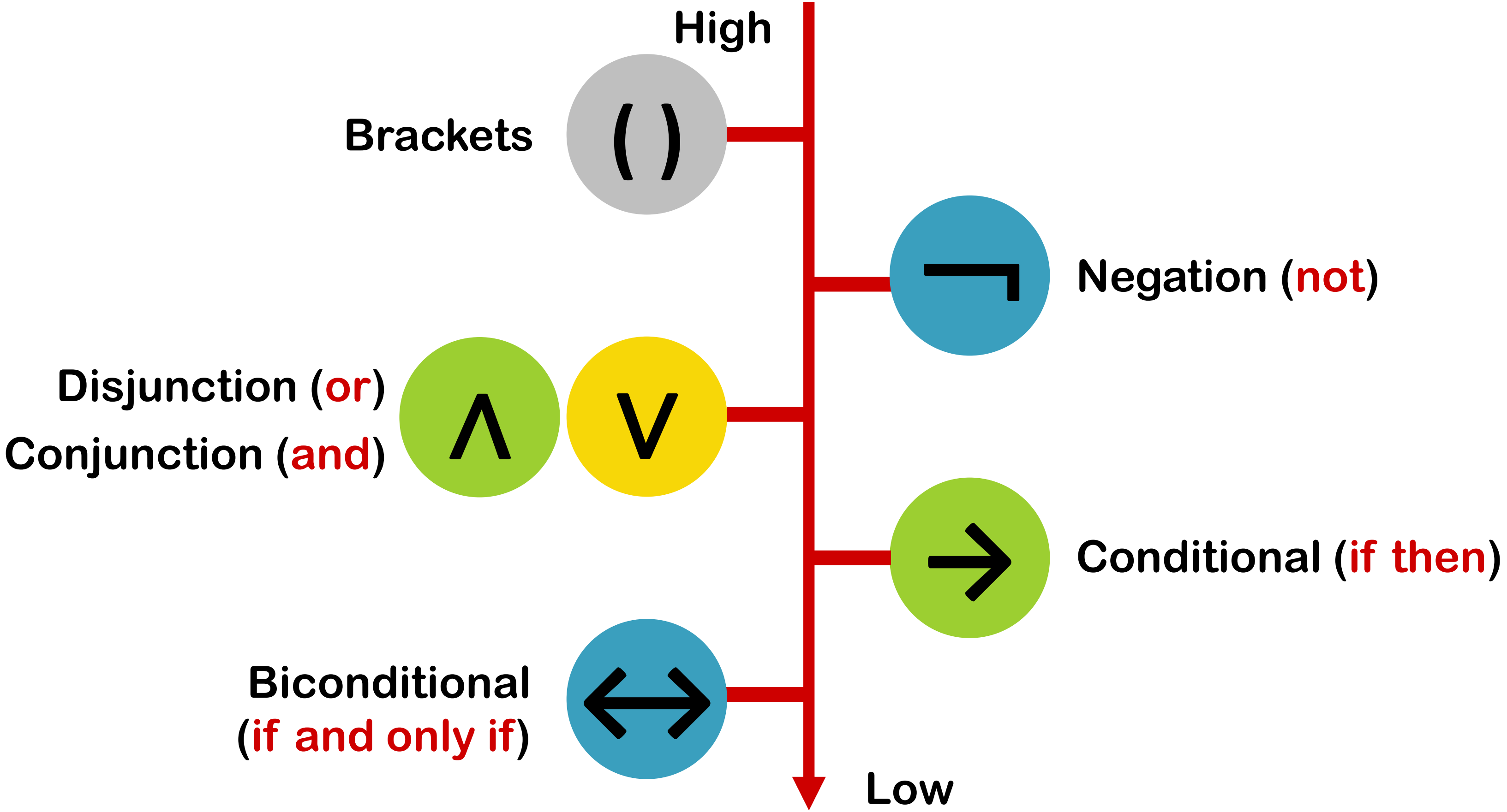
These three statements are equivalent:

$$\neg h \wedge \neg b \equiv \neg b \wedge \neg h \equiv \neg(b \vee h)$$

has same truth table

b h	$\neg b$	$\neg h$	$b \vee h$	$\neg h \wedge \neg b$	$\neg b \wedge \neg h$	$\neg(b \vee h)$
T T	F	F	T	F	F	F
T F	F	T	T	F	F	F
F T	T	F	T	F	F	F
F F	T	T	F	T	T	T

Operator Precedence: High to Low



Example

Prove or disprove the following statement:

$$p \to q \equiv \neg p \wedge q.$$

Disproof

Set $p = T$ & $q = T$

then LHS $\neq$ RHS

(T) (F)

Truth table

p	q	$p \to q$	$\neg p \wedge q$
T	T	T	F
T	F	F	
F	T	T	
F	F	T	

Avoid this approach!

Disproof

conversion thm

$$LHS = p \to q \equiv \neg p \vee q$$

$$\not\equiv \neg p \wedge q = RHS$$

Justification is incomplete.

Example

Prove or disprove the following statement:

$$(p \vee r) \to (p \wedge q) \equiv (p \to q) \wedge (r \to q).$$

Try

p	q	r	$p \vee r$	$p \wedge q$	$(p \vee r) \to (p \wedge q)$	$(p \to q) \wedge (r \to q)$
F	F	T	T	F	F	F

Idea: set $\boxed{q = T}$. Then RHS $= T$ $(* \to T = T)$

can we make LHS $= F$ when $q = T$?

want $p \vee r = T$ & $p \wedge q = F$ $\Rightarrow \boxed{p = F}$

$\Rightarrow \boxed{r = T}$

Example

Prove or disprove the following statement: ✓ not equivalent

$$(p \lor r) \to (p \land q) \equiv (p \to q) \land (r \to q).$$

p	q	r	$p \lor r$	$p \land q$	$(p \lor r) \to (p \land q)$	$p \to q$	$r \to q$	$(p \to q) \land (r \to q)$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	T	F	F	F	F	F
T	F	F	T	F	F	F	T	F
F	T	T	T	F	F	T	T	T
F	T	F	F	F	T	T	T	T
F	F	T	T	F	F	T	F	F
F	F	F	F	F	T	T	T	T

Arguments: Valid Argument

An **argument** is a sequence of statements. The last statement is called the **conclusion**. All the previous statements are called **premises** (or **assumptions**/**hypotheses**).

A **valid argument** is an argument where the conclusion is true if the premises are all true.

A critical row with a false conclusion is a **counterexample**.

Arguments: Valid Argument Template

$$\begin{array}{r} p \to q; \\ p; \\ \therefore\ q \end{array}$$

$p \to q;$ $p;$ — **Premises**

$\therefore\ q$ — **Conclusion**

- **Critical rows** are rows with all premises true.
- If in all critical rows the conclusion is true, then the **argument is valid** (otherwise it is invalid).

p	q	$p \to q$	$(p \to q) \land p$	$((p \to q) \land p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

No need to calculate

Example

Decide whether or not the following argument is valid:

need to be T

$p \leftrightarrow q$;
$q \rightarrow r$; ✓
$r \vee s$; ✓
s; ✓
$\therefore r \wedge s$;

Hunt for a counterexample

p	q	r	s
F	F	F	T

← counterexample

want $r \wedge s = F$
since $s = T$
must have $r = F$

$q \rightarrow r \Rightarrow q = F$

$\therefore$ argument is not valid.

Inference Rules: Definition

A **rule of inference** is a logical construct which takes premises, analyses their syntax and returns a conclusion.

We already saw…

$$\begin{array}{r} p \to q; \\ p; \\ \therefore \quad q \end{array}$$

Modus Ponens
(Method of Affirming)

$$\begin{array}{r} p \to q; \\ \neg\, q; \\ \therefore \neg\, p \end{array}$$

Modus Tollens
(Method of Denying)

Inference Rules: More Inference Rules

Conjunctive Simplification (Particularising)

$p \land q;$

$\therefore p$

Disjunctive Syllogism (Case Elimination)

$p \lor q;$

$\neg p;$

$\therefore q$

Conjunctive Addition (Specialising)

$p;$

$q;$

$\therefore p \land q$

Rule of Contradiction

$\neg p \rightarrow C;$

$\therefore p$

Disjunctive Addition (Generalisation)

$p;$

$\therefore p \lor q$

Alternative Rule of Contradiction

$\neg p \rightarrow F;$

$\therefore p$

Example

Decide whether or not the following argument is valid:

$(p \vee q) \rightarrow \neg r;$
$\neg r \rightarrow s;$
$p;$
$\therefore s$

1. $(p \vee q) \rightarrow \neg r$
2. $\neg r \rightarrow s$
3. p

assume true

$\therefore$ 4. $p \vee q$ (Generalisation of 3.)

$\therefore$ 5. $\neg r$ (Modus ponens 1 & 4)

$\therefore$ s (Modus ponens on 2 & 5)