Discrete Mathematics

MH1812

Topics 5 and 6
Summary

Principle of Counting: Filling r Slots With n Choices

There are n elements, with which to fill r slots.

Can Be Repeated

When elements **can be** repeated using the principle of counting:
$n*n*\ldots*n = n^r$ **choices**.

n n . . . n
___ ___ ___ . . . ___
1 2 . .

Cannot Be Repeated

When elements **cannot be** repeated:

- n choices for first slot
- $n - 1$ choices for second slot
- $n - (r - 1)$ choices for last slot
- In total: $n(n - 1)(n - 2)\ldots(n - r + 1)$ choices

n n−1 . . . .
___ ___ ___ . . . ___
1 2 . . r

Permutations: $P(n,r)$

$\underset{\_\_\_}{n}$ $\underset{\_\_\_}{n-1}$ $\ldots$ $\_\_\_$

Number of permutations of n objects

$$n*(n-1)*(n-2)\ldots*2*1 = n!$$

where $n!$ is called **n factorial**.

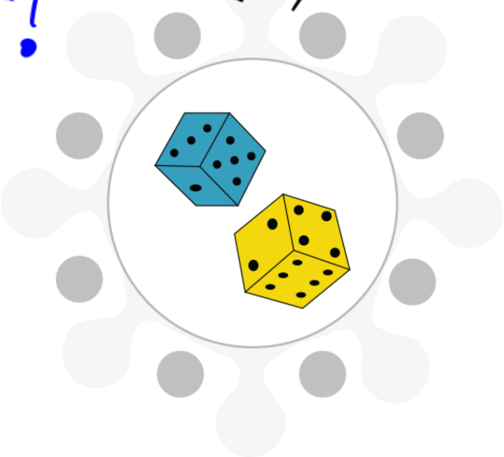
$1 = 0!$ ✓

$C(n,r) = \frac{P(n,r)}{r!} = \binom{n}{r}$

Number of permutations of n objects taken r at a time (n objects, the number of ways in which r items can be ordered):

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1) = n!/(n-r)!$$

where $n! = n*(n-1)*(n-2)* \ldots *2*1$ (called **n factorial**).



$3! = 3 \cdot 2 \cdot 1$

$2! = 2 \cdot 1$

$1! = 1$

$0! =$ nothing!

Example

no repeated digits

> 0

How many positive integers less than *1000* have distinct digits?

173 ✓

252 ✗

1, 2, 3, …, 9

10 … 99

~~999~~

1 digit

$\underset{9}{\_}$

2 digits

$\underset{9}{\_}\ \underset{9}{\_}$

3 digits

$\underset{9}{\_}\ \underset{9}{\_}\ \underset{8}{\_}$

(cannot be same as first digit & can be 0)

$$\text{Total} = 9 + 9 \cdot 9 + 9 \cdot 9 \cdot 8 = 738$$

2 digits

Claim: # 2digit number w/ distinct digits = 81

10	20	30		91
~~11~~	21	⋮		92
⋮	~~22~~	~~33~~		⋮
	⋮	⋮	. . .	
19	29	39		~~99~~

Total = $9 \cdot 10 - 9$

$= 81$

Permutations: Distinguishable Permutations

In general, the number of distinguishable permutations from a collection of objects, where the first object appears (repeats) k_1 times, the second object k_2 times, ... for r distinct objects:

$$n!/(k_1!\, k_2! \ldots k_r!)$$

"MISSISSIPPI"

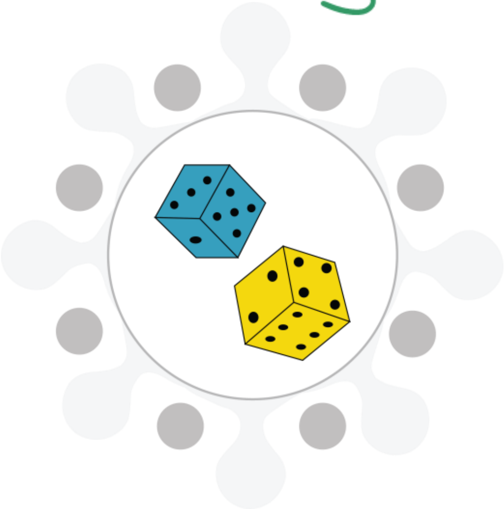
E.g., distinguishable permutations of "BOB"

$\ell_1 \ell_2 \ell_3$

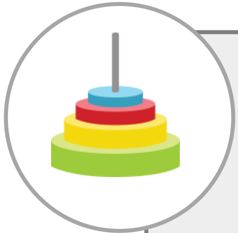
$\ell_1 \ell_2 \ell_3$ BOB
$\ell_1 \ell_3 \ell_2$ BBO
$\ell_2 \ell_1 \ell_3$ OBB
$\ell_2 \ell_3 \ell_1$ OBB
$\ell_3 \ell_1 \ell_2$ BBO
$\ell_3 \ell_2 \ell_1$ BOB

distinguishable

BOB
BBO
OBB



Introduction to Recurrence Relation: Definition



A **recurrence relation** is an equation that **recursively defines a sequence**, i.e., each term of the sequence is defined as a function of the preceding terms.

A recursive formula must be accompanied by **initial conditions** (information about the beginning of the sequence).

r_n = "expression in term of earlier elements"

Want: expression for r_n in terms of n

Backtracking: Solving Recurrence Relation

Backtracking is a technique for finding explicit formula for recurrence relation.

$a_1=2; \; a_2=5; \; a_3=8, \ldots$

tidy up

Example

$a_n = a_{n-1} + 3$ and $a_1 = 2$

$a_{n-1} = a_{n-2} + 3$

Limitation: gets messy

$$a_n = a_{n-1} + 3 = (a_{n-2} + 3) + 3 = a_{n-2} + 2*3$$
$$= (a_{n-3} + 3) + 2*3 = a_{n-3} + 3*3$$
$$= (a_{n-4} + 3) + 3*3 = a_{n-4} + 4*3$$
$$\ldots$$
$$= a_1 + (n - 1)*3$$

find Pattern

$$a_n = 2 + (n - 1)*3$$

Characteristic Equation: Homogenous Relation of Degree *d*

$a_n = a_{n-1}^2$ ✗

no constant term

A **linear homogeneous relation** of degree *d* is of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d}$$

constants

typically $d=1$, $d=2$

E.g.,:

- The Fibonacci sequence
- The relation: $a_n = 2a_{n-1}$ (degree *1*)
- But **not** the relation: $a_n = 2a_{n-1} + 1$ ✗

not homogeneous

The **characteristic equation** of the above relation is:

$$x^d = c_1 x^{d-1} + c_2 x^{d-2} + \ldots + c_d$$

Characteristic Equation: Theorem

$d=2$

If the characteristic equation $x^2 - c_1 x - c_2 = 0$ (of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$) has:

Step 1

- two distinct roots s_1, s_2, then the explicit formula for the sequence a_n is

$$a_n = us_1^n + vs_2^n$$

Step 2

quadratic formula
get s_1, s_2

- a single root s, then the explicit formula for a_n is

$$a_n = us^n + vns^n$$

Step 3

where u and v are determined by initial conditions. Find u & v

Step 3 Page 17, Topic G

E.g., $\frac{u+v}{2} + \frac{u-v}{2}\sqrt{5} = 2$ (1)

$\frac{3(u+v)}{2} + \frac{u-v}{2}\sqrt{5} = 3$ (2)

(2) − (1): $u+v = 1$ (3)

3(1) − (2): $(u-v)\sqrt{5} = 3$ (4)

(4) + $\sqrt{5}$(3): $2\sqrt{5}\,u = \sqrt{5} + 3$

$\Rightarrow u = \frac{3+\sqrt{5}}{2\sqrt{5}}$

Plug u into (3): $v = \frac{\sqrt{5}-3}{2\sqrt{5}}$

Proof (characteristic equation method works)

For sequence $(a_0, a_1, \ldots)$ with $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and $a_0 =$ "given", $a_1 =$ "given",

WTS: $\forall n \in \mathbb{N},\ a_n = u s_1^n + v s_2^n$ (This is the case when $s_1 \neq s_2$)

where s_1, s_2 are roots of $x^2 - c_1 x - c_2 = 0$

and $a_0 = u + v$, $a_1 = u s_1 + v s_2$

$\boxed{\text{By induction:}}$ Base cases, $n = 0$ & $n = 1$.

$a_0 = u s_1^0 + v s_2^0$ ✓

$a_1 = u s_1^1 + v s_2^1$ ✓

Inductive hypothesis: assume $\underbrace{a_k = u s_1^k + v s_2^k}_{P(k)}$ & $\underbrace{a_{k-1} = u s_1^{k-1} + v s_2^{k-1}}_{P(k-1)}$

It remains to show that $P(k) \wedge P(k-1) \rightarrow P(k+1)$

Start with LHS of $P(k+1)$:

$$a_{k+1} = c_1 a_k + c_2 a_{k-1}$$

Now use inductive hypothesis on a_k & a_{k-1}

$$= c_1\left(u s_1^k + v s_2^k\right) + c_2\left(u s_1^{k-1} + v s_2^{k-2}\right)$$

$$= u\left(c_1 s_1^k + c_2 s_1^{k-1}\right) + v\left(c_1 s_2^k + c_2 s_2^{k-1}\right) \quad (*)$$

Now, recall that $s_1^2 - c_1 s_1 - c_2 = 0$

$$\Rightarrow \quad s_1^2 = c_1 s_1 + c_2$$

same for s_2

$$\Rightarrow \quad s_1^{k-1} \cdot s_1^2 = c_1 s_1 \cdot s_1^{k-1} + c_2 s_1^{k-1}$$

$$s_1^{k+1} = c_1 s_1^k + c_2 s_1^{k-1}$$

So $(*)$ becomes

$$a_{k+1} = u s_1^{k+1} + v s_2^{k+1}.$$

This completes the proof by induction.

General sequences (Bonus material) ← not assessed! (even in final)

- 1
- 11
- 21
- 1211
- 111221
- 312211

What is the next term in the sequence?

"Look and Say"

Series (Bonus material)

A **series** is the *sum* of terms in a sequence.

E.g., given the sequence $a_0, a_1, a_2, a_3, a_4, \ldots$

the corresponding series is $S(n) = \sum_{k=0}^{n} a_k$

Example

- $0,1,2,3,4,\ldots \quad (a_k = k)$
- $1,-1,1,-1,1,-1,\ldots \quad (a_k = (-1)^k)$

$S(n) = \sum_{k=0}^{n} k$

$S(n) = \sum_{k=0}^{n} (-1)^k$

The Geometric Series (Bonus material)

The geometric series: $S(n) = \sum_{k=0}^{n} a_k,$ **where** $a_k = r^k$ **for some** r

Example

- $1,1,1,1,1,\ldots$ $(a_k = 1)$
- $1,-1,1,-1,1,-1,\ldots$ $(a_k = (-1)^k)$
- $1,2,4,8,16,32,\ldots$ $(a_k = 2^k)$

$\sum_{k=0}^{n} 1 = 1+1+1\ldots+1 = n+1$

$\sum_{k=0}^{n} (-1)^k = ?$

$\sum_{k=0}^{n} 2^k = ?$

We can show that $S(n) = \sum_{k=0}^{n} r^k = \dfrac{1 - r^{n+1}}{1 - r}$

for $r \neq 1$

WTS: $S(n) = \sum_{k=0}^{n} r^k = \dfrac{1-r^{n+1}}{1-r}$

Direct proof

$$(1-r)S(n) = S(n) - rS(n)$$

$$= 1 + \cancel{r} + \cancel{r^2} + \dots + \cancel{r^n}$$

$$- (\cancel{r} + \cancel{r^2} + \dots + \cancel{r^n} + r^{n+1})$$

$$= 1 - r^{n+1}$$

for $r \neq 1$

$$\Rightarrow S(n) = \frac{1-r^{n+1}}{1-r}$$

Proof (by induction)

WTS: $S(n) = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$

$P(n)$

Base case $n=0$

LHS: $\sum_{k=0}^{0} r^k = r^0 = 1$

RHS: $\frac{1-r^{0+1}}{1-r} = \frac{1-r}{1-r} = 1$ ✓

Inductive step: Assume $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$

LHS for $P(n+1)$: $\sum_{k=0}^{n+1} r^k = \sum_{k=0}^{n} r^k + r^{n+1}$

$$= \frac{1-r^{n+1}}{1-r} + r^{n+1}$$

$$= \frac{1-r^{n+1} + (1-r)r^{n+1}}{1-r}$$

$$= \frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r} = \frac{1-r^{n+2}}{1-r}$$