MH1820 Introduction to Probability and Statistical Methods

Tutorial 8 (Week 9) $\boxed{\textbf{Solution}}$

Problem 1 (Joint PDF, Marginal PDF) Let $f(x, y) = (3/16)xy^2$, $0 \le x \le 2$, $0 \le y \le 2$, be the joint PDF of X and Y.

(a) Find $f_X(x)$ and $f_Y(y)$, the marginal PDF of X and Y respectively.

(b) Are the two random variables independent? As in the discrete case, two continuous-type random variables X and Y are independent provided $f(x, y) = f_X(x)f_Y(y)$.

(c) Compute the mean μ_X and variance σ_X^2 of X.

(d) Find $\mathbb{P}(X \le Y)$.

$\boxed{\textbf{Solution}}$

(a)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 (3/16)xy^2\,dy = (3/16)\left[\frac{xy^3}{3}\right]_0^2 = \frac{3}{16}\left(\frac{x(2)^3}{3} - 0\right) = \frac{x}{2},$$

for $0 \le x \le 2$.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 (3/16)xy^2\,dx = (3/16)\left[\frac{x^2y^2}{2}\right]_0^2 = \frac{3}{16}\left(\frac{(2)^2y^2}{2} - 0\right) = \frac{3y^2}{8},$$

for $0 \le y \le 2$.

(b) Notice that $f(x, y) = (3/16)xy^2 = (x/2)(3y^2/8) = f_X(x) \cdot f_Y(y)$ for $0 \le x \le 2$, $0 \le y \le 2$. Thus, X and Y are independent.

(c)

$$\mu_X = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^2 x \cdot \frac{x}{2}\,dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}.$$

$$\sigma_X^2 = \mathbb{E}[X^2] - \mu_X^2 = \int_{\infty}^{\infty} x^2 f_X(x)\,dx - \left(\frac{4}{3}\right)^2 = \int_0^2 x^2 \cdot \frac{x}{2}\,dx - \frac{16}{9} = \left[\frac{x^4}{8}\right]_0^2 - \frac{16}{9} = \frac{2}{9}.$$

(d) The event $X \leq Y$ is given by the set of points (x, y), where $0 \leq x \leq 2$, $x \leq y \leq 2$.

$$
\begin{aligned}
\mathbb{P}(X \leq Y) &= \int_0^2 \int_x^2 \frac{3}{16} x y^2 \, dy \, dx \\
&= \frac{3}{16} \int_0^2 x \left[\frac{y^3}{3} \right]_x^2 dx \\
&= \frac{3}{16} \int_0^2 x \left(\frac{8}{3} - \frac{x^3}{3} \right) dx \\
&= \frac{1}{16} \int_0^2 8x - x^4 \, dx \\
&= \frac{1}{16} \left[\frac{8x^2}{2} - \frac{x^5}{5} \right]_0^2 \\
&= \frac{3}{5}.
\end{aligned}
$$

□

Problem 2 (Joint PDF, Marginal PDF, Conditional PDF) Let $f(x, y) = 1/40$, $0 \leq x \leq 10$, $10 - x \leq y \leq 14 - x$ be the joint PDF of X and Y.

(a) Sketch the region of the points (x, y) satisfying the inequalities $0 \leq x \leq 10$, and $10 - x \leq y \leq 14 - x$.

(b) Find $f_X(x)$, the marginal PDF of X.

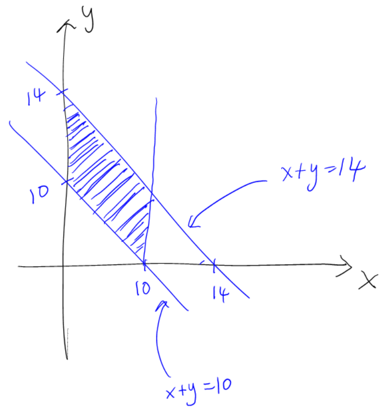
(c) Determine $h(y|x)$, the conditional PDF of Y, given that $X = x$.

(d) Calculate $\mathbb{E}[Y|X = x]$, the conditional mean of Y, given that $X = x$.

Solution

(a) The lower bound $10 - x \leq y$ for y means that the points (x, y) lie above (or on) line $10 = x + y$. The upper bound $y \leq 14 - x$ means that the points (x, y) lie below (or on) the line $14 = x + y$. The region required is given by

$$\{(x, y) : 0 \leq x \leq 2, 10 - x \leq y \leq 14 - x\}$$



(b)

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\
&= \int_{10-x}^{14-x} \frac{1}{40}\,dy \\
&= \frac{1}{40}\,[y]_{10-x}^{14-x} \\
&= \frac{1}{40}\,((14-x)-(10-x)) \\
&= \frac{1}{10},
\end{aligned}
$$

for $0 \le x \le 10$.

(c)

$$
h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1/40}{1/10} = \frac{1}{4},
$$

for $10 - x \le y \le 14 - x$.

(d) The conditional mean of Y given $X = x$ is

$$
\begin{aligned}
\mu_{Y|x} &= \mathbb{E}[Y|X=x] \\
&= \int_{10-x}^{14-x} y \cdot \frac{1}{4}\,dy \\
&= \frac{1}{4}\left[\frac{y^2}{2}\right]_{10-x}^{14-x} \\
&= \frac{1}{4}\left[\frac{(14-x)^2}{2} - \frac{(10-x)^2}{2}\right] \\
&= 12 - x,
\end{aligned}
$$

for $0 \le x \le 10$.

$\square$

Problem 3 (Conditional PDF, Conditional Expectation)

Let X and Y be continuous random variables with joint PDF

$$f(x,y) = \begin{cases} x + \frac{3}{2}y^2, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the conditional PDF $f(x|y)$ for all x, y.

(b) Compute the conditional expectation $E[X|Y = y]$ for all y.

(c) Find the conditional probabilities (i) $P\left(X \le \frac{1}{2}|Y = \frac{1}{2}\right)$ and (ii) $P\left(\frac{1}{4} \le X \le \frac{3}{4}|Y = \frac{1}{2}\right)$.

Solution

(a) The marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 x + \frac{3}{2}y^2\,dx = \left[\frac{x^2}{2} + \frac{3}{2}y^2x\right]_0^1 = \frac{1}{2} + \frac{3}{2}y^2.$$

By definition, we have

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x + \frac{3}{2}y^2}{\frac{3}{2}y^2 + \frac{1}{2}} = \frac{2x + 3y^2}{3y^2 + 1}$$

for $0 \le x \le 1$ and $0 \le y \le 1$ and $f(x|y) = 0$ otherwise.

(b) We compute

$$E[X|Y = y] = \int_0^1 xf(x|y)dx = \int_0^1 x\left(\frac{2x + 3y^2}{3y^2 + 1}\right)dx = \frac{9y^2 + 4}{6(3y^2 + 1)}$$

for $0 \le y \le 1$ and $E[X|Y = y] = 0$ otherwise.

(c) We compute

$$P\left(X \le \frac{1}{2}\middle|Y = \frac{1}{2}\right) = \int_{-\infty}^{1/2} f\left(t\middle|\frac{1}{2}\right)dt = \int_0^{1/2} \frac{2x + \frac{3}{4}}{\frac{3}{4} + 1}dx = \frac{5}{14},$$

$$P\left(\frac{1}{4} \le X \le \frac{3}{4}\middle|Y = \frac{1}{2}\right) = \int_{1/4}^{3/4} \frac{2x + \frac{3}{4}}{\frac{3}{4} + 1}dx = \frac{1}{2}.$$

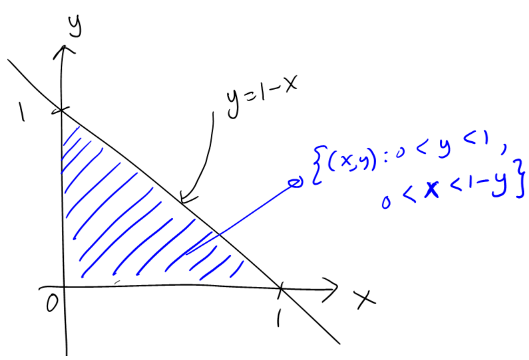
Problem 4 (Joint PDF, Marginal PDF, Conditional probability)
Let X and Y have the joint PDF $f(x,y) = cx(1-y)$, $0 < y < 1$, and $0 < x < 1-y$, where c is a constant.

(a) Determine c.

(b) Compute $\mathbb{P}(Y < X \mid X \leq 1/4)$.

$\boxed{\textbf{Solution}}$

Since $0 < y < 1$ and $0 < x < 1-y$, the support, i.e. the region of (x,y) where $f(x,y) > 0$ is given as follows (excluding the boundary):



(a) We must have $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$. So

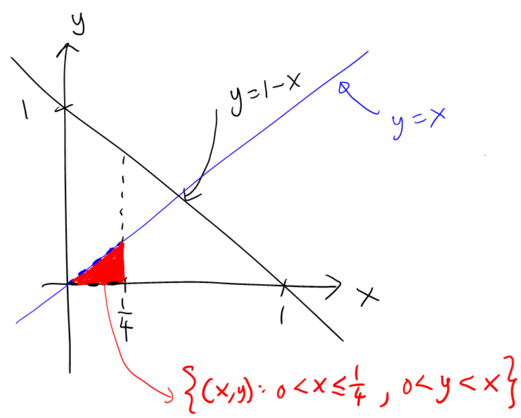
$$\begin{aligned}
1 &= \int_0^1\int_0^{1-y} cx(1-y)\,dx\,dy \\
&= \int_0^1 c(1-y)\left[\frac{x^2}{2}\right]_0^{1-y} dy \\
&= \int_0^1 \frac{c}{2}(1-y)^3\,dy \\
&= \frac{c}{2}\left[\frac{(1-y)^4}{-4}\right]_0^1 \\
&= \frac{c}{2}\left[0 + \frac{1}{4}\right] = \frac{c}{8}.
\end{aligned}$$

This implies that $c = 8$.

(b) Note that

$$\mathbb{P}(Y < X | X \leq 1/4) = \frac{\mathbb{P}((Y < X)\ \&\ (X \leq 1/4))}{\mathbb{P}(X \leq 1/4)} = \frac{\mathbb{P}(0 < Y < X \leq 1/4)}{\mathbb{P}(X \leq 1/4)}$$

The event $0 < Y < X \leq 1/4$ is given by the following region of (x,y):



$$
\begin{aligned}
\mathbb{P}(0 < Y < X \leq 1/4) &= \int_0^{1/4} \int_0^x 8x(1-y)\,dy\,dx \\
&= \int_0^{1/4} 8x \left[\frac{(1-y)^2}{-2}\right]_0^x dx \\
&= \int_0^{1/4} 4x(1-(1-x)^2)\,dx \\
&= \int_0^{1/4} 4x(2x-x^2)\,dx \\
&= \left[\frac{8x^3}{3} - x^4\right]_0^{1/4} \\
&= \frac{29}{24 \times 32}.
\end{aligned}
$$

On the other hand,

$$
f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{1-x} 8x(1-y)\,dy = 8x\left[\frac{(1-y)^2}{-2}\right]_0^{1-x} = 4x(1-x^2),
$$

for $0 < x < 1$. So

$$
\begin{aligned}
\mathbb{P}(X \leq 1/4) &= \int_0^{1/4} f_X(x)\,dx \\
&= \int_0^{1/4} 4x(1-x^2)\,dx \\
&= \left[2x^2 - x^4\right]_0^{1/4} \\
&= \frac{31}{32 \times 8}
\end{aligned}
$$

Hence,

$$
\mathbb{P}(Y < X | X \leq 1/4) = \frac{\mathbb{P}(0 < Y < X \leq 1/4)}{\mathbb{P}(X \leq 1/4)} = \frac{\frac{29}{24\times 32}}{\frac{31}{32\times 8}} = \frac{29}{31 \times 3} = \frac{29}{93}.
$$

$\square$

Answer Keys.

1(a) $f_X(x) = x/2$ for $0 \le x \le 2$, $f_Y(y) = \frac{3y^2}{8}$ for $0 \le y \le 2$ **1(b)** Yes **1(c)** $\mu_X = 4/3$, $\sigma_X^2 = 2/9$ **1(d)** 3/5 **2(b)** $f_X(x) = \frac{1}{10}$ for $0 \le x \le 10$. **2(c)** $h(y|x) = \frac{1}{4}$ for $10 - x \le y \le 14 - x$ **2(d)** $\mu_{Y|x} = 12 - x$ for $0 \le x \le 10$. **3(a)** $\frac{2x+3y^2}{3y^2+1}$ **3(b)** $\frac{9y^2+4}{6(3y^2+1)}$ **3(c)** (i) 5/4 (ii) 1/2 **4(a)** $c = \frac{1}{8}$ **4(b)** 29/93