Name:			
Matric. no.:		Tutor group:	

February 2022 **CA1** TIME ALLOWED: 50 minutes

QUESTION 1. **(15 marks)**

(a) [5 marks] Let $S = \{\text{integers congruent to 1011 modulo 2022}\}$ and Δ be addition. Is S closed under Δ? Justify your answer.

(b) [5 marks] Find all $a \in \{0, 1, \ldots, 7\}$ such that the statement below is true.

$$\exists x \in \{0, 1, \ldots, 7\} \text{ such that } x^2 \equiv a \pmod 8$$

(c) [5 marks] Find all $n \in \mathbb{N}$ such that $2^n + 5$ is a perfect square, i.e., $\exists x \in \mathbb{Z}$ such that $2^n + 5 = x^2$. Justify your answer.

Solution:

(a) No. $1011 \in S$, but $1011 + 1011 = 2022 \equiv 0 \pmod{2022}$.

[Distribution: 2 marks for No, and 3 marks for the justification]

(b) $a = 0$, $a = 1$, and $a = 4$.

[Distribution: 1 marks for each $a = 0, 1$, 2 marks for $a = 4$ and 1 mark for no mistakes]

(c) If $2^n + 5 = x^2$ then the same should be true of the remainders modulo 8. That is, $2^n + 5 \equiv x^2 \pmod 8$. Here, we see that if $n \geqslant 3$ then $2^n + 5 \equiv 5 \pmod 8$. But by Part (b), we know that $2^n + 5 = x^2$ must be congruent to 0,1 or 4 modulo 8. Therefore, n must be equal to 1 or 2. When $n = 1$ we have $2^n + 5 = 7$, which is not a square. The only possibility is $n = 2$ when $2^n + 5 = 9 = 3^2$.

[Distribution: 2 marks for $n = 2$, and 3 marks for the justification]

For graders only:	Question	1(a)	1(b)	1(c)	2(a)	2(b)	2(c)	3(a)	3(b)	3(c)	Total
	Marks										

QUESTION 2. **(15 marks)**

Let $S = \{\text{integers congruent to 1 modulo 7}\}$, $\mathbb{Z}$ denote the set of integers, and $\mathbb{Q}$ denote the set of rational numbers. Determine the truth value of the following statements. Justify your answers.

(a) [5 marks] $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Z},\ \exists z \in S,\ x(y+z) \in \mathbb{Z}$;

(b) [5 marks] $\exists x \in \mathbb{Q},\ \exists y \in \mathbb{Z},\ \forall z \in S,\ (x+y)z \in S$;

(c) [5 marks] $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Z},\ \exists z \in S,\ xyz \in S$.

Solution:

(a) True. Let $x \in \mathbb{Q}$. Then $x = p/q$ where $p, q \in \mathbb{Z}$. Now take $y = q - 1$ and $z = 1$. Then

$$\begin{aligned} x(y+z) &= \frac{p}{q}(q-1+1) \\ &= \frac{p}{q}q \\ &= p \in \mathbb{Z} \end{aligned}$$

(b) True. Take $x = 1$ and $y = 0$. Then $(x+y)z = z \in S$ for all $z \in S$.

(c) False. When $x = 0 \in \mathbb{Q}$, we have $xyz = 0 \notin S$ no matter which $y \in \mathbb{Z}$ and $z \in S$ we pick.

[Distribution: 2 marks for each correct truth value, and 3 marks for each justification]

QUESTION 3. **(20 marks)**

(a) [5 marks] Prove the following equivalence using De Morgan's law, double negation, the conversion theorem, and distributivity (noting where each is used).

$$(\neg p \to q) \to r \equiv (p \to r) \wedge (q \to r)$$

(b) [10 marks] Decide whether or not the following argument is valid:

$$\begin{aligned} &\neg p \to r \wedge s; \\ &t \to s; \\ &u \to \neg p; \\ &\neg w; \\ &u \vee w; \\ \therefore\ &\neg t. \end{aligned}$$

Briefly justify your answer.

(c) [5 marks] In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humour and love of logical puzzles. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements (a–e, below) and challenged the reader to use them to figure out the location of the treasure.

(a) If this house is next to a lake, then the treasure is not in the kitchen.
(b) If the tree in the front yard is an elm, then the treasure is in the kitchen.
(c) If the tree in the back yard is an oak, then the treasure is in the garage.
(d) The tree in the front yard is an elm or the treasure is buried under the flagpole.
(e) This house is next to a lake.

Where is the treasure hidden? Briefly justify your answer.

Solution:

(a)

$$\begin{aligned} (\neg p \to q) \to r &\equiv \neg(\neg p \to q) \vee r && \text{conversion theorem} \\ &\equiv \neg(\neg\neg p \vee q) \vee r && \text{conversion theorem} \\ &\equiv \neg(p \vee q) \vee r && \text{double negation} \\ &\equiv (\neg p \wedge \neg q) \vee r && \text{De Morgan} \\ &\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \text{distributivity} \\ &\equiv (p \to r) \wedge (q \to r) && \text{conversion theorem} \end{aligned}$$

[Distribution: about 1 mark for each line of justification]

(b) The argument is invalid.

Counterexample: $p = F$, $r = T$, $s = T$, $t = T$, $u = T$, $w = F$.

Alternatively, one can show that the argument is invalid using a truth table.

[Distribution: 5 marks for the correct answer, and 5 marks for the justification]

(iii) The treasure is buried under the flagpole.

Denote by

$$
\begin{aligned}
l &= \text{“this house is next to a lake”}\\
k &= \text{“the treasure is in the kitchen”}\\
e &= \text{“the tree in the front yard is an elm”}\\
b &= \text{“the tree in the back yard is an oak”}\\
g &= \text{“the treasure is in the garage”}\\
f &= \text{“the treasure is buried under the flagpole”}
\end{aligned}
$$

The corresponding argument has the form.

$$
\begin{aligned}
& l \rightarrow \not k;\\
& e \rightarrow k;\\
& b \rightarrow g;\\
& e \vee f;\\
& l;\\
\therefore\ & f.
\end{aligned}
$$

(1) $l \rightarrow \neg k$

(2) $e \rightarrow k$

(3) $b \rightarrow g$

(4) $e \vee f$

(5) l

(6) $\therefore \neg k$ from (5) and (1)

(7) $\therefore \neg e$ from (6) and (2)

(8) $\therefore f$ from (7) and (4)

[Distribution: 2 marks for the correct answer, and 3 marks for the justification]