MH1820 Week 6

1. Moment generating functions

2. Bivariate distribution (Joint PMF, CDF and Marginal PMF)

Moment generating functions

Moment generating functions

- A distribution of a random variable X is determined by its CDF or by its PDF/PMF.
- If a random variable is defined by some expression (e.g. $X = \frac{1}{10}(X_1 + \cdots + X_{10})$, then it may be tedious to compute the CDF/PDF/PMF directly.
- Moment generating functions sometimes can be used in these cases to identify the distribution of X indirectly in a much quicker way.

Let X be a random variable. Its **moment generating function (MGF)** is defined by

$$M_X(t) = \mathbb{E}[e^{tX}], \quad t \in \mathbb{R}.$$

- Continuous case: $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$, where $f(x)$ is the PDF of X.
- Discrete case: $M_X(t) = \sum_x e^{tx} p(x)$, where $p(x)$ is the PMF of X.

Example 1

Let X be a random variable with PDF $f(x) = e^x$ for $0 \leq x \leq \ln 2$, and $f(x) = 0$ otherwise. Compute the moment generating function $M_X(t)$ of X for $t \neq -1$.

Solution.

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} e^x \, dx \\ &= \int_0^{\ln 2} e^{(t+1)x} \, dx \\ &= \left[\frac{e^{(t+1)x}}{t+1} \right]_0^{\ln 2} \quad (\text{since } t \neq -1) \\ &= \frac{e^{(t+1)\ln 2} - 1}{t+1} = \frac{2^{t+1} - 1}{t+1}. \end{aligned}$$

Example 2

Let X be a discrete random variable with PMF $p(x)$ given as follows:

x	0	1	2	3
$p(x)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

We compute the moment generating function $M_X(t)$ of X for all $t \in \mathbb{R}$.

$$
\begin{aligned}
M_X(t) &= \sum_{x=0}^{3} e^{tx} p(x) \\
&= \frac{3}{8} e^{t \cdot 0} + \frac{3}{8} \cdot e^{t \cdot 1} + \frac{1}{8} \cdot e^{t \cdot 2} + \frac{1}{8} \cdot e^{t \cdot 3} \\
&= \frac{1}{8}(3 + 3e^{t} + e^{2t} + e^{3t}).
\end{aligned}
$$

$\square$

Theorem 3 (Properties of MGF – Part I)

Let X, Y be random variables with $M_X(t) < \infty$, $M_Y(t) < \infty$ for $-h < t < h$. Then

(a) $\mathbb{E}[X^n] = M_X^{(n)}(0)$, *where* $M_X^{(n)}(t) = \frac{d^n}{dt^n} M_X(t)$, *the n-th derivative of* $M_X(t)$.

(b) *(Inversion Theorem) If $M_X(t) = M_Y(t)$ for all t, then X and Y have the same distribution, i.e. they have the same CDF/PDF.*

$\mathbb{E}[X^n]$ is called the n-**th moment** of X. E.g. the first moment is the same as the expected value (or mean).

Theorem 4 (Properties of MGF – Part II)

(c) *If $Y = aX + b$, where $a, b \in \mathbb{R}$, then*

$$M_Y(t) = e^{tb} M_X(at)$$

(d) *If X and Y are independent, then*

$$M_{X+Y}(t) = M_X(t) M_Y(t).$$

MGFs of common distributions.

Distribution	MGF
$Bernoulli(p)$	$pe^t + 1 - p$
$Geom(p)$	$\dfrac{pe^t}{1-(1-p)e^t}$ for $t < -\ln(1-p)$
$Binomial(n, p)$	$(pe^t + 1 - p)^n$
$Poisson(\lambda)$	$e^{\lambda(e^t-1)}$
$U(a, b)$	$\dfrac{e^{tb} - e^{ta}}{t(b-a)}$ for $t \neq 0$, 1 for $t = 0$
$N(\mu, \sigma^2)$	$e^{\mu t + \sigma^2 t^2/2}$
$Gamma(\alpha, \theta)$	$(1-\theta t)^{-\alpha}$ for $t < \frac{1}{\theta}$
$Exp(\theta)$	$(1-\theta t)^{-1}$ for $t < \frac{1}{\theta}$

Example 5

Let $X \sim Binomial(n, p)$. Show that the MGF of X is $(pe^t + 1 - p)^n$.

Solution. Let $Y \sim Bernoulli(p)$. We first show that the MGF of Y is $pe^t + 1 - p$. Let $p_Y(x)$ be the PMF of Y. Then

$$
\begin{aligned}
M_Y(t) &= \sum_y e^{tx} p_Y(x) \\
&= e^{t \cdot 0} p_Y(0) + e^{t \cdot 1} p_Y(1) \\
&= e^{t \cdot 0}(1 - p) + e^{t \cdot 1} p \\
&= pe^t + 1 - p.
\end{aligned}
$$

We now apply Property (d) of MGF. Since $X = \sum_{i=1}^{n} Y_i$, where $Y_i \sim Bernoulli(p)$, and Y_i's are independent, we deduce that

$$\begin{aligned} M_X(t) &= M_{Y_1+\cdots+Y_n}(t) \\ &= M_{Y_1}(t)M_{Y_2}(t)\cdots M_{Y_n}(t) \\ &= (pe^t + 1 - p)\cdots(pe^t + 1 - p) \\ &= (pe^t + 1 - p)^n. \end{aligned}$$

$\square$

Example 6

Let $X \sim Exp(\theta)$. Derive the MGF $M_X(t)$ of X, for $t < \frac{1}{\theta}$, and use it to find the mean and variance of X.

Solution. Let $f(x)$ be the PDF of X.

$$
\begin{aligned}
M_X(t) &= \mathbb{E}[e^{tX}] = \int_{0}^{\infty} e^{tx} \cdot \frac{1}{\theta} e^{-x/\theta}\, dx \\
&= \frac{1}{\theta} \int_0^{\infty} e^{(t-\frac{1}{\theta})x}\, dx \\
&= \frac{1}{\theta} \left[\frac{e^{(t-\frac{1}{\theta})x}}{t - \frac{1}{\theta}} \right]_0^{\infty} \\
&= \frac{1}{\theta} \left(- \frac{1}{t - \frac{1}{\theta}} \right) = \frac{1}{1 - \theta t}, \quad (\textbf{since } t < \frac{1}{\theta}).
\end{aligned}
$$

Differentiating the MGF, we have

$$M_X^{(1)}(t) = \frac{d}{dt} M_X(t) = \theta(1 - \theta t)^{-2}$$

$$M_X^{(2)}(t) = \frac{d^2}{dt^2} M_X(t) = 2\theta^2(1 - \theta t)^{-3}$$

By Property (a) of MGF, we have

$$\mathbb{E}[X] = M_X^{(1)}(0) = \theta, \quad \mathbb{E}[X^2] = M_X^{(2)}(0) = 2\theta^2.$$

The mean of X is $\mathbb{E}[X] = \theta$.

The variance of X is

$$\begin{aligned}\mathrm{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= 2\theta^2 - \theta^2 \\ &= \theta^2.\end{aligned}$$

$\square$

Example 7

Suppose $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$. Use MGF to find the distribution of $X_1 + X_2$.

Solution. From the Table of MGF,

$$M_{X_1}(t) = e^{\mu_1 t + \sigma_1^2 t^2/2}, \quad M_{X_2} = e^{\mu_2 t + \sigma_2^2 t^2/2}.$$

By Property (c) of MGF,

$$M_{X+Y}(t) = M_X(t) M_Y(t) = e^{(\mu_1+\mu_2)t + \frac{(\sigma_1^2+\sigma_2^2)}{2} t^2}$$

From the Table of MGF and Property (d) of MGF, we deduce that

$$X + Y \sim N\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right),$$

i.e. $X + Y$ is normally distributed with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. $\square$

Bivariate distribution (Joint PMF, CDF and Marginal PMF)

Motivating Example: 2 balls are drawn from a box which contains 2 blue, 3 red, and 4 yellow balls.

- X = number of blue balls drawn
- Y = number of red balls drawn

For each possible pair of values of (x, y), we are interested in the probability that $X = x$, $Y = y$ occur simultaneously, i.e.

$$\mathbb{P}(X = x, Y = y).$$

Here, we require $0 \le x \le 2$, $0 \le y \le 2$ and $0 \le x + y \le 2$.

The **joint PMF** of X and Y is given by

$$p(x,y) = \mathbb{P}(X = x, Y = y) = \frac{\binom{2}{x}\binom{3}{y}\binom{4}{2-x-y}}{\binom{9}{2}}$$

$x \setminus y$	0	1	2
0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
1	$\frac{2}{9}$	$\frac{1}{6}$	0
2	$\frac{1}{36}$	0	0

The distribution given by the joint PMF is called the **joint distribution** of X and Y.

Let X, Y be discrete random variables.

- The **joint probability mass distribution (joint PMF)** of X and Y is defined by
$$p(x, y) = \mathbb{P}(X = x, Y = y).$$

- The **joint cumulative density function (joint CDF)** of X and Y is defined by
$$F(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \sum_{s \leq x} \sum_{t \leq y} p(s, t).$$

Example 8

Roll a pair of fair dice. For each of the 36 sample points with probability $1/36$, let X denote the smaller and Y the larger outcome on the dice. If both numbers of the dice are the same, then X and Y take on the same value.

Find the joint PMF of X and Y.

Solution. We represent the outcome on the dice by (a, b). If $x < y$, then the event $X = x$ and $Y = y$ occurs twice (as (x, y) and (y, x)) with probability $\frac{2}{36}$. Otherwise, the event $X = Y = a$ occurs with with probability $\frac{1}{36}$ since it occurs once as (a, a).

The joint PMF of X and Y is

$$p(x, y) = \begin{cases} \frac{1}{36} & \text{if } x = y \\ \frac{2}{36} & \text{if } x < y \\ 0 & \text{if } x > y \end{cases}$$

$\square$

$x \setminus y$	1	2	3	4	5	6
1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
2	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
3	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
4	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
5	0	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$
6	0	0	0	0	0	$\frac{1}{36}$

Let X and Y have the joint probability mass function $f(x, y)$.

- The probability mass function of X alone, which is called the **marginal probability mass function** of X, is defined by

$$p_X(x) = \sum_{y} p(x, y) = \mathbb{P}(X = x).$$

  $p_X(x)$, $p_Y(y)$ are the called the marginal PMF of X and Y.

- If $u(X, Y)$ is a function of X and Y, then

$$\mathbb{E}[u(X, Y)] = \sum_{x} \sum_{y} u(x, y) p(x, y)$$

  is the **expected value** of $u(X, Y)$.

Theorem 9 (Independence via marginals)

The random variables X and Y are **independent** *if and only if*

$$p(x, y) = p_X(x)p_Y(y) \quad \text{for all } x,\ y.$$

Otherwise, X and Y are said to be **dependent**.

Example 10

A dice is rolled 2 times. Let

- $X =$ number of rolls that are 1
- $Y =$ number of rolls that are 2

(i) Find $F(2,1)$ where $F(x,y)$ is the joint CDF of X and Y.

(ii) Find the marginal PMF $p_X(x)$, where $p(x,y)$ is the joint PMF of X and Y.

(iii) Are X and Y independent?

Solution. Joint PMF:

$x \setminus y$	0	1	2
0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$
1	$\frac{8}{36}$	$\frac{2}{36}$	0
2	$\frac{1}{36}$	0	0

(i)

$$F(2,1) = \mathbb{P}(X \leq 2, Y \leq 1) = \frac{16}{36} + \frac{8}{36} + \frac{8}{36} + \frac{2}{36} + \frac{1}{36} = \frac{35}{36}.$$

(ii) The marginal PMF of X is given by

$$p_X(x) = \sum_y p(x, y).$$

So

- $p_X(0) = \sum_y p(0, y) = \frac{16}{36} + \frac{8}{36} + \frac{1}{36} = \frac{25}{36}$.
- $p_X(1) = \sum_y p(1, y) = \frac{8}{36} + \frac{2}{36} + 0 = \frac{10}{36}$.
- $p_X(2) = \sum_y p(2, y) = \frac{1}{36} + 0 + 0 = \frac{1}{36}$.

(iii) Note that

$$p(2,2) = 0, \quad p_X(2) = \frac{1}{36}, \quad p_Y(2) = \frac{1}{36}.$$

Since

$$p(2,2) \neq p_X(2)p_Y(2),$$

the random variables X and Y are dependent.

$\square$

Example 11

A manufactured item is classified as good, fair or defective with probabilities $6/10$, $3/10$, and $1/10$, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of fair items, and $15 - X - Y$ the number of defective items.

(i) Find the joint PMF of X and Y.

(ii) Find the marginal PMF $p_X(x)$ and $p_Y(y)$.

(iii) Find $\mathbb{P}(X \leq 11)$.

(iV) Are X and Y independent?

Solution. (i) Let x and y be fixed. Consider the different ways of having x good items, y fair items and $15 - x - y$ defective items,

There are $\binom{15}{x}$ possible ways of selecting x items out of 15 to be good, $\binom{15-x}{y}$ possible ways of selecting y items out of the remaining $15 - x$ items to be fair, and one way (having chosen x good items and y fair items) of selecting the rest to be defective.

Hence, the PMF is given by

$$p(x, y) = \mathbb{P}(X = x, Y = y) = \binom{15}{x}\binom{15-y}{y}(0.6)^x(0.3)^y(0.1)^{15-x-y}.$$

(ii) We will find the marginal PMFs directly. Indeed, $X \sim Binomial(15, 0.6)$, $Y \sim Binomial(15, 0.3)$, that is

$$p_X(x) = \mathbb{P}(X = x) = \binom{15}{x}(0.6)^x(0.4)^{15-x}.$$

$$p_Y(y) = \mathbb{P}(X = y) = \binom{15}{y}(0.3)^y(0.7)^{15-y}.$$

(iii)

$$
\begin{aligned}
\mathbb{P}(X \leq 11) &= 1 - \mathbb{P}(X \geq 12) \\
&= 1 - \left(\sum_{x=12}^{15} \binom{15}{x} (0.6)^x (0.4)^{15-x} \right) \\
&= 0.9095.
\end{aligned}
$$

(iv) Notice that $p_X(0) = 0.4^{15}$, $p_Y(0) = 0.7^{15}$, $p(0,0) = (0.1)^{15}$. Hence, $p(0,0) \neq p_X(0)p_Y(0)$. So X and Y are <u>NOT</u> independent. $\square$