MH1820 Introduction to Probability and Statistical Methods

Tutorial 1 (Week 2) $\boxed{\text{Solution}}$

In this tutorial, we will compute some probabilities. We assume that all possible outcomes in the (finite) sample space have the same probability. Therefore, the probability of an event $E \subseteq \Omega$ can be found using the formula

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}.$$

Problem 1 (n-tuples, Multiplication principle)

(a) How many 7-digit numbers are there all of whose digits are odd?

(b) How many 5-digit numbers are there all of whose digits are primes? (a prime is an integer $p \geq 2$ whose only divisors are 1 and p; note that 0 and 1 are not primes)

(c) How many 5-digit numbers are there? (all digits are allowed, but the first digit must not be 0)

(d) If a 5-digit number is chosen randomly, what is the probability that all its digits are primes?

(e) How many 6-digit numbers are there with no repeated digits?

(f) If a 6-digit number is chosen randomly, what is the probability that it has no repeated digits?

$\boxed{\textbf{Solution}}$ (a) We can create a 7-digit number in seven steps, chosing one digit in each step. Since the digits have to be odd, there are 5 choices in each step (we can choose one of the digits 1, 3, 5, 7, 9). Thus, by the multiplication principle, there are 5^7 7-digit numbers are there all of whose digits are odd.

(b) The digits that are primes are exactly $2, 3, 5, 7$. Thus we create the 5-digit number in 5 steps with 4 choices in each step. By the multiplication principle, there are exactly 4^5 5-digit numbers all of whose digits are primes.

(c) For the first digit, there are 9 choices (we can choose one of the digits 1,2,3,...,9) and for each of the other 4 digits there are 10 choices (we can choose one of the digits 0,1,2,3,...,9). Hence, by the multiplication principle, the total number of 5-digits numbers is $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 90000$. Of course, this answer also can be obtained directly by noting that the 5-digits numbers are exactly 10000, 10001, ..., 99999. These are 90000 numbers in total (if a, b are integers with $a < b$, then the range $a, a+1, \ldots, b$ contains exactly $b - a + 1$ integers).

(d) For this statistical experiment, the sample space Ω is the set of 5-digit numbers. By part (c), we have $|\Omega| = 90000$. Let E be set of 5-digit numbers all of whose digits are primes. By part (b), we have $|E| = 4^5$. Hence the probability that a randomly chosen 5-digit number only has digits that are primes is

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{4^5}{90000} \approx 0.011.$$

(e) We create the 6-digit number one digit at a time. For the first digit, there are 9 choices (0 is not allowed as first digit). For the second digit, there again are 9 choices (all digits that are different from the first one). For the third digit, there are 8 choices (all digits different from the first and second), for the fourth digit there are 7 choices, etc. By the multiplication principle, there are exactly $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 136080$ 6-digit numbers with no repeated digits.

(f) For this statistical experiment, the sample space Ω is the set of 6-digit numbers. Similar to part (c), we get $|\Omega| = 9 \cdot 10^5$. Let E be the set of 6-digit numbers with no repeated digits. By part (e), we have $|E| = 136080$. Hence, if a 6-digit number is chosen randomly, the probability that it has no repeated digits is

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{136080}{9 \cdot 10^5} = 0.1512.$$

Problem 2 (Permutations)

(a) How many possibilities are there to line up 5 persons in a queue?

(b) Suppose there are 2000 spectators in a soccer-stadium. During halftime, a queue of 5 spectators forms in front of a coffee shop. How many possibilities are there for such a queue to form from the spectators?

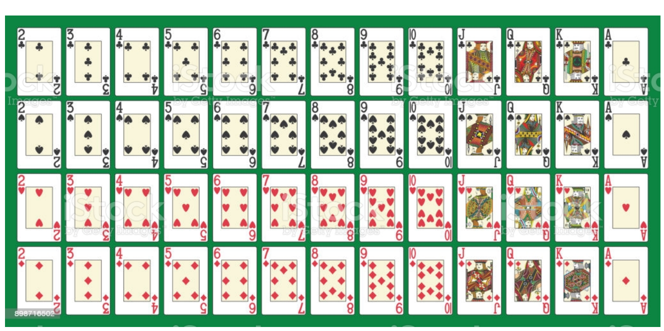
Solution (a) This is the number of permutations of a 5-element set which is $5! = 120$.

(b) This is number of 5-permutations of a 2000-element set which is

$$2000 \cdot 1999 \cdot 1998 \cdot 1997 \cdot 1996 = 31840279800048000.$$

Problem 3 (Combinations)

We consider a standard poker deck with 52 cards:



Note that there are 4 suits (clubs, spades, hearts, diamonds) and 13 ranks (ace, king queen, etc.). For the definition of poker hands (flush, straight, etc.) see
`https://en.wikipedia.org/wiki/List_of_poker_hands`

(a) How many ways are there to select a poker hand of 5 cards from the deck of 52 cards?

(b) What is the probability that a randomly chosen poker hand of 5 cards forms a
(i) flush (excluding straight flush) (ii) pair, (iii) two pairs, (iv) full house?

(c) What is the probability that a randomly chosen hand forms a full house without any aces or kings?

Solution (a) This is the number of 5-element combinations from an 52-set, which is

$$\binom{52}{5} = 2,598,960.$$

Note that in a poker hand, the order of the cards does *not* matter. Hence we count combinations (and not permutations).

(b) (i) For a fixed suit, say club, the number of flushes (including straight flushes) in clubs is $\binom{13}{5}$ (5 cards have to be chosen from the 13 cards of clubs). The number of straight flushes in clubs is 10 (the lowest card in a straight can be ace, 2, 3 ,. . . , 10). Hence the number of flushes (excluding the straights) in clubs is $\binom{13}{5} - 10$. Since there are 4 suits, the total number of flushes (excluding the straights) is

$$4 \cdot \left(\binom{13}{5} - 10 \right) = 5,108.$$

Hence the probability of a flush (excluding straight flush) is

$$\frac{5,108}{2,598,960} \approx 0.002.$$

(ii) We first compute the number of ways to pick two cards that form a pair. Let R be the rank the cards in the pair have. There are 13 ways to choose R. We need to pick 2 from the 4 cards of rank R. Hence the total number of ways to pick to cards that form a pair is

$$13\binom{4}{2} = 78.$$

Now the remaining three cards need be chosen such that we do not get two pairs, three of kind, four of kind, or a full house. This means the remaining three cards must be from different ranks and their rank must be different from R. Hence the number of ways to choose the ranks of the remaining three cards is $\binom{12}{3}$. Once the ranks for the remaining three cards are determined, we have 4 choices for each card, since there are 4 cards of each rank. Thus, by the multiplication principle, there are $\binom{12}{3}4^3$ choices for the remaining three cards.

In summary, the number of ways to choose the pair is 78 and the number of ways to choose the remaining three cards is $\binom{12}{3}4^3$. Hence the total number of hands that form a pair is

$$78 \cdot \binom{12}{3}4^3 = 1,098,240$$

and the probability of a pair is

$$\frac{1,098,240}{2,598,960} \approx 0.42.$$

(iii) The two pairs must come from two ranks and there are $\binom{13}{2}$ ways to choose these two ranks. Once the ranks are chosen, there are $\binom{4}{2}$ ways to pick a pair from the 4 cards of each rank. Thus, by the multiplication principle, there are $\binom{13}{2}\binom{4}{2}^2$ ways to pick the two pairs. The remaining card must fberom one of remaining 11 ranks and there are $11 \cdot 4$ ways to pick that card. Hence the total number number of hands that form two pair is

$$\binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4 = 123,552$$

and the probability of two pairs is

$$\frac{123,552}{2,598,960} \approx 0.048.$$

(iv) A full house contains three cards of the same rank. There are 13 ways to choose this rank and $\binom{4}{3}$ to pick the 3 cards once the rank has been chosen. Thus the total number of ways to choose three cards of the same rank is $13 \cdot \binom{4}{3}$. The remaining two cards must form a pair of a different rank. There are 12 ranks left from which the pair can be choosen and $\binom{4}{2}$ pairs can be picked from each rank. Hence there are $12 \cdot \binom{4}{2}$ ways to choose the last two cards. In total, there are

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3,744$$

hands that form a full house and the probability of a full house is

$$\frac{3,744}{2,598,960} \approx 0.0014.$$

(c) We count the number of hands that form a full house without any ace or king involved. The argument is essentially the same as in part (b) (iv), with the only difference that the number of ranks has been reduced from 13 to 11 (since we need to exclude ace and king). Hence, in total, there are

$$11 \cdot \binom{4}{3} \cdot 10 \cdot \binom{4}{2} = 2,640.$$

hands that form a full house without any ace or king involved, and so the required probability is

$$\frac{2,640}{2,598,960} \approx 0.001.$$

Problem 4 (Permutations of multisets)

(a) How many words can be formed from the letters in `successlessness`? (here every permutation of these letters counts as a "word" even if does not make sense)

(b) In an orchid show, 11 orchids are to be placed along one side of the greenhouse. There are 5 lavender orchids, 4 white orchids, and 2 yellow orchids. Considering only the color of the orchids, find the number of different color displays?

(c) How many possibilities are there to split up 12 players into Team A, B, C with 5, 4, 3 players respectively?

$\boxed{\textbf{Solution}}$ (a) The letters of the `successlessness` form the multiset

$$\{c, c, e, e, e, l, n, s, s, s, s, s, s, s, u\}$$

with 15 elements and multiplicities $2, 3, 1, 1, 7, 1$. According to the lecture, the number of permutations of this multiset is

$$\binom{15}{2, 3, 1, 1, 7, 1} = \frac{15!}{2!\,3!\,1!\,1!\,7!\,1!} = 21621600.$$

(b) The multiset is

$$\{l, l, l, l, l, w, w, w, w, y, y\}$$

where l is the lavender, w is white and y is yellow. The number of permutations of this multiset is

$$\binom{11}{5, 4, 2} = \frac{11!}{5!4!2!} = 6,930.$$

(c) We label the players by 1, 2, ..., 12. Each possibility can be represented by a permutation of the multiset $\{A, A, A, A, A, B, B, B, B, C, C, C\}$. For example, $AABBAABBCCCA$ means that Player 1,2,5,6,12 belong to Team A, Player 3,4,7,8 belong to Team B, and Player 9,10,12 belong to Team C.
So the total number of possibilities is the permutation of this multiset:

$$\binom{12}{5, 4, 3} = \frac{12!}{5!4!3!} = 27,720.$$

Answer Keys. 1. (a) 5^7 (b) 4^5 (c) $90,000$ (d) 0.011 (e) $136,080$ (f) 0.1512 2. (a) 120 (b) 31840279800048000 3. (a) $2,598,960$ (b) (i) 0.002 (ii) 0.42 (iii) 0.048 (iv) 0.0014 (c) 0.001 4. (a) $21,621,600$ (b) $6,930$ (c) $27,720$.