MH1820 Week 7

1. Bivariate Distribution (Joint PDF, CDF and Marginal PDF)

2. Conditional Distributions

Recap: Joint PMF of X, Y (discrete).

$p(x,y)$

$x \backslash y$	y	$p_X(x)$
x	$\cdots\ p(x,y)$	$p_X(x)$
	$p_Y(y)$	

$$\sum_y \sum_x p(x,y) = 1$$

Marginal pmF of X: $p_X(x) = \sum_y p(x,y)$

" " " Y: $p_Y(y) = \sum_x p(x,y)$

Week 6.

$n = 15$.

Example 11

A manufactured item is classified as good, fair or defective with probabilities 6/10, 3/10, and 1/10, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of fair items, and $15 - X - Y$ the number of defective items.

(i) Find the joint PMF of X and Y. $p(x, y)$

(ii) Find the marginal PMF $p_X(x)$ and $p_Y(y)$.

(iii) Find $\mathbb{P}(X \leq 11)$.

(iV) Are X and Y independent?

$p(x,y)$ = prob. of having
x good items &
y fair items.

(the rest $15-x-y$ are
defective)

	# ways	prob.
good.	$\binom{15}{x}$	$(0.6)^x$.
fair	$\binom{15-x}{y}$	$(0.3)^y$
defective.	$\binom{15-x-y}{15-x-y}$ $=1$	$(0.1)^{15-x-y}$.

(a)

$$p(x,y) = \binom{15}{x}\binom{15-x}{y}\cdot 1 \cdot 0.6^{x}\, 0.3^{y}\, (0.1)^{15-x-y}$$

(b)

$$p_X(x) = \sum_{y} p(x,y)$$

$$= \sum_{y} \binom{15}{x}\binom{15-x}{y} (0.6)^{x} (0.3)^{y} (0.1)^{15-x-y}$$

$$= \binom{15}{x}(0.6)^{x} \sum_{y=0}^{15-x} \binom{15-x}{y} (0.3)^{y} (0-1)^{15-x-y}$$

$$= \binom{15}{x} (0.6)^{x} (0.4)^{15-x}$$

Recall:

$$(a+b)^m = \sum \binom{m}{i} a^i b^{m-i}$$

$$0.4^{15-x} = (0.3+0.1)^{15-x} = \sum \binom{15-x}{y} 0.3^y \, 0.1^{15-x-y}$$

Similarly:

$$P_Y(y) = \binom{15}{y} (0.3)^y (0.7)^{15-y}.$$

$X \sim \text{Binomial}(15, 0.6)$ ✓

$Y \sim \text{Binomial}(15, 0.3)$.

Recall:

X, Y <u>independent</u>.

PROVIDED: $p(x,y) = p_X(x) \cdot p_Y(y)$

for all x, y.

Solution. (i) Let x and y be fixed. Consider the different ways of having x good items, y fair items and $15 - x - y$ defective items,

There are $\binom{15}{x}$ possible ways of selecting x items out of 15 to be good, $\binom{15-x}{y}$ possible ways of selecting y items out of the remaining $15 - x$ items to be fair, and one way (having chosen x good items and y fair items) of selecting the rest to be defective.

Hence, the PMF is given by

$$p(x, y) = \mathbb{P}(X = x, Y = y) = \binom{15}{x}\binom{15-y}{y}(0.6)^x(0.3)^y(0.1)^{15-x-y}.$$

(ii) We will find the marginal PMFs directly. Indeed, $X \sim Binomial(15, 0.6)$, $Y \sim Binomial(15, 0.3)$, that is

$$p_X(x) = \mathbb{P}(X = x) = \binom{15}{x}(0.6)^x(0.4)^{15-x}.$$

$$p_Y(y) = \mathbb{P}(X = y) = \binom{15}{y}(0.3)^y(0.7)^{15-y}.$$

$n=15$.

(iii)

$$
\begin{aligned}
\mathbb{P}(X \le 11) &= 1 - \mathbb{P}(X \ge 12) \\
&= 1 - \left(\sum_{x=12}^{15} \binom{15}{x} (0.6)^x (0.4)^{15-x} \right) \\
&= 0.9095.
\end{aligned}
$$

(iv) Notice that $p_X(0) = 0.4^{15}$, $p_Y(0) = 0.7^{15}$, $p(0,0) = (0.1)^{15}$. Hence, $p(0,0) \neq p_X(0)p_Y(0)$. So X and Y are NOT independent. □

$p(0,0) = (0.1)^{15}$

$P_X(0) = 0.4^{15}$

$P_Y(0) = 0.7^{15}$

Bivariate Distribution (Joint PDF, CDF and Marginal PDF)

For continuous bivariate distributions, the definitions are really the same as the those in the discrete case except that integrals replace summations.

The **joint probability density function (joint PDF)** of two continuous-type random variables is an integrable function $f(x, y)$ with the following properties:

For joint PMF :

(a) $f(x, y) \geq 0$.

(a) $p(x,y) \geq 0$

(b) $\underline{\underline{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}}} f(x, y)\, dx\, dy = 1$

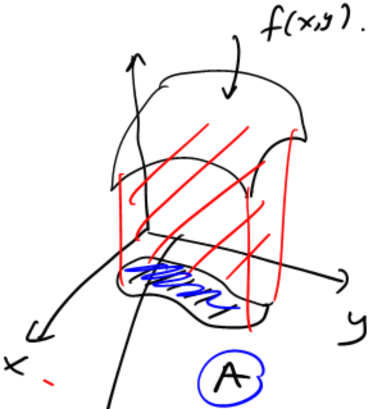
(b) $\underline{\underline{\sum_y \sum_x}} p(x,y) = 1$

(c) $\mathbb{P}((X, Y) \in A) = \iint_A f(x, y)\, dx\, dy$, where A is an event defined by a region on the xy-plane.

(c). A event.

$\mathbb{P}(A) = \sum_{(x,y) \in A} p(x,y)$

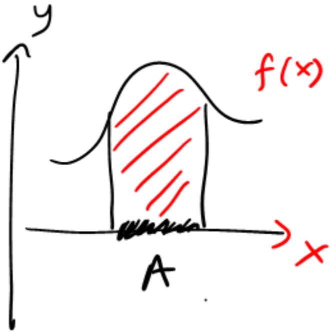
$f(x,y)$ joint PDF of X & Y.



$$\iint_A f(x,y)\,dx\,dy$$

$f(x)$ PDF of X

$$\mathbb{P}(A) = \int_A f(x)\,dx$$



The **joint cumulative density function (joint CDF)** of X and Y is given by

$$F(x,y) = \mathbb{P}(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds,$$

where $f(x,y)$ is the joint PDF of X and Y.

Compare this to single variable case:

$$F(x) = \mathbb{P}(X \leqslant x) = \int_{-\infty}^{x} f(s)\,ds.$$

The respective **marginal PDF** of continuous-type random variables X and Y are given by

- $$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy,$$

- $$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

marginal PMF

$$p_X(x) = \sum_{y} p(x,y)$$

$$p_Y(y) = \sum_{x} p(x,y)$$

Example 1

Let X and Y have the joint PDF

$$f(x,y) = \frac{4}{3}(1 - xy), \quad \boxed{0 \le x \le 1, 0 \le y \le 1.}$$

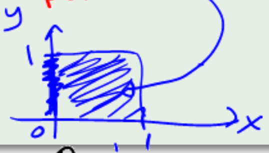
Find

(a) the marginal PDFs;

(b) $\mathbb{P}(\underline{Y \le X/2})$;

(c) the mean and variance of X.

Domain.



A

$A = \{Y \le \frac{X}{2}\}$

$$f_X(x) = \int_{-\infty}^{\infty} \underbrace{f(x,y)}\, dy$$

regard x as constant.

Solution.

(a)

$$f_X(x) = \int_0^1 \underbrace{\frac{4}{3}(1 - xy)}\, dy = \frac{4}{3}\left[y - \frac{xy^2}{2}\right]_{0=0}^{1=y} = \frac{4}{3}\left(1 - \frac{x}{2}\right).$$

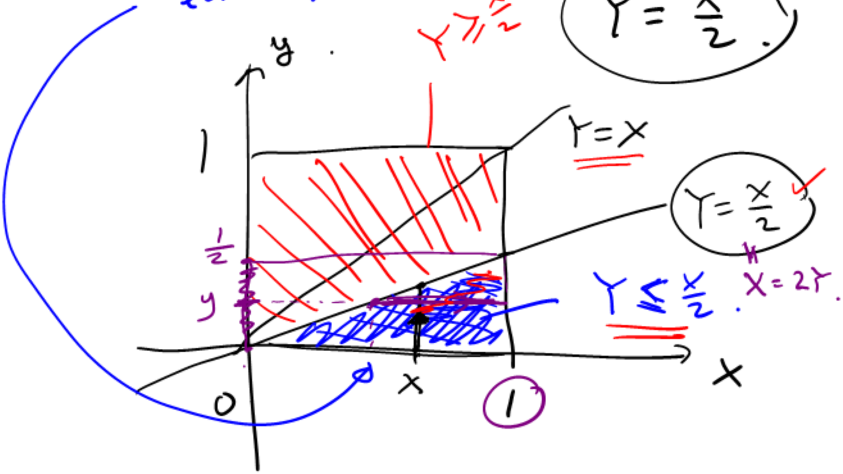
$$f_Y(y) = \int_0^1 \frac{4}{3}(1 - xy)\, dx = \frac{4}{3}\left[x - \frac{x^2 y}{2}\right]_{0=x}^{1=x} = \frac{4}{3}\left(1 - \frac{y}{2}\right).$$

$$f_Y(y) = \int_{-\infty}^{\infty} \underbrace{f(x,y)}\, dx$$

regard y as constant.

(b) $\mathbb{P}\left(\underbrace{Y \leq \frac{X}{2}}_{\text{event } A}\right)$

$Y = \frac{X}{2}.$



(b)

$$\mathbb{P}(A) = \iint_A f(x,y)\,dx\,dy.$$

$$\begin{aligned}
\mathbb{P}(Y \le X/2) &= \int_0^1 \int_0^{x/2} \frac{4}{3}(1-xy)\,dy\,dx \\
&= \frac{4}{3}\int_0^1 \left[y - \frac{xy^2}{2}\right]_0^{x/2} dx \\
&= \frac{4}{3}\int_0^1 \frac{x}{2} - \frac{x^3}{8}\,dx \\
&= \frac{7}{24}.
\end{aligned}$$

$$\mathbb{P}\left(Y \leq \frac{X}{2}\right) = \underbrace{\int_0^{\frac{1}{2}} \left(\int_{2y}^{1}}_{A} \frac{4}{3}(1-xy)\, dx \right) dy$$

$$= \quad \vdots$$

$$= \frac{7}{24} \quad \#$$

(c)

$$
\begin{aligned}
\text{Mean of } X = \mu_X &= \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \\
&= \int_0^1 x \frac{4}{3}\left(1 - \frac{x}{2}\right)\, dx \\
&= \frac{4}{3}\left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^1 \\
&= \frac{4}{9}.
\end{aligned}
$$

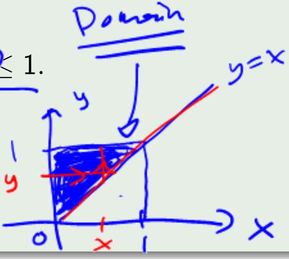
$$
\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[X^2] - \mu_X^2 \\
&= \int_0^1 x^2 \frac{4}{3}\left(1 - \frac{x}{2}\right)\, dx - \left(\frac{4}{9}\right)^2 \\
&= \frac{4}{3}\left[\frac{x^3}{3} - \frac{x^4}{8}\right]_0^1 - \left(\frac{4}{9}\right)^2 \\
&= \frac{13}{162}.
\end{aligned}
$$

$\square$

Example 2

Let X and Y have the joint PDF

$$f(x,y) = 2, \quad \text{for } 0 \le x \le y \le 1.$$

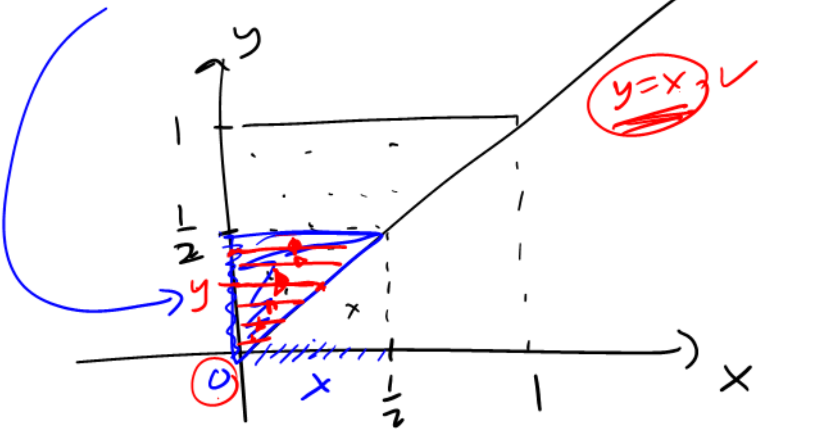


(a) Find $\mathbb{P}(X \le \frac{1}{2}, Y \le \frac{1}{2})$.

(b) Find the marginal PDFs $f_X(x)$, $f_Y(y)$.

Solution. The condition that $0 \le x \le y \le 1$ means that $f(x,y) = 2$ whenever (x,y) comes from the triangular region bounded by x-axis, the line $y = x$ and vertical line $x = 1$; and $f(x,y) = 0$ otherwise.

(i) $P(\underbrace{X \le \tfrac{1}{2}, Y \le \tfrac{1}{2}}_{\text{event } A})$



$$\mathbb{P}\left(X \le \tfrac{1}{2},\ Y \le \tfrac{1}{2}\right) = \iint_A 2 \, \underline{\underline{dx\,dy}}$$

or

$$= \iint_A 2 \, dy\,dx$$

$$= \int_0^{\frac{1}{2}} \int_0^{y} 2 \, \underline{\underline{dx\,dy}}.$$

or $\rightarrow$ $$\int_0^{\frac{1}{2}} \int_x^{\frac{1}{2}} 2 \, \underline{\underline{dy\,dx}}$$

(a)

$$
\begin{aligned}
\mathbb{P}\left(X \le \frac{1}{2}, Y \le \frac{1}{2}\right) &= \mathbb{P}\left(0 \le X \le \frac{1}{2}, X \le Y \le \frac{1}{2}\right) \\
&= \int_0^{1/2} \left(\int_x^{1/2} 2\, dy \right) dx \\
&= \int_0^{1/2} [2y]_x^{1/2}\, dx \\
&= \int_0^{1/2} 1 - 2x\, dx \\
&= \left[x - x^2\right]_0^{1/2} \\
&= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.
\end{aligned}
$$

$$\int_0^{\frac{1}{2}} \int_0^{y} 2 \quad dx\, dy$$

$$= \int_0^{\frac{1}{2}} \left[2x\right]_0^{y} dy$$

$$= \int_0^{\frac{1}{2}} 2y \; dy$$

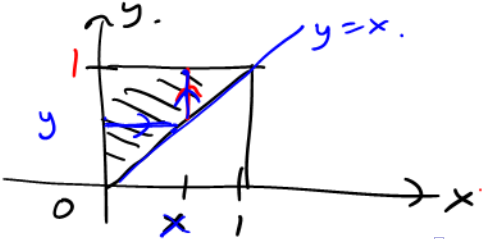
$$= \left[y^2\right]_0^{\frac{1}{2}} = \frac{1}{4}.$$

$\square$

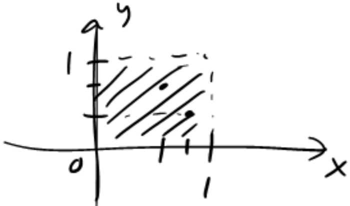
(b)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 2\,dy = 2(1-x), \quad 0 \le x \le 1.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 2\,dx = 2y, \quad 0 \le y \le 1.$$



Ex 1: $0 \le x \le 1, \quad 0 \le y \le 1$



$y = 0.7$

$1 \le x \le 0.7$

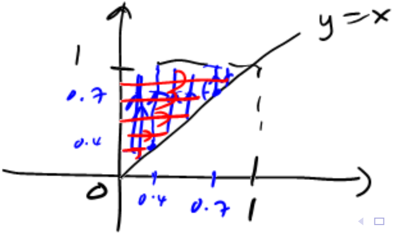
Ex 2: $0 \le x \le y \le 1$.

$x = 0.7$.

$0.7 \le y \le 1$

$x = 0.4$

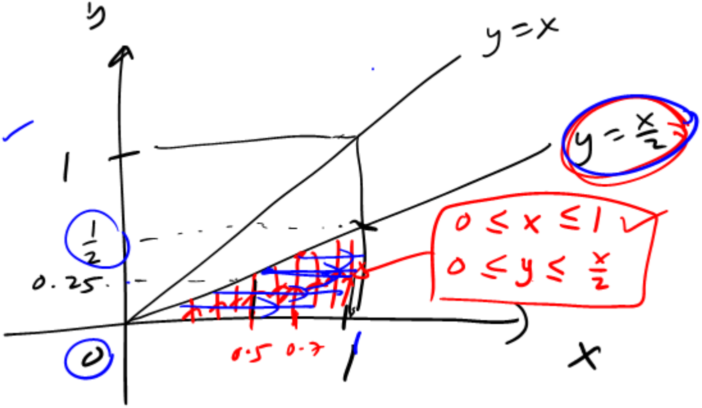
$0.4 \le y \le 1$



Ex 1.

$0 \le y \le \frac{1}{2}$

$2y \le x \le 1$



(b) Event $\boxed{Y \le \frac{X}{2}}$

$x = 0.5$

$y \le \frac{x}{2} = \frac{0.5}{2}$

$x = 0.7$

$y \le \frac{x}{2} = \frac{0.7}{2} = 0.35$

Conditional Distributions

Suppose $f(x,y)$ is the joint PMF/PDF of X and Y, and $f_X(x)$ and $f_Y(y)$ are the marginal PMFs/PDFs.

- The **conditional PMF/PDF** of X, given that $Y = y$, is defined by

$$g(x|y) = \frac{f(x,y)}{f_Y(y)}. \quad = \frac{\text{Joint PDF}}{\text{marginal PDF of } Y}$$

- The **conditional PMF/PDF** of Y, given that $X = x$, is defined by

$$h(y|x) = \frac{f(x,y)}{f_X(x)}. \quad = \frac{\text{Joint PDF}}{\text{marginal of } X}.$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}.$$

We can use conditional PDF/PMF to compute conditional probabilities.

conditioning

Discrete case:

$$\mathbb{P}(a \le X \le b | Y = y) = \sum_{x:a \le x \le b} g(x|y).$$

$$\mathbb{P}(a \le Y \le b | X = x) = \sum_{y:a \le y \le b} h(y|x).$$

Continuous case:

$$\mathbb{P}(a \le X \le b | Y = y) = \int_a^b g(x|y)\, dx.$$

$$\mathbb{P}(a \le Y \le b | X = x) = \int_a^b h(y|x)\, dy.$$

- **Conditional mean** of Y given $X = x$:

$$\mu_{Y|x} = \mathbb{E}[Y|X = x]$$

- **Conditional variance** of Y given $X = x$:

$$\sigma^2_{Y|x} = \mathbb{E}[Y^2|X = x] - \left(\mu_{Y|x}\right)^2$$

Remark: $\mu_{X|y}$ and $\sigma^2_{X|y}$ are defined similarly.

Example 3

Suppose X and Y have the joint PMF

$$p(x, y) = \frac{x + y}{21}, \quad x = 1, 2, 3, \quad y = 1, 2.$$

(a) Find the conditional PMF $g(x|y)$ of X given $Y = y$, and $h(y|x)$ of Y given $X = x$.

(b) Find $\mu_{Y|x}$ and $\sigma^2_{Y|x}$ when $x = 3$.

conditional PMF = $\dfrac{\text{Joint PMF}}{\text{marginal}}$.

Solution. (a) Note that

$$p_Y(y) = \sum_x p(x,y) = \frac{1+y}{21} + \frac{2+y}{21} + \frac{3+y}{21} = \frac{2+y}{7}.$$

$$p_X(x) = \sum_y p(x,y) = \frac{x+1}{21} + \frac{x+2}{21} = \frac{2x+3}{21}.$$

Hence,

$$g(x|y) = \frac{p(x,y)}{p_Y(y)} = \frac{(x+y)/21}{(2+y)/7} = \frac{x+y}{6+3y}.$$

$$h(y|x) = \frac{p(x,y)}{p_X(x)} = \frac{(x+y)/21}{(2x+3)/21} = \frac{x+y}{2x+3}.$$

(b)

$$
\begin{aligned}
\mu_{Y|x} &= \mathbb{E}[Y|X=x] \\
&= \sum_y y h(y|x) \\
&= 1 \cdot h(1|x) + 2 \cdot h(2|x) \\
&= 1 \cdot \frac{x+1}{2x+3} + 2 \cdot \frac{x+2}{2x+3} = \frac{3x+5}{2x+3}.
\end{aligned}
$$

Hence,

$$
\mu_{Y|3} = \frac{3(3)+5}{2(3)+3} = \frac{14}{9}.
$$

$x=3$

$\sigma^2_{Y|X}$

$$
\begin{aligned}
\sigma^2_{Y|3} &= \mathbb{E}[Y^2|X=3] - \mu^2_{Y|3} \\
&= \sum_y y^2 h(y|3) - (14/9)^2 \\
&= 1^2 \cdot h(1|3) + 2^2 \cdot h(2|3) - (14/9)^2 \\
&= 1 \cdot \frac{3+1}{2(3)+3} + 2^2 \cdot \frac{3+2}{2(3)+3} - (14/9)^2 = \frac{20}{81}.
\end{aligned}
$$

□

Same as Exmple 2.

Example 4

Let X and Y have the joint PDF

$$f(x, y) = 2, \quad \text{for } 0 \leq x \leq y \leq 1.$$

Find

(a) the conditional mean of Y given $X = x$.

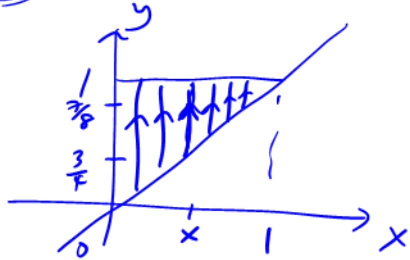
(b) the conditional variance of Y given $X = x$.

(c) $\mathbb{P}\left(\frac{3}{4} \leq Y \leq \frac{7}{8} \;\middle|\; X = \frac{1}{4}\right)$

ϕ

Solution. (a) By Example 2, the marginal PDF of X is $f_X(x) = 2(1-x)$, $0 \le x \le 1$. So the conditional PDF of Y given $X = x$ is

$$\underline{\underline{h(y|x)}} = \frac{f(x,y)}{f_X(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}, \quad \underbrace{0 \le x \le 1, x \le y \le 1.}$$



The conditonal mean of Y given $X = x$ is

$$
\begin{aligned}
\mu_{Y|x} &= \mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y \cdot \underbrace{h(y|x)}\, dy \\
&= \int_x^1 y \frac{1}{1-x}\, dy \\
&= \frac{1}{1-x}\left[\frac{y^2}{2}\right]_x^1 \\
&= \frac{1}{1-x}\left(\frac{1}{2} - \frac{x^2}{2}\right) \\
&= \frac{1+x}{2},
\end{aligned}
$$

for $0 \leq x \leq 1$.

(b) The conditional variance is

$$\int_{-\infty}^{\infty} y^2 h(y|x)\, dy.$$

$$
\begin{aligned}
\sigma^2_{Y|x} &= \mathbb{E}[Y^2|X=x] - \mu^2_{Y|x} \\
&= \int_x^1 y^2 \frac{1}{1-x}\, dy - \left(\frac{1+x}{2}\right)^2 \\
&= \frac{1}{1-x}\left(\frac{1}{3} - \frac{x^3}{3}\right) - \frac{(1+x)^2}{4} \\
&= \frac{(1-x)^2}{12},
\end{aligned}
$$

for $0 \leq x \leq 1$.

(c)

$$
\begin{aligned}
\mathbb{P}\left(\frac{3}{4} \leq Y \leq \frac{7}{8} \,|X = \frac{1}{4}\right) &= \int_{3/4}^{7/8} h(y|1/4)\, dy \\
&= \int_{3/4}^{7/8} \frac{1}{1-(1/4)}\, dy \\
&= \frac{4}{3}\int_{3/4}^{7/8} 1\, dy \\
&= \frac{4}{3}\cdot\frac{1}{8} = \frac{1}{6}.
\end{aligned}
$$

$\square$

$$\mathbb{P}(a \leq Y \leq b \mid X = x) = \int_a^b h(y/x)\, dy.$$

$$h(y/x) = \frac{1}{1-x}.$$

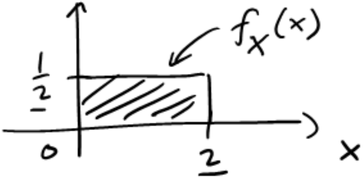
Example 5

Let X have a uniform distribution $U(0, 2)$, i.e. $f_X(x) = 1/2$ if $0 < x < 2$ and 0 otherwise. Let the conditional distribution of Y, given that $X = x$, be $U(0, x^2)$.

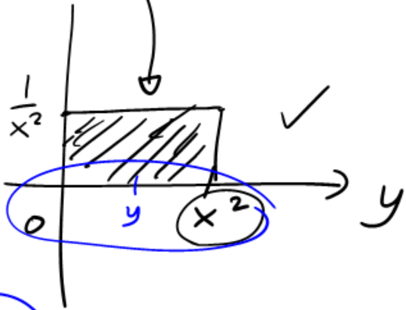
(a) Find the joint PDF $f(x, y)$ of X and Y. Sketch the region where $f(x, y) > 0$.

(b) Find the marginal PDF $f_Y(y)$ of Y.

marginal of X



conditional distribution $h(y|x)$



Know: $h(y|x) = \dfrac{\text{joint PDF } f(x,y)}{f_X(x)}$

$U(0, x^2)$

Solution. (a) Given $0 < x < 2$, we have

$$\frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{1}{x^2} & 0 < y < x^2 \\ 0 & \text{otherwise.} \end{cases}$$

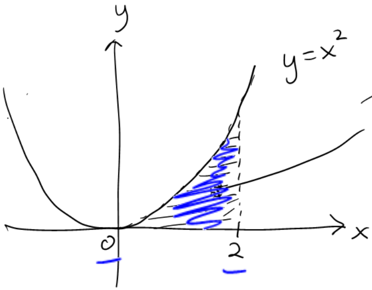
$U(0,2)$

Since $f_X(x) = 1/2$ if $0 < x < 2$, and 0 otherwise, we have

$$f(x,y) = \begin{cases} \frac{1}{2x^2} & 0 < y < x^2 \\ 0 & \text{otherwise,} \end{cases}$$

for $0 < x < 2$.

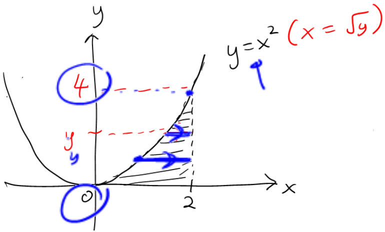
$$0 \le y \le x^2$$



$$\boxed{\begin{array}{l} 0 < x < 2 \\ 0 < y < x^2 \end{array}}$$

Domain.

y



(b)

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{\sqrt{y}}^{2} \frac{1}{2x^2}\,dx \\
&= \left[-\frac{1}{2}x^{-1}\right]_{\sqrt{y}}^{2} \\
&= -\frac{1}{4} + \frac{1}{2\sqrt{y}} = \frac{2-\sqrt{y}}{4\sqrt{y}},
\end{aligned}
$$

for $0 < y < 4$.