MH1820 Week 3

Discrete Random Variables, PMF and CDF

Random Variable

Motivating example: A fair dice is rolled 4 times.

$$\Omega = \text{set of all 4-tuples } (x_1, x_2, x_3, x_4) \text{ with } x_i \in \{1, 2, 3, 4, 5, 6\}.$$

Consider the following functions X and Y:

- $X =$ sum of the rolls. E.g. $X((1, 2, 5, 6)) = 1 + 2 + 5 + 6 = 14$.
- $Y =$ maximum among the four numbers. E.g. $Y((1, 2, 5, 6)) = 6$.

These functions are called random variables on Ω.

A **random variable** on Ω is a function X that assigns a real number $X(\omega)$ to every outcome ω.

Random variables provide an efficient and intuitive way to specify events.

E.g. Using the random variable X above,

$$X = 5 \iff E = \{(1,1,1,2),(1,1,2,1),(1,2,1,1),(2,1,1,1)\} \subseteq \Omega$$

Example 1

A fair coin is tossed three times.

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Consider the random varibles X and Y defined by

- $X =$ number of heads that occur
- $Y =$ number of tails that occur

- $\mathbb{P}(X = 3) = \mathbb{P}(\{HHH\}) = \frac{1}{8}$.
- $\mathbb{P}(X \leq 1) = \mathbb{P}(\{HTT, THT, TTH, TTT\}) = \frac{4}{8}$.
- $\mathbb{P}(X \in \{0, 3\}) = \mathbb{P}(\{HHH, TTT\}) = \frac{2}{8}$.
- $\mathbb{P}(X > Y) = \mathbb{P}(\{HHH, HHT, HTH, THH\}) = \frac{4}{8}$.

A **discrete random variable** is a random variable whose set of possible values is finite or countably infinite.

A dice is thrown repeatedly.

Consider the following random variables

- X: number of 6's among the first 10 throws
- Y: number of throws until the first 6 is thrown

Set of possible values of X: $\{0, 1, 2, \ldots, 10\}$ (finite set)

Set of possible values of Y: $\{1, 2, \ldots\}$ (countably infinite set)

PMF

Let X be a discrete random variable. The **probability mass function (PMF)** of X is defined as

$$p_X(x) = \mathbb{P}(X = x)$$

for all real numbers x.

Note:

- $p_X(x) = 0 \Leftrightarrow x$ is not a possible value of X.
- the PMF of X uniquely determines the probabilities of all events involving X.
- sometimes we just write $p(x)$ instead of $p_X(x)$.

Example 2

A fair coin is tossed 3 times. Let $X =$ number of heads that occur. The PMF is given by

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

□

CDF

If X is a discrete random variable with PMF $p(x)$, then the **Cumulative Density Function (CDF)** of X is defined by

$$F(x) = \mathbb{P}(X \leq x) = \sum_{t \leq x} p(t), \quad -\infty < x < \infty$$

where the sum runs over all numbers $t \leq x$.

Example 3

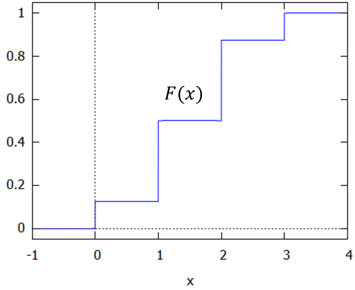
A fair coin is tossed 3 times. Let $X =$ number of heads that occur.

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Let F be the CDF of X. Then

- $F(-1) = \sum_{t \leq -1} p(t) = 0$.
- $F(0) = \sum_{t \leq 0} p(t) = p(0) = \frac{1}{8}$.
- $F(1) = \sum_{t \leq 1} p(t) = p(0) + p(1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$.
- $F(2) = \sum_{t \leq 2} p(t) = p(0) + p(1) + p(2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$.
- $F(3) = \sum_{t \leq 3} p(t) = p(0) + p(1) + p(2) + p(3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$.

$$F(x) = \begin{cases} 0, x < 0 \\ 1/8, 0 \leq x < 1 \\ 1/2, 1 \leq x < 2 \\ 7/8, 2 \leq x < 3 \\ 1, x \geq 3. \end{cases}$$



Note: At every x with $p(x) > 0$ there is a jump by $p(x)$.

The CDF has the following properties:

- $F(x)$ is a non-decreasing function of x, for $-\infty < x < \infty$.
- $F(x)$ ranges from 0 to 1.
- If a is the minimum possible value of X, then $F(a) = p_X(a)$. If $c < a$ then $F(c) = 0$.
- If b is the maximum possible value of X, then $F(b) = 1$.
- Also called the distribution function.

Expected Values and Variance

If a fair coin is tossed 1000 times, we expect around 500 heads.
If a dice is rolled 6000 times, around 1000 sixes are expected.

Both statements can be expressed in terms of random variables:

- Let X be the number of heads among 1000 tosses. Then $\mathbb{E}[X] = 500$ (expected value of X)
- Let Y be the number of sixes among 6000 throws. Then $\mathbb{E}[Y] = 1000$

The definition of expected values formalizes this.

Expected Value of Randon Variable

The **expected value** (or **mean**) of a discrete random variable X with PMF $p(x)$ is

$$\mathbb{E}[X] = \sum_{x} xp(x)$$

where the sum runs over all numbers x with $p(x) > 0$.

Intuitive interpretation: $\mathbb{E}[X]$ is the sum of all possible values of X, weighted by their probabilities.

Remark: If c is a constant, then $\mathbb{E}[c] = c$.

Example 4

A fair coin is tossed 3 times. Let X = number of heads that occur.

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\mathbb{E}[X] = \sum_x xp(x) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = \frac{3}{2}.$$

$\square$

Example 5

Two balls are randomly selected without replacement from an urn containing 5 balls numbered 1 through 5. Let X denote the larger number among the two balls selected. Find $\mathbb{E}[X]$.

Solution. Note that $|\Omega| = \binom{5}{2} = 10$. If $X = x$, then there are exactly $x - 1$ choices for the number of the other ball selected. So

$$p(x) = \frac{x-1}{10}.$$

Hence,

$$\mathbb{E}[X] = \sum_{x=1}^{5} xp(x) = 1(0) + 2(1/10) + 3(2/10) + 4(3/10) + 5(4/10) = 4.$$

$\square$

Expected Value of Function of Random Variable

Let X be a discrete random variable with PMF $p(x)$, and $g(X)$ be a function of X (e.g $g(X) = X^2$, $g(X) = e^X$ etc.) Then

$$\mathbb{E}[g(X)] = \sum_x g(x)p(x).$$

Example 6

A fair coin is tossed 3 times. Let $X =$ number of heads that occur.

x	0	1	2	3
$p_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$g(X) = X^2$$

$$\mathbb{E}[X^2] = \sum_x x^2 p_X(x) = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} = \frac{24}{8} = 3.$$

$\square$

Linearity of Expected Values

Theorem 7 (Linearity of Expected Values)

Let X_1, ..., X_n be random variables such that $\mathbb{E}[X_i]$ exists for all $i = 1, \ldots, n$. Let a_1, ..., a_n be real numbers (constants). Then

$$\mathbb{E}[a_1X_1 + \cdots a_nX_n] = a_1\mathbb{E}[X_1] + \cdots + a_n\mathbb{E}[X_n].$$

Rules:

- constants can be pulled out of expected values
- expected value of a sum is the sum of expected values of the summands

Example 8

Suppose X, Y, Z are random variables with

$$\mathbb{E}[X] = -10, \quad \mathbb{E}[Y] = 20, \quad \mathbb{E}[Z] = 5000.$$

Then

$$\begin{aligned}
\mathbb{E}[3X - 2Y + 5Z] &= 3\mathbb{E}[X] - 2\mathbb{E}[Y] + 5\mathbb{E}[Z] \\
&= 3(-10) - 2(20) + 5(5000) \\
&= 24930.
\end{aligned}$$

$\square$

Example 9

A firm purchases X number of computers each year, where X has the following probability distributions:

x	0	1	2	3
$f(x)$	$1/10$	$3/10$	$2/5$	$1/5$

If the cost of the computer is 1200 per unit and at the end of this year a rebate of $50X^2$ dollars will be issued, how much can this firm expect to spend on new computers during this year?

Give it a try!

Variance

The **variance** of a random variable X is defined as

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

Interpretation:

- $X - \mathbb{E}[X]$: deviation of X from its expected value
- $(X - \mathbb{E}[X])^2$: measures (squared) deviation from expected value
- $\mathbb{E}[(X - \mathbb{E}[X])^2]$ measure average (squared) deviation of X from its expected value. So variance measures how 'spread out' X is from its mean.

The **standard deviation** of a random variable X is defined as

$$\sigma_X = \sqrt{\mathrm{Var}(X)}.$$

Theorem 10 (Formula for Variance)

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Proof. Write $\mu = \mathbb{E}[X]$. Note that μ is a constant.

$$\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[(X-\mu)^2] \\
&= \mathbb{E}[X^2 - 2\mu X + \mu^2] \\
&= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mathbb{E}[\mu^2] \quad \text{(by lineariry of expected values)} \\
&= \mathbb{E}[X^2] - 2\mu^2 + \mu^2 \quad \text{(expected value of constant)} \\
&= \mathbb{E}[X^2] - \mu^2 \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2.
\end{aligned}$$

□

Discrete Distribution: Bernoulli, Binomial and Geometric

Bernoulli distribution

We say that a random variable X has a **Bernoulli distribution**, denoted by $X \sim Bernoulli(p)$ if X only takes value 0 (failure) and 1 (success) with $\mathbb{P}(X = 1) = p$. That is, its PMF is given by

$$p(x) = \begin{cases} 1 - p & \text{if } x = 0 \\ p & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 11 (Bernoulli)

If $X \sim Bernoulli(p)$, then

$$\mathbb{E}[X] = p, \quad \mathrm{Var}[X] = p(1 - p).$$

It follows that the standard deviation of X is $\sqrt{p(1-p)}$.

Some applications of Bernoulli distribution

- Experiments with only two outcomes, e.g. $X = 1$ if coin toss is head and $X = 0$ for tail
- Yes-no-questions, e.g., $X = 1$ if person voted for candidate A and $X = 0$ otherwise
- True-false conditions, e.g., $X = 1$ if total of 4 dice rolls is ≥ 20 and $X = 0$ otherwise

Binomial distribution

A random variable X has a **Binominal distribution**, denoted by $X \sim Binomial(n, p)$ if X is a sum of n independent Bernoulli random variables $Bernoulli(p)$.

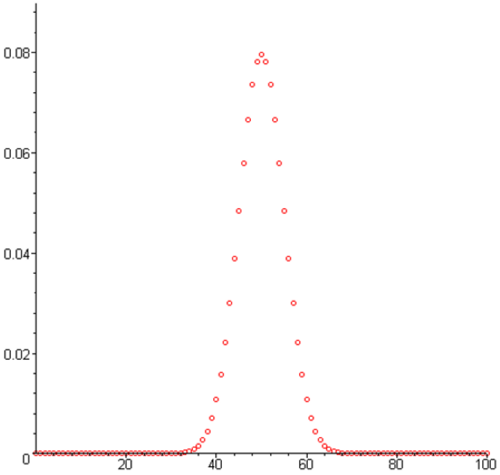
Interpretation: $X =$ number of successes among n independent experiments with success probability p.

Theorem 12 (Binomial dsitribution)

If $X \sim Binomial(n, p)$, then

$$\text{PMF: } \; p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n.$$

$$\mathbb{E}[X] = np, \quad \text{Var}[X] = np(1-p).$$



PMF of Binomial(100,0.5)

Example 13

A dice is rolled 10 times. let X be the number of 6's rolled. Then

$$X \sim \mathit{Binomial}\left(10, \frac{1}{6}\right)$$

$$\mathbb{P}(X = 2) = p(2) = \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8 \approx 0.29.$$

Example 14

In a production line, 10% of the items produced are defective. In a particular test, five items are independently selected from the production line and are tested. Let X denote the number of defective items among the five items.

(i) Find the expected value and variance of X.

(ii) What is probability that at most one item is defective?

Solution.

Note: defective = success. $X \sim \mathit{Binomial}(5, 0.1)$.

(i) $\mathbb{E}[X] = np = 5 \times (0.1) = 0.5$,
$\mathrm{Var}[X] = np(1-p) = 5(0.1)(0.9) = 0.45$.

(ii)

$$\mathbb{P}(X \leq 1) = \binom{5}{0}(0.1)^0(0.9)^5 + \binom{5}{1}(0.1)^1(0.9)^4 = 0.9185.$$

$\square$

Geometric distribution

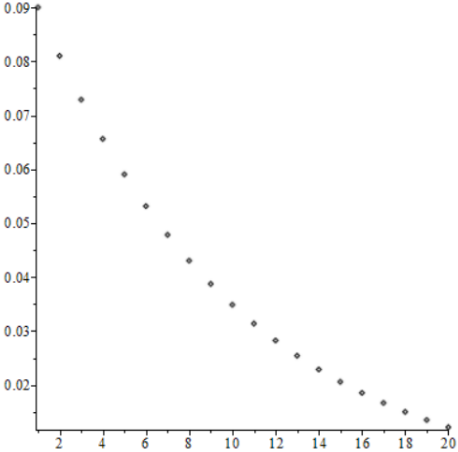
A random variable X has a **Geometric distribution**, denoted by $X \sim Geom(p)$, if X counts the number of experiments until the first success in a sequence of independent experiments with success probability p.

Theorem 15 (Geometric distribution)

If $X \sim Geom(p)$, then

$$\text{PMF:} \;\; p(x) = (1-p)^{x-1}p, \quad x = 1, 2, \ldots,$$

$$\mathbb{E}[X] = \frac{1}{p}, \quad \mathrm{Var}[X] = \frac{1-p}{p^2}.$$



PMF of Geom(0.1)

Example 16

A fair dice is rolled repeatedly. What is the probability that the 5th roll is the first roll for which a 1 or 6 occurs?

Solution. Success $=$ get a roll of 1 or 6. So $p = \frac{2}{6} = \frac{1}{3}$.

Let X be the number of rolls until the first success. Then $X \sim Geom(1/3)$.

We want to calculate $\mathbb{P}(X = 5)$. Let $p(x)$ be the PMF. Then

$$\mathbb{P}(X = 5) = p(5) = \left(\frac{2}{3}\right)^{5-1} \left(\frac{1}{3}\right) \approx 0.066.$$

□