

Tutorial 2

Matrix Algebra

1. Consider an $m \times n$ matrix A and an $n \times p$ matrix B . Show that if the columns of B are linearly dependent, then so are the columns of AB .
2. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible.
3. Suppose A, B and X are $n \times n$ matrices with A, X and $A - AX$ invertible. Also, suppose $(A - AX)^{-1} = X^{-1}B$. Solve for X . If you need to invert a matrix, explain why that matrix is invertible.
4. a. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system with n linear equations and n unknowns. If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, show that for any positive integer k , the system $A^k\mathbf{x} = \mathbf{0}$ also has only the trivial solution.
 b. Let $A\mathbf{x} = \mathbf{b}$ be any consistent system of linear equations and let \mathbf{x}_1 be a fixed solution. Show that every solution to the system can be written in the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$ where \mathbf{x}_0 is a solution to $A\mathbf{x} = \mathbf{0}$.
5. a. If the columns of an $n \times n$ matrix A are linearly independent, show that the columns of A^2 span \mathbb{R}^n .
 b. Show that if AB is invertible, so is A .
6. Using the notion of pivots and free variables only, answer the following questions:
 - a. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n . Explain why the equation $A\mathbf{x} = \mathbf{b}$ has in fact exactly one solution.
 - b. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each \mathbf{b} in \mathbb{R}^n .
7. Solve the equation $A\mathbf{x} = \mathbf{b}$ using the LU factorization given for A :

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}.$$

8. (*Spectral Factorization*) Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is some invertible 3×3 matrix and D is the

diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$. Find A^2 and A^3 and hence, a simple formula for A^k (where k is a positive integer). This factorization is useful when computing high powers of A .

Answers

7. $\mathbf{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

8. $A^k = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2^k & 0 \\ 0 & 0 & 1/3^k \end{bmatrix} P^{-1}$

End