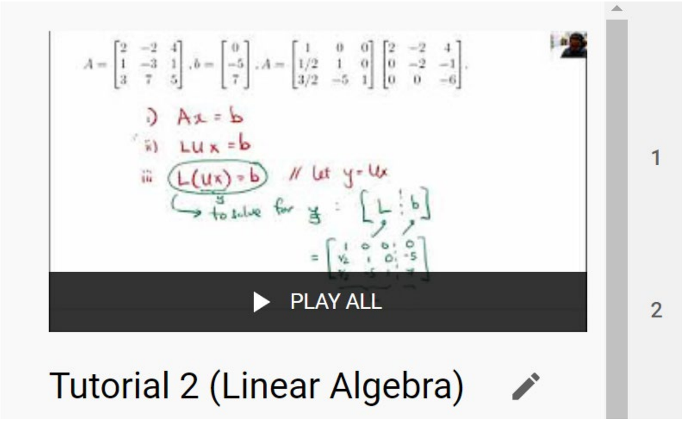
Maths/LA/Tut3

14 Sep 2020

Tutorial 3 Help links



Youtube link: playlist

https://www.youtube.com/channel/UCBzG5jg3huxiPkCt-Serrjw/playlists

PDF: writeup of Sol:

https://www.dropbox.com/s/5rhy9ja11ljshbb/Tut3_ces.pdf?dl=0

Determinant Background

- What is determinant?
  - 3Blue1Brown: https://www.youtube.com/watch?v=Ip3X9LOh2dk&vl=en
  - LeiosOS: https://www.youtube.com/watch?v=vvR3JSXO2fo
  - Purdue:https://www.youtube.com/watch?v=ktAgSCcRYfo

2 Important properties:

a) det(A) = 0 -> A is singular matrix (not invertible)

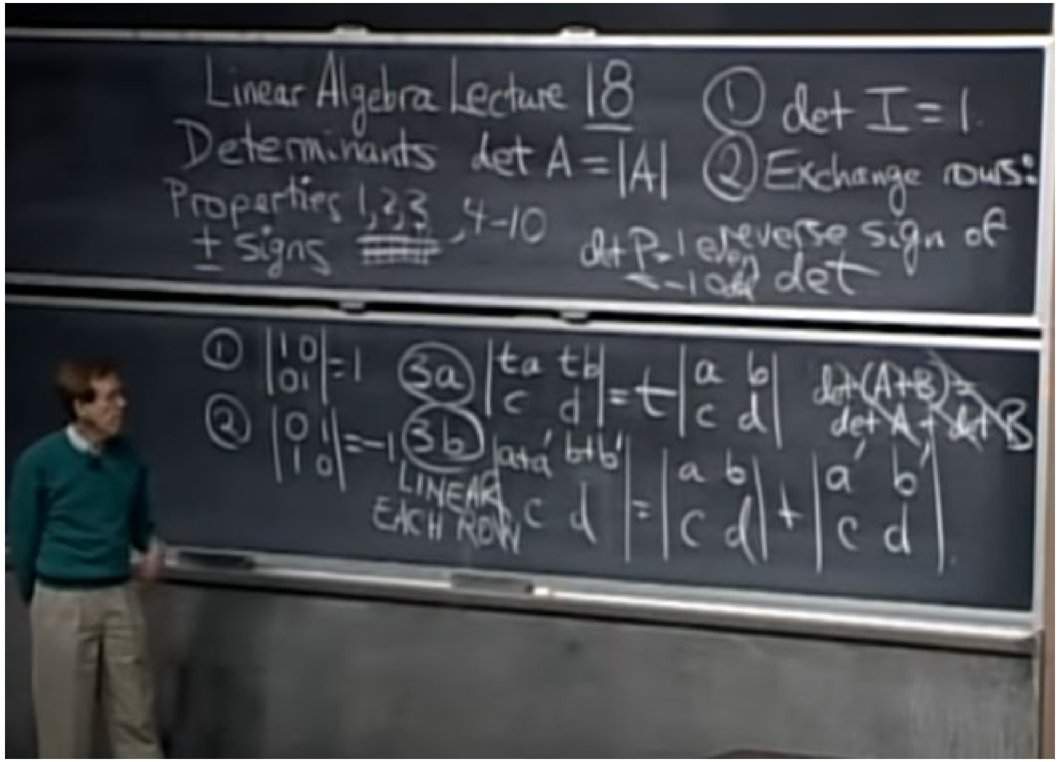
b) det(AB) = det(A)det(B)

Proof: http://www.math.lsa.umich.edu/~kesmith/ProofDeterminantTheorem.pdf
https://www.math.upenn.edu/~moose/240S2013/slides7-16.pdf

MIT’s lecture

- https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/least-squares-determinants-and-eigenvalues/properties-of-determinants/MIT18_06SCF11_Ses2.5sum.pdf

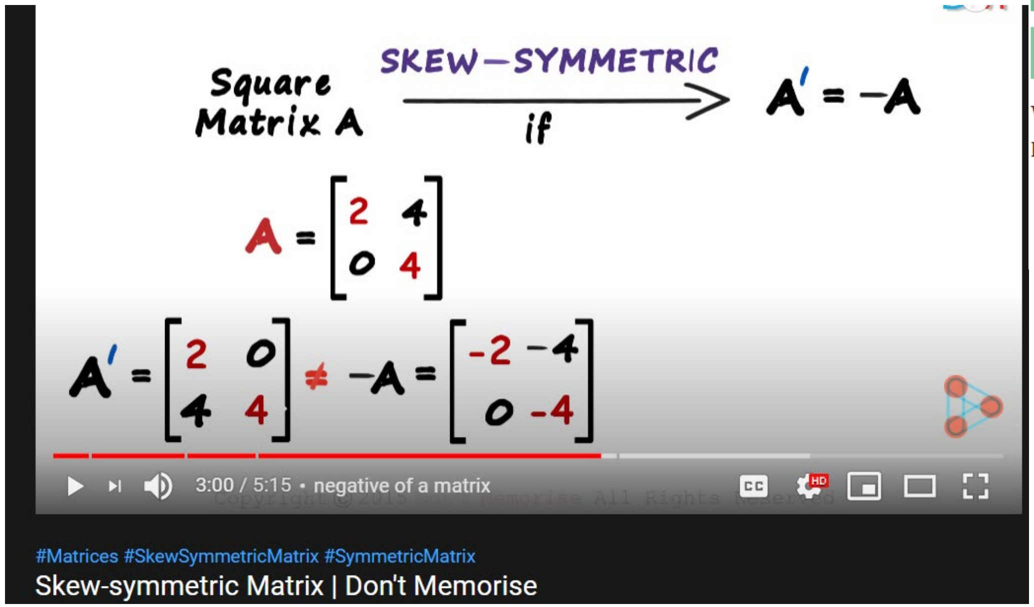
- https://www.youtube.com/watch?v=srxexLishgY



Q3

- **Background:**
  - Skew Symmetric: https://www.youtube.com/watch?v=uKPmyG18N7I
  - Determinant of matrix when row scaled by a scalar: https://www.khanacademy.org/math/linear-algebra/matrix-transformations/determinant-depth/v/linear-algebra-determinant-when-row-multiplied-by-scalar

- **Answer:** https://yutsumura.com/the-determinant-of-a-skew-symmetric-matrix-is-zero/



Proof.

Properties of Determinants

We will use the following two properties of determinants of matrices. For any $n \times n$ matrix A and a scalar c, we have

1. $\det(A) = \det(A^{\mathrm{T}})$,
2. $\det(cA) = c^n \det(A)$.

Main Part of the Proof

Suppose that n is an odd integer and let A be an $n \times n$ skew-symmetric matrix. Thus, we have

$$A^{\mathrm{T}} = -A$$

by definition of skew-symmetric.

Then we have

$$\begin{aligned} \det(A) &= \det(A^{\mathrm{T}}) && \text{by property 1} \\ &= \det(-A) && \text{since } A \text{ is skew-symmetric} \\ &= (-1)^n \det(A) && \text{by property 2} \\ &= -\det(A) && \text{since } n \text{ is odd.} \end{aligned}$$

Therefore, it yields that $2\det(A) = 0$, and hence $\det(A) = 0$.

Comment.

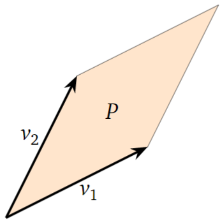
The result implies that every odd degree skew-symmetric matrix is not invertible, or equivalently singular.

Also, this means that each odd degree skew-symmetric matrix has the eigenvalue 0.

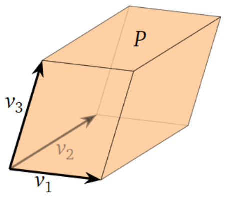
Q4: volume and parallel piped

- https://textbooks.math.gatech.edu/ila/determinants-volumes.html

Example (Parallelograms). When $n = 2$, a paralellepiped is just a paralellogram in $\mathbf{R}^2$. Note that the edges come in parallel pairs.



Example. When $n = 3$, a parallelepiped is a kind of a skewed cube. Note that the faces come in parallel pairs.



Determinants and Volumes

The key observation above is only the beginning of the story: the volume of a parallelepiped is *always* a determinant.

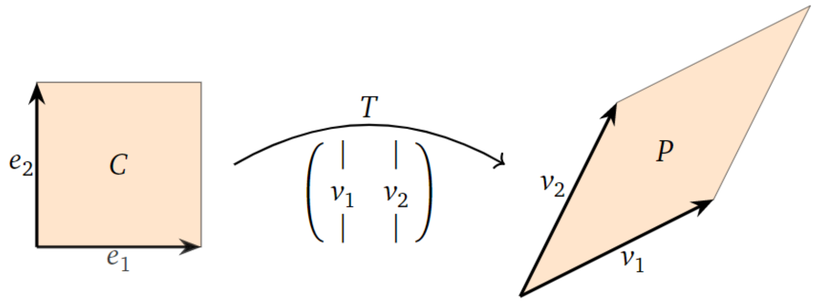
Theorem (Determinants and volumes). *Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbf{R}^n$, let P be the parallelepiped determined by these vectors, and let A be the matrix with rows $v_1, v_2, \ldots, v_n$. Then the absolute value of the determinant of A is the volume of P:*

$$|\det(A)| = \mathrm{vol}(P).$$

Q4) Determinant and Transformation

Volumes of Regions

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the associated matrix transformation $T(x) = Ax$. Then $T(e_1) = v_1$ and $T(e_2) = v_2$, so T takes the unit cube C to the parallelepiped P determined by $v_1, v_2, \ldots, v_n$:



Since the unit cube has volume 1 and its image has volume $|\det(A)|$, the transformation T scaled the volume of the cube by a factor of $|\det(A)|$. To rephrase:

> If A is an $n \times n$ matrix with corresponding matrix transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, and if C is the unit cube in $\mathbf{R}^n$, then the volume of $T(C)$ is $|\det(A)|$.

The notation $T(S)$ means the image of the region S under the transformation T. In set builder notation, this is the subset

$$T(S) = \{T(x) \mid x \text{ in } S\}.$$

In fact, T scales the volume of *any* region in $\mathbf{R}^n$ by the same factor, even for curvy regions.

Theorem. *Let A be an $n \times n$ matrix, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the associated matrix transformation $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then*

$$\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S).$$

Proof. Let C be the unit cube, let $v_1, v_2, \ldots, v_n$ be the columns of A, and let P be the paralellepiped determined by these vectors, so $T(C) = P$ and $\text{vol}(P) = |\det(A)|$. For $\varepsilon > 0$ we let εC be the cube with side lengths ε, i.e., the paralellepiped determined by the vectors $\varepsilon e_1, \varepsilon e_2, \ldots, \varepsilon e_n$, and we define εP similarly. By the second defining property, T takes εC to εP. The volume of εC is ε^n (we scaled each of the n standard vectors by a factor of ε) and the volume of εP is $\varepsilon^n |\det(A)|$ (for the same reason), so we have shown that T scales the volume of εC by $|\det(A)|$.

Q5) Determinant and Linear Transform

- https://mathinsight.org/determinant_linear_transformation

The linear transformation

$$\mathbf{T}(\mathbf{x}) = A\mathbf{x}, \qquad A = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$$

should change orientation as $\det(A) = (-1)(3) - (-1)(1) = -2$. It should also increase area by a factor of $|\det(A)| = 2$.

Again, we visualize the transformation $\mathbf{T}$ by looking at how it maps the unit square $[0,1] \times [0,1]$. As shown below, it maps the square into a parallelogram. Inspection of the paralleogram reveals that it indeed has area 2, so $\mathbf{T}$ doubles area as claimed. the orientation is also reversed. Moving counterclockwise around the perimeter of the parallelogram leads to the opposite color order red, blue, yelow, green. There is no way to stretch and move the original unit square into the parallelogram without taking it out of the plane and flipping it (or somehow moving the region through itself).