MH1820 Week 4

1. Discrete distribution: Poisson
2. Continuous random variables, PDF and CDF
3. The Exponential and Gamma distribution

The Poisson distribution

The Poisson distribution

Some experiments result in counting the **number of times** particular events occur/arrive during a given **time interval**.

Examples:

- Number of phone calls between 9AM and 10AM; the number of customers that arrive at a ticket window between 12noon and 2pm.
- Number of typos on a 10-page report. (here: 10-page is like the "time interval")

Usually, need a parameter λ that measures the average particular events occur per unit time.

A discrete random variable X has a **Poisson distribution**, denoted by $X \sim \mathit{Poisson}(\lambda)$, if its PMF is of the form

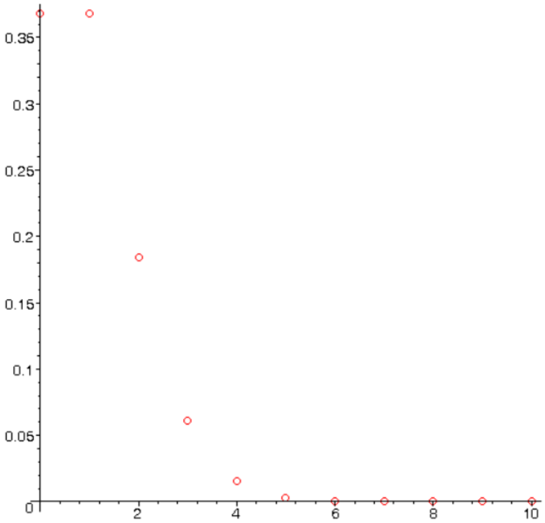
$$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where $\lambda > 0$.

Theorem 1 (Poisson)

If $X \sim \mathit{Poisson}(\lambda)$, then

$$\mathbb{E}[X] = \lambda, \quad \mathrm{Var}[X] = \lambda.$$



PMF of Poisson(1)

Example 2

On average, there are 2 supernovae in the milky way per century. Assuming Possion distribution, what is the probability that there are 2 supernovae in the milky way within one decade?

Solution.

- $X =$ number of supernovae in the milky way in one decade.
- $\mathbb{E}[X] = \frac{2}{10} = \frac{1}{5} = \lambda$, so $X \sim \mathit{Poisson}\left(\frac{1}{5}\right)$.
- $\mathbb{P}(X = 2) = e^{-\lambda}\frac{\lambda^x}{x!} = e^{-1/5}\frac{(1/5)^2}{2!} \approx 0.016$.

$\square$

Example 3

In a city, telephone calls to 911 come on the average of two every 3 minutes. If one assumes a Poisson distribution, what is the probability of five or more calls arriving in a 9-minute period?

Solution.

Let X = number of calls in a 9-minute period.

Then $\mathbb{E}[X] = 2 \times 3 = 6 = \lambda$. So $X \sim \textit{Poisson}(6)$.

$$
\begin{aligned}
\mathbb{P}(X \geq 5) &= 1 - \mathbb{P}(X \leq 4) \\
&= 1 - \sum_{x=0}^{4} e^{-6} \frac{6^x}{x!} \\
&= 1 - 0.285 = 0.715.
\end{aligned}
$$

$\square$

The PMF of a binomial distribution $Binomial(n, p)$ can be approximated by that of $Possion(\lambda)$ with $\lambda = np$.

This works well if $np < 10$ and $n > 50$.

Example 4

Let $X \sim Binomial(100, 0.02)$.

- $\mathbb{P}(X = 2) = \binom{100}{2} 0.02^2 0.98^{98} \approx 0.273.$
- Approximation by $Possion(100 \times 0.02)$,

$$\mathbb{P}(X = 2) = e^{-2}\frac{2^2}{2!} = 0.271.$$

Continuous random variables, PDF and CDF

A **continuous random variable** typically is a random variable whose set of possible values is an interval of real numbers or a union of such intervals.

Example: An air sample is analyzed and the fraction X of oxygen in the sample is determined (e.g. $X = 0.15$ means that 15% of the volume is taken up by oxygen).

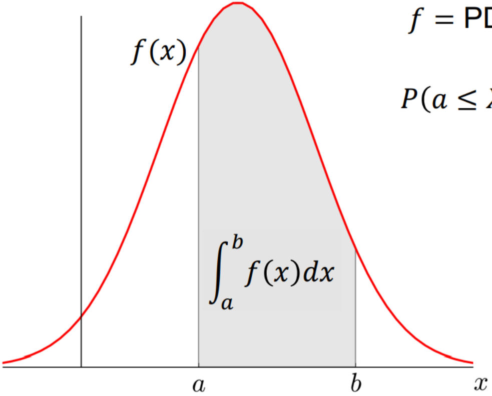
Set of possible values of $X \in [0, 1]$ (interval of real numbers x with $0 \leq x \leq 1$).

A function f that assigns a nonnegative real number $f(x)$ to each real number x is a **probability density function (PDF)** for a continuous random variable X if

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\, dx$$

for all real numbers a, b, $a \leq b$.

- Note $\int_a^b f(x)\, dx$ is the area between the graph of f and the segment of the x-axis between a and b.
- If necessary, we write f_X instead of f to indicate that f belongs to X.



f = PDF of X

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Example 5 (Uniform distribution)

The random variable X has a **uniform distribution** if its PDF $f(x)$ is equal to a constant on its support. In particular, if the support is the interval $[a, b]$, then

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

We shall denote it by $X \sim U(a, b)$.

Example: If $X \sim U(0, 1)$, then

$$\mathbb{P}\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right) = \int_{1/4}^{3/4} f(x)\, dx = \int_{1/4}^{3/4} 1\, dx = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}.$$

Example 6

Suppose X has PDF

$$f(x) = e^{-x-1}, \quad -1 \leq x < \infty.$$

Compute $\mathbb{P}(X \leq 1)$ and $\mathbb{P}(X \geq 1)$.

Give it a try!

If X is a continuous random variable with PDF $f(x)$, then the **Cumulative Density Function (CDF)** of X is defined by

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(t)\, dt.$$

Note:

- $F(x)$ is nondecreasing
- $0 \leq F(x) \leq 1.$
- $F'(x) = \frac{dF}{dx} = f(x)$ (PDF)

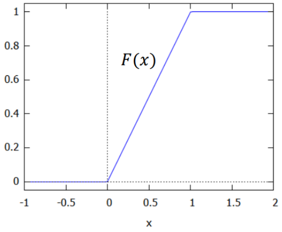
Example 7

Let $X \sim U(0,1)$ be the uniform distribution on $[0,1]$. Its CDF is given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Note: For $0 \leq x \leq 1$, we have

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x 1\,dt = x.$$



Example 8

Suppose X has PDF

$$f(x) = \begin{cases} \frac{1}{10} e^{-x/10} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the CDF of X.

Solution. For $x \leq 0$, $F(x) = 0$. For $x \geq 0$,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x \frac{1}{10} e^{-t/10}\,dt = \frac{1}{10}\left[\frac{e^{-t/10}}{-\frac{1}{10}}\right]_0^x = 1 - e^{-x/10}.$$

So

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-x/10} & \text{if } x \geq 0. \end{cases}$$

$\square$

Let X be a continuous random variable with PDF $f(x)$.

- Its **expected value** or **mean** is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$$

- If $g(X)$ is a function of X, then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

- Similar to discrete random variables, the **variance** and **standard deviation** σ of X can be calculated as follows (where $\mu = \mathbb{E}[X]$):

$$\mathrm{Var}[X] = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mu^2, \quad \sigma = \sqrt{\mathrm{Var}[X]}.$$

Example 9

The total amount of medical claims (in millions) of the employees of a company has the PDF given by

$$f(x) = 30x(1-x)^4, \quad 0 < x < 1.$$

Find

(i) The mean and the standard deviation of the total in dollars.

(ii) The probability that the total exceeds $0.2 millions.

Solution.
(i)

$$\text{mean} \quad \mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(30x)(1-x)^4\,dx = 0.286 \text{ millions.}$$

$$\text{variance} = \mathbb{E}[X^2] - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - (0.286)^2 = \int_0^1 x^2(30x)(1-x)^4\,dx - (0.286)^2 = 0.107 - (0.286)^2 = 0.025204 \text{ millions.}$$

$$\sigma = \sqrt{\mathrm{Var}[X]} = \sqrt{0.025204} = 0.159 \text{ millions.}$$

(ii)

$$
\begin{aligned}
\mathbb{P}(X > 0.2) &= \int_{0.2}^{\infty} 30x(1-x)^4\,dx \\
&= \int_{0.2}^{1} 30x(1-x)^4\,dx \\
&= 0.6554.
\end{aligned}
$$

$\square$

The Exponential and Gamma distribution

The Exponential distribution

We now turn to a continuous random variable that is related to Poisson distribution.

- Let λ be the mean/average number of occurrences per unit interval.
- Let $X \sim Poisson(\lambda w)$ be the random variable that counts the number of occurences in an interval of size w.
- Then $\mathbb{P}(\text{no occurences in } [0, w]) = \mathbb{P}(X = 0) = e^{-\lambda w}$.

Let W = waiting time until the first occurence. Then its CDF $F(w)$ is given by

$$F(w) = \mathbb{P}(W \leq w) = 1 - \mathbb{P}(W > w)$$
$$= 1 - \mathbb{P}(\text{no occurences in } [0, w]) = 1 - e^{-\lambda w}$$

Note that W is nonnegative. For $w \geq 0$, the PDF of W is

$$\frac{dF}{dw} = f(w) = \lambda e^{-\lambda w}.$$

We often let $\lambda = \frac{1}{\theta}$, and say that the random variable X has an **exponential distribution**, denoted by $X \sim Exp(\theta)$, if its PDF is defined by

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 \leq x < \infty.$$

Theorem 10 (Exponential distribution)

If $W \sim Exp(\theta)$, then

$$\mathbb{E}[W] = \theta, \quad \mathrm{Var}[W] = \theta^2.$$

Example 11

Customers arrive in a certain shop according to a Poisson process at mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer?

Solution. Let $W =$ the waiting time in minutes until the first customer arrives. Goal: Compute $\mathbb{P}(W > 5)$.

Expected number of arrivals per minute: $\lambda = \frac{20}{60} = \frac{1}{3} \implies \theta = \frac{1}{\lambda} = 3$.

$$\mathbb{P}(W > 5) = \int_5^\infty \frac{1}{3} e^{-x/3}\, dx = e^{-5/3} = 0.1889.$$

□

Let W denote the waiting time until the αth occurrence in a Poisson process with $\lambda = \frac{1}{\theta}$. Then W has a **Gamma distribution** with **shape** parameter α and **scale** parameter θ, denoted by $W \sim Gamma(\alpha, \theta)$, with PDF given by

$$f(w) = \frac{1}{\Gamma(\alpha)\theta^\alpha} w^{\alpha-1} e^{-w/\theta}, \quad 0 \leq w < \infty.$$

Here:

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy, \quad \alpha > 0.$$

For our purpose, α is usually a positive integer, and so

$$\Gamma(\alpha) = (\alpha - 1)!.$$

Theorem 12 (Gamma distribution)

If $W \sim Gamma(\alpha, \theta)$, then

$$\mathbb{E}[W] = \alpha\theta, \quad \mathrm{Var}[W] = \alpha\theta^2.$$

Example 13

Suppose the number of customers per hour arriving at a shop follows a Poisson distribution with mean 30. What is the probability that the shopkeeper will wait for more than 5 minutes until the second customer arrives?

Solution. W = waiting time (in minutes) until the second customer arrives. Then

$$W \sim \textit{Gamma}\left(\alpha = 2, \theta = 2\right). \quad (\text{why?})$$

Want to compute $\mathbb{P}(W > 5)$.

$$
\begin{aligned}
\mathbb{P}(W > 5) &= \int_5^\infty \frac{1}{\Gamma(2)\,(2)^2} w^{2-1} e^{-w/2}\,dw \\
&= \frac{1}{4}\int_5^\infty we^{-w/2}\,dw \\
&= \frac{1}{4}\left(\left[-2we^{-w/2}\right]_5^\infty - \int_5^\infty (-2)e^{-w/2}\,dw\right) \\
&\quad \text{(by integration-by-parts)} \\
&= \frac{1}{4}\left(10e^{-5/2} + 2\left[(-2)e^{-w/2}\right]_5^\infty\right) \\
&= \frac{1}{4}\left(10e^{-5/2} + 4e^{-5/2}\right) \\
&= 0.2873.
\end{aligned}
$$

$\square$