MH1812 Discrete Mathematics: Quiz (CA) 2

Name: Tutorial Group:

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There are 3 (THREE) questions, please try all of them, and justify all your answers! Best of luck!

Question 1 (40 points)

a) Prove or disprove the following set equality (20 points):

$$A - (B \cup C) = (A - B) \cap (A - C).$$

Solution. Recall that $A - B = A \cap \bar{B}$, thus

$$A - (B \cup C) = A \cap \overline{(B \cup C)} = A \cap (\bar{B} \cap \bar{C})$$

using De Morgan law for sets. Then since $A \cap A = A$, we further have

$$A \cap (\bar{B} \cap \bar{C}) = A \cap \bar{B} \cap A \cap \bar{C} = (A - B) \cap (A - C).$$

b) If you toss 3 fair coins, what is the probability of getting at least 2 heads? (20 points).

Solution. The sample space is $\{HHH, THH, HTH, TTH, HHT, THT, HTT, TTT\}$ therefore out of the 8 events, 4 correspond to at least 2 heads, HHT, HTH, THH and HHH, so the probability is $\frac{4}{8} = \frac{1}{2}$.

Question 2 (40 points)

a) Find $a, b \in \mathbb{R}$ which satisfy the following equation (20 points):

$$-3 + a = \frac{4}{i} - i + ib$$

where $i = \sqrt{-1}$.

Solution. Since $a, b \in \mathbb{R}$, the real part of this equation is

$$-3 + a = 0 \Rightarrow a = 3.$$

The imaginary part is

$$-4i - i + ib = 0 \Rightarrow -5 + b = 0 \Rightarrow b = 5.$$

b) Consider the following system of linear equations. Write it in matrix form, and determine its solutions, if any (20 points):

$$\begin{cases} x_1 + 2x_2 & = & 3 \\ 2x_1 + 4x_2 & = & -1 \end{cases}$$

Solution. In matrix form, we have

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ -1 \end{pmatrix}}_{b}$$

Since the rank of the matrix A is 1, and that of $(A|b)$ is 2, there is no solution.

Question 3 (20 points)

Consider the following two recurrence relations:

$$a_n = 3a_{n-1},\ a_1 = 4$$

and

$$b_n = 4b_{n-1} - 3b_{n-2},\ b_1 = 0,\ b_2 = 12.$$

Choose and solve ONE OF THE TWO (namely, pick the one you prefer and solve it, you do NOT need to solve both of them).

Solution. The first one is easier to solve using backtracking:

$$a_n = 3a_{n-1} = 3(3a_{n-2}) = 9(3a_{n-3}) = 3^i a_{n-i} = 3^{n-1}a_1 = 4 \cdot 3^{n-1}.$$

We can check it by mathematical induction: For $n = 1$, we have $a_1 = 4$ as needed. Then suppose $a_n = 4 \cdot 3^{n-1}$.

$$a_{n+1} = 3a_n = 3(4 \cdot 3^{n-1}) = 4 \cdot 3^n$$

as needed.

The second one is easier to solve using the characteristic equation:

$$x^n = 4x^{n-1} - 3x^{n-2} \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0$$

therefore

$$b_n = c3^n + d$$

with

$$3c + d = 0,\ 9c + d = 12.$$

Thus $c = 2$, $d = -6$ and

$$b_n = 2 \cdot 3^n - 6.$$