MH1820 Week 2

Properties of probability

Properties of probability

The following properties can be deduced from the definition of probability.

(a) $\mathbb{P}(\emptyset) = 0$.

(b) $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A)$.

(c) If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(d) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

(e) $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \overline{B})$.

Example 1

Find the probability that a randomly chosen card from a 52-card poker deck is of hearts or from one of the ranks ace, king, queen. Assume all outcomes in Ω have the same probability.

- Ω: set of all 52 cards, $|\Omega| = 52$
- A: set of all cards that are hearts, $|A| = 13$
- B: set of all cards that are aces, kings or queens, $|B| = 12$
- $A \cap B = \{$ace of heart, king of heart, queen of heart$\}$, $|A \cap B| = 3$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26}.$$

$\square$

Example 2

Of a group of small-business owners, 30% consult both an accountant and a planner and 10% consult neither. The probability of consulting an accountant exceeds the probability of consulting a planner by 20%.

What is the probability of a randomly selected person X from this group consulting an accountant?

Define the following events:
A: X consults an accountant
B: X consults a planner

Given that

$$\mathbb{P}(A \cap B) = 0.3, \quad \mathbb{P}(\overline{A} \cap \overline{B}) = 0.10, \quad \mathbb{P}(A) = \mathbb{P}(B) + 0.20.$$

Goal: Find $\mathbb{P}(A)$.

$$
\begin{aligned}
\mathbb{P}(A \cup B) &= 1 - \mathbb{P}(\overline{A \cup B}) \\
&= 1 - \mathbb{P}(\overline{A} \cap \overline{B}) \\
&= 1 - 0.10 = 0.90 \qquad (1)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A) + (\mathbb{P}(A) - 0.2) - 0.3 \\
&= 2\mathbb{P}(A) - 0.5. \qquad (2)
\end{aligned}
$$

Combining (1) and (2):

$$
\mathbb{P}(A) = \frac{1}{2}(0.5 + 0.9) = 0.7.
$$

Conditional Probability

Motivation: We are interested in the probability of the event B given that another event A has already occured.

Under this condition, A becomes the new sample space and event $B \subseteq \Omega$ is represented by $A \cap B$ in the new sample space

We use the notation $\mathbb{P}(B|A)$ for the probability that an event B occurred under the condition that A occured.

For illustration, if we assume that all outcomes have the same probability in the finite sample space Ω, then

$$\mathbb{P}(B|A) = \frac{|A \cap B|}{|A|}.$$

But we can write this as

$$\begin{aligned}\mathbb{P}(B|A) &= \frac{|A \cap B|}{|A|} \\ &= \frac{|A \cap B|/|\Omega|}{|A|/|\Omega|} \\ &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.\end{aligned}$$

This motivates the followng definition of conditional probability.

Let $A, B \subseteq \Omega$ be events with $\mathbb{P}(A) > 0$. The **conditional probability** of B given A is

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

Note that $\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A)$. This formula often can be used to compute $\mathbb{P}(A \cap B)$.

Example 3

Consider a poker deck of 52 cards. What is the probability that 5 randomly chosen cards form a four of a kind under the assumption that they do not contain any ace, king, or queen?

$\Omega =$ set of all combinations of 5 cards taken from 52 cards, $\quad |\Omega| = \binom{52}{5}$.

$A =$ event that the 5 cards do not contain any ace, king, or queen,

$$|A| = \binom{52-12}{5} = \binom{40}{5}$$

$B =$ event that 5 cards form a four of a kind

$A \cap B =$ event that 5 cards form four of a kind and contain no ace, king, or queen

$$|A \cap B| = 10 \cdot 36 \quad \text{WHY?}$$

$$
\begin{aligned}
\mathbb{P}(B|A) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \\
&= \frac{\frac{|A \cap B|}{|\Omega|}}{\frac{|A|}{|\Omega|}} \\
&= \frac{|A \cap B|}{|A|} \\
&= \frac{10 \cdot 36}{\binom{40}{5}} \approx 0.0005
\end{aligned}
$$

$\square$

Example 4

An organ transplant operation succeeds with probability 0.65. Given that the operation succeeded, the probability that the body rejects the organ is 0.2. What is the probability that a randomly selected patient is treated successfully?

- A: event that a transplant succeds
- B: event that body does not reject organ
- $A \cap B$: event that a patient is treated successfully

$$\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A) = 0.8 \cdot 0.65 = 0.52.$$

□

Independent Events

Usually steps of an experiment are conducted independently.

For instance, if a dice is rolled 3 times, then the result of the one of the rolls does not influence the results of the other rolls.

In other words: If we know the result of one roll, this does not give us any information on the results of the other rolls. Events with this property are called **independent**.

Formally, events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

In this case,

- $\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)} = \mathbb{P}(B).$
- $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$

That is, information that A occurs does not change probability that B occurs and vice versa.

Example 5

A dice is rolled two times

- A: first roll is a 1
- B: second roll is a 6

A and B are independent:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

Example 6

A red die and a white die are rolled. Assume the dice are fair. Consider the events

- $C = \{5 \text{ on red die}\}$
- $D = \{\text{sum of dice is } 11\}$.

Are the event C and D independent?

Note: $C = \{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$, $|C| = 6$, and $D = \{(5,6),(6,5)\}$, $|D| = 2$. So $C \cap D = \{(5,6)\}$, $|C \cap D| = 1$.

$$\mathbb{P}(C \cap D) = \frac{1}{36} \neq \mathbb{P}(C)(D) = \frac{6}{36}\frac{2}{36}.$$

Hence, C and D are dependent events. $\square$

Independence for several events

Events A_1, …, A_n are **mutually independent** if

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{1_i}) \cdots \mathbb{P}(A_{i_k})$$

for every nonempty subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ with $k \geq 2$.

Intuitive interpretation: Knowledge of any particular event A_i does not give information whether the other event A_j, where $i \neq j$.

Example: Events A_1, A_2, A_3 are mutually independent if all of the following hold:

- $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_3)\mathbb{P}(A_3)$
- $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$
- $\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_3)$
- $\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2)\mathbb{P}(A_3)$

Example 7

A fair six-sided die is rolled 6 independent times. Let A_i be the event that side i is observed on the ith roll, called a match on the ith trial, $i = 1, \ldots, 6$. What is the probability that at least one match occurs?

Let B: event that at least one match occurs. Then

$$\mathbb{P}(B) = 1 - \mathbb{P}(\overline{B}) = 1 - \mathbb{P}(\overline{A_1} \cap \cdots \cap \overline{A_6})$$

Events $\overline{A_i}$ are mutually independent. So

$$\mathbb{P}(B) = 1 - \mathbb{P}(\overline{A_1}) \cdots \mathbb{P}(\overline{A_6}) = 1 - \frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6} = 1 - \left(\frac{5}{6}\right)^5.$$

□

Bayes' Theorem

Law of Total Probability

Suppose the sample space Ω is partitioned as

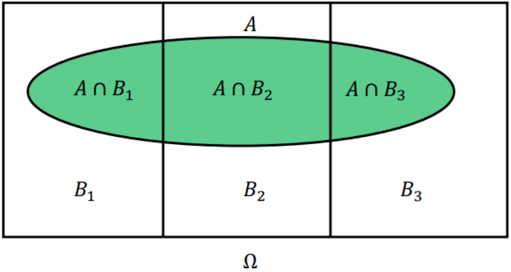
$$\Omega = B_1 \cup \cdots \cup B_n$$

where the B_i are disjoint events. Then

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(A \cap B_1) + \cdots + \mathbb{P}(A \cap B_n) \\ &= \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \cdots + \mathbb{P}(A|B_n)\mathbb{P}(B_n) \end{aligned}$$

for every event A.

Intuition behind law of total probability:



Theorem 8 (Bayes' Theorem)

Let A, B be events with $\mathbb{P}(A), \mathbb{P}(B) > 0$. Then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Proof.

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \implies \mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A).$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$

$\square$

Bayes' Theorem allows us to flip what is given.

Example 9

Consider a lab test for a disease:

- It is 95% effective at detecting the disease
- It has a false positive rate of 1%
- The rate of occurence of the disease in the general population is 0.5%

I take the screening test and get a positive result. What is the likelihood I have the disease?

Carefully define our events:

- T: I tested positive
- D: I have the disease

We want to find $\mathbb{P}(D|T)$.

Information given:

$$\mathbb{P}(T|D) = 0.95, \quad \mathbb{P}(T|\overline{D}) = 0.01, \quad \mathbb{P}(D) = 0.005.$$

By Bayes' Theorem

$$\mathbb{P}(D|T) = \frac{\mathbb{P}(T|D)\mathbb{P}(D)}{\mathbb{P}(T)} = \frac{0.95 \cdot 0.005}{\mathbb{P}(T)} = \frac{0.00475}{\mathbb{P}(T)}.$$

By law of total probability:

$$\begin{aligned}\mathbb{P}(T) &= \mathbb{P}(T|D)\mathbb{P}(D) + \mathbb{P}(T|\overline{D})\mathbb{P}(\overline{D}) \\ &= (0.95)(0.005) + (0.01)(1 - 0.005) \\ &= 0.0147.\end{aligned}$$

Hence,

$$\mathbb{P}(D|T) = \frac{0.00475}{\mathbb{P}(T)} = \frac{0.00475}{0.0147} \approx 0.323$$

$\square$