NANYANG TECHNOLOGICAL UNIVERSITY

MIDTERM II (CA2)

MH1812 – Discrete Mathematics

March 2017 TIME ALLOWED: 40 minutes

Name:

Matric. no.: Tutor group:

INSTRUCTIONS TO CANDIDATES

1. **DO NOT TURN OVER PAPER UNTIL INSTRUCTED.**
2. This midterm paper contains **FOUR (4)** questions.
3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. This **IS NOT** an **OPEN BOOK** exam.
6. Candidates should clearly explain their reasoning when answering each question.

QUESTION 1. **(25 marks)**

Solve the following linear recurrences:

(a) $a_n = 7a_{n-1}$, with initial condition $a_1 = 5$;

(b) $b_n = 20b_{n-1} - 51b_{n-2}$, with initial conditions $b_0 = 5$ and $b_1 = 6$.

Solution:

(a) The characteristic equation is $x - 7 = 0$. So the general solution for a_n is $a_n = u7^n$. From the initial condition, we see that $a_1 = 7u = 5$ and hence $u = 5/7$. So $a_n = 5 \cdot 7^{n-1}$.

(b) The characteristic equation is $x^2 - 20x + 51 = 0 = (x - 17)(x - 3)$. So the general solution for b_n is $b_n = u17^n + v3^n$. From the initial conditions, we see that $b_0 = u + v = 5$ and $b_1 = 17u + 3v = 6$. Solving this gives $u = -9/14$ and $v = 79/14$. So $b_n = \frac{79}{14}3^n - \frac{9}{14}17^n$.

QUESTION 2. **(20 marks)**

Prove that

$$\sum_{j=0}^{n} j!j = (n+1)! - 1 \qquad \forall n \in \mathbb{N}.$$

Solution: Set $P(n)$ be the predicate $\sum_{j=0}^{n} j!j = (n+1)! - 1$. For the base case we check $P(1)$ where we have LHS: $0!0+1!1 = 1$ and RHS: $(1+1)!-1 = 1$. So $P(1)$ is true.

Next we have the inductive step. Assume that $P(n)$ is true, for all $n \in \{1, \ldots, k\}$. Now we want to show $P(k+1)$ is true, that is, we want to show

$$\sum_{j=0}^{k+1} j!j = (k+2)! - 1.$$

On the left hand side we have

$$\begin{aligned}
\sum_{j=0}^{k+1} j!j &= \sum_{j=0}^{k} j!j + (k+1)!(k+1) \\
&= (k+1)! - 1 + (k+1)!(k+1) \\
&= (k+1)!(1+k+1) - 1 \\
&= (k+2)! - 1.
\end{aligned}$$

Hence $P(k+1)$ is true. As required.

QUESTION 3. **(25 marks)**

Let $S = \{1, \ldots, n\}$ be a finite set and let $\mathcal{P}(S)$ denote the power set of S. Set $A = \{s \in \mathcal{P}(S) : |s| \text{ is even }\}$ and $B = \{s \in \mathcal{P}(S) : |s| \text{ is odd }\}$. Using the binomial theorem, or otherwise, prove that the cardinalities of A and B are equal, that is, prove that $|A| = |B|$.

Solution: The set A is just the set of all subsets of S that have an even number of elements and the set B is the set of subsets of S that have an odd number of elements. The binomial coefficient $\binom{n}{k}$ is equal to the number of subsets of S that have k elements. So we have

$$\begin{aligned} |A| &= \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} \\ |B| &= \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k}. \end{aligned}$$

Now, the binomial theorem says

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

If we substitute $x = 1$ and $y = -1$ then this equation becomes

$$\begin{aligned} 0 &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \\ &= \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} (-1)^k + \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} (-1)^k \\ &= \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} - \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} \\ &= |A| - |B|. \end{aligned}$$

Therefore $|A| = |B|$, as required.

QUESTION 4. **(30 marks)**

(a) Prove, for the sets A, B, C, D, that

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D).$$

(b) Does equality hold? Justify your answer with either a proof or a counterexample.

Solution:

(a) Take $x \in (A \times B) \cap (C \times D)$. Then $x = (x_1, x_2)$ where $x_1 \in A$, $x_2 \in B$, and $x_1 \in C$, $x_2 \in D$. So $x_1 \in A \cap C$ and $x_2 \in B \cap D$. Hence $x \in (A \cap C) \times (B \cap D)$. Therefore

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D),$$

as required.

(b) Equality holds. For the proof it suffices to show that

$$(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D).$$

Take $x \in (A \cap C) \times (B \cap D)$. Then $x = (x_1, x_2)$ where $x_1 \in A \cap C$ and $x_2 \in B \cap D$. In particular we have that $x_1 \in A$, $x_2 \in B$, and $x_1 \in C$, $x_2 \in D$. Which means that $x \in (A \times B) \cap (C \times D)$. This shows that

$$(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D),$$

as required.