AY 21/22 MH1820 Midterm Test 1

1. [2 marks] How many 4-digit numbers are there all of whose digits are odd?

Solution: There are 5 odd digits. By the multiplication principle, there are 5^4 numbers with 4 digits all of which are odd.

2. [2 marks] Five persons $P_1, \dots, P_5$ are randomly assigned to five car seats $S_1, \dots, S_5$. What is the probability that P_1 is seated at S_1 and P_2 is seated at S_2?

Solution: There are 5! ways to seat the persons and 3! ways to seat them such that P_1 is seated at S_1 and P_2 is seated at S_2. Thus the required probability is $3!/5! = 1/20$.

3. [2 marks] Three cards are randomly chosen from a standard poker deck of 52 cards. Which of the following events has the highest probability?

○ Exactly two of the cards are kings.
○ Exactly one of the cards is a king and exactly one is a queen.
○ All three cards are of spades.

Solution:

- The number of ways to choose 3 cards such that exactly two of them are kings is $\binom{4}{2} \cdot 48 = 288$.
- Exactly one of the cards is a king and exactly one is a queen: Here the number of choices is $\binom{4}{1}\binom{4}{1} \cdot 44 = 704$.
- All three cards are of spades: There are $\binom{13}{3} = 286$ choices.

Hence the event "Exactly one of the cards is a king and exactly one is a queen" has the highest probability.

4. [2 marks] A fair dice is rolled 3 times. What is the probability that the total rolled is at most 4 under the condition that the first roll is a 1?

Solution: Let A be the event that the total rolled is at most 4 and let B be the event that the first roll is a 1. We have $|B| = 36$ and thus $P(B) = 36/216 = 1/6$. Moreover, $A \cap B$ is the event that the first roll is a 1 and the total is at most 4. There are 3 outcomes like this $(1,1,1), (1,1,2), (1,2,1)$. Hence $P(A \cap B) = 3/216$ and

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{216}}{\frac{1}{6}} = \frac{3}{36} = \frac{1}{12}.$$

5. [2 marks] A fair coin is tossed 5 times. Let X be the total number of heads that occur and let $F(x)$ be the CDF of X. Which of the following is equal to $F(1)$?

Solution: We have

$$F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{32} + \frac{5}{32} = \frac{3}{16}.$$

6. [5 marks] A ball is drawn from one of 2 boxes. The boxes contain the following number of balls of colors blue (B) and red (R).

	B	R
Box 1	1	4
Box 2	6	2

The following procedure is used to draw the ball.

- One of the boxes is chosen at random: Box 1 is chosen with probability 0.2 and Box 2 with probability 0.8.
- A ball is drawn from the chosen box (each ball in the box is chosen with the same probability).

(a) What is the probability that a blue ball is drawn?

(b) If a blue ball is drawn, what is the probability that it was drawn from Box 1?

Solution: We first define the relevant events.

$$\begin{aligned} B_1, B_2 &: \text{ball is drawn from box } 1, 2, \text{ respectively,} \\ A &: \text{a blue ball is drawn.} \end{aligned}$$

Since the probability that the ball is drawn from box 1 is 0.2, we have $P(B_1) = 0.2$. Similarly, $P(B_2) = 0.8$.

If the ball is drawn from box 1, then the probability that it is blue is 0.2, since 1 out of 5 balls in box 1 is blue. Hence $P(A|B_1) = 0.2$. Similarly, we get $P(A|B_2)$. In summary,

$$\begin{aligned} P(A|B_1) &= 0.2, \\ P(A|B_2) &= 0.75. \end{aligned}$$

(a) By the Law of Total Probability, the probability that a blue ball is drawn is

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) = 0.2 \cdot 0.2 + 0.75 \cdot 0.8 = 0.64.$$

(b) By Bayes' Theorem, under the condition that a blue ball was drawn, the probability that it was drawn from box 1 is

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{0.2 \cdot 0.2}{0.64} = 0.0625.$$

7. [5 marks] Let X be a continuous random variable with PDF given by

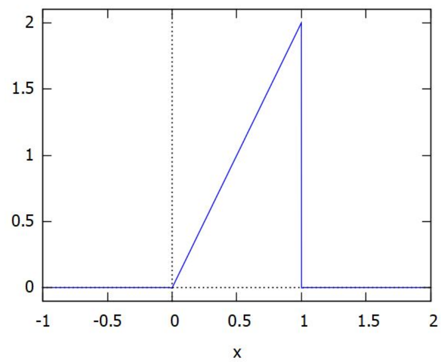
$$f(x) = 2x \text{ for } 0 \le x \le 1 \text{ and } f(x) = 0 \text{ otherwise.}$$

(a) Draw a graph of f.

(b) Compute the CDF F of X and draw of graph of F.

(c) Compute $E[X]$.

Solution: (a)



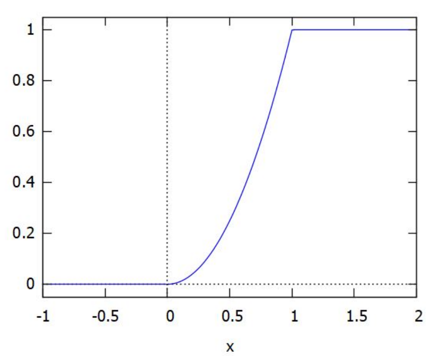
(b) Let $F(x)$ denote the CDF of X. Let $0 \le x \le 1$. We compute

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt = \int_0^x 2t dt \\ &= \left[t^2\right]_0^x = x^2. \end{aligned}$$

Hence

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ x^2 & \text{for } 0 \le x \le 1, \\ 1 & \text{for } x > 1. \end{cases}$$

Plot of the CDF:



(c) We have

$$E[X] = \int_{-\infty}^{\infty} f(x)x\,dx = \int_0^1 2x^2 dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}.$$

8. [5 marks] Let X and Y be a independent discrete random variables, both with PMF $f(x)$ given by

x	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

and $f(x) = 0$ otherwise.

(a) Compute $E[X]$ and $Var[X]$.

(b) Compute $E[XY]$ and $Var[2X - Y]$.

Solution: (a) We compute

$$\begin{aligned} E[X] &= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} = \frac{5}{4}, \\ E[X^2] &= 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{2} = \frac{9}{4}, \\ Var[X] &= E[X^2] - E[X]^2 = \frac{9}{4} - \left(\frac{5}{4}\right)^2 = \frac{11}{16}. \end{aligned}$$

(b) Since X and Y have the same distribution, we have $E[Y] = E[X]$ and $Var[Y] = Var[X]$. Since X and Y are independent, we get

$$\begin{aligned} E[XY] &= E[X]E[Y] = E[X]^2 = \frac{25}{16}, \\ Var[2X - Y] &= 2^2 Var[X] + (-1)^2 Var[Y] = 5Var[X] = \frac{55}{16}. \end{aligned}$$