MH1820 Week 5

$N(\mu, \sigma^2)$ μ = mean
σ = s.d

The Normal distribution

Table.

- PDF μ, σ
- $\mathbb{P}(a \leq X \leq b)$ etc.

$N(\mu, \sigma^2) \xrightarrow[\text{transformation.}]{} N(0,1)$

- percentiles of $N(0,1)$

The Normal distribution

Central Limit Theorem.

The **Normal distribution** is the most important continuous probability distribution in the entire field of statistics. Gauss

It is also known as the **Gaussian distribution**.

Its graph is called the **normal curve** and is a bell-shaped curve.

PDF.

μ, σ

The probability density function (PDF) of a **normal random variable** X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \underline{exp} \left[-\frac{(x-\mu)^2}{2\sigma^2} \right], \quad \underline{-\infty < x < \infty.}$$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\underline{\underline{e = 2.718\ldots}}$$

Notation:

$$\boxed{X \sim N(\mu, \sigma^2)}$$

Motivation :

$N(\mu, \sigma^2)$ approximates Binomial(n, p)

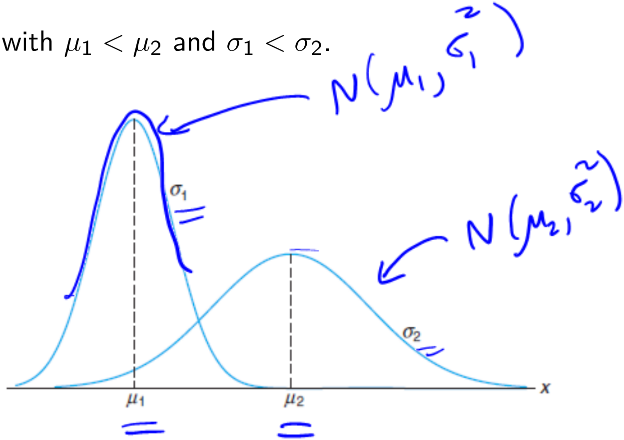
$$\mu = np$$
$$\sigma^2 = np(1-p).$$

Theorem 1 (Normal distribution)

If $X \sim N(\mu, \sigma^2)$*, then*

$$\mathbb{E}[X] = \mu \quad \mathrm{Var}[X] = \sigma^2.$$

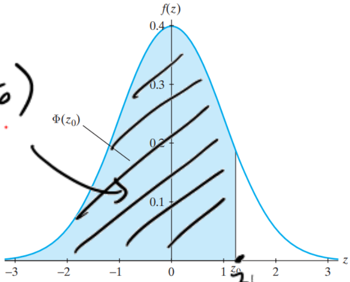
Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$.



If $\mu = 0$ and $\sigma = 1$, then $Z \sim N(0,1)$ is called a **standard normal random variable**, and its distribution is called a **standard normal distribution**. The CDF of Z is denoted by

$$\Phi(z) = \mathbb{P}(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

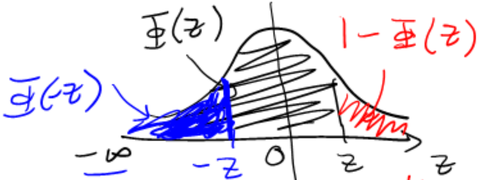
$\Phi(z_0)$



$-\infty$

z_0

Figure 3.3-1 Standard normal pdf



It is useful to know that for any z, we have

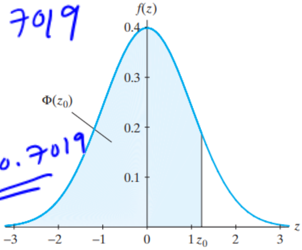
$$\Phi(-z) = 1 - \Phi(z).$$

The CDF $\Phi(z)$ cannot be calculated by hand, so in this course we will just look them up in the table.

Table Va The Standard Normal Distribution Function

$\Phi(0.53) = 0.7019$

$\Phi(0.5271) \approx \Phi(0.53) = 0.7019$



$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$\Phi(-z) = 1 - \Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

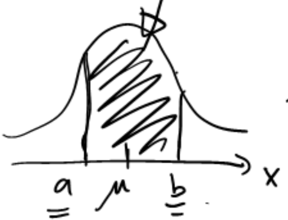
Theorem 2 (Transforming to standard normal)

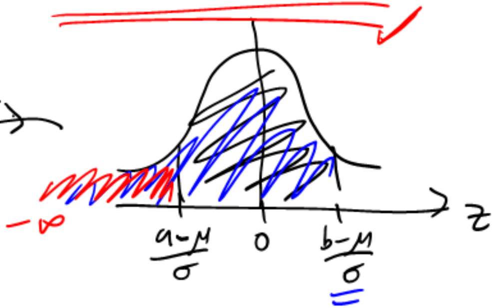
If $X \sim N(\mu, \sigma^2)$, *then*

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

$$\mathbb{P}(a \le X \le b) = \mathbb{P}\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$$

$$\overset{\star}{=} \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$





Example 3

If $X \sim N(3, 16)$, find $\mathbb{P}(X \geq 5)$ and $\mathbb{P}(4 \leq X \leq 8)$

$\mu = 3$ $\quad \sigma^2 = 16$

$\sigma = 4$

$$\mathbb{P}(X \geq 5) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \geq \frac{5-\mu}{\sigma}\right)$$
$$= \mathbb{P}\left(Z \geq \frac{5-3}{4} = 0.5\right)$$
$$= 1 - \mathbb{P}(Z \leq 0.5)$$
$$= 1 - \Phi(0.5)$$
$$= 1 - 0.6915$$
$$= 0.3085 \ \#$$

$$
\begin{aligned}
P(4 \leq X \leq 8) &\overset{*}{=} \Phi\left(\frac{8-\mu}{\sigma}\right) - \Phi\left(\frac{4-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{8-3}{4}\right) - \Phi\left(\frac{4-3}{4}\right) \\
&= \Phi(1.25) - \Phi(0.25) \\
&= 0.8944 - 0.5987 \\
&= 0.2957 \ \#.
\end{aligned}
$$

Example 4

If $X \sim N(25, 36)$, find c such that

$\mu = 25$, $\sigma^2 = 36$, $\sigma = 6$.

$$\mathbb{P}(|X - 25| \leq c) = 0.9544.$$

$$|X - 25| \leq c$$

$$-c \leq X - 25 \leq c$$

$$-\frac{c}{6} \leq \underbrace{\frac{X-25}{6}}_{Z \sim N(0,1)} \leq \frac{c}{6}.$$

$$\mathbb{P}(|X-25| \leq c) = 0.9544$$

$$\mathbb{P}\left(-\frac{c}{6} \leq Z \leq \frac{c}{6}\right) = 0.9544$$

$$\Phi\left(\frac{c}{6}\right) - \Phi\left(-\frac{c}{6}\right) = 0.9544$$

$$\mathbb{P}\left(Z \leq \frac{c}{6}\right) - \mathbb{P}\left(Z \leq -\frac{c}{6}\right)$$

$$\boxed{\Phi(-z) = 1 - \Phi(z)}$$

$$\Phi\left(\frac{c}{6}\right) - \left(1 - \Phi\left(\frac{c}{6}\right)\right) = 0.9544$$

$$2\Phi\left(\frac{c}{6}\right) - 1 = 0.9544$$

$$\Phi\left(\frac{c}{6}\right) = \frac{1.9544}{2} = 0.9772$$

From Table:

$$\frac{c}{6} = 2,$$

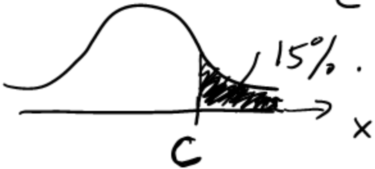
$$c = 2 \times 6 = 12 \quad \#.$$

Example 5

Suppose X, the grade on a midterm exam, is normally distributed with mean 70 and standard deviation 10. The instructor wants to give 15% of the class an A (the best grade). What cutoff should the instructor use to determine who gets an A?

$$X \sim N(\mu = 70, \sigma^2 = 10^2)$$

C = cutoff for A.



$$\mathbb{P}(X \geq c) = 0.15.$$

$$\mathbb{P}\Big(\underbrace{\frac{X-\mu}{\sigma}}_{Z} \geqslant \underbrace{\frac{c-\mu}{\sigma}}_{\frac{c-70}{10}}\Big) = 0.15$$

$$\mathbb{P}\left(Z \geq \frac{c-70}{10}\right) = 0.15$$

$$1 - \mathbb{P}\left(Z \leqslant \frac{c-70}{10}\right) = 0.15$$

$$\mathbb{P}\left(Z \leqslant \frac{c-70}{10}\right) = 0.85.$$

$$\Phi\left(\frac{c-70}{10}\right) = \underline{\underline{0.85}}$$

From Table:

$$\boxed{\frac{c-70}{10} = 1.04.}$$

since $\Phi(1.04) \approx 0.85$

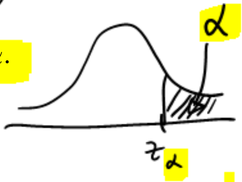
$$c = 70 + 10.4$$

$$= \underline{\underline{80.4}}.$$

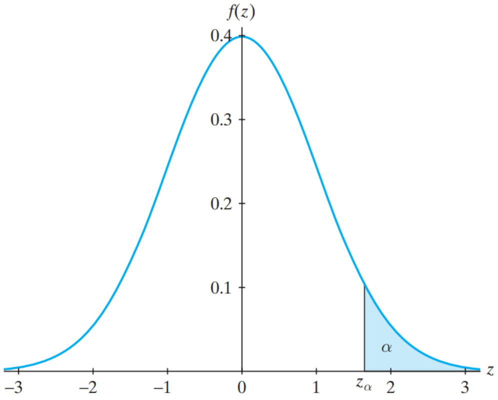
In statistical applications, we are often interested in the numbers called percentiles. Let $Z \sim N(0,1)$. Given α, $0 < \alpha < 1$

- The $100(1-\alpha)$**th percentile** (or **the upper** 100α**th percentage point**) for the standard normal is the number z_α such that

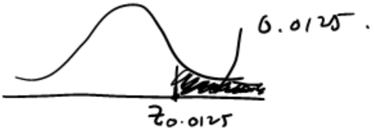
$$\mathbb{P}(Z > z_\alpha) = \mathbb{P}(Z \geq z_\alpha) = \alpha.$$



- The 100α**th percentile** is the number $z_{1-\alpha}$.



Note that by symmetry we have $\mathbb{P}(Z \leq -z_\alpha) = \mathbb{P}(Z \geq z_\alpha) = \alpha$.
For a table of $\mathbb{P}(Z \geq z_\alpha)$, see see NTULearn $->$ Content $->$ TABLES.pdf.



Example 6

Find $z_{0.0125}$. z_α $\alpha = 0.0125$.

Solution. Note that

$$\mathbb{P}(Z > z_{0.0125}) = 0.0125.$$

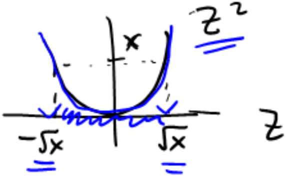
From the table, we have $z_{0.0125} = 2.24$. □

Example 7

Let $X = Z^2$, where $Z \sim N(0,1)$. Compute the CDF of X. Hence, deduce the PDF of X.

Let $F(x) = \text{CDF of } X$

$$\begin{aligned} &= \mathbb{P}(X \leq x) \\ &= \mathbb{P}(Z^2 \leq x) \\ &= \mathbb{P}(-\sqrt{x} \leq Z \leq \sqrt{x}) \\ &= \Phi(\sqrt{x}) - \Phi(-\sqrt{x}) \end{aligned}$$



know $\Phi(-z) = 1 - \Phi(z)$

$$F(x) = \Phi(\sqrt{x}) - \left(1 - \Phi(\sqrt{x})\right)$$

$$= 2\Phi(\sqrt{x}) - 1.$$

Recall: $\frac{dF}{dx}$ = PDF of X

$$\text{PDF of } X = \frac{d}{dx}\left(F(x)\right) = \frac{d}{dx}\left(2\Phi(\sqrt{x}) - 1\right).$$

$$= 2 \cdot \frac{d}{dx}\Phi(\sqrt{x})$$

$$= 2\,\frac{d}{du}\Phi(u) \cdot \frac{du}{dx}$$ (chain Rule) $u = \sqrt{x}$

$$= \cancel{2}\,\underbrace{\frac{1}{\sqrt{2\pi}} e^{-u^2/2}}_{\text{PDF of } N(0,1) \text{ in } u.} \cdot \frac{1}{\cancel{2}\sqrt{x}}$$

PDF of $X = \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{1}{\sqrt{x}}$

$$= \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-x/2}$$

#

$X = Z^2 \qquad Z \sim N(0,1)$.