SC1004 Part 2

Lectured by Prof Guan Cuntai
(teaching materials by Prof Chng Eng Siong)

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Quiz 2 and Exam:

1. Quiz 2

- Coverage: Ch 6 – 8.1.2

- Time & Venue:

Full-time students:
- **Time/Date**: Week 13, last lecture time (10:30-11.20am, 17th April 2024, Wed)
- **Venue**: LT1A, LT 7, LT 10, LT 11, LT 17

Part-time students:
- **Time/Date**: Week 13, 7:30-8.20 pm, 16th April 2024, Tuesday
- **Venue**: LHN-TR+15

2. Final Exam

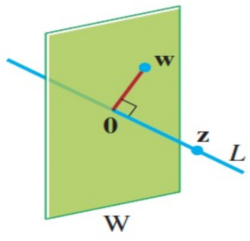
- **Coverage**: Ch 6, 7, 8 (Q3 & Q4)
- **Time/Venue**: 2 May 2024 (Thursday), 1.00-3.00 pm, Hall W.

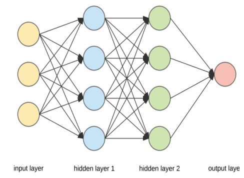
(Ch 9 will not be tested)

Syllabus for Part 2

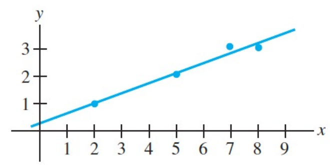
Chapter	Topics	Week (Lecture)	Week (Tut)
6	Orthogonality	8-9	9-10
7	Least Squares	9-10	10-11
8	EigenValue and Eigenvectors	11-12	12-13
9	Singular Value Decomposition (SVD)	13	

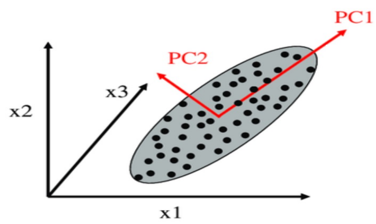
Table 1: schedule





$$\mathbf{A}\mathbf{x} = \mathbf{b}$$





Online Video learning Schedule

https://www.youtube.com/channel/UCBzG5jg3huxiPkCt-Serrjw

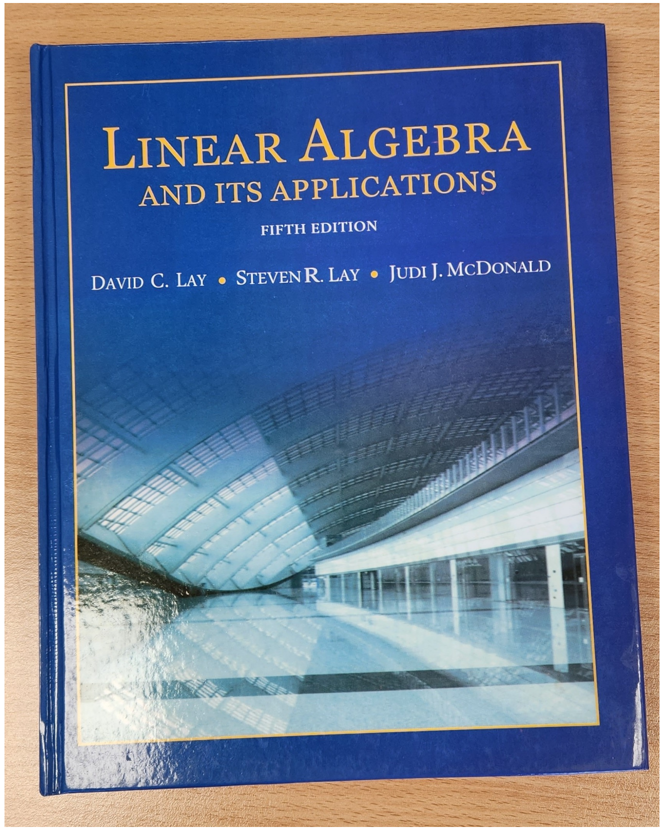
Week	Part	Topic	Notes
8	6.1.1-6.2.3	Orthogonality, Normalization, Dot-Product, Inequalities,	Lecture 1: **6.1.1 - 6.1.3** Lecture 2: **6.1.4 - 6.2.3**
9	6.2.4-6.3.2	Orthogonal/Orthonormal Sets, Basis, Gram Schmidt and QR Decomposition	Lecture 3: **6.2.4** Lecture 4: **6.2.5 – 6.3.2**
10	7.1.1-7.2.1	Least Squares and Normal Eqn, Projection Matrix, Applications	Lecture 5: **7.1.1 – 7.1.3** Lecture 6: **7.1.4 – 7.2.1**
11	8.1.1-8.1.2	Eigenvectors, Eigen-values, Characteristics Eqn	Lecture 7: **8.1.1** Lecture 8: **8.1.2**
12	8.1.3-8.1.5	Diagonalisation, Power of A, Change of basis	Lecture 9: **8.1.3** Lecture 10: **8.1.4 – 8.1.5**
13	9.1.1-9.2	Introduction to SVD and PCA (Not examined in quiz/exam)	Lecture 11: **9.1.1 – 9.2** Lecture 12: **Quiz 2**

How will we conduct the course?

1) Before the lectures, watch the videos according to the schedule in Table 1
   - You can watch past years zoom video recordings at
   https://www.youtube.com/@linearalgebra1884/playlists?view=50&sort=dd&shelf_id=2

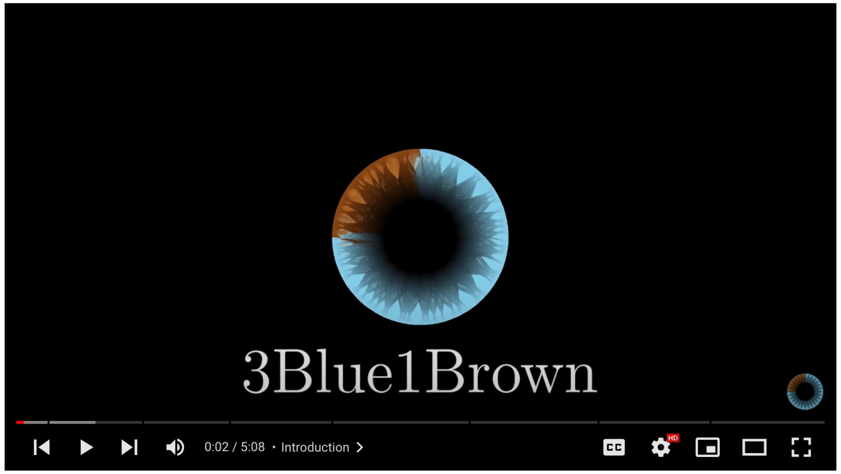
2) During lecture hours –
   - We will summarize the lectures and highlight the key points
   - Q&A.

References



Linear Algebra and Its Applications
by David Lay, Steven Lay, Judi McDonald

3Blue1Brown on YouTube



Essence of linear algebra preview

https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab

Lecture (Week 13)

(Chapter 9 & Revision)

9. Introduction to SVD & PCA

- Singular Value Decomposition (SVD)
- Principal Component Analysis (PCA)

9.1.1 Singular Value Decomposition (SVD)

$A_{n\times n} = PDP^{-1}$

- Definition
  - For any matrix $A_{m\times n}$ with rank r
    - There exists a matrix $\Sigma_{m\times n}$ and $D_{r\times r}$, for which the diagonal entries in $D_{r\times r}$ are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ (2)
    - There exist an orthogonal matrix $U_{m\times m}$ and an orthogonal matrix $V_{n\times n}$ such that:

$$A = U\Sigma V^T$$

where $\Sigma = \begin{bmatrix} D & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_r \end{bmatrix}$

$m \times n$

    - V contains the eigenvectors $\boldsymbol{v}_i$ of A^TA, U contains $\boldsymbol{u}_i = \frac{1}{\sigma_i} A\boldsymbol{v}_i$

Singular values: $\sigma_i = \sqrt{\lambda_i}$ are the square roots of the eigenvalues of the matrix A^TA. σ_i are sorted in descending order

- Compare with Eigenvalue Decomposition:
  - Eigenvalue decomposition is for $n\times n$ invertible matrix.
  - Singular values decomposition is for any $m\times n$ matrix.

9.1.2 Find Singular Value Decomposition

- Example: $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$ 2x3
  - Step 1: Find an orthogonal diagonalization of A^TA
    - $A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$

      3x2x2x3 → 3x3
    - Find the eigenvalues of A^TA: $\lambda_1 = 360,\ \lambda_2 = 90,\ \lambda_3 = 0$
    - Find the eigenvectors of A^TA: $\boldsymbol{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$
  - Step 2: Form V and Σ
    - $V = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3] = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$
    - $D = \begin{bmatrix} \sqrt{360} & 0 \\ 0 & \sqrt{90} \end{bmatrix}$ (only non−zero singular values),
    - $\Sigma = \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix}$
  - Step 3 Construct U
    - $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{360}} \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$
    - $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{\sqrt{90}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$
  - Finally:

$$A = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

$$A = U\Sigma V^T$$

9.2 Principal Component Analysis (PCA)

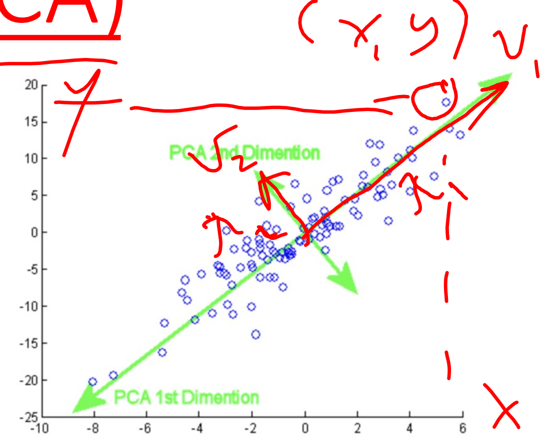
- Definition
  - Principal Component Analysis (PCA): to identify patterns and reduce the dimensionality of a large dataset by transforming the variables into a new set of uncorrelated variables called principal components.

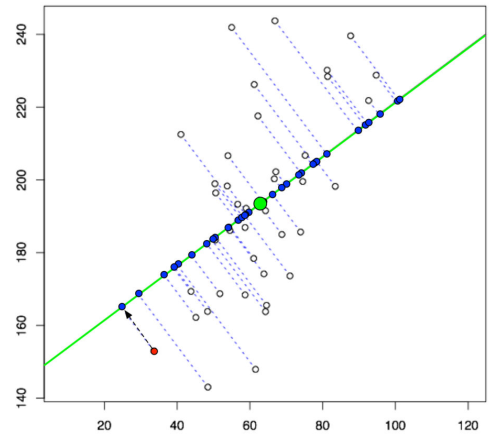
$Ax = \lambda x$

  - Process
    - Calculate the Covariance Matrix S (square matrix)
    - Calculate the Eigenvectors and Eigenvalues of the Covariance Matrix S

      $V = [\boldsymbol{v}_1 \boldsymbol{v}_2 \ldots \boldsymbol{v}_n]$ and $\lambda_1, \lambda_2 \ldots \lambda_n$
    - Identify the Principal Components
      - Ranking the eigenvectors in order of their eigenvalues λ_i , highest to lowest
      - Principal Components are the eigenvectors of the covariance pointing to the directions of the axes where there is the most variance (most information)
      - Eigenvalues give the amount of variance carried in each Principal Component
    - Select the first r eigenvectors (corresponding to r largest eigenvalues)

      $\tilde{V} = [\boldsymbol{v}_1 \boldsymbol{v}_2 \ldots \boldsymbol{v}_p]$, which takes the first p eigenvectors
    - Recast the data along the principal component axes
      - $\boldsymbol{y} = \tilde{V}^T \boldsymbol{x}$
      - Dimension: $\boldsymbol{x} \in R^n$, $\mathbf{y} \in R^p$, reduced from n-dimension to p-dimension $(p \leq n)$

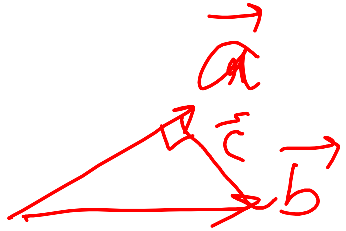




Revision

- Apr/May 2021

(b) Given two vectors: $a = (1,4,1,1)^T$ and $b = (4,5,-2,-3)^T$. Find vector $c \in \mathbb{R}^4$ that is orthogonal to vector a, so that $Span\{a, b\} = Span\{a, c\}$.



$$c = \operatorname{proj}_a b = \frac{b \cdot a}{a \cdot a}\, a$$

Revision

- Apr/May 2021

(b) Given vectors $a = \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$ and $b = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$. Verify whether they are orthogonal. Find vector c that is orthogonal to both vector a and vector b.

$$a \cdot b = a^T b = 0 \qquad a \cdot c = 0 \;\&\; b \cdot c = 0$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \begin{cases} -4c_1 + 5c_2 - c_3 = 0 \\ -2c_1 - c_2 + 3c_3 = 0 \end{cases} \qquad \left[\begin{array}{cccc} -4 & 5 & -1 & 0 \\ -2 & -1 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc} -4 & 5 & -1 & 0 \\ \hline 0 & -1 & 1 & 0 \end{array}\right] \qquad c_2 = c_3 \qquad \rightarrow c_1 = c_3$$

$$-4c_1 + 5c_2 - c_3 = 0 \qquad c_1 = c_3$$

$$c = c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Revision

- Apr/May 2021

(d) Verify whether $\lambda = -2$ is an eigenvalue of the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 9 & -6 & 4 \\ 9 & -5 & 3 \end{bmatrix}$

If true, find an eigenvector associated with the eigenvalue $\lambda = -2$.

$(A - \lambda I)X = 0$

$\boxed{A - \lambda I} = \begin{bmatrix} 1+2 & -2 & 2 \\ 9 & -6+2 & 4 \\ 9 & -5 & 3+2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 2 \\ 9 & -4 & 4 \\ 9 & -5 & 5 \end{bmatrix}$

$X = X_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 & -2 & 2 & 0 \\ 9 & -4 & 4 & 0 \\ 9 & -5 & 5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \Rightarrow 3X_1 = 0$

$-X_2 + X_3 = 0$

Revision

$A_{n \times n}$: n

- Nov/Dec 2021

(d) Answer True/False for the followings with respect to a square matrix A.

(i) If A is invertible, it will always be diagonalizable. F

(ii) All diagonalizable matrices are invertible. F

(iii) If A has unique eigenvalues, it is always diagonalizable. T

(iv) The null space of $(A - \lambda I)$ is spanned by A's eigenvectors. T

(v) If λ is the eigen value of A, then λ^{-1} is an eigen value of A^{-1}. T

2×2

$\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ inv ✓, × diag

$\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$ × invt. diag

$$Ax = \lambda x$$
$$(A - \lambda I)x = 0$$

$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$(A - \lambda I) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

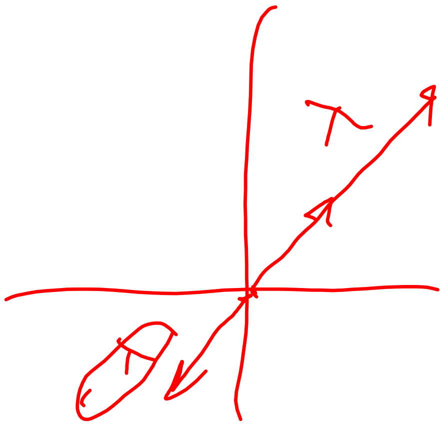
$\begin{bmatrix} 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda = a$: $x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$b x_2 = 0$

Revision

- Nov/Dec 2022



(b) Answer True/False or fill in the blank, for the followings with respect to a square matrix A.

(i) The eigenvalues of an n x n dimension ____triangular____ (what type of matrix) are its diagonal elements.

(ii) The number of eigen values including multiplicity is always equal to the number of independent eigen vectors. F

(iii) An invertible matrix is always diagonalizable. F

(iv) If the eigen values are distinct, then the eigen vectors are independent. T

(v) The eigen value of a matrix cannot be the scalar 0. F

(vi) If λ is an eigenvalue of invertible matrix A, then ______ is an eigen value of A^{-1}.

(vii) The normalized eigenvector associated with an eigen value is unique.

Revision

Nov/Dec 2021

3. Given the following matrix,

$$A = \begin{bmatrix} -2/\sqrt{2} & -3/\sqrt{2} & 0 \\ -2/\sqrt{2} & +3/\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) (i) Show that the columns of A form an orthogonal set and then normalize these columns to form an orthogonal matrix Q.

(6 marks)

(ii) Let $y = [1\ \ 2\ \ 3]^T$. Two columns of Q (from Q3a(i)) are used to form U to approximate y by $Ux = \hat{y}$, where $\hat{y}$ is the least squares approximation to y. Determine which 2 columns of Q should be selected. Provide your reasons and/or workings.

(5 marks)

(iii) Using your chosen Q, calculate the least squares solution x, $\hat{y}$, and the norm of the residual error.

(6 marks)

(b) Given a 3×2 matrix U with orthonormal columns spanning subspace W, comment on UU^T matrix's properties in terms of rank, orthogonality, dimension, type of matrix, and space that it spans.

(8 marks)

<u>(a-i)</u>. $\boldsymbol{u}_1 = \begin{bmatrix} -2/\sqrt{2} \\ -2/\sqrt{2} \\ 0 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} -3/\sqrt{2} \\ 3/\sqrt{2} \\ 0 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 3 - 3 = 0,\ \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = 0,\ \boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = 0,$

To normalize: $\hat{\boldsymbol{u}}_1 = \frac{1}{\|\boldsymbol{u}_1\|}\begin{bmatrix} -2/\sqrt{2} \\ -2/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix},\quad \hat{\boldsymbol{u}}_2 = \frac{1}{\|\boldsymbol{u}_2\|}\begin{bmatrix} -3/\sqrt{2} \\ 3/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix},\ \hat{\boldsymbol{u}}_3 = \frac{1}{\|\boldsymbol{u}_3\|}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$

<u>(a-ii)</u> in order to chose 2 columns from Q to approximate $\boldsymbol{y}$, to find the weight $\boldsymbol{x}$ in $Q\boldsymbol{x} = \boldsymbol{y}$,

Left-multiply $Q^{-1}Q\boldsymbol{x} = Q^{-1}\boldsymbol{y} \Rightarrow \boldsymbol{x} = Q^{-1}\boldsymbol{y} = Q^T\boldsymbol{y} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3/\sqrt{2} \\ 1/\sqrt{2} \\ 3 \end{bmatrix}$, we choose the components corresponding to larger $|x_i|$-- the weights of the components.

$|x_1| = 3/\sqrt{2};\ |x_2| = 1/\sqrt{2};\ |x_1| = 3;$ ➔ $|x_2|$ is the smallest, so we choose $\boldsymbol{u}_1$and $\boldsymbol{u}_3$

$U = \begin{bmatrix} -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$

<u>(a-iii)</u>. $\boldsymbol{x} = U^T\boldsymbol{y} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3/\sqrt{2} \\ 3 \end{bmatrix},\ \ \hat{\boldsymbol{y}} = U\boldsymbol{x} = \begin{bmatrix} -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3/\sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Residual:$\boldsymbol{e} = \boldsymbol{y} - \hat{\boldsymbol{y}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},\ \|\boldsymbol{e}\| = 1$

End