Name: ______________________________ Tutorial group: ________________

Matriculation number: | | | | | | | | | |

31 October 2020 **MH1812 Continuous Assessment 2** 60 minutes

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QUESTION 1. **(30 marks)**

Define the symmetric difference of sets A and B as $A \triangle B = (A - B) \cup (B - A)$.

(a) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6, 7, 8\}$,

(i) (5 marks) find the set $A \triangle B$,

(ii) (5 marks) find the set $A \triangle B \triangle C$.

[**Solution**] $A \triangle B = \{1, 2, 5, 6\}$, $A \triangle B \triangle C = \{1, 2, 7, 8\}$.

[For grading: no partial marks.]

(b) (20 marks) Prove, for any sets A, B, and C, if $A \triangle C = B \triangle C$ then $A = B$. (Any proof technique[1] is allowed)

[**Solution**] It is easy to use a membership table, see below. It follows immediately from the critical rows that $A = B$.

A	B	C	$A \triangle C$	$B \triangle C$	
0	0	0	0	0	critical row
0	0	1	1	1	critical row
0	1	0	0	1	
0	1	1	1	0	
1	0	0	1	0	
1	0	1	0	1	
1	1	0	1	1	critical row
1	1	1	0	0	critical row

[For grading: For proofs based on membership table: a correct membership table without indication of critical rows is worth 15 points; a membership containing at most 3 errors is worth 5 points. The possible scores based on membership table are 0, 5, 15, 20.

For other methods, if the proof contains a fatal mistake (e.g., suggesting that $A \cup C = B \cup C$ or $A \setminus C = B \setminus C$ would imply $A = B$) or is insensible to the grader, no marks will be given.]

[1] Set identity laws to use include: identity, domination, idempotent, double complement, commutative, associative, distributive, De Morgan's.

For graders only	Question	1(a)	1(b)	2	3	4(a)	4(b)	Bonus	Total
	Marks								

QUESTION 2. **(20 marks)**

Solve the recurrence relation

$$a_n = a_{n-1} + 4a_{n-2} - 4a_{n-3}$$

with $a_0 = 3, a_1 = 0, a_2 = 6$.

[**Solution**] The characteristic equation is

$$x^3 = x^2 + 4x - 4$$

which factors to $(x-2)(x+2)(x-1) = 0$ with three distinct roots $2, -2, 1$. Then $a_n = u \cdot 2^n + v \cdot (-2)^n + w \cdot 1^n$, and

$$\begin{aligned} a_0 &= 3 = u + v + w \\ a_1 &= 0 = 2u - 2v + w \\ a_2 &= 6 = 4u + 4v + w. \end{aligned}$$

Solving the linear system leads to $u = 0, v = 1, w = 2$, hence $a_n = (-2)^n + 2$.

[For grading: the characteristic equation is worth 4 marks, the roots 4 marks, the general form of a_n 4 marks, the system of linear equations 4 marks, the solution to the linear system and the final answer 4 marks. No partial marks are given for the characteristic equation or the general form (except for typo-level mistakes such as writing -2^n for $(-2)^n$).]

QUESTION 3. **(15 marks)**

How many numbers in the range $[1812, 2020]$ (inclusive of both 1812 and 2020) are integer multiples of **exactly** one of the two factors 4 and 5? Justify your answer.

[**Solution**] Let $S_4(n)$, $S_5(n)$, and $S_{20}(n)$ denote the set of integers in the range $[1, n]$ that are multiple of 4, 5, and 20, respectively. Then being multiple of exactly one factor $T(n) = |S_4(n)| + |S_5(n)| - 2 \cdot |S_{20}(n)| = \lfloor n/4 \rfloor + \lfloor n/5 \rfloor - 2 \cdot \lfloor n/20 \rfloor$. Our question corresponds to

$$\begin{aligned} T(2020) - T(1811) &= (\lfloor 2020/4 \rfloor + \lfloor 2020/5 \rfloor - 2 \cdot \lfloor 2020/20 \rfloor) - (\lfloor 1811/4 \rfloor + \lfloor 1811/5 \rfloor - 2 \cdot \lfloor 1811/20 \rfloor) \\ &= (505 + 404 - 2 \cdot 101) - (452 + 362 - 2 \cdot 90) \\ &= 73 \end{aligned}$$

[For grading: each correct number for the set is worth 2 marks, other combination might be possible. Or, let $S_k([a, b])$ denote the set of integers in the range $[a, b]$ that are integer multiple of k. The cardinalitiy of $S_k([a, b])$ equals $\lfloor \frac{b}{k} \rfloor - \lfloor \frac{a-1}{k} \rfloor$ (also equals $\lfloor \frac{b}{k} \rfloor - \lceil \frac{a}{k} \rceil + 1$, or $\lfloor \frac{b-a'}{k} \rfloor + 1$ where $a' = \min\{m \in [a, b] \mid k|m\}$). Each correct number for the cardinalities of $S_4([1812, 2020])$, $S_5([1812, 2020])$, $S_{20}([1812, 2020])$, *i.e.*, 53, 42, 11, is worth 4 marks. Subtracting two times of the cardinality of intersection (*i.e.*, $2 \cdot |S_{20}([1812, 2020])|$) from the sum (*i.e.*, $|S_4([1812, 2020])| + S_5([1812, 2020])|$) is worth 3 marks.]

QUESTION 4. (35 marks)

Let $A = \{1, 2, 3, \ldots, n\}$ for some positive integer n.

(a) (20 marks) How many relations R are defined on A such that R is simultaneously an equivalence relation and a partial order? Justify your answer.

[**Solution**] There is **1** such relation, that is $R = \{(x, x) | x \in A\} = \{(1, 1), (2, 2), \ldots, (n, n)\}$.

(i) Equivalence relation requires R to be symmetric, i.e., $(x, y), (y, x)$ shall belong to R simultaneously if one of them does. However, a partial order requires that at most one out of the pair $(x, y), (y, x)$ belongs to R, which leaves the only choice that $(x, y) \notin R$, and $(y, x) \notin R$ for all $x \neq y$.

(ii) Both equivalence relation and partial order requires R to be reflexive, i.e., $(x, x) \in R$ for all $x \in A$, and transitive.

(iii) Further check shows this only option is indeed transitive.

[for grading: a correct answer is worth 5 marks; Step (i) is worth 10 marks and Step (ii) 5 marks; Step (iii) is a giveaway for the current grading. No partial marks are given for each grading component.]

(b) (15 marks) Let R, S be partial orders defined on A. Is $R \cap S$ also a partial order? Justify your answer.

[**Solution**] The answer is YES.

(i) Reflexive: since both R and S are partial orders, hence they both are reflexive, i.e., for all $x \in A$ $(x, x) \in R$ and $(x, x) \in S$, hence $(x, x) \in R \cap S$.

(ii) Transitive: for all $x, y, z \in A$, if $(x, y), (y, z) \in R \cap S$, then $(x, y), (y, z) \in R$. Since R is a partial order, it is also transitive, hence $(x, z) \in R$. Similarly, $(x, z) \in S$, hence $(x, z) \in R \cap S$.

(iii) Anti-Symmetric: for all $x, y \in A$, if $(x, y), (y, x) \in R \cap S$, then $(x, y), (y, x) \in R$. Since R is a partial order, hence anti-symmetric, hence $x = y$.

[for grading: correct answer worths 6 marks, justification for each property worth 3 marks. While counter-example can be used to disprove something, examples can NOT be used as proof. To check the individual property, the assumption should be consistent with that from the definition, e.g., for transitivity one assumes (x,y) and (y,z) in that relation ($R \cap S$) and see if (x,z) is also in that relation; similarly, for anti-symmetry one assumes (x,y) and (y,x) in that relation and see if this assumption leads to x=y, discussions by cases are allowed but should be exhaustive and cover all the cases including when both (x,y) and (y,x) are not in the relation.]

BONUS QUESTION. (10 marks)

[Points will be given to *fully* correct solutions only. The total mark of this test is capped at 100 marks.]

Suppose that n is a positive integer. Find an explicit expression[2] for

$$\sum_{k=0}^{n} \binom{n}{k} 3^k (n-k).$$

Show your working.

The answer is: $n \cdot 4^{n-1}$.

[**Solution**] We can just manipulate the sum.

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} 3^k (n-k) &= \sum_{k=0}^{n-1} \binom{n}{n-k} 3^k (n-k) \\
&= \sum_{k=0}^{n-1} \frac{n}{n-k} \binom{n-1}{n-k-1} 3^k (n-k) \\
&= n \sum_{k=0}^{n-1} \binom{n-1}{k} 3^k \\
&= n \cdot 4^{n-1}.
\end{aligned}$$

[**Solution 2**] We can also use double counting method with the following scenario. There are n people who go on a wine tour. Since they are responsible adults, one person is selected to be the designated driver and he must stay sober. Each of the remaining $n-1$ adults can pick independently one out of 4 options from a menu, where the last menu item is the non-alcoholic option.

For the solution $n \cdot 4^{n-1}$, we first choose a k-subset of people who prefers one of the first three options. They then pick which of these they like. The designated driver cannot, of course, be among these k people, and is selected among the remaining $(n-k)$ people — all of who also pick the non-alcoholic option. By summing over all possible values of k, we see that the situation agrees with the solution.

[2] which means that one can calculate the expression using a constant number of "well-known" standard operations, independent of n. The "well-known" standard operations include addition, subtraction, multiplication, division, exponentiation, logarithm, trignometric functions, inverse trignometric functions and integer factorials.