NANYANG TECHNOLOGICAL UNIVERSITY

MIDTERM II (CA2)

MH1812 – Discrete Mathematics

April 2018 TIME ALLOWED: 40 minutes

Name:			
Matric. no.:		Tutor group:	

INSTRUCTIONS TO CANDIDATES

1. **DO NOT TURN OVER PAPER UNTIL INSTRUCTED.**
2. This midterm paper contains **THREE (3)** questions.
3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. This **IS NOT** an **OPEN BOOK** exam.
6. Candidates should clearly explain their reasoning when answering each question.

QUESTION 1. **(30 marks)**

Solve the following linear recurrences, that is, write a_n and b_n in terms of n:

(a) $a_n = 10a_{n-1} - 21a_{n-2}$ for $n \geqslant 2$, with initial conditions $a_0 = 3$, $a_1 = 5$;

(b) $b_n = b_{n-1} + 2$ for $n \geqslant 1$, with initial condition $b_0 = 2$.

Justify your answers.

Solution:

(a) The recurrence is homogeneous so we can use the characteristic equation method. The characteristic equation is $x^2 - 10x + 21 = 0$, which has roots $\alpha_1 = 3$ and $\alpha_2 = 7$. Therefore $a_n = u\alpha_1^n + v\alpha_2^n = 3^n u + 7^n v$. Using the initial conditions, we see that $u + v = 3$ and $3u + 7v = 5$. Hence $u = 4$ and $v = -1$, and so $a_n = 4 \cdot 3^n - 7^n$.

(b) The recurrence is not homogeneous. Using backtracking, we see that

$$b_n = b_{n-1} + 2 = b_{n-2} + 4 = \dots .$$

We guess that $b_n = b_{n-i} + 2i$ for all $i \in \{1, \dots, n\}$. Then for $i = n$, we have $b_n = b_0 + 2n = 2(n + 1)$.

Now we prove that $b_n = 2(n+1)$, by induction. Let $P(k)$ be the predicate $b_k = 2(k + 1)$. The basis case $P(0)$ is true, since $b_0 = 2$.

Suppose that $P(k)$ is true (the induction hypothesis). We want to prove that $P(k + 1)$ is true. The LHS of $P(k + 1)$ is $b_{k+1} = b_k + 2$. By the induction hypothesis we have $b_k = 2(k + 1)$. Hence $b_k = 2(k + 1) + 2 = 2(k + 2)$. Therefore $P(k + 1)$ is true.

QUESTION 2. **(30 marks)**

(a) Prove that

$$\sum_{j=1}^{n} j(3j-1) = n^2(n+1), \qquad \forall n \in \mathbb{N}.$$

(b) Let $A = \{0, 1\}$ and $B = \{4, 5\}$.

(i) Write out all elements of the set $A \times B$.

(ii) What is the cardinality of the power set of $A \times B$?

Solution:

(a) We prove by induction. Let $P(k)$ be the predicate $\sum_{j=1}^{k} j(3j-1) = k^2(k+1)$. Basis case: $P(1)$ has LHS $3 - 1 = 2$ equal to the RHS $1 + 1 = 2$. Assume the induction hypothesis that $P(k)$ is true for $k \geqslant 1$. We want to prove that $P(k+1)$ is true.

The LHS of $P(k+1)$ is

$$\begin{aligned}
\sum_{j=1}^{k+1} j(3j-1) &= \sum_{j=1}^{k} j(3j-1) + (k+1)(3(k+1)-1) \\
&= k^2(k+1) + (k+1)(3(k+1)-1) \\
&= (k+1)(k^2 + 3(k+1) - 1) \\
&= (k+1)(k^2 + 3k + 2) \\
&= (k+1)^2(k+2),
\end{aligned}$$

which is equal to the RHS of $P(k+1)$.

(b) Let $A = \{0, 1\}$ and $B = \{4, 5\}$.

(i) $(0, 4)$, $(0, 5)$, $(1, 4)$, $(1, 5)$.

(ii) 16.

QUESTION 3. **(40 marks)**

(a) Let A, B, and C be sets.

(i) Prove that $(\overline{A \cap B}) \cap C = (C - A) \cup (C - B)$;

(ii) Is $(C - A) \cup (C - B) = C$? If yes, prove it, if no, give a counterexample.

(b) Let $S = \{3a + 6b \,|\, a, b \in \mathbb{Z}\}$.

(i) Show that $S \subseteq \mathbb{Z}$;

(ii) Is $S = \mathbb{Z}$? If yes, prove it, if no, give a counterexample.

Solution:

(a) (i) We prove by using set identities.

$$\begin{aligned}
(\overline{A \cap B}) \cap C &= (\overline{A} \cup \overline{B}) \cap C && \text{De Morgan} \\
&= (\overline{A} \cap C) \cup (\overline{B} \cap C) && \text{Distributivity} \\
&= (C \cap \overline{A}) \cup (C \cap \overline{B}) && \text{Commutativity} \\
&= (C - A) \cup (C - B)
\end{aligned}$$

(ii) No. Counterexample: $A = B = C = \{1\}$.

(b) (i) Take $x \in S$. We want to show that $x \in \mathbb{Z}$. Since $x \in S$, we must have that $x = 3a + 6b$ for some integers a and b. Since $\mathbb{Z}$ is closed under multiplication and addition we have that $x \in \mathbb{Z}$.

(ii) No. $1 \in \mathbb{Z}$ and $1 \notin S$. Indeed, each element of S is an integer that is divisible by 3.