MH1820 Introduction to Probability and Statistical Methods

Tutorial 3 (Week 4) Solution

Problem 1 (discrete random variables, PMF, CDF)
For the following random variables X and Y, compute their PMF, CDF, expected value, and variance. Draw graphs of the CDFs.

(a) A fair 4-sided dice (with faces numbered 1 through 4) is rolled twice independently. Let X be the sum of the two numbers obtained.

(b) Let a chip be taken at random from a bowl that contains six white chips, three red chips, and one blue chip. Let the random variable $X = 1$ if the outcome is a white chip, let $X = 5$ if the outcome is a red chip, and let $X = 10$ if the outcome is a blue chip.

Solution

(a) Notice that $|\Omega| = 4^2 = 16$. The PMF is given by

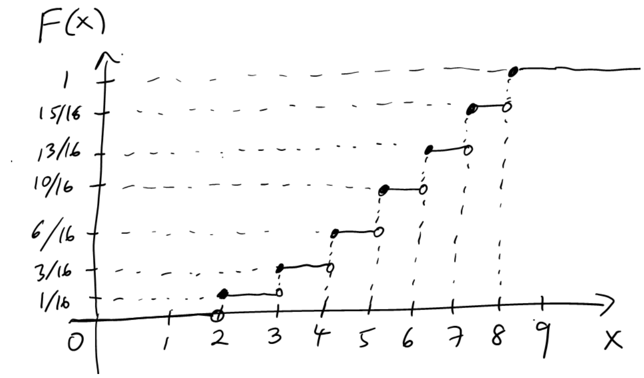
x	2	3	4	5	6	7	8
$p(x)$	1/16	2/16	3/16	4/16	3/16	2/16	1/16

For example, if $X = 5$, there are 4 possible outcomes: $(1,4)$, $(4,1)$, $(2,3)$, $(3,2)$. So $p(5) = \frac{4}{16}$,

For the CDF of X, we have

$$F(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{16}, & 2 \le x < 3 \\ \frac{3}{16}, & 3 \le x < 4 \\ \frac{6}{16}, & 4 \le x < 5 \\ \frac{10}{16}, & 5 \le x < 6 \\ \frac{13}{16}, & 6 \le x < 7 \\ \frac{15}{16}, & 7 \le x < 8 \\ 1, & x \ge 8 \end{cases}$$

Plot of CDF:



– Expected value:

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_x xp(x) \\
&= 2(1/16) + 3(2/16) + 4(3/16) + 5(4/16) + 6(3/16) + 7(2/16) + 8(1/16) \\
&= 5
\end{aligned}
$$

– Variance:

$$
\begin{aligned}
\mathbb{E}[X^2] &= \sum_x x^2 p(x) \\
&= 2^2(1/16) + 3^2(2/16) + 4^2(3/16) + 5^2(4/16) + 6^2(3/16) + 7^2(2/16) + 8^2(1/16) \\
&= 27.5
\end{aligned}
$$

Hence,

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 27.5 - (5)^2 = 2.5.$$

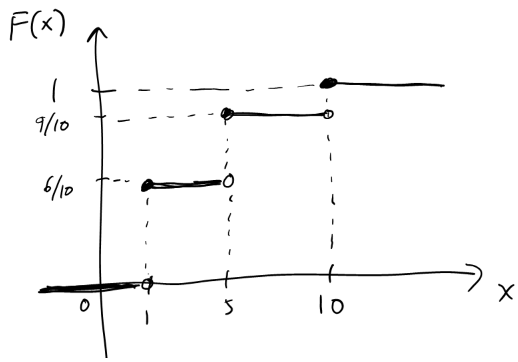
(b) Notice that $|\Omega| = 10$. The PMF is given by

x	1	5	10
$p(x)$	6/10	3/10	1/10

For the CDF of X, we have

$$
F(x) = \begin{cases} 0, & x < 1 \\ \frac{6}{10}, & 1 \le x < 5 \\ \frac{9}{10}, & 5 \le x < 10 \\ 1, & x \ge 10 \end{cases}
$$

Plot of CDF:



- Expected value:

$$\begin{aligned}\mathbb{E}[X] &= \sum_x xp(x) \\ &= 1(6/10) + 5(3/10) + 10(1/10) \\ &= 3.1\end{aligned}$$

- Variance:

$$\begin{aligned}\mathbb{E}[X^2] &= \sum_x x^2p(x) \\ &= 1^2(6/10) + 5^2(3/10) + 10^2(1/10) \\ &= 18.1\end{aligned}$$

Hence,

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 18.1 - (3.1)^2 = 8.49.$$

□

Problem 2 (linearity of expected values)
Find the expectated value of the sum obtained when

(a) 10 fair dice are rolled.

(b) n fair dice are rolled.

Solution Let X_i be the value on the die i. Let X be the sum obtained when n fair dice are rolled. Then

$$X = X_1 + \cdots + X_n.$$

So by linearity of expected values,

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Since $\mathbb{E}[X_i] = 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6) = 7/2$, we deduce that

(a) The expected value when 10 dice are rolled is

$$\sum_{i=1}^{10} \mathbb{E}[X_i] = 10(7/2) = 35.$$

(b) The expected value when n dice are rolled is

$$\sum_{i=1}^{n} \mathbb{E}[X_i] = n(7/2) = 3.5n.$$

Problem 3 (linearity of expected values)
A fair four-sided die has two faces numbered 0 and two faces numbered 2. Another fair four-sided die has its faces numbered 0, 1, 4, and 5. The two dice are rolled. Let X and Y be the respective outcomes of the roll. Find the expected value of $X + Y$.

Solution
The PMF of X and Y are given as follows:

x	0	2
$f_X(x)$	$\frac{1}{2}$	$\frac{1}{2}$

y	0	1	4	5
$f_Y(y)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Hence,

$$\begin{aligned} \mathbb{E}[X+Y] &= \mathbb{E}[X] + \mathbb{E}[Y] \\ &= 0 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} + 5 \cdot \frac{1}{4} \\ &= 3.5 \end{aligned}$$

□

Problem 4 (expected values, variance)
Let X equal the larger outcome when a pair of four-sided dice (with faces numbered from 1 through 4) is rolled. If both dice give the same number, then X is equal to that number. Find the expected value, variance and standard deviation of X.

Solution The PMF of X is given as follows:

x	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

For example, there are 7 outcomes whose larger number is 4: $(4,4)$, $(4,3)$, $(4,2)$, $(4,1)$, $(1,4)$, $(2,4)$, $(3,4)$. Since $|\Omega| = 4 \times 4 = 16$, we have $f(4) = \frac{7}{16}$.

$$\text{expected value} = \mathbb{E}[X] = 1 \cdot \frac{1}{16} + 2 \cdot \frac{3}{16} + 3 \cdot \frac{5}{16} + 4 \cdot \frac{7}{16} = 3.125.$$

$$\mathbb{E}[X^2] = 1^2 \cdot \frac{1}{16} + 2^2 \cdot \frac{3}{16} + 3^2 \cdot \frac{5}{16} + 4^2 \cdot \frac{7}{16} = 10.625.$$

$$\text{variance} = \mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 10.625 - 3.125^2 = 0.8594.$$

$$\text{standard deviatrion} = \sqrt{\mathrm{Var}[X]} = \sqrt{0.8594} = 0.9270.$$

□

Problem 5 (Discrete random variables: Bernoulli, Binomial and Geometric)
Apples are packaged automatically in 3-pound bags. Suppose that 4% of the time the bag of apples weighs less than 3 pounds. If you select bags randomly and weigh them in order to discover one underweight bag of apples, find the probability that the number of bags that must be selected is

(a) exactly 20

(b) at least 20

You may assume the following formula for a geometric sum with ratio r: $\sum_{i=1}^{n} r^{i-1} = \frac{1-r^n}{1-r}$.

Solution Let X be the number of bags selected until the first one which is underweight (with weight less than 3 pounds). Then $X \sim \mathit{Geom}(p)$, with $p = 0.04$.

(a)
$$\mathbb{P}(X = 20) = (1-p)^{19}p = 0.96^{19} \times 0.04 = 0.01842.$$

(b)

$$\begin{aligned}
\mathbb{P}(X \geq 20) = 1 - \mathbb{P}(X < 20) & \\
&= 1 - (\mathbb{P}(X=1) + \mathbb{P}(X=2) + \cdots + \mathbb{P}(X=19)) \\
&= 1 - \sum_{i=1}^{19} 0.96^{i-1}(0.04) \\
&= 1 - 0.04 \sum_{i=1}^{19} 0.96^{i-1} \\
&= 1 - 0.04\left(\frac{1 - 0.96^{19}}{1 - 0.96}\right) \\
&= 0.4604.
\end{aligned}$$

$\square$

Problem 6 (Discrete random variables: Bernoulli, Binomial and Geometric)
It is believed that approximately 75% of American youth now have insurance due to the health care law. Suppose this is true, and let X equal the number of American youth in a random sample of $n = 15$ with health insurance.

(a) How is X distributed?

(b) Find the probability that X is at most 13.

(c) Give the mean, variance, and standard deviation of X.

Solution

(a) $X \sim Binomial(15, 0.75)$.

(b)

$$\begin{aligned}
\mathbb{P}(X \le 13) &= 1 - \mathbb{P}(X > 13) \\
&= 1 - (\mathbb{P}(X = 14) + \mathbb{P}(X = 15)) \\
&= 1 - \binom{15}{14} 0.75^{14} 0.25^{1} - \binom{15}{15} 0.75^{15} 0.25^{0} \\
&= 1 - 0.0668 - 0.0134 = 0.9198.
\end{aligned}$$

(c)

$$\text{mean} = \mathbb{E}[X] = np = 15 \times 0.75 = 11.25.$$

$$\text{variance} = \text{Var}[X] = np(1-p) = 15 \times 0.75 \times 0.25 = 2.8125.$$

$$\text{standard deviation} = \sqrt{\text{Var}[X]} = \sqrt{2.8125} = 1.6771.$$

$\square$

Problem 7 (Discrete random variables: Bernoulli, Binomial and Geometric)
Your stockbroker is free to take your calls about 60% of the time; otherwise, he is talking to another client or is out of the office. You call him at five random times during a given month. (Assume independence.)

(a) What is the probability that he will accept exactly three of your five calls?

(b) What is the probability that he will accept at least one of the calls?

Solution
Let $p = 0.6$, and $n = 5$. Then $X \sim Binomial(5, 0.6)$.

(a) $\mathbb{P}(X = 3) = \binom{5}{3}(0.6)^3(0.4)^2 = 0.3456$.

(b)

$$\begin{aligned}\mathbb{P}(X \geq 1) &= 1 - \mathbb{P}(X = 0) \\ &= 1 - \binom{5}{0}(0.6)^0(0.4)^5 \\ &= 1 - 0.4^5 = 0.98976.\end{aligned}$$

□

Problem 8 (Discrete random variables: Bernoulli, Binomial and Geometric)
It is known that 2% of people whose luggage is screened at an airport have questionable objects in their luggage.

(a) What is the probability that a string of 15 people pass through screening successfully before an individual is caught with a questionable object?

(b) What is the expected number of people to pass through before an individual is caught with a questionable object?

Solution Let X be the number of people screened until the first person is caught with a questionable object. Then $Y = X - 1$ is the number of people to pass through before an inidiviudal is caught with a questionable object.

Note that $X \sim Geom(p)$ where $p = 0.02$.

(a) $\mathbb{P}(Y = 15) = \mathbb{P}(X = 16) = (0.98)^{15}(0.02) = 0.01477$.

(b) Since $\mathbb{E}[X] = \frac{1}{p} = \frac{1}{0.02} = 50$, we have

$$\mathbb{E}[Y] = \mathbb{E}[X - 1] = \mathbb{E}[X] - 1 = 50 - 1 = 49.$$

□

Answer Keys. 1(a). $\mathbb{E}[X] = 5$, $\text{Var}[X] = 2.5$ 1(b). $\mathbb{E}[X] = 3.1$, $\text{Var}[X] = 8.49$ 2(a). 35 2(b). $3.5n$ 3. 3.5 4. $\mathbb{E}[X] = 3.125$, $\text{Var}[X] = 0.8594$ 5(a). 0.01842 5(b). 0.4604 6(b). 0.9198 6(c). mean = 11.25, variance = 2.8125, standard deviation = 1.6771 7(a). 0.3456 7(b). 0.98976 8(a) 0.01477 8(b) 49.