QUESTION 1. **(15 marks)**

(a) Solve the following linear recurrence, that is, write a_n in terms of n: $a_n = 10a_{n-1} - 25a_{n-2}$ for $n \geqslant 2$, with initial conditions $a_0 = 7$, $a_1 = 10$.

(b) Using induction, prove that

$$\sum_{i=0}^{n-1}(2i+1) = n^2, \qquad \forall n \in \mathbb{N}.$$

Solution

(a) The characteristic equation is

$$\begin{aligned} x^2 - 10x + 25 &= 0 \\ (x-5)^2 &= 0. \end{aligned}$$

This equation has a repeated root $s = 5$. Hence $a_n = u5^n + nv5^n$ for some u and v. Using the initial conditions, we find that $u = 7$ and $v = -5$. Thus $a_n = 7 \cdot 5^n - n5^{n+1}$.

[Distribution: 4 marks for correct expression for a_n and 3 marks for the justification]

(b) Let $P(k)$ denote the predicate $\sum_{i=0}^{k-1}(2i+1) = k^2$. First we check the base case $P(1)$. Here the LHS, $(2 \cdot 0 + 1) = 1$ is equal to the RHS $1^2 = 1$.

Now we want to prove the proposition $\forall k \in \mathbb{N}, P(k) \to P(k+1)$. For our inductive hypothesis, assume $P(k)$ is true for some $k \in \mathbb{N}$. The LHS of $P(k+1)$, $\sum_{i=0}^{k}(2i+1)$, is equal to $2k+1$ plus by the LHS of $P(k)$. Hence, using the inductive hypothesis, we have

$$\begin{aligned} \sum_{i=0}^{k}(2i+1) &= \sum_{i=0}^{k-1}(2i+1) + (2k+1) \\ &= k^2 + (2k+1) \\ &= (k+1)^2. \end{aligned}$$

Thus, we have shown that $P(k+1)$ follows from $P(k)$, as required.

[Distribution: 2 marks for correct predicate, 2 marks for base case, 2 marks for inductive hypothesis, 2 marks for correctly using induction.]

QUESTION 2. **(15 marks)**

Let A, B, and C be sets.

(a) Prove that $A \cap \left(\overline{(C \cup B)} \cup (\overline{B} \cap C)\right) = A \cap \overline{B}$.

(b) Show that $(A - B) - C \subseteq A - (B - C)$.

(c) Is $(A - B) - C = A - (B - C)$? If yes, prove it, if no, give a counterexample.

Solution

(a) It suffices to show $\left(\overline{(C \cup B)} \cup (\overline{B} \cap C)\right) = \overline{B}$.

$$\begin{aligned}
\overline{(C \cup B)} \cup (\overline{B} \cap C) &= (\overline{C} \cap \overline{B}) \cup (\overline{B} \cap C) \\
&= \left(\overline{C} \cup (\overline{B} \cap C)\right) \cap \left(\overline{B} \cup (\overline{B} \cap C)\right) \\
&= \left((\overline{C} \cup \overline{B}) \cap (\overline{C} \cup C)\right) \cap \left(\overline{B} \cup (\overline{B} \cap C)\right) \\
&= (\overline{C} \cup \overline{B}) \cap \left(\overline{B} \cup (\overline{B} \cap C)\right) \\
&= (\overline{C} \cup \overline{B}) \cap \overline{B} \\
&= \overline{B}
\end{aligned}$$

[Distribution: 5 marks for justification, any set equality technique is OK.]

(b) Take $x \in (A - B) - C$. Then $x \in A - B$ and $x \notin C$. Hence $x \in A$ and $x \notin B$. Since $x \notin B$ we have $x \notin B - C$. Thus, $x \in A - (B - C)$.

[Distribution: 5 marks for justification, also OK to use other methods, e.g., membership table. Venn diagrams can be used as a guide but should not be the only justification.]

(c) Set $A = \{1, 2\}$, $B = C = \{1\}$. Then $(A - B) - C = \{1\}$ but $A - (B - C) = \{1, 2\}$.

[Distribution: 5 marks for counterexample. Any valid set assignment for a counterexample is OK.]

QUESTION 3. **(20 marks)**

(a) Consider the sets $A = \{0, 2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$.

(i) Write out each subset of A that has cardinality 2.

(ii) Write out the cardinalities of $A \cup B$, $A - B$, and $A \times B$.

(iii) Find the number of subsets of $A \times B$ that have at most 2 elements.

No justification is necessary for the answers of part (a).

(b) Consider the *distinguishable* permutations of the number 436314. (Note that each permutation has six digits.)

(i) How many are there in total?

(ii) How many are odd numbers?

Briefly justify your answers.

Solution

(a) Consider the sets $A = \{0, 2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$. Find

(i) $\{0, 2\}$, $\{0, 4\}$, $\{0, 6\}$, $\{2, 4\}$, $\{2, 6\}$, $\{4, 6\}$.
[Distribution: 4 marks total, at least 2 marks if half the sets are found or if the right sets are found but the notation is wrong.]

(ii) $|A \cup B| = 6$, $|A - B| = 2$, and $|A \times B| = 16$.
[Distribution: 1 mark for correct cardinality, 1 bonus mark if all are correct]

(iii) $1 + 16 + 120 = 137$.
[Distribution: 4 marks for 137, 3 marks for 136 (e.g., missed the empty set)]

(b) (i) $6!/(2!2!)$
[Distribution: 2 marks for correct number, 2 marks for justification]

(ii) The number of distinguishable permutations where the last digit is 1: $5!/(2!2!)$.

The number of distinguishable permutations where the last digit is 3: $5!/(2!)$.

Thus the total is $5!/(2!2!) + 5!/(2!)$.

[Distribution: 2 marks for correct number, 2 marks for justification]