

System of Linear Equations

Overview and Learning Outcomes

- Linear Equations
 - Determine whether a given equation is linear
 - Determine whether a linear system is consistent or inconsistent
- Homogeneous Linear Systems
 - Recognize homogeneous linear systems
 - Understand the types of solutions for homogeneous linear systems
- Linear Systems and Matrices
 - Represent linear systems in matrix form

Overview and Learning Outcomes

- Existence of a Solution of Linear Systems
 - Find the augmented matrix of a linear system
 - Perform elementary row operations on an augmented matrix
- Solving Linear Systems
 - Use Gaussian elimination and Gauss-Jordan elimination to find the general solution of a linear system
- Linear Combination of Vectors
 - Write a given vector as a linear combination of other vectors
 - Determine the span of a set of vectors

Overview and Learning Outcomes

- Solution of Linear Systems in the form $A\mathbf{x} = \mathbf{b}$
 - Find parametric vector equation of line and plane
 - Find solution as a sum of a particular solution and of a homogeneous system
- Linear independence
 - Determine if a set of vectors is linearly independent
- Linear Transformations
 - Find the image of a vector under a linear transformation
 - Determine the matrix corresponding to a linear transformation

1.1 Linear Equations

Definition. A **linear equation** in n variables x_1, x_2, \dots, x_n is one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where a_1, a_2, \dots, a_n and b are constants and the a 's, also called **coefficients**, are not all zero.

A general **linear system** of m equations in n unknowns (variables) x_1, x_2, \dots, x_n can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\quad \vdots && \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

In the special case where the $b_1 = b_2 = \dots = b_m = 0$, the linear system is called a **homogeneous** system; otherwise, it is called **non-homogeneous**.

Example:

Linear system of two equations:

$$\begin{array}{rcl} 5x + y & = & 3 \\ 2x - y & = & 4 \end{array} \quad \text{Solution: } x = 1, y = -2$$

Non-linear equations:

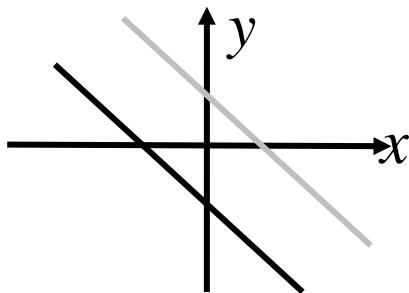
$$\begin{array}{rcl} x + 3y^2 & = & 4 \\ \sin x + y & = & 0 \end{array}$$

A **solution** of a linear system in n unknowns x_1, x_2, \dots, x_n is a sequence of n numbers s_1, s_2, \dots, s_n for which the substitution $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ makes each equation a true statement.

A linear system is **consistent** if it has at least one solution and **inconsistent** if it has no solutions.

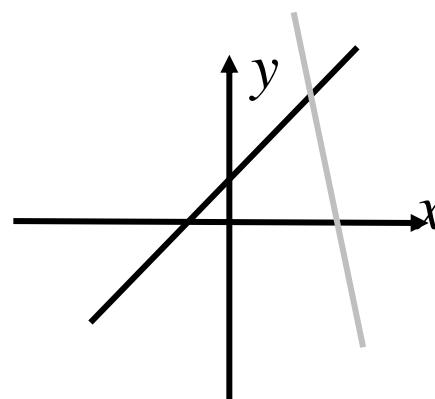
Linear system in two unknowns

Every system of linear equations has *zero, one, or infinitely many* solutions. There is no other possibility (applicable to *any number of unknowns*)



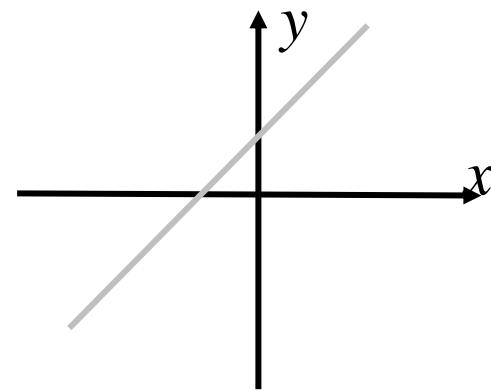
No solution

$$\begin{aligned}x - y &= 4 \\3x - 3y &= 6\end{aligned}$$



One solution

$$\begin{aligned}x - y &= 1 \\2x + y &= 6\end{aligned}$$



Infinitely many solutions
(coincident lines)

$$\begin{aligned}4x - 2y &= 1 \\16x - 8y &= 4\end{aligned}$$

1.2 Homogeneous Linear Systems

A **homogeneous** linear system has the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots &\quad \vdots && \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

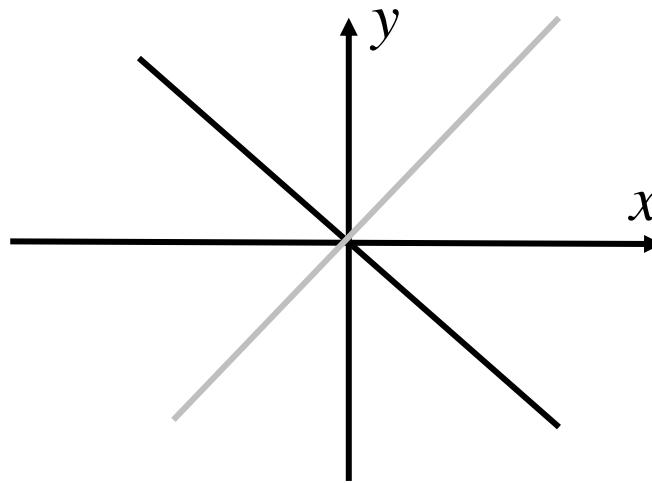
Every homogeneous system of linear equations is **consistent** because all such systems have $x_1 = 0, x_2 = 0, \dots, x_n = 0$ as a solution. This solution is called the **trivial solution**; if there are other solutions, they are called **nontrivial solutions**.

Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

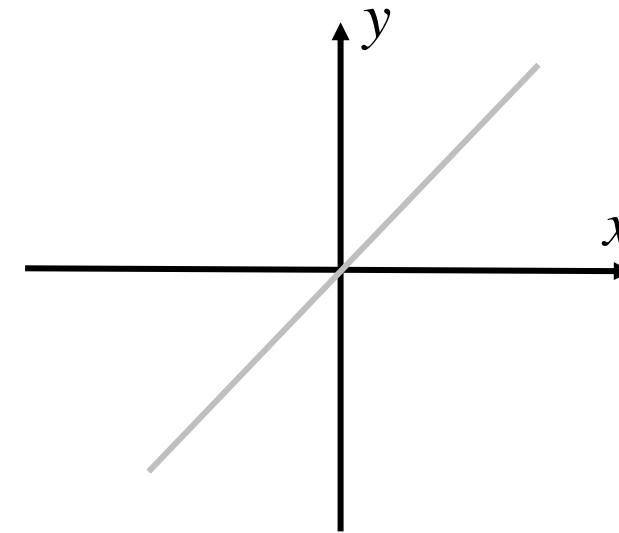
- The system has only the trivial solution
- The system has infinitely many solutions in addition to the trivial solution

Homogeneous linear system of two equations in two unknowns:

$$\begin{aligned} a_1x + b_1y &= 0 & (a_1, b_1 \text{ not both zero}) \\ a_2x + b_2y &= 0 & (a_2, b_2 \text{ not both zero}) \end{aligned}$$



Only the trivial solution



Infinitely many solutions
(coincident lines)

Matrix notation

The essential information of a linear system is captured in a rectangular array called a **matrix**.

Example:

Linear system of two equations:

$$\begin{array}{rcl} 5x + y & = & 3 \\ 2x - y & = & 4 \end{array}$$

Coefficient matrix : $\begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}$

Augmented matrix : $\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$

Size of a matrix of m rows and n columns is written as $m \times n$

1.3 Two fundamental questions about a linear system

1. Is the system consistent, i.e., does at least one solution *exist*?
2. If a solution exists, is it the only one, i.e., is the solution *unique*?

To answer these questions: convert augmented matrix into a **row equivalent** matrix

Definition. Two matrices are **row equivalent** if there is a sequence of elementary row operations (EROs) that transforms one matrix into another.

Elementary Row Operations on a matrix/augmented matrix:

1. Multiply a row through by a non-zero constant
2. Interchange two rows
3. Add a constant times one row to another

Two row equivalent forms:

Row Echelon Form and Reduced Row Echelon Form

A matrix is in Row Echelon Form if it has the following properties:

1. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
2. In any two consecutive rows that do not consist entirely of zeros, the first nonzero element, called the *pivot*, in the lower row occurs farther to the right than the pivot in the higher row.

A matrix is in Reduced Row Echelon Form if it has the following properties:

1. The matrix is in row echelon form.
2. In every column containing a pivot, the pivot has value 1 and all other elements in the column are 0.

Row Echelon Form and Reduced Row Echelon Form (contd.)

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & 9 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Row Echelon Form}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Reduced Row Echelon Form}$$

Gaussian elimination: A matrix that is not in row echelon form **can be transformed** into one through a **sequence** of EROs. A different sequence may lead to a different matrix.

Gauss-Jordan elimination: A matrix that is not in reduced row echelon form **can be transformed** into one through a **sequence** of EROs.

Row Echelon Form and Reduced Row Echelon Form (contd.)

Exercise: Transform matrix to row echelon form (Gaussian elimination)

$$\begin{array}{c}
 \left[\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 1 & -4 \\ 7 & 0 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -1 & -7 \\ 7 & 0 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -1 & -7 \\ 0 & -14 & -19 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -1 & -7 \\ 0 & 0 & 79 \end{array} \right] \\
 r2 \leftarrow r2 - r1 \qquad r3 \leftarrow r3 - 7r1 \qquad r3 \leftarrow r3 - 14r2 \qquad \square : \text{pivot}
 \end{array}$$

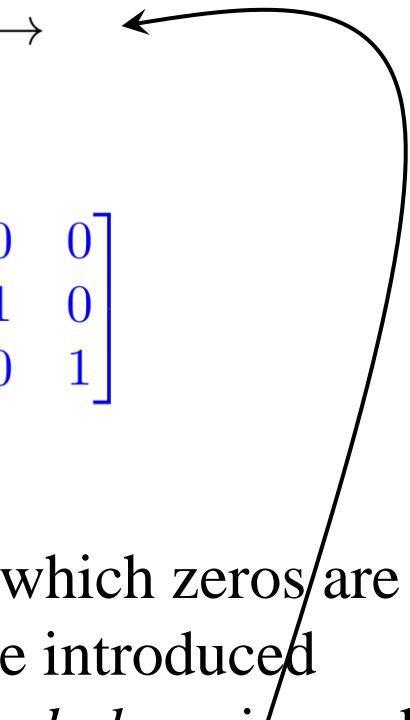
↑ ↑ ↑

Pivot columns

$$\begin{array}{c}
 \left[\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 1 & -4 \\ 7 & 0 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 1 & -4 \\ 1 & 2 & 3 \\ 7 & 0 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 1 & -4 \\ 0 & 1 & 7 \\ 7 & 0 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 1 & -4 \\ 0 & 1 & 7 \\ 0 & -7 & 30 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 1 & -4 \\ 0 & 1 & 7 \\ 0 & 0 & 79 \end{array} \right] \\
 r2 \leftrightarrow r1 \qquad r2 \leftarrow r2 - r1 \qquad r3 \leftarrow r3 - 7r1 \qquad r3 \leftarrow r3 + 7r2
 \end{array}$$

Row Echelon Form and Reduced Row Echelon Form (contd.)

Exercise: Transform matrix to reduced row echelon form (Gauss-Jordan elimination)

$$\begin{array}{c}
 \left[\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 1 & -4 \\ 7 & 0 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -1 & -7 \\ 7 & 0 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -1 & -7 \\ 0 & -14 & -19 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -1 & -7 \\ 0 & 0 & 79 \end{array} \right] \rightarrow \\
 \text{r2} \leftarrow r2 - r1 \qquad \qquad \qquad \text{r3} \leftarrow r3 - 7r1 \qquad \qquad \qquad \text{r3} \leftarrow r3 - 14r2 \qquad \qquad \qquad \text{r1} \leftarrow r1 + 2r2 \\
 \left[\begin{array}{ccc} 1 & 0 & -11 \\ 0 & -1 & -7 \\ 0 & 0 & 79 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 0 & -11 \\ 0 & -1 & -7 \\ 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & -7 \\ 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]
 \end{array}$$


Gauss-Jordan elimination consists of two parts – a **forward phase** in which zeros are introduced below the pivots and a **backward phase** in which zeros are introduced above the pivots and the pivots are transformed to 1. If *only the forward phase is* used, then the procedure produces a row echelon form only and is called Gaussian elimination.

1. Is the system consistent, i.e., does at least one solution *exist*?
2. If a solution exists, is it the only one, i.e., is the solution *unique*?

Exercise:

Determine if the following system is consistent:

$$\begin{array}{rcl} & x_2 - 4x_3 = 8 \\ \begin{matrix} 2x_1 - 3x_2 + 2x_3 = 1 \\ 4x_1 - 8x_2 + 12x_3 = 1 \end{matrix} & & \end{array}$$

Solution:

The augmented matrix is $\left[\begin{array}{cccc} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array} \right]$

$$\underbrace{\left[\begin{array}{cccc} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array} \right]}_{r2 \leftrightarrow r1} \rightarrow \underbrace{\left[\begin{array}{cccc} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array} \right]}_{r3 \leftarrow r3 - 2r1} \rightarrow \underbrace{\left[\begin{array}{cccc} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{array} \right]}_{r3 \leftarrow r3 + 2r2} \rightarrow \left[\begin{array}{cccc} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array} \right]$$

Corresponding linear system:

$$\begin{array}{rcll} 2x_1 & - & 3x_2 & + & 2x_3 = 1 \\ & & x_2 & - & 4x_3 = 8 \\ 0x_1 & + & 0x_2 & + & 0x_3 = 15 \end{array}$$

Last equation is $0 = 15$. Therefore, Inconsistent, i.e., no solutions.

1. Is the system consistent, i.e., does at least one solution *exist*?
2. If a solution exists, is it the only one, i.e., is the solution *unique*?

1.4 Solutions of Linear Systems

Row reduction of the augmented matrix directly leads to a description of the solution set of a linear system.

Exercise:

$$\begin{array}{lcl} x_1 + 3x_2 - 2x_3 + 2x_5 & = & 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 & = & -1 \\ 5x_3 + 10x_4 + 15x_6 & = & 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 & = & 6 \end{array} \Rightarrow M = \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array} \right]$$

$$\underbrace{\left[\begin{array}{cccccc} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array} \right]}_{\substack{r2 \leftarrow r2 - 2r1 \\ r4 \leftarrow r4 - 2r1}} \rightarrow \underbrace{\left[\begin{array}{cccccc} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right]}_{r2 \leftarrow -1r2} \rightarrow \underbrace{\left[\begin{array}{cccccc} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right]}_{\substack{r3 \leftarrow r3 - 5r2 \\ r4 \leftarrow r4 - 4r2}}$$

$$\underbrace{\left[\begin{array}{cccccc} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array} \right]}_{\substack{r3 \leftrightarrow r4 \\ r3 \leftarrow \frac{1}{6}r3}} \rightarrow \underbrace{\left[\begin{array}{cccccc} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]}_{r2 \leftarrow r2 - 3r3} \rightarrow \underbrace{\left[\begin{array}{cccccc} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]}_{r1 \leftarrow r1 + 2r2}$$

Row echelon form

Solutions of Linear Systems (contd.)

$$\rightarrow \left[\begin{array}{cccccc} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \text{Reduced row echelon form}$$

From the reduced row echelon form, the corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 + 4x_4 + 2x_5 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_6 &= \frac{1}{3} \end{aligned}$$

x_1 , x_3 , and x_6 are called **leading variables**. The other variables are called **free variables**. Solving for the leading variables

$$x_1 = -3x_2 - 4x_4 - 2x_5$$

$$x_3 = -2x_4$$

$$x_6 = \frac{1}{3}$$

The general solution of the system is expressed parametrically by assigning the free variables x_2 , x_4 and x_5 arbitrary values r , s , and t , respectively. That is,

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$$

1. Is the system consistent, i.e., does at least one solution *exist*?
2. If a solution exists, is it the only one, i.e., is the solution *unique*?

Procedure

Using row reduction to solve a linear system

1. Write the augmented matrix.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in row echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced row echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Gaussian Elimination and Back-Substitution

For small linear systems that are solved by hand, Gauss-Jordan elimination is good. However, for large linear systems, it is more efficient to use Gaussian elimination with back-substitution.

Here, we start with the row echelon form

$$\left[\begin{array}{ccccccc} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ x_3 + 2x_4 &+ 3x_6 = 1 \\ &x_6 = \frac{1}{3} \end{aligned}$$

Solving for the leading variables

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \frac{1}{3} \end{aligned}$$

Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.

Gaussian Elimination and Back-Substitution (contd.)

Substituting $x_6 = \frac{1}{3}$ into the second equation

$$x_1 = -3x_2 + 2x_3 - 2x_5$$

$$x_3 = -2x_4$$

$$x_6 = \frac{1}{3}$$

Substituting $x_3 = -2x_4$ into the first equation

$$x_1 = -3x_2 - 4x_4 - 2x_5$$

$$x_3 = -2x_4$$

$$x_6 = \frac{1}{3}$$

Assigning arbitrary values r, s and t to the free variables x_2, x_4 and x_5 , we obtain the same solution:

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$$

1.5 Linear combination of vectors

Definition. Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbb{R}^n and given scalars c_1, c_2, \dots, c_p , the vector \mathbf{y} defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ with **weights** c_1, c_2, \dots, c_p .

Exercise:

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether \mathbf{b} can be written as a linear combination of \mathbf{a}_1 and \mathbf{a}_2 . ■

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is $[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n \ \mathbf{b}]$. In particular, \mathbf{b} can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the above augmented matrix.

1.6 Span

Definition. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are in \mathbb{R}^n , then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ and is called the **subset of \mathbb{R}^n spanned (or generated) by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$** .

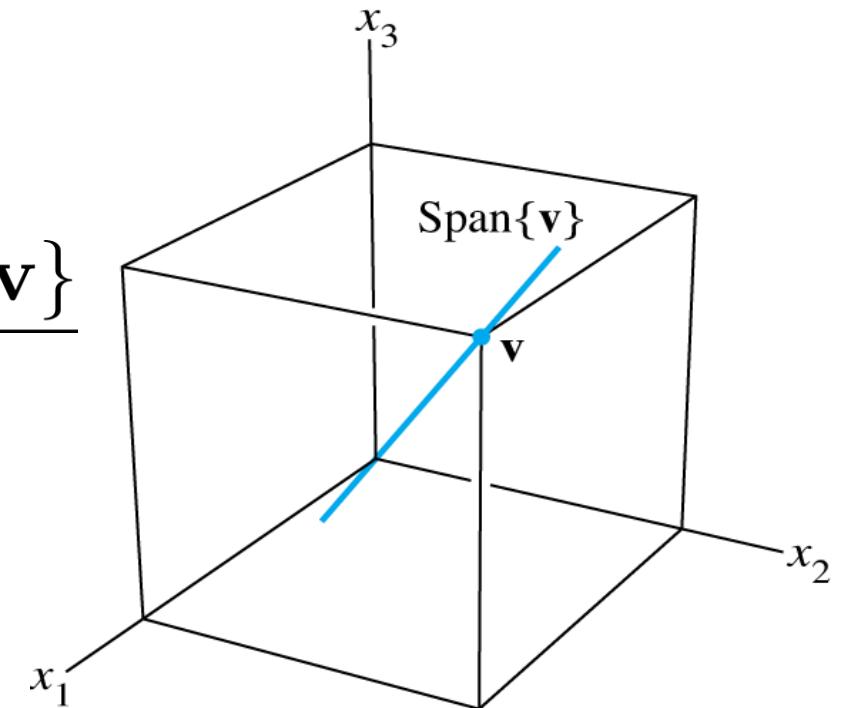
That is, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$

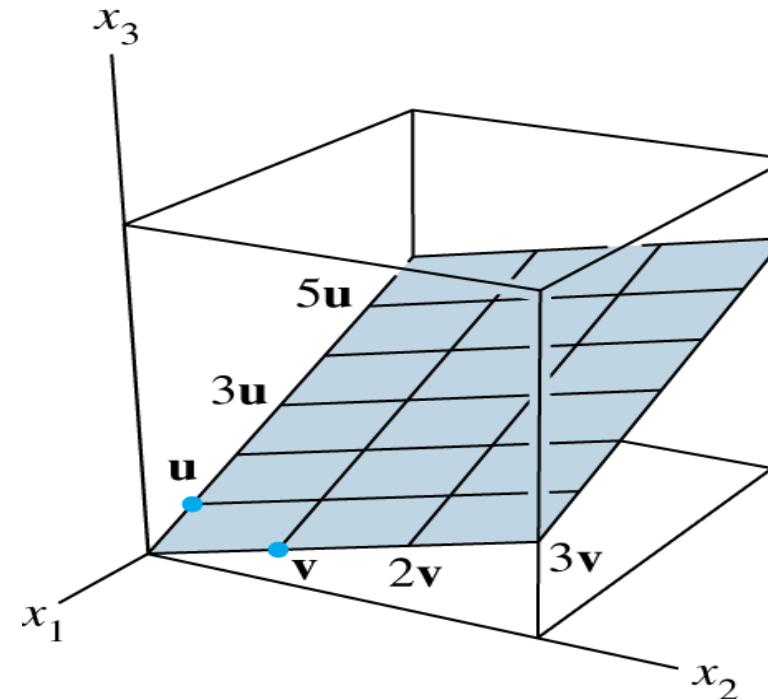
with scalars c_1, c_2, \dots, c_p .

Geometric view of $\text{Span}\{\mathbf{v}\}$ and $\text{Span}\{\mathbf{u}, \mathbf{v}\}$

Let \mathbf{v} be a nonzero vector in \mathbb{R}^3 . $\text{Span}\{\mathbf{v}\}$ is the set of all scalar multiples of \mathbf{v} , which is the set of points in the line in \mathbb{R}^3 through \mathbf{v} and $\mathbf{0}$.



If \mathbf{u} and \mathbf{v} are nonzero vectors in \mathbb{R}^3 , with \mathbf{v} not a multiple of \mathbf{u} , then $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is the plane in \mathbb{R}^3 that contains \mathbf{u} , \mathbf{v} and $\mathbf{0}$.



Exercise:

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Determine if \mathbf{b} is in the $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$. ■

1.7 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

A general linear system of m equations in n unknowns (variables) x_1, x_2, \dots, x_n can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\quad \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

which can be written in matrix notation as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \text{that is, } A\mathbf{x} = \mathbf{b}$$

where A is the matrix of coefficients, \mathbf{x} is the vector of unknowns/variables and \mathbf{b} is the vector of constant terms.

Definition. If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and if \mathbf{x} is in \mathbb{R}^n , then the **product of A and \mathbf{x}** , denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in \mathbf{x} as weights**, i.e.,

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n$$

Theorem 1.1. If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and if \mathbf{b} is in \mathbb{R}^m , then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n \quad \mathbf{b}]$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if \mathbf{b} is a linear combination of the columns of A .

Exercise:

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible values of b_1, b_2, b_3 ?

For $A\mathbf{x} = \mathbf{b}$ to be consistent, $b_1 - \frac{1}{2}b_2 + b_3 = 0$.

The columns of $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ span a plane through $\mathbf{0}$ in \mathbb{R}^3 . ■

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^m **spans** (or **generates**) \mathbb{R}^m if every vector in \mathbb{R}^m is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$, i.e., if $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \mathbb{R}^m$.

Theorem 1.2. *Let A be an $m \times n$ matrix. Then the following statements are logically equivalent, i.e., for a particular A , either they are all true statements or they are all false.*

- a. *For each \mathbf{b} in \mathbb{R}^m , the equation $Ax = \mathbf{b}$ has a solution.*
- b. *Each \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A .*
- c. *The columns of A span \mathbb{R}^m .*
- d. *A has a pivot position in every row.*

Theorem 1.3. *If A is an $m \times n$ matrix, \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^m , and c is a scalar, then:*

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- $A(c\mathbf{u}) = c(A\mathbf{u})$

Proof of Theorem 1.2

(a), (b) and (c) are logically equivalent.

For an arbitrary matrix A , show that (a) and (d) are either both true or both false.

If U is an echelon form of A and \mathbf{b} is in \mathbb{R}^m , then we can row reduce $[A \quad \mathbf{b}]$ to $[U \quad \mathbf{d}]$ for some \mathbf{d} in \mathbb{R}^m .

Assume (d) is true. Then each row of U contains a pivot position \Rightarrow No pivot in augmented column $\Rightarrow A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b} \Rightarrow$ (a) is true.

Assume (d) is false. Then last row of U is all zeros. Let \mathbf{d} be a vector with a 1 in the last entry. Then $[U \quad \mathbf{d}]$ represents an *inconsistent* system. Since row operations are reversible, $[U \quad \mathbf{d}]$ can be transformed into $[A \quad \mathbf{b}] \Rightarrow A\mathbf{x} = \mathbf{b}$ is also inconsistent \Rightarrow (a) is false.

1.8 Solution Sets of Linear Systems

- Solutions of Homogeneous Systems

$Ax = b$: If $b = \mathbf{0}$ \Rightarrow homogeneous equations and $b \neq \mathbf{0}$ \Rightarrow non-homogeneous equations

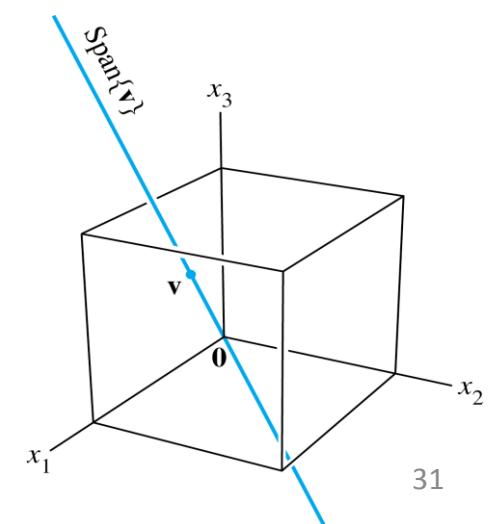
RECALL: For $Ax = \mathbf{0}$, the solution $x = \mathbf{0}$ is called the *trivial solution*. If there is a non zero vector x that satisfies $Ax = \mathbf{0}$, it is called a *non trivial solution*.

Exercise:

Determine if the following homogeneous system has a nontrivial solution.

$$\begin{array}{rclcl} 3x_1 & + & 5x_2 & - & 4x_3 = 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 = 0 \\ 6x_1 & + & x_2 & - & 8x_3 = 0 \end{array}$$

Every solution of $Ax = \mathbf{0}$ in this case is a scalar multiple of $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$



- Solutions of Nonhomogeneous Systems

Exercise:

Describe all solutions of $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$.

Solution:

Row operations on $[A \quad \mathbf{b}]$:

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]. \quad \begin{array}{rcl} x_1 & - \frac{4}{3}x_3 & = -1 \\ x_2 & & = 2 \\ 0 & & = 0 \end{array}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{c} \uparrow \\ \mathbf{p} \end{array} \quad \begin{array}{c} \uparrow \\ \mathbf{v} \end{array}$$

Parametric vector form of the solution set of $A\mathbf{x} = \mathbf{b}$:

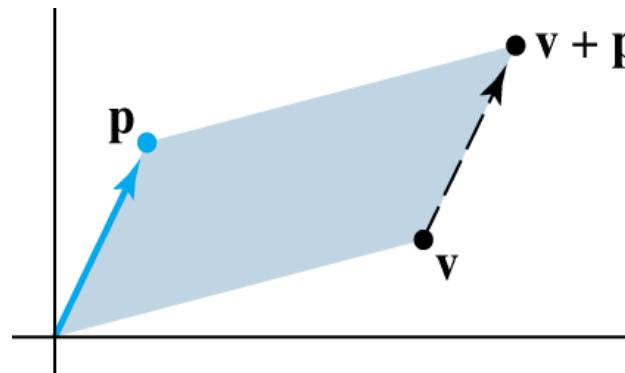
$$\mathbf{x} = \mathbf{p} + x_3 \mathbf{v} \quad \text{or} \quad \mathbf{x} = \mathbf{p} + t \mathbf{v}, \quad t \in \mathbb{R}$$

Previous exercise (slide 31): Parametric vector form of the solution set of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t \mathbf{v}$.

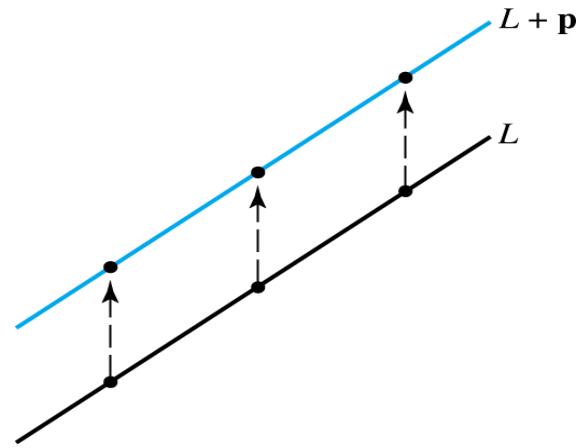
Solutions of $A\mathbf{x} = \mathbf{b}$ obtained by adding \mathbf{p} to the solutions of $A\mathbf{x} = \mathbf{0}$.

\mathbf{p} is one particular solution of $A\mathbf{x} = \mathbf{b}$ (corresponding to $t = 0$). ■

Geometrically, vector addition as *translation* $\implies \mathbf{v}$ is translated by \mathbf{p} to $\mathbf{v} + \mathbf{p}$



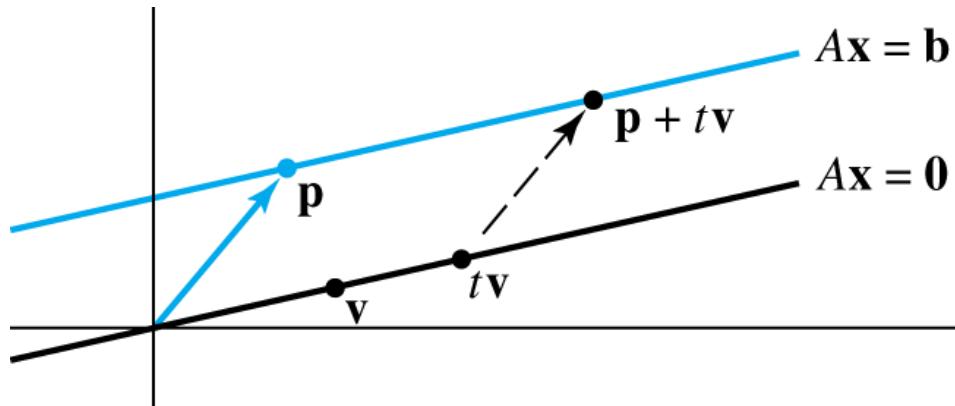
Each point on a line translated by a vector \mathbf{p}



Parametric vector form of the solution set of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t\mathbf{v}$.
This is a line L through $\mathbf{0}$ and \mathbf{v} .

Adding \mathbf{p} to each point on L , we get the **the equation of the line through \mathbf{p} parallel to \mathbf{v}** .

The solution set of $A\mathbf{x} = \mathbf{b}$ is a line through \mathbf{p} parallel to the solution set of $A\mathbf{x} = \mathbf{0}$.



Theorem 1.4. Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given \mathbf{b} , and let \mathbf{p} be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where \mathbf{v}_h is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

1.9 Linear Independence

Definition. An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights c_1, \dots, c_p , not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

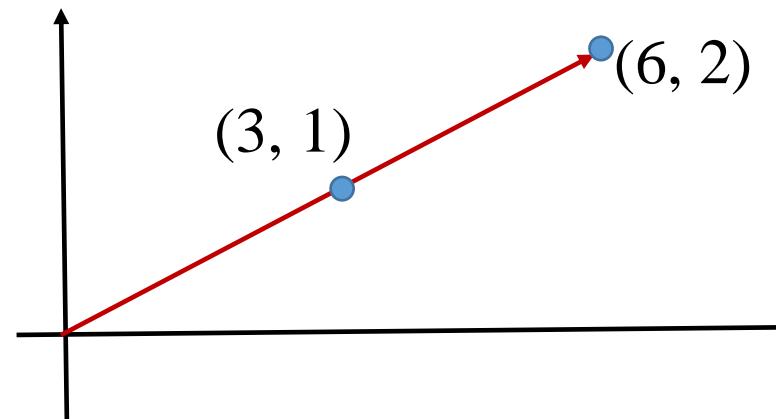
Exercise:

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. If not, what is the linear dependence relation among them? ■

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

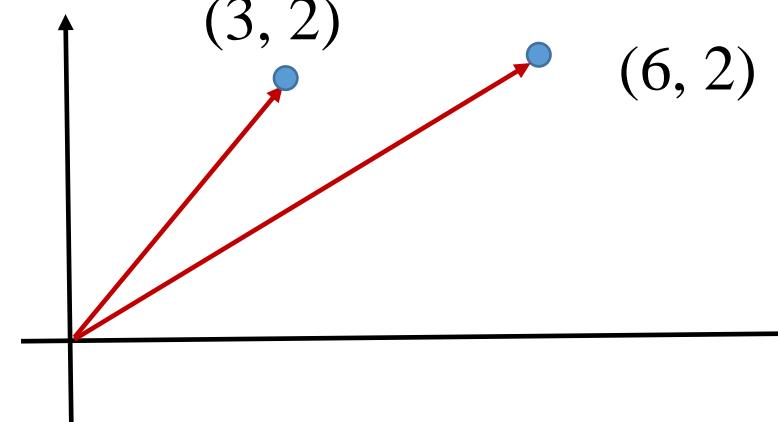
By inspection for simple cases

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$



Linearly Dependent

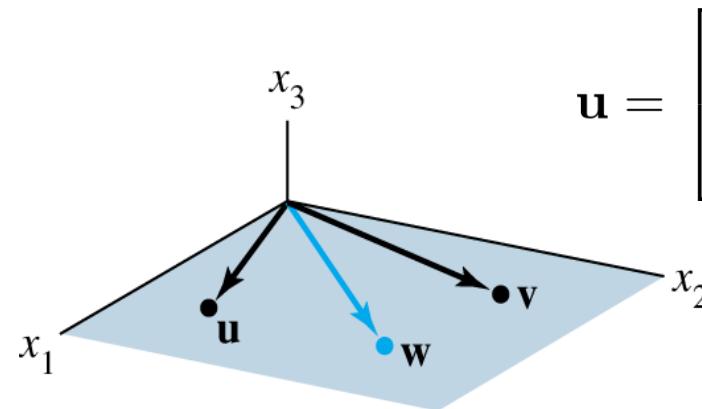
$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$



Linearly Independent

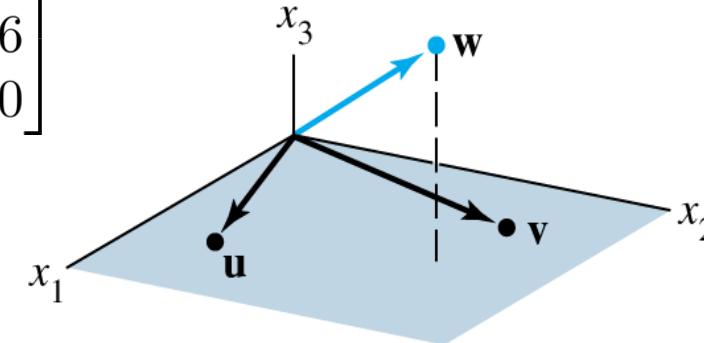
Characterization of linearly dependent sets

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some \mathbf{v}_j (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.



Linearly dependent,
 w in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$



Linearly independent,
 w not in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$

w is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

Theorem 1.5. If a set contains more vectors than there are entries in each vector, then the set is linearly dependent, i.e., any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is linearly dependent if $p > n$.

Proof

Let $A = [\mathbf{v}_1 \ \dots \ \mathbf{v}_p]$. A is $n \times p$.

$A\mathbf{x} = \mathbf{0}$ corresponds to n equations in p unknowns.

$p > n \implies$ more variables than equations \implies there must be a free variable \implies

$A\mathbf{x} = \mathbf{0}$ has non trivial solution \implies columns of A are linearly dependent. ■

$$n \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}^p$$

If $p > n$, the columns are linearly dependent.

Example

The vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly dependent. Note: none of the vectors is a multiple of one of the other vectors.

Theorem 1.6. *If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n contains the zero vector, then the set is linearly dependent.*

Proof

By renumbering the vectors, we may suppose $\mathbf{v}_1 = \mathbf{0}$. Then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_p = \mathbf{0}$ shows that S is linearly dependent. ■

1.10 Introduction to Linear Transformations

Think of $A\mathbf{x} = \mathbf{b}$ as :

matrix A as an object that “acts” on a vector \mathbf{x} by multiplication to produce a new vector called $A\mathbf{x}$.

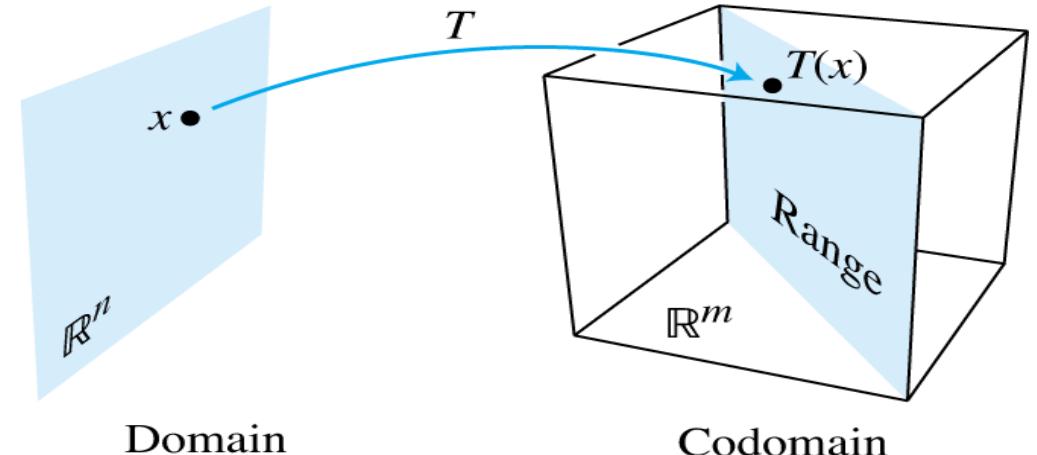
A **transformation** T from \mathbb{R}^n to \mathbb{R}^m (denoted by $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$) is a rule that assigns to each vector \mathbf{x} in \mathbb{R}^n a vector $T(\mathbf{x})$ in \mathbb{R}^m .

\mathbb{R}^n - **domain** of T

\mathbb{R}^m - **codomain** of T

$T(\mathbf{x})$ - **image** of \mathbf{x}

Set of all images $T(\mathbf{x})$ - **range** of T



**Domain, codomain, and range
of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.**

Exercise:

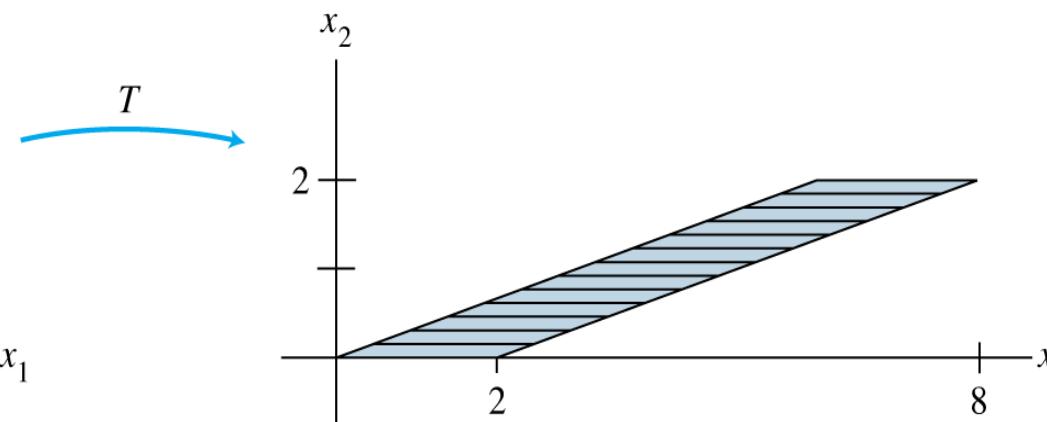
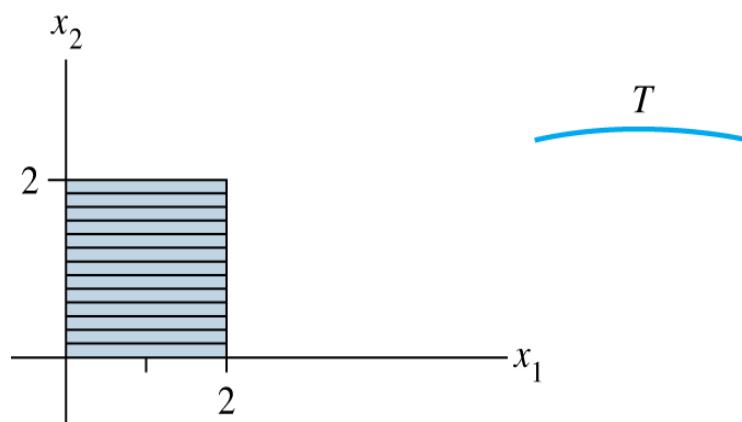
Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

- a. Find $T(\mathbf{u})$.
- b. Find an \mathbf{x} in \mathbb{R}^2 whose image under T is \mathbf{b} .
- c. Is there more than one \mathbf{x} whose image under T is \mathbf{b} ?
- d. Determine if \mathbf{c} is in the range of T .

Examples:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. What is this transformation?

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. What is this transformation?



1.11 Linear Transformations

Definition. A transformation T is **linear** if:

- i. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in the domain of T ;
- ii. $T(c\mathbf{u}) = cT(\mathbf{u})$

Generalization:

$$T(c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \dots + c_pT(\mathbf{v}_p)$$

Examples:

- $T(\mathbf{x}) = r\mathbf{x}$, $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is called a **contraction** when $0 \leq r \leq 1$ and a **dilation** when $r > 1$.

- Show that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a linear transformation by finding the images of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

1.12 The Matrix of a Linear Transformation

Given the description of a linear transformation ‘in words’, how to obtain the corresponding matrix?

Key observation: T is completely determined by what it does to the columns of the $n \times n$ identity matrix I_n .

Exercise:

Let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transfor-

mation from \mathbb{R}^2 to \mathbb{R}^3 such that $T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. Find the formula for the image of an arbitrary \mathbf{x} in \mathbb{R}^2 .

Theorem 1.7. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose j^{th} column is the vector $T(\mathbf{e}_j)$, where \mathbf{e}_j is the j^{th} column of the identity matrix in \mathbb{R}^n :

$$A = [T(\mathbf{e}_1) \quad \dots \quad T(\mathbf{e}_n)].$$

■

A is called the **standard matrix for the linear transformation T** .

Exercise:

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation that rotates each point in \mathbb{R}^2 about the origin through an angle Φ , with counterclockwise rotation for a positive angle. Find the standard matrix A for this transformation.

***** END OF CHAPTER *****