Discrete Mathematics

MH1812

Topic 3 - Predicate Logic
Summary

Quantification: Order of Nesting Matters

Is $\forall x \in D, \exists y \in D, P(x,y) \equiv \exists y \in D, \forall x \in D, P(x,y)$ in general?

$\forall$ = for All

$\exists$ = Exist

LHS

$\forall x \in D, \exists y \in D, P(x,y)$

y can **vary** with *x*

RHS

$\exists y \in D, \forall x \in D, P(x,y)$

y is **fixed**, but *x* **varies**

Let $P(x,y)$ = "x admires y"

"Every person admires someone"

"Some people are admired by everyone"

Example

$y = kx, \exists k \in \mathbb{Z}$

Consider the domains $X = \{2,3\}$ and $Y = \{2,4,6\}$, and the predicate $P(x,y)$ = "x **divides** y".

$x \mid y$

$\cancel{x/y}$ ✗

$x \div y$ ✗

What are the truth values of

1. $\forall y \in Y, \exists x \in X, P(x,y)$. T
2. $\exists x \in X, \forall y \in Y, P(x,y)$. T

For $y=2$, take $x=2$ then $x \mid y$ ✓
For $y=4$, take $x=2$ then $x \mid y$ ✓
For $y=6$, take $x=3$ then $x \mid y$ ✓

Try $x=2$: for $y=2$, have $x \mid y$ ✓
for $y=4$, have $x \mid y$ ✓
for $y=6$ have $x \mid y$ ✓

Example

Consider the domains $X = \{2,3\}$ **and** $Y = \{2,6,9\}$, **and the predicate** $P(x,y)$ = "x **divides** y".

What are the truth values of

1. $\forall y \in Y, \exists x \in X, P(x,y)$. T For $y=2$, take $x=2$ then $x \mid y$
For $y=6$, take $x=2$ then $x \mid y$
For $y=9$, take $x=3$ then $x \mid y$

2. $\exists x \in X, \forall y \in Y, P(x,y)$. F

2. Try $x=2$: for $y=2$, have $x \mid y$
for $y=6$, have $x \mid y$
for $y=9$, then $x \nmid y$ ✗

Try $x=3$: for $y=2$ then $x \nmid y$ ✗

Quantification: Order of Nesting Matters

Consider (arbitrary) domains X and Y with m and n members respectively.

$X = \{x_1, x_2, \ldots, x_m\}$

$Y = \{y_1, y_2, \ldots, y_n\}$

Then $\exists x \in X, \exists y \in Y, P(x,y) \equiv \exists y \in Y, \exists x \in X, P(x,y)$

and $\forall x \in X, \forall y \in Y, P(x,y) \equiv \forall y \in Y, \forall x \in X, P(x,y)$

$(\exists y \in Y, P(x_2, y))$

$$\exists x \in X, \exists y \in Y, P(x,y) \equiv [\exists y \in Y, P(x_1,y)] \vee \ldots \vee [\exists y \in Y, P(x_m,y)]$$

$$\equiv [P(x_1, y_1) \vee \ldots \vee P(x_1, y_n)] \vee \ldots \vee [P(x_m, y_1) \vee \ldots \vee P(x_m, y_n)]$$

$a \vee b \equiv b \vee a$

commutativity

$$\equiv [P(x_1, y_1) \vee \ldots \vee P(x_m, y_1)] \vee \ldots \vee [P(x_1, y_n) \vee \ldots \vee P(x_m, y_n)]$$

$$\equiv [\exists x \in X, P(x, y_1)] \vee \ldots \vee [\exists x \in X, P(x, y_n)]$$

$$\equiv \exists y \in Y, \exists x \in X, P(x,y)$$

Determining Truth Values: Method of Case

Positive Example to Prove Existential Quantification

Let $\mathbb{Z}$ denote all integers.

Is $\exists x \in \mathbb{Z}, x^2 = x$ true or false?

Take $x = 0$ or 1 and we have it.

one example suffices

Positive Example

It is **not** a proof of universal quantification.

Counterexample to Disprove Universal Quantification

Let $\mathbb{R}$ denote all reals.

Is $\forall x \in \mathbb{R}, x^2 > x$ true or false?

Take $x = 0.3$ as a counterexample.

$x = 0$

Negative Example

It is **not** disproof of existential quantification.

Conditional Quantification: Definitions

Given a conditional quantification such as…

$$\forall x{\in}A\ (P(x) \rightarrow Q(x))$$

$\forall x \in \mathbb{Z}$ $P(x)$

if last digit of x is 4

then $\boxed{x \text{ is even}}$

$Q(x)$

Then, we define…

Contrapositive	$\forall x{\in}A,\ \neg Q(x) \rightarrow \neg P(x)$
Converse	$\forall x{\in}A,\ Q(x) \rightarrow P(x)$
Inverse	$\forall x{\in}A,\ \neg P(x) \rightarrow \neg Q(x)$

Note: a conditional proposition is logically equivalent to its contrapositive.

Conditional Quantification: Negation

What is $\neg(\forall x \in X,\ P(x) \rightarrow Q(x))$**?**

$\neg(\forall x \in X,\ R(x))$

$\equiv \exists x \in X,\ \neg R(x)$

$\neg(\forall x \in X,\ P(x) \rightarrow Q(x))$

$\equiv \exists x \in X,\ \neg(P(x) \rightarrow Q(x))$ **Negation of Quantified Statements**

$\equiv \exists x \in X,\ \neg(\neg P(x) \vee Q(x))$ **Conversion of Conditionals**

$\equiv \exists x \in X,\ P(x) \wedge \neg Q(x)$ **De Morgan**

Example

Consider the domains $X = \{2,3\}$ **and** $Y = \{2,4,6\}$, **and the predicate** $P(x,y)$ = "x **divides** y".

What are the truth values of

1. $\neg(\exists x \in X, \underbrace{\exists y \in Y\, P(x,y)}_{Q(x)})$. F   take $x=2$ & $y=2$

$\equiv \forall x \in X, \neg Q(x) \equiv \forall x \in X, \neg(\exists y \in Y, P(x,y))$

$\equiv \forall x \in X, \forall y \in Y, \neg P(x,y) = F$

counterexample: $x=2, y=2$

2. $\neg(\forall y \in Y, \exists x \in X, P(x,y))$. $= F$

$\equiv \exists y \in Y, \neg(\exists x \in X, P(x,y))$

$\equiv \exists y \in Y, \forall x \in X, \neg P(x,y)$

Basic Inference Rules:

not assessed

in any assessment!

$\forall\exists$

$P(c)$ for **any arbitrary** c **from the domain** D.

$\therefore\ \forall x \in D,\ P(x)$

$\forall\exists$

$P(c)$

$\therefore\ \exists x \in D,\ P(x)$

for c **some specific element of the domain** D.

$\forall\exists$

$\forall x \in D,\ P(x)$

$\therefore\ P(c)$

where c **is any element of the domain** D.

$\forall\exists$

$\exists x \in D,\ \mathrm{P}(x)$

$\therefore\ P(c)$ **for some** c **in the domain** D.

EN I

- number systems
- mod
- operator closure

Midterm 1

Logic

proposition

$\wedge$: conjunction (and)

$\vee$: disjunction (or)

$\neg$: negation (not, alternatively ~)

$p \to q$: conditional (if then)

$p \leftrightarrow q$: biconditional (if and only if)

equivalence laws (e.g. De Morgan, Conversion Theorem, Distributivity)

valid argument (premises and conclusion)

inference rules, e.g. Modus ponens/tollens

$$\begin{array}{r} p \to q; \\ p; \\ \therefore q \end{array}$$

$$\begin{array}{r} p \to q; \\ \neg q; \\ \therefore \neg p \end{array}$$

predicate

Quantification:

- Universal $\forall$
- Existential $\exists$
- Nested
- Negation
- Conditional
- Negation of conditional

Inference rules:

- Universal generalization/instantiation
- Existential generalization/instantiation

not assessed