Exercises for Chapter 9

Exercise 83. Consider the set $A = \{a, b, c\}$ with power set $P(A)$ and $\cap$: $P(A) \times P(A) \to P(A)$. What is its domain? its co-domain? its range? What is the cardinality of the pre-image of $\{a\}$?

Solution. Its domain is the cartesian product $P(A) \times P(A)$, its co-domain is $P(A)$. Its range is $P(A)$: indeed, for any subset X of A, $X \cap X = X$, therefore every element of $P(A)$ has a pre-image. The pre-image of $\{a\}$ is the set of elements in $P(A) \times P(A)$ which are mapped to $\{a\}$, that is, pairs (X, Y) of subsets of A whose intersection is $\{a\}$. Now $\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$ are all the subsets containing $\{a\}$, so this gives 2^4 possible pairs, but among them, not all are suitable: we have to remove those with bigger intersection. So we can intersect $\{a\}$ with all of them:

$$(\{a\}, \{a\}),\ (\{a\}, \{a, b\}),\ (\{a\}, \{a, c\}),\ (\{a\}, \{a, b, c\}),$$

or $\{a, c\}$ with $\{a, b\}$. Note that the ordering of a pair matters, thus all those pairs give rise to another pair, apart for $(\{a\}, \{a\})$ thus a total of 9.

Exercise 84. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not one-to-one.

Solution. We have that $\sin(0) = \sin(\pi) = 0$ but $\pi \neq 0$, which contradicts the definition of one-to-one, since there exist $x_1 = 0, x_2 = \pi$ such that $\sin(x_1) = \sin(x_2)$ but $x_1 \neq x_2$.

Exercise 85. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not onto, but $\sin : \mathbb{R} \to [-1, 1]$ is.

Solution. It is not onto because $\exists y \in \mathbb{R}$, say $y = 2$, such that for all $x \in \mathbb{R}$, $f(x) \neq 2$.

Exercise 86. Is $h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = 4n - 1$, onto (surjective)?

Solution. No, it is not. For example, take $y = 1$. Then it is not possible that $1 = 4n - 1$ for n an integer, because this equation means that $n = 1/2$.

Exercise 87. Is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, a bijection (one-to-one correspondence)?

Solution. Injectivity: suppose $f(x_1) = f(x_2)$, then $x_1^3 = x_2^3$ and it must be that $x_1 = x_2$. Surjectivity: take $y \in \mathbb{R}$, and $x = \sqrt[3]{y} \in \mathbb{R}$, then $f(x) = y$, so surjectivity holds. Therefore it is a bijection.

Exercise 88. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + 5$. What is $g \circ f$? What is $f \circ g$?

Solution. We have

$$g(f(x)) = g(x^2) = x^2 + 5,\ f(g(x)) = f(x + 5) = (x + 5)^2.$$

Exercise 89. Consider $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ and $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$. What is $g \circ f$? What is $f \circ g$?

Solution. We have

$$g(f(n)) = g(n + 1) = (n + 1)^2,\ f(g(n)) = f(n^2) = n^2 + 1.$$

Exercise 90. Given two functions $f : X \to Y$, $g : Y \to Z$. If $g \circ f : X \to Z$ is one-to-one, must both f and g be one-to-one? Prove or give a counter-example.

Solution. It is not true. For example, take $f : X \to Y$ and $g : Y \to Z$ as follows, $X = \{a, b, c\}$, $Y = \{w, x, y, z\}$, $Z = \{1, 2, 3\}$:

$$f(a) = x,\ f(b) = y,\ f(c) = z,\ g(w) = 1,\ g(x) = 1,\ g(y) = 2,\ g(z) = 3.$$

Then $g \circ f$ is one-to-one, but g is not.

Exercise 91. Show that if $f : X \to Y$ is invertible with inverse function $f^{-1} : Y \to X$, then $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_Y$.

Solution. Take $x \in X$, with $y = f(x)$. Then $f^{-1}(f(x)) = f^{-1}(y) = x$ by definition of inverse, and $x = i_X(x)$ for all $x \in X$ therefore $f^{-1} \circ f = i_X$. Similarly take $y \in Y$ and $x = f^{-1}(y)$. Then $f(f^{-1}(y)) = f(x) = y$ by definition of inverse, and $y = i_Y(y)$ for all $y \in Y$ therefore $f \circ f^{-1} = i_Y$.

Exercise 92. Prove or disprove $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$, for x, y two real numbers.

Solution. This is wrong. Indeed, take $x = y = 1/2$. Then

$$\lceil x + y \rceil = \lceil 1 \rceil = 1$$

however

$$\lceil x \rceil + \lceil y \rceil = \lceil 1/2 \rceil + \lceil 1/2 \rceil = 2.$$

Exercise 93. If you pick five cards from a deck of 52 cards, prove that at least two will be of the same suit.

Solution. If you pick 5 cards, then the first one will be of a given suit (say heart), if the second is also heart, then you got two of the same suit. If the second is not heart (say diamond), then take a 3rd. If it is either heart or diamond, then you got at least two of the same suit, if not, say it is club, pick a 4rth card. Again, if the 4rth card is heart, diamond of club, you got at least two of the same suit, if not, it must be that this 4rth card is spade. But now all the 4 possible choices of suits are picked, so no matter what is the next suit of the card, it will be one that has already appeared. This shows that you will always get at least two cards of the same suit. This is an application of the pigeonhole principle: you have 4 suits, and 5 cards, therefore 2 cards must be of the same suit.

Exercise 94. If you have 10 black socks and 10 white socks, and you are picking socks randomly, you will only need to pick three to find a matching pair.

Solution. Pick the first sock, it is say white. Pick the second sock, if it white, then you got a matching pair. If not, pick a third one. But by now, you have already one white and one black sock, so no matter which is the color of the third one, ou will have a matching pair. This is an application of the pigeonhole principle: you have 2 colors, and 3 socks, therefore 2 socks must be of the same color.

Exercise 95. Prove that the set of all integers is countable.

Exercise 96. Intuitively, we can list all integers in a sequence by starting with 0, and alternating between positive and negative integers:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

Now to get a proof, we formalize this idea by creating a one-to-one correspondence between the set of positive integers and the set of all integers. Take $f : \mathbb{Z}_{>0} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} n/2 & n \text{ even} \\ -(n-1)/2 & n \text{ odd} \end{cases}$$

Note that $f(n) = n/2$ when n is even is already enough to make sure that all positive integers are obtained, since $2, 4, 6, 8, 10, \ldots$ are mapped to

$1, 2, 3, 4, 5, \ldots$. Now when $n = 1$, we get $f(1) = 0$ and we are left to check that the negative integers are obtained. But $3, 5, 7, 9, 11, \ldots$ are mapped to $-1, -2, -3, -4, \ldots$. So this function seems to do what we want! So let us give a proof. We have to show that $f(n)$ is injective. So suppose $f(n) = f(m)$ for some positive integers n, m. So either $f(n) = f(m)$ is positive, then we have

$$n/2 = m/2 \Rightarrow n = m$$

or $f(n) = f(m)$ is negative and

$$-(n-1)/2 = -(m-1)/2 \Rightarrow n = m.$$

So this shows the function is injective. Now let us proof that it is surjective (or onto). We need to show that for any arbitrary m which is an integer, there exists a positive integer n such that $f(n) = m$. If $m > 0$, pick n to be $n = 2m$ (note that n defined like that is indeed a positive integer), and $f(n) = 2m/2 = m$. If $m = 0$, pick $n = 1$ and $f(1) = 0$. If $m < 0$, then pick $n = -2m + 1$. We have $f(n) = -[(-2m + 1) - 1]/2 = m$ and because m is negative, $-2m + 1$ is indeed a positive integer.

Exercises for Chapter 10

Exercise 97. Prove that if a connected graph G has exactly two vertices which have odd degree, then it contains an Euler path.

Solution. Suppose that v and w are the vertices of G which have odd degrees, while all the other vertices have an even degree. Create a new graph G', formed by G, with one more edge e, which connects v and w. Every vertex in G' has even degree, so by the theorem on Euler cycles, there is an Euler cycle. Say this Euler cycle is

$$v, e_1, v_2, e_2, \ldots, w, e, v$$

then

$$v, e_1, v_2, e_2, \ldots, w$$

is an Euler path.

Exercise 98. Draw a complete graph with 5 vertices.