MH1820 Introduction to Probability and Statistical Methods

Tutorial 6 (Week 7) Solution

Problem 1 (MGF) (a) Let X be a discrete random variable with PMF p given by

$$\begin{array}{c|c|c|c} x & -1 & 0 & 1 \\ \hline p(x) & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array} \text{ and } p(x) = 0 \text{ for } x \notin \{-1, 0, 1\}.$$

Compute the MGF of X and use the properties of MGFs from the lecture to compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

(b) Let X be a continuous random variable with PDF f given as follows.

$$f(x) = 2x \text{ for } 0 \leq x \leq 1 \text{ and } f(x) = 0 \text{ otherwise.}$$

Compute the MGF $\mathbb{E}[e^{tX}]$ of X for $t \neq 0$.

(c) Let X be a random variable with $M_X(t) = \dfrac{e^t}{2 - e^t}$ for all $t < \ln 2$. What is the distribution of X? Hint: Check the table for MGF of common distributions.

Solution (a) We get the following.

$$\begin{aligned}
M_X(t) &= \mathbb{E}[e^{tX}] \\
&= \sum_{x=-1}^{1} e^{tx} p(x) \\
&= e^{-t} p(-1) + e^0 p(0) + e^t p(1) \\
&= \frac{1}{2} e^{-t} + \frac{1}{4} + \frac{1}{4} e^t \\
M_X^{(1)}(t) = \frac{dM_X(t)}{dt} &= -\frac{1}{2} e^{-t} + \frac{1}{4} e^t \\
M_X^{(2)}(t) = \frac{d^2 M_X(t)}{dt^2} &= \frac{1}{2} e^{-t} + \frac{1}{4} e^t \\
\mathbb{E}[X] &= M_X^{(1)}(0) = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4} \\
\mathbb{E}[X^2] &= M_X^{(2)}(0) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}
\end{aligned}$$

(b) Using integration by parts at one step, we get the following.

$$\begin{aligned}M_X(t) = \mathbb{E}[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_0^1 2xe^{tx}dx \\ &= \left[\frac{2xe^{tx}}{t}\right]_0^1 - \int_0^1 \frac{2e^{tx}}{t}dx = \left[\frac{2xe^{tx}}{t}\right]_0^1 - \left[\frac{2e^{tx}}{t^2}\right]_0^1 \\ &= \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2}, \quad \text{provided } t \neq 0.\end{aligned}$$

(c) Recall that if $X \sim Geom(p)$, then its MGF is given by

$$\frac{pe^t}{1-(1-p)e^t}, \quad \text{for } t < -\ln(1-p).$$

Note that $M_X(t)$ is the MGF of $Geom(\frac{1}{2})$. Hence $X \sim Geom(\frac{1}{2})$. □

Problem 2 (MGF) Let $X \sim Poisson(\lambda)$. Using the definition of MGF, verify that the MGF of X is given by

$$e^{\lambda(e^t-1)}.$$

Hence, find the mean and variance of X.

Hint: You may assume that $e^z = \sum_{x=0}^{\infty} \frac{z^x}{x!}$. This is the Maclaurin series for e^z.

Solution

$$\begin{aligned}M_X(t) &= \mathbb{E}[e^{tX}] \\ &= \sum_x e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda(e^t-1)}.\end{aligned}$$

Differentiating the MGF yields (by Chain Rule)

$$M_X^{(1)}(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t.$$

Differentiating again (via product rule),

$$\begin{aligned}M_X^{(2)}(t) &= e^{\lambda(e^t-1)} \cdot \frac{d}{dt}\lambda e^t + \lambda e^t \frac{d}{dt} e^{\lambda(e^t-1)} \\ &= e^{\lambda(e^t-1)} \cdot \lambda e^t + \lambda e^t (e^{\lambda(e^t-1)} \cdot \lambda e^t) \\ &= e^{\lambda(e^t-1)} \cdot \lambda e^t + (\lambda e^t)^2 \cdot e^{\lambda(e^t-1)} \\ &= e^{\lambda(e^t-1)}(\lambda e^t)(\lambda e^t + 1)\end{aligned}$$

Hence,

$$\mathbb{E}[X] = M_X^{(1)}(0) = \lambda, \quad \mathbb{E}[X^2] = \lambda(\lambda + 1).$$

So the mean is $\mathbb{E}[X] = \lambda$, and the variance is

$$\begin{aligned} \sigma^2 &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \lambda(\lambda + 1) - \lambda^2 \\ &= \lambda. \end{aligned}$$

□

Problem 3 (MGF) Suppose X and Y are independent Poisson random variables with means λ_1 and λ_2 respectively. What is the distribution of $X + Y$?

$\boxed{\textbf{Solution}}$

$$\begin{aligned} M_{X+Y}(t) &= M_X(t)M_Y(t) \\ &= M_X(t)M_Y(t) \\ &= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} \\ &= e^{(\lambda_1+\lambda_2)(e^t-1)} \end{aligned}$$

Hence, $X + Y$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$. □

Problem 4 (Joint PMF, CDF, Marginal PMF) Let the joint PMF of X and Y be defined by

$$p(x, y) = \frac{x + y}{32},$$

for $x = 1, 2,\ y = 1, 2, 3, 4$.

(a) Find $p_X(x)$, the marginal PMF of X.

(b) Find $p_Y(y)$, the marginal PMF of Y.

(c) Find $\mathbb{P}(X > Y)$

(d) Find $\mathbb{P}(Y = 2X)$

(e) Find $\mathbb{P}(X + Y = 3)$.

(f) Find $\mathbb{P}(X \leq 3 - Y)$.

(g) Are X and Y independent or dependent? Why or why not?

Solution (a)

$$p_X(x) = \sum_{y=1}^{4} p(x,y) = \frac{x+1}{32} + \frac{x+2}{32} + \frac{x+3}{32} + \frac{x+4}{32} = \frac{4x+10}{32},$$

for $x = 1, 2$.
(b)

$$p_Y(y) = \sum_{x=1}^{2} p(x,y) = \frac{1+y}{32} + \frac{2+y}{32} = \frac{3+2y}{32},$$

for $y = 1, 2, 3, 4$.
(c) The event $X > Y$ occurs only when $x = 2$ and $y = 1$. So

$$\mathbb{P}(X > Y) = p(2,1) = \frac{3}{32}.$$

(d) The event $Y = 2X$ is given by $\{(1,2), (2,4)\}$. So

$$\mathbb{P}(Y = 2X) = p(1,2) + p(2,4) = \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32}.$$

(e) The event $X + Y = 3$ is given by $\{(1,2), (2,1)\}$. So

$$\mathbb{P}(X + Y = 3) = p(1,2) + p(2,1) = \frac{1+2}{32} + \frac{2+1}{32} = \frac{6}{32}.$$

(f) The event $X + Y \leq 3$ is given by $\{(1,1), (1,2), (2,1)\}$. So

$$\mathbb{P}(X \leq 3 - Y) = \mathbb{P}(X + Y \leq 3) = p(1,1) + p(1,2) + p(2,1) = \frac{1+1}{32} + \frac{1+2}{32} + \frac{2+1}{32} = \frac{8}{32}.$$

(g) Note that $p(1,1) = \frac{1+1}{32} = \frac{2}{32}$, $p_X(1) = \frac{14}{32}$, $p_Y(1) = \frac{5}{32}$. Since $p(1,1) \neq p_X(1)p_Y(1)$, X and Y are not independent.

Problem 5 (Joint PMF, CDF, Marginal PMF) From a standard poker deck of 52 cards, 3 cards are chosen randomly. Let X be number of clubs among the 3 cards and let Y be the number of hearts among the 3.

(a) Find the joint PMF of X and Y.

(b) Find the marginal PMFs of X and Y.

(c) Are X and Y independent?

Solution (a) Note that $X = x$ and $Y = y$ mean that exactly x of the 3 cards are clubs and exactly y of the 3 cards are hearts. There are $\binom{13}{x}$ ways to choose x cards from the 13 hearts and $\binom{13}{y}$ ways to choose y cards of hearts. There are $3 - x - y$ remaining that have to be chosen from the 26 cards that not clubs and not hearts. Hence there are $\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}$ ways to choose the 3 cards so that x of them are clubs and y of them are hearts. We conclude that the joint PMF of X and Y is given by

$$p(x,y) = \mathbb{P}(X = x, Y = y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}}{\binom{52}{3}} \text{ for } x, y = 0, 1, 2, 3,$$

and $p(x,y) = 0$ otherwise. The following table shows the values of the joint PMF.

$x \setminus y$	0	1	2	3
0	$\frac{200}{1700}$	$\frac{325}{1700}$	$\frac{156}{1700}$	$\frac{22}{1700}$
1	$\frac{325}{1700}$	$\frac{338}{1700}$	$\frac{78}{1700}$	0
2	$\frac{156}{1700}$	$\frac{78}{1700}$	0	0
3	$\frac{22}{1700}$	0	0	0

(b) We could compute the marginal PMFs of X and Y by summing over the rows and columns of the above table, respectively. However, we alternatively can find the marginal PMFs directly as follows.

$$p_X(x) = P(X = x) = \frac{\binom{13}{x}\binom{39}{3-x}}{\binom{52}{3}},$$
$$p_Y(y) = P(Y = y) = \frac{\binom{13}{y}\binom{39}{3-y}}{\binom{52}{3}}$$

for $0 \leq x \leq 3$ and $0 \leq y \leq 3$ and $p_X(x) = 0$ and $p_Y(y) = 0$ otherwise. For example,

$$p_X(0) = \frac{\binom{13}{0}\binom{39}{3}}{\binom{52}{3}} = \frac{703}{1700},$$

which coincides with the sum of the PMF values in the second row of the table.
(c) Intuitively, X and Y cannot be independent, since information on X also gives information on Y. For instance, if $X = 3$, then all 3 cards are clubs and thus $Y = 0$. For a rigorous argument, we compute, for instance,

$$p(3,1) = 0,$$
$$p_X(3) = \frac{22}{1700},$$
$$p_Y(1) = 741/1700.$$

Hence $p(3,1) \neq p_X(3)p_Y(1)$ which proves that X and Y are not independent. $\square$

Problem 6 (Joint PMF, CDF, Marginal PMF) There are eight similar chips in a bowl: three marked $(0,0)$, two marked $(1,0)$, two marked $(0,1)$, and one marked $(1,1)$. A player selects a chip at random and is given the sum of the two coordinates in dollars. What is the expected payoff?

Solution Let X and Y represent the x- and y-coordinate of the marking on the chips. The joint PMF is given by

$x \setminus y$	0	1
0	$\frac{3}{8}$	$\frac{2}{8}$
1	$\frac{2}{8}$	$\frac{1}{8}$

The payoff is $X+Y$. So the expected payoff is

$$\begin{aligned}\mathbb{E}[X+Y] &= \sum_{x=0}^{1}\sum_{y=0}^{1}(x+y)p(x,y) \\ &= (0+0)\frac{3}{8} + (0+1)\frac{2}{8} + (1+0)\frac{2}{8} + (1+1)\frac{1}{8} \\ &= \frac{10}{8} = 0.75.\end{aligned}$$

The expected payoff is \$0.75. □

Answer Keys. **1(a)** $M_X(t) = \frac{1}{2}e^{-t} + \frac{1}{4} + \frac{1}{4}e^{t}$, $-1/4$, $3/4$. **1(b)** $M_X(t) = \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2}$ **1(c)** $X \sim Geom(1/2)$. **3** $X+Y$ is Poisson with mean $\lambda_1 + \lambda + 2$. **4(a)** $\frac{4x+10}{32}$ **4(b)** $\frac{3+2y}{32}$ **4(c)** 3/32 **4(d)** 9/32 **4(e)** 6/32 **4(f)** 8/32 **4(g)** Dependent. **5(a)** $p(x,y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}}{\binom{52}{3}}$ **5(b)** $p_X(x) = \frac{\binom{13}{x}\binom{39}{3-x}}{\binom{52}{3}}$, $p_Y(y) = \frac{\binom{13}{y}\binom{39}{3-y}}{\binom{52}{3}}$ **5(c)** Dependent **6** \$0.75.