AY 23/24 MH1820 Midterm Test – SOLUTION

Q1. Answer: $\dfrac{8!}{2!2!}$.
The letter S and I appear 2 times, and all other letters appear exactly once. So the number of permutations is $\dfrac{8!}{2!2!}$.

Q2. Answer: 36
The total number of choices without any restriction is $\binom{8}{5} = 56$. The number of choices where X and Y attend together is $\binom{6}{3} = 20$. So the number of choices where X and Y will not attend together is $56 - 20 = 36$.

Q3. Answer: $\dfrac{4}{5}$.
Let (a, b) denote the numbers obtained on dice A and B respectively. The total number of outcomes where $a \neq b$ is $6 \times 5 = 30$. The outcomes where $a \neq b$ and both a and b are odd are $(1, 3)$, $(3, 1)$, $(1, 5)$, $(5, 1)$, $(3, 5)$, $(5, 3)$. So the conditional probability that at least one of the dice is even given that the two dice landed on different numbers is $1 - \frac{6}{30} = 1 - \frac{1}{5} = \frac{4}{5}$.

Q4. Answer: $1 - \displaystyle\sum_{x=0}^{9} e^{-6}\frac{6^x}{x!}$.
Let X be the number of hits per 2 minutes. Then $X \sim Poisson(\lambda = \frac{180}{60} \times 2 = 6)$. So $\mathbb{P}(X \geq 10) = 1 - \mathbb{P}(X \leq 9) = 1 - \sum_{x=0}^{9} e^{-6}\frac{6^x}{x!}$.

Q5. Answer: 14
Recall that $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. So $5 = \mathbb{E}[X^2] - 1^2 \implies \mathbb{E}[X^2] = 6$. So $\mathbb{E}[(2 + X)^2] = \mathbb{E}[4 + 4X + X^2] = 4 + 4\mathbb{E}[X] + \mathbb{E}[X^2] = 4 + 4(1) = 6 = 14$.

Q6. (a) Given a bet, the number X of times it appears on the dice follows a binomial distribution with $n = 3$, $p = 1/6$. Thus $\mathbb{P}(X = 2) = \binom{3}{2}(1/6)^2(5/6) = 15/216 = 5/72 = 0.0694$. [3 marks]
(b) Let W be the winning. Then

X	0	1	2	3
W	-2	2	4	6
$\mathbb{P}(W) = \mathbb{P}(X)$	$\binom{3}{0}(5/6)^3 = \frac{125}{216}$	$\binom{3}{1}(1/6)(5/6)^2 = \frac{75}{216}$	$\binom{3}{2}(1/6)^2(5/6) = \frac{15}{216}$	$\binom{3}{3}(1/6)^3 = \frac{1}{216}$

The expected winning is $\mathbb{E}[W] = (-2)(5/6)^3 + (2)(3/6)(5/6)^2 + (4)(3/6^2)(5/6) + (6)1/(6^3) = -34/6^3 = -17/108 = -\$0.1574$. [3 marks]

Q7. (a) For $0 \leq x \leq 1$, $F(x) = \int_0^x 3t^2\,dt = [t^3]_0^x = x^3$. For $x < 0$, $F(x) = 0$; and for $x > 1$, $F(x) = 1$. [3 marks]
(b) For $1 \leq y \leq e$ (i.e. $0 \leq \ln y \leq 1$), the CDF of Y is $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(e^X \leq y) = \mathbb{P}(X \leq \ln y) = F(\ln y) = (\ln y)^3$ (by part (a)). For $y < 1$, $F_Y(y) = \mathbb{P}(e^X \leq y) = 0$; and for $y > e$ i.,e. $\ln y > 1$, $F_Y(y) = F(\ln y) = 1$. Differentiating $F_Y(y)$ with respect to y, we obtain the PDF of Y given by $f_Y(y) = \frac{3}{y}(\ln y)^2$ if $1 \leq y \leq e$; $f_Y(y) = 0$ otherwise. [3 marks]

Q8. (a) Let A (R respectively) be the event that the student is accepted (rejected) respectively. Let M, T, W be the event that the student receives mail on Mon, Tue, Wed respectively. It is given that $\mathbb{P}(A) = 0.6$ (so $\mathbb{P}(R) = 0.4$).

$$\mathbb{P}(M) = \mathbb{P}(M|A)\mathbb{P}(A) + \mathbb{P}(M|R)\mathbb{P}(R) = 0.15(0.6) + 0.05(0.4) = 0.11. \quad \text{[2 marks]}$$

(b) Similar to part (a), we have $\mathbb{P}(T) = 0.20(0.6)+0.10(0.4) = 0.16$, $\mathbb{P}(W) = 0.25(0.6)+0.10(0.4) = 0.19$. By Bayes Theorem,

$$\mathbb{P}(T|\overline{M}) = \frac{\mathbb{P}(\overline{M}|T)\mathbb{P}(T)}{\mathbb{P}(\overline{M})} = \frac{1 \cdot 0.16}{1 - 0.11} = \frac{0.16}{0.89} = \frac{16}{89} = 0.1798. \quad \text{[3 marks]}$$

(c) By Bayes Theorem,

$$\begin{aligned}
\mathbb{P}(A|\overline{M \cup T \cup W}) &= \frac{\mathbb{P}(\overline{M \cup T \cup W}|A)\mathbb{P}(A)}{\mathbb{P}(\overline{M \cup T \cup W})} = \frac{(1 - \mathbb{P}(M \cup T \cup W|A))\mathbb{P}(A)}{1 - \mathbb{P}(M \cup T \cup W)} \\
&= \frac{(1 - (0.15 + 0.20 + 0.25))(0.6)}{1 - (0.11 + 0.16 + 0.19)} = \frac{24}{54} = \frac{12}{27} = \frac{4}{9} = 0.4444. \quad \text{[3 marks]}
\end{aligned}$$