For graders only:	Question	1(a)	1(b)	2(a)	2(b)	2(c)	3(a)	3(b)	3(c)	Total
	Marks									

MIDTERM II (CA2)

MH1812 – Discrete Mathematics

April 2024 TIME ALLOWED: 50 minutes

Name:

Matric. no.: Tutor group:

INSTRUCTIONS TO CANDIDATES

1. **DO NOT TURN OVER PAPER UNTIL INSTRUCTED.**
2. This midterm paper contains **THREE (3)** questions.
3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
4. Read the question carefully to see how to write your answers.
5. Clearly indicate your answers. Unclear or ambiguous answers will receive **zero marks**.
6. For questions that require you to **circle** to indicate your answer, the choice that you circle will be interpreted as your answer.
7. This **IS NOT** an **OPEN BOOK** exam.
8. Calculators are allowed.

QUESTION 1. **(10 marks)**

(a) Consider the recurrence relation $a_n = 13a_{n-1} - 42a_{n-2}$ for $n \geqslant 2$, with initial conditions $a_0 = 2$, $a_1 = 11$.

(i) [1 mark] $a_7 = \boxed{16265}$

(ii) [4 marks] We can write $a_n = us_1^n + vs_2^n$ where $s_1 > s_2$. Complete the table:

u	v	s_1	s_2
-1	3	7	6

First, s_1 and s_2 are roots of the equation: $x^2 - 13x + 42 = (x-6)(x-7) = 0$. Hence, $s_1 = 7$ and $s_2 = 6$. Furthermore, $a_n = us_1^n + vs_2^n$. Plug in the initial conditions: $a_0 = 2 = u + v$ and $a_1 = 11 = 7u + 6v$. It follows that $u = -1$ and $v = 3$.

[mark distribution: 1 mark for each correct number. Maximum 3.5 marks if $s_1 < s_2$. E.g., award 3.5 marks for solutions with u and v swapped and s_1 and s_2 swapped.]

(b) Use induction to show that, for each $n \in \mathbb{N} - \{0\}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Denote the above predicate by $P(n)$.

(i) [2 marks] Base case. Show that $P(1)$ is true:
The LHS of $P(1)$ is $1^3 = 1$. The RHS of $P(1)$ is $1^2(1+1)^2/4 = 1$.

(ii) [3 marks] Inductive step. Show that $P(k) \rightarrow P(k+1)$:
Assume $P(k)$, i.e., $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$.
Now we start with the LHS of $P(k+1)$:

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= (1^3 + 2^3 + 3^3 + \cdots + k^3) + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3, \end{aligned}$$

using $P(k)$. It remains to show that $\frac{k^2(k+1)^2}{4} + (k+1)^3$ is equal to the RHS of $P(k+1)$. We have

$$\begin{aligned} \frac{k^2(k+1)^2}{4} + (k+1)^3 &= \frac{k^2(k+1)^2 + 4(k+1)^2(k+1)}{4} \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \text{RHS of } P(k+1). \end{aligned}$$

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QUESTION 2. **(9 marks)**

In this question **no justification is required**. For each part, **give an explicit number** as your answer, not an expression.

(a) Find the number of

(i) [1 mark] distinguishable permutations of the word "BANANA": $\boxed{60}$

$\frac{6!}{1!2!3!} = 60$

(ii) [1 mark] elements in the power set of $\{a, b, c, d, e, f\}$: $\boxed{64}$

$2^6 = 64$

(iii) [1 mark] subsets of $\{1, 2, 3, 4\}$ that have cardinality at most 2: $\boxed{11}$

$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} = 11$

[1/2 a mark for 10 (assuming the empty set was missed).]

(b) Consider all ternary strings of length 6. E.g., 012102.

(i) [1 mark] How many are there in total? $\boxed{729}$

$3^6 = 729$

(ii) [1 mark] How many contain at least five '0's? $\boxed{13}$

$1 + 2(6) = 13$

(iii) [1 mark] How many begin with a '1' or a '2'? $\boxed{486}$

$3^5 + 3^5 = 286$

(c) Consider all distinguishable permutations of the digits of 123213.

(i) [1 mark] How many are divisible by 3? $\boxed{90}$

All permutations are divisible by 3 (the sum of their digits is a multiple of 3) $\frac{6!}{2!2!2!} = 90$

[This question has been made a bonus question. You still get the mark if you answered it correctly. But the maximum score of the test is now 28, instead of 29.]

(ii) [1 mark] How many are odd? $\boxed{60}$

$90 - \frac{5!}{2!2!1!} = 60$

(iii) [1 mark] How many are less than 200000? $\boxed{30}$

$\frac{5!}{2!2!1!} = 30$

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QUESTION 3. **(10 marks)**

For this question, recall that $P(A)$ denotes the power set of the set A.

(a) Let A, B, and C be sets. Determine the truth value of the following statements. (**Circle** "T" or "F" to indicate your answer.)

(i) [1 mark] If $A \subseteq B$ and $B \subseteq C$ then $A \in P(C)$. $\boxed{\text{T}}$

(ii) [1 mark] If $A \cap B = C$ then $C \in P(A) \cap P(B)$. $\boxed{\text{T}}$

(iii) [1 mark] If $A \subseteq B \cup C$ then $A \in P(B) \cup P(C)$. $\boxed{\text{F}}$

No justification is required.

(b) Consider the sets

$A = \{a, b\}$, $B = \{b, c\}$, and $C = \{a, c\}$.

Write out the elements of each of the following sets.

(i) [2 marks] $((A \cup B) - C) \cup (A \cap B)$: $\boxed{b}$

(ii) [2 marks] $P(B \cup C) \cap P(A \cap B)$: $\boxed{\emptyset, \{b\}}$

[mark distribution: for 2 marks, must write $\{b\}$, note that b is not an element of $P(B \cup C) \cap P(A \cap B)$.]

No justification is required.

(c) [3 marks] Let A, B, and C be sets. Show that $(A - B) \times C \subseteq (A \times C) - (B \times C)$:

Let $x \in (A - B) \times C$. Then $x = (x_1, x_2)$ where $x_1 \in A - B$ and $x_2 \in C$. Hence, $x_1 \in A$ and $x_1 \notin B$. Which implies, $x \in A \times C$ and $x \notin B \times C$. Therefore $x \in (A \times C) - (B \times C)$.

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