Discrete Mathematics
MH1812

Topic 9 - Functions
Summary

Introduction to Functions: Definition

$f(x)$

Let X and Y be sets. A **function** f from X to Y is a rule that assigns every element x of X to a unique y in Y. We write $f: X \to Y$ and $f(x) = y$.

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

$y = f(x)$

the image of x

X =	Domain
Y =	Codomain
y = the	**Image** of x under f
x = a	**Preimage** of y under f
Range =	Subset of Y with preimages

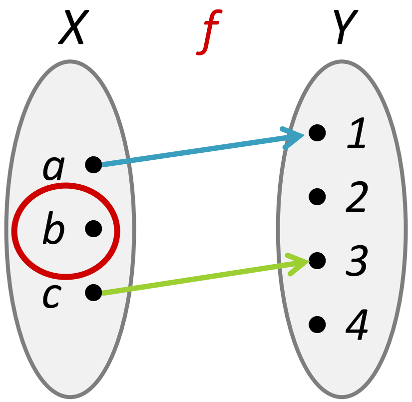
Introduction to Functions: Functions vs. Non-functions

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

≥ 1 arrow out of each element in X

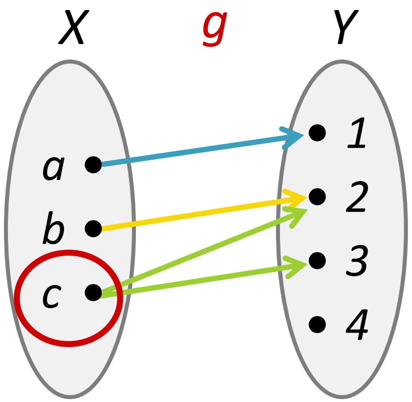
≤ 1 arrow out of each element in X

X = {*a*,*b*,*c*} **to** *Y* = {*1*,*2*,*3*,*4*}



No!

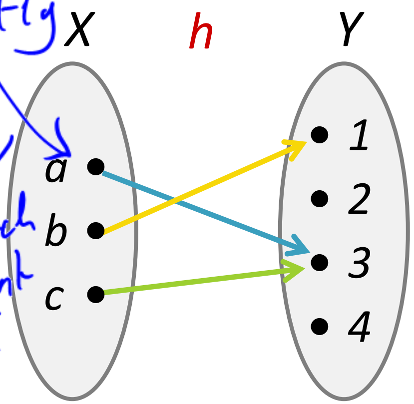
(*b* has no image)



No!

(*c* has two images)

exactly 1 arrow for each element in X



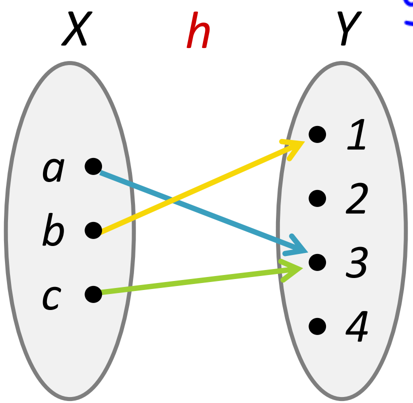
Yes!

(Each element of *X* has exactly one image)

Introduction to Functions: Functions vs. Non-functions

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \land (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

X = {*a*,*b*,*c*} **to** *Y* = {*1*,*2*,*3*,*4*}



the Preimage of 3?

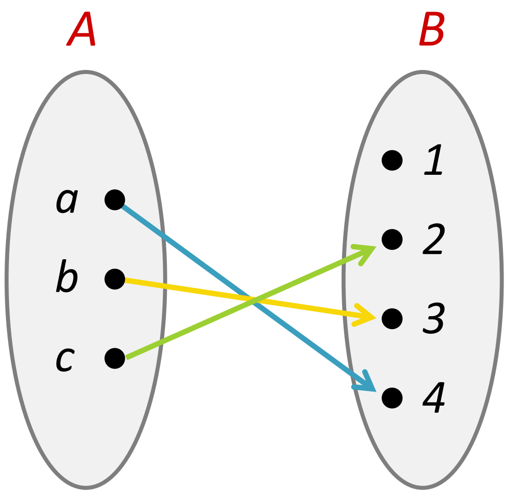
$3 = h(a)$ & $3 = h(c)$

def^n the Preimage of $y \in Y$

$= \{ x \in X \mid y = f(x) \}$

Preimage of $3 = \{a, c\}$

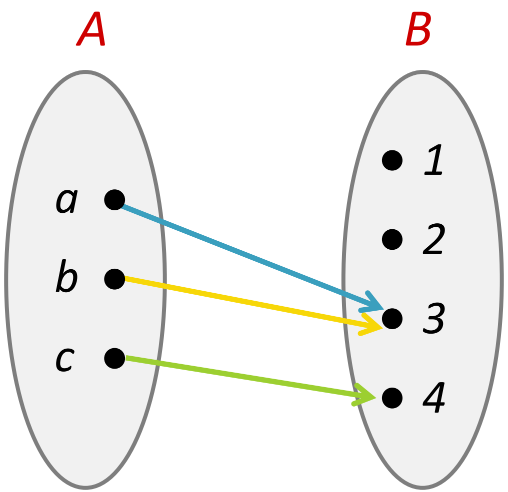
Injectivity: One-to-one Example



One-to-one

(All elements in A have a different image)

$\forall$ element in Codomain has ≤ 1 arrow



Not one-to-one

(a and b have the same image)

Example

$$f: \mathbb{R} - \{0\} \to \mathbb{R},\ f(x) = \frac{x+1}{x}$$

Is f injective? Yes

WTS: $x_1 = x_2$

Suppose $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1+1}{x_1} = \frac{x_2+1}{x_2}$$

$$\Rightarrow \cancel{x_2 x_1} + x_2 = \cancel{x_1 x_2} + x_1$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is injective

Example

Note $g: \mathbb{Z} \to \mathbb{R}$, $g(x) = \frac{x}{x^2+1}$ is injective

$$f: \mathbb{R} \to \mathbb{R},\ f(x) = \frac{x}{x^2+1}$$

Is f injective? No

$\exists\ x_1 \neq x_2$ s.t.

$f(x_1) = f(x_2)$

Suppose $f(x_1) = f(x_2)$

$$\frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1(x_2^2+1) = x_2(x_1^2+1)$$

$$\Rightarrow x_1 x_2^2 - x_2 x_1^2 + x_1 - x_2 = 0$$

$$\Rightarrow (x_1 - x_2)(1 - x_1 x_2) = 0$$

$(x_1 - x_2)\, x_1 x_2$

$= x_1^2 x_2 - x_1 x_2^2$

$\Rightarrow x_1 = x_2$ or $x_1 x_2 = 1$

To find $x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$,

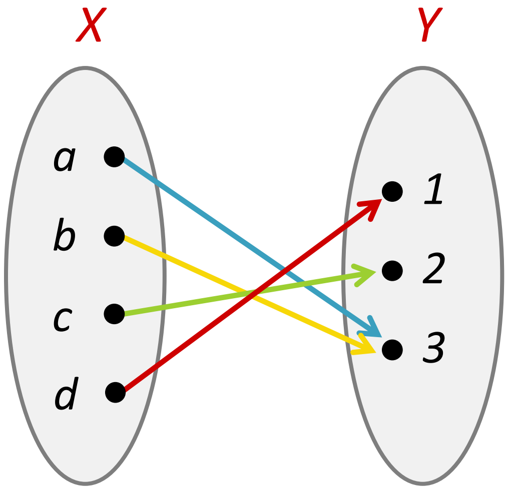
fix x_1 and let $x_2 = \frac{1}{x_1}$

Try $x_1 = 1$ and $x_2 = \frac{1}{1} = 1$ ✗

Try $x_1 = 2$ and $x_2 = \frac{1}{2}$

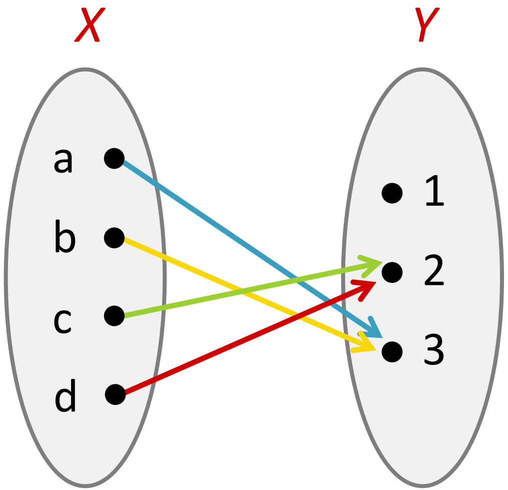
$f(2) = \frac{2}{4+1} = \frac{2}{5}$, $f(\frac{1}{2}) = \frac{\frac{1}{2}^{\times 4}}{\frac{1}{4}+1^{\times 4}} = \frac{2}{5}$ ✓

Surjectivity: Onto Example



Onto

(All elements in *Y* have a preimage)



Not onto

(*1* has no preimage)

Example

$$f: \mathbb{R} - \{0\} \to \mathbb{R} - \{1\},\ f(x) = \frac{x+1}{x}$$

Is f surjective?

Does $4 \in \mathbb{R} - \{1\}$ have a preimage? Yes $f(\frac{1}{3}) = 4$

$$4 = \frac{x+1}{x} \Rightarrow 4x = x+1 \Rightarrow 3x = 1 \Rightarrow x = \frac{1}{3}$$

WTS: $\forall y \in \mathbb{R} - \{1\},\ \exists x \in \mathbb{R} - \{0\},\ y = f(x)$ onto

Take $y \in \mathbb{R} - \{1\}$

Suppose $y = f(x) \Rightarrow y = \frac{x+1}{x}$

$$\Rightarrow xy = x+1$$

$$\Rightarrow xy - x = 1$$

$$\Rightarrow x(y-1) = 1$$

$$\Rightarrow x = \frac{1}{y-1}$$

$\therefore$ y has premage $\frac{1}{y-1}$

$\therefore$ f is surjective.

Example

$f: \mathbb{R} \to \mathbb{R}, f(x) = \frac{x}{x^2+1}$

Is f surjective?

No

Does 0 have a preimage?

$0 = \frac{x}{x^2+1} \Rightarrow x = 0$ ✓

Does 1 have a preimage?

Suppose $f(x) = 1$

$\Rightarrow \frac{x}{x^2+1} = 1$

$$\Rightarrow x = x^2 + 1$$

$$\Rightarrow x^2 - x + 1 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1-4}}{2} \in \mathbb{R} ? \text{ No}$$

not real!

$\therefore$ 1 does not have a preimage

Example

$$f, g: \mathbb{R} \to \mathbb{R},\ f(x) = x + 3; \qquad g(x) = -x^3$$

Find $f^{-1}, g^{-1},\ g \circ f$, and $f \circ g$.

f^{-1} suppose $y = f(x)$

$= x + 3$

$\Rightarrow x = y - 3 = f^{-1}(y)$

g^{-1} $y = g(x) = -x^3$

$\Rightarrow x = \sqrt[3]{-y} = g^{-1}(y)$

$g \circ f$

$$g \circ f(x) = g(f(x)) = g(x+3) = -(x+3)^3$$

$f \circ g$

$$f \circ g(x) = f(g(x)) = f(-x^3) = -x^3 + 3$$

Characteristic Equation: Example

1011011 ✓

101001 ✗

Determine the number of bit strings (i.e., comprising *0*s and *1*s) of length *n* that contains **no adjacent *0*s**.

Suppose string ends in a 1

$\underbrace{\_\ \_\ \_\ \_\ \cdots\ \_}_{n-1}\ 1$

C_{n-1} of these

- C_n = the number of such bit strings
- A binary string with no adjacent *0*s is constructed by:
  - Adding "*1*" to any string *w* of length $n - 1$ satisfying the condition, or
  - Adding "*10*" to any string *v* of length $n - 2$ satisfying the condition
- Thus $C_n = C_{n-1} + C_{n-2}$ where $C_1 = 2$ (*0,1*), $C_2 = 3$ (*01, 10, 11*)

Suppose string ends in a 0

$$\underbrace{\_\ \_\ \_\ \cdots\ \_}_{n-2}\ 1\ 0$$

must be 1

c_{n-2} of these