Quantum Mechanics Study Group Math Primer

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1. Sets
   (a) equivalence, subsets
   (b) functions
   (c) open and closed sets
   (d) metric spaces
   (e) balls and spheres
   (f) external measure
   (g) continuity and sequential characterization of the limit
   (h) measurable functions
2. Complex Numbers
   (a) the imaginary unit
      i. the complex plane
      ii. powers of i
      iii. roots of unity
   (b) modulus and argument
      i. complex conjugate
      ii. De Moivre's Theorem
      iii. polar representation
      iv. trigonometric identities
   (c) Complex functions of a single real variable
   (d) Complex functions of a single complex variable
      i. analytic functions
      ii. homogenous polynomials, $p(\alpha z) = \alpha^n p(z)$
   (e) Exponential Functions of a complex argument
   (f) Consider the complex-valued function $f(z) = \dfrac{az+b}{cz+d}$. For some $a,\ b,\ c,\ d \in \mathbb{C}$ find the fixed points $\{z_o | f(z_o) = z_o\}$.
3. Linear Algebra
   (a) Matrix Definition
      i. arithmetic
      ii. norm
   (b) Examples and exercises with commutative and non-commutative groups of matrices.
   (c) Matrix representation of complex numbers
      i. isomorphism of the two

ii. determinant of a matrix representation of a complex number

iii. exponential of a matrix representation of a complex number

(d) Action of $M_{n\times n}$ on $M_{n\times m}$ for any M.

example with eigenvector, confirm result is parallel

(e) Vectors are $M_{n\times 1}$.

action of gamma matrices on $\mathbb{C}_{2\times 1}$

prove that every linear transformation is a composition of a scaling and a rotation, which commute.

(f) eigenvectors

Diagonal matrix

(g) vector spaces

basis

othonormal basis

(h) inner product and dual space

inner product

linear independence and orthogonality

angle

example of matrix multiplication of vector, find angle between

example of an inner product on a vector space

dual space is a vector space

Dual space of $M_{n\times 1}$ is matrix product with some $\mathbf{v} \in M_{1\times n}$.

prove the Pauli vector identity $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$ where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

(i) Linearity

(j) Functions of a Complex Variable

(k) Exponential Functions

(l) Eigenvectors and Eigenvalues

(m) Hermitian Matrices

definition

spectral theorem

eigenvectors form an orthonormal basis

Example of a Hermitian matrix

Show it's Hermitian

Express a vector in that basis

Two Hermitian matrices, find a matrix T that changes the basis from one to the other.

Pauli Matrices a basis for 2 x 2 Hermitian Matrices

Prove that the span of the Pauli Matrices is a vector space (subspace of SL(2)?)

Prove that a linear combination of the Pauli Matrices is Hermitian

Hermitian operators on a Hilbert space

quaternions isomorphic to $span(i\sigma_1, i\sigma_2, i\sigma_3)$ as (vector space?).

(n) Unitary Matrices
   i. The Hermitian conjugate is the inverse
   ii. Eigenvectors of different eigenvalues are orthogonal
   iii. Eigenvectors span the space
   iv. T

4. Differential Calculus
   (a) the derivative
   (b) chain and product rules
   (c) Taylor's Theorem
   (d) Convergence in the complex plane
   (e) The complex derivative
   (f) curl and divergence of scalar fields

5. Integral Calculus
   (a) Definition and Riemann sums
   (b) The mean value theorem
   (c) Integration by Substitution
   (d) Integration by Parts
   (e) Integration in series form
   (f) distributions, the dirac delta function

6. Algebraic objects
   (a) groups
      i. roots of unity

   ii. rotations and
   iii. translations of the plane
   (b) commutativity and associativity
   (c) rings and fields

7. Abstract vector spaces
   (a) complex numbers
   (b) square integrable functions
   (c) solutions of LDE
   (d) separable vector spaces

8. Integral Calculus of Probability Theory
   (a) discrete and continuous random variables
   (b) PDFs and CDFs
   (c) Expected values and variance
   (d) independence
   (e) For each of the following functions
      i. normalize.
      ii. find the expected value and the standard deviation
      iii. prove that f is a PDF (positive, sums to 1, subadditive)

9. Differential Equations
   (a) definition
   (b) solutions
   (c) linear differential equations
   (d) separation of variables
   (e) solution of a system of linear differential equations
   (f) series solutions of LDEs

10. Partial Derivatives and Partial Differential Equations
   (a) definition
   (b) solutions
   (c) gradient
   (d) linearity
   (e) example solution of the wave equation
   (f) potential functions
   (g) harmonic polynomials $\Delta p = \sum_{i=1}^{3} \frac{\partial f(x)}{\partial x_i} = 0$