Discrete Proper Time and the Tempo Lapse: Gravitational Dynamics from the Core Axioms

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Abstract

Starting from the Core Axioms as formulated in Fujiyama (2025) [1]—discrete proper time, unitary one-step (single-step) evolution on a principal band, and band-limited kinematics—this note shows that the same axioms furnish a discrete-time, constraint-driven formulation that *retranslates* general relativity (GR) in that it closes smoothly at the discrete level and returns to standard GR in the $\tau \to 0$ limit. Nothing beyond those axioms is added, and no alternative dynamics are posited. Time is sampled at $t = n\tau$, dynamics proceed via the unitary one-step map U on a Hilbert space as organised in [1], and the Cayley–exponential correspondence yields a bounded discrete generator with a quantitative continuum limit. Within this kinematic backbone, a discrete variational (implicit midpoint) scheme appears that is symplectic, constraint-preserving, and consistent with $U = \exp(-i\tau K(H))$ on band-limited windows. A constructive general-solution theorem with $O(\tau^2)$ convergence is obtained on finite intervals. Cosmological backgrounds with linear perturbations also fit into the same band-limited framework, with leading $O((\tau\omega)^2)$ corrections vanishing as $\tau \to 0$. The result is a step-by-step *retranslation* back to GR derived from the stipulated minimal axioms.

1 Core Axioms and Kinematic Starting Point

The starting point follows exactly the Core Axioms as laid out in [1]:

- **Discrete proper time.** $t = n\tau$ with step $\tau > 0$.
- **Unitary one-step evolution (principal band).** A unitary map U on a Hilbert space induces Heisenberg-type updates $A_n = U^{-n} A U^n$ for bounded observables A; the principal band structure is as organised in [1].
- **Cayley–exponential correspondence.**

$$U = \exp(-i\tau K(H)), \qquad K(H) = \frac{2}{\tau}\arctan\Big(\frac{\tau}{2}H\Big), \tag{1.1}$$

  with $K(H)$ self-adjoint and $\operatorname{spec} K \subset (-\pi/\tau, \pi/\tau]$, yielding on bounded windows

$$K(H) = H - \frac{\tau^2}{12}H^3 + O(\tau^4). \tag{1.2}$$

- **Band-limit (Nyquist) scale.** The maximal angular frequency in coordinate time satisfies $\omega_{\max} = \pi/\tau$; locally in proper time, $\omega_{\max}(x) = \pi/(\Theta(x)\tau)$ once a tempo lapse Θ is present.

All subsequent statements below are consequences of these axioms; no additional assumptions are introduced.

2 Tempo Lapse as Lagrange-Multiplier Density (Terminology from [1])

In a time-orthogonal gauge ($g_{ti} = 0$),

$$ds^2 = -c^2\,\Theta(t,\mathbf{x})^2\,dt^2 + \gamma_{ij}(t,\mathbf{x})\,dx^i dx^j, \qquad d\tau_{\text{prop}} = \Theta\,dt. \tag{2.1}$$

The descriptive term *tempo lapse* (tempo field) follows [1] and refers to a positive lapse-density–type Lagrange multiplier. It enters linearly to impose the Hamiltonian constraint, thereby fixing proper-time sampling; importantly, it does not introduce any propagating degrees of freedom.[1]

3 Discrete Variational Principle (Implicit Midpoint) and Updates

Let (γ_{ij}, π^{ij}) be the spatial metric and its canonical momentum; (ψ, π_ψ) the matter fields and their momenta. At $t_n = n\tau$, define the midpoint discrete action

$$\begin{aligned} S_d = \sum_n \int d^3x\, \Big[&\pi^{ij}_{n+\frac{1}{2}}(\gamma_{ij,n+1} - \gamma_{ij,n}) + \pi_{\psi,n+\frac{1}{2}}(\psi_{n+1} - \psi_n) \\ &- \tau\,\Theta_{n+\frac{1}{2}}\,\mathscr{H}\big(\gamma_{ij,n+\frac{1}{2}}, \pi^{ij}_{n+\frac{1}{2}};\ \psi_{n+\frac{1}{2}}, \pi_{\psi,n+\frac{1}{2}}\big)\Big], \end{aligned} \tag{3.1}$$

with $X_{n+\frac{1}{2}} = \frac{1}{2}(X_{n+1} + X_n)$. Variation yields implicit midpoint updates:

$$\frac{\gamma_{ij,n+1} - \gamma_{ij,n}}{\tau} = \Theta_{n+\frac{1}{2}} \frac{\partial \mathscr{H}}{\partial \pi^{ij}}\Big|_{n+\frac{1}{2}}, \qquad \frac{\pi^{ij}_{n+1} - \pi^{ij}_n}{\tau} = -\,\Theta_{n+\frac{1}{2}} \frac{\partial \mathscr{H}}{\partial \gamma_{ij}}\Big|_{n+\frac{1}{2}}, \tag{3.2}$$

$$\frac{\psi_{n+1} - \psi_n}{\tau} = \Theta_{n+\frac{1}{2}} \frac{\partial \mathscr{H}}{\partial \pi_\psi}\Big|_{n+\frac{1}{2}}, \qquad \frac{\pi_{\psi,n+1} - \pi_{\psi,n}}{\tau} = -\,\Theta_{n+\frac{1}{2}} \frac{\partial \mathscr{H}}{\partial \psi}\Big|_{n+\frac{1}{2}}, \tag{3.3}$$

and the per-step Hamiltonian constraint

$$\boxed{\mathscr{H}\big(\gamma_{ij,n+\frac{1}{2}}, \pi^{ij}_{n+\frac{1}{2}};\ \psi_{n+\frac{1}{2}}, \pi_{\psi,n+\frac{1}{2}}\big) = 0} \qquad \Big(\delta S_d/\delta\Theta_{n+\frac{1}{2}} = 0\Big). \tag{3.4}$$

Because (3.1) is variational, the midpoint update scheme is symplectic.

4 Constraint Propagation (Discrete Noether Identity)

A standard discrete Noether argument yields:

Lemma 4.1 (Propagation up to truncation order)**.** *If* (3.4) *holds at* $n + \frac{1}{2}$ *and* (3.2)–(3.3) *are satisfied, then*

$$\mathscr{H}\big(\gamma_{ij,n+\frac{3}{2}}, \pi^{ij}_{n+\frac{3}{2}};\ \psi_{n+\frac{3}{2}}, \pi_{\psi,n+\frac{3}{2}}\big) = \mathscr{H}\big(\gamma_{ij,n+\frac{1}{2}}, \pi^{ij}_{n+\frac{1}{2}};\ \psi_{n+\frac{1}{2}}, \pi_{\psi,n+\frac{1}{2}}\big) + O(\tau^3) = 0.$$

[1] ADM form $ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$ with $N^i = 0$ in the time-orthogonal gauge gives $N = c\,\Theta$.

5 Constructive General Solution (Existence, Uniqueness, Convergence)

Theorem 5.1 (Consequences of the axioms)**.** *Assume: (i) initial data* $(\gamma_{ij}, \pi^{ij};\ \psi, \pi_\psi)|_{t_0}$ *satisfy the continuum constraints and lie in a Sobolev class where* $\mathscr{H}$ *is* C^1*-Lipschitz on bounded sets; (ii) relevant spectra remain in a bounded window so that (1.1)–(1.2) apply; (iii) physical modes obey* $\omega \ll \omega_{\max} = \pi/(\Theta\tau)$*. Then there exists* $\tau_0 > 0$ *such that for* $\tau < \tau_0$ *the midpoint system (3.2)–(3.4) admits a unique solution at each step on finite intervals, propagates the constraint in the discrete Noether sense, and satisfies a global error bound* $\|\mathbf{X}^{(\tau)}(t_n) - \mathbf{X}(t_n)\| \le C\,\tau^2$*. Hence* $\mathbf{X}^{(\tau)} \to \mathbf{X}$ *(the corresponding continuum GR solution) as* $\tau \to 0$*.*

Sketch. The implicit midpoint map

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \tau\,\Theta_{n+\frac{1}{2}}\, J\,\nabla\mathscr{H}\Big(\frac{\mathbf{X}_{n+1}+\mathbf{X}_n}{2}\Big), \tag{5.1}$$

$$0 = \mathscr{H}\Big(\frac{\mathbf{X}_{n+1}+\mathbf{X}_n}{2}\Big), \tag{5.2}$$

is a contraction for $\tau < \tau_0$ by the C^1-Lipschitz property of $\nabla\mathscr{H}$. Symplecticity follows from discrete variational calculus. The $O(\tau^2)$ global error bound follows from midpoint consistency and Grönwall, using (1.2) on bounded windows and the band-limit assumption to control aliasing.[2] □

6 Strong Closure via Standard Backward-Error Analysis

A standard variational backward-error analysis yields a modified Hamiltonian density

$$\tilde{\mathscr{H}} = \mathscr{H} + \tau^2 \Delta\mathscr{H} + \tau^4 \Delta\mathscr{H}^{(2)} + \cdots, \tag{6.1}$$

whose exact time-τ flow interpolates the midpoint step on the principal window, with $\|\Delta\mathscr{H}\| = O(H^3)$ and $\tilde{\mathscr{H}} \to \mathscr{H}$ as $\tau \to 0$.

Discrete Poisson bracket. Let ω_d be the discrete symplectic two-form induced by (3.1). The discrete Poisson bracket $\{\cdot,\cdot\}_d$ is then defined via ω_d^{-1} on functionals of midpoint data.

Theorem 6.1 (Strong (Dirac) closure)**.** *With* $\tilde{\mathscr{H}}$ *as above and* $\tilde{\mathcal{D}}_i = \mathcal{D}_i + \tau^2\Delta\mathcal{D}_i + \cdots$*, one has*

$$\{\tilde{H}[N],\, \tilde{H}[M]\}_d = \tilde{D}[\gamma^{ij}(N\,\partial_j M - M\,\partial_j N)], \tag{6.2}$$

$$\{\tilde{D}[\vec{N}],\, \tilde{H}[M]\}_d = \tilde{H}[\mathcal{L}_{\vec{N}} M], \tag{6.3}$$

$$\{\tilde{D}[\vec{N}],\, \tilde{D}[\vec{M}]\}_d = \tilde{D}[[\vec{N}, \vec{M}]]\,, \tag{6.4}$$

and $\tilde{\mathscr{H}} \to \mathscr{H}$*,* $\tilde{\mathcal{D}}_i \to \mathcal{D}_i$ *as* $\tau \to 0$*. In other words, the discrete constraint algebra converges to the continuum Dirac algebra.*

Corollary. If $\tilde{\mathscr{H}} = \tilde{\mathcal{D}}_i = 0$ initially, then they vanish exactly at all midpoints. Hence $\mathscr{H}, \mathcal{D}_i = O(\tau^2)$ along the evolution, and both tend to 0 as $\tau \to 0$.

[2] Midpoint is A-stable (not L-stable); accuracy requires $(\Omega\,\Delta t) \ll \pi$ for mode frequency Ω.

7 Discrete Raychaudhuri Inequality and Horizon Area

With proper-time increment $\Delta\tau_{\text{prop}} = \Theta_{n+\frac{1}{2}}\,\tau$,

$$\theta_{n+1} - \theta_n \;\le\; -\,\frac{\Delta\tau_{\text{prop}}}{2}\,\theta_n^2 \;-\; \Delta\tau_{\text{prop}}\,\sigma_n^2 \;-\; 8\pi G\,\Delta\tau_{\text{prop}}\,T_{\mu\nu}k^\mu k^\nu \;+\; O(\tau^3)\,. \tag{7.1}$$

Under the Null Energy Condition (NEC), a future outer trapping horizon satisfies $A_{n+1} \ge A_n - C\,\tau^2$, and hence $A_{n+1} \ge A_n$ as $\tau \to 0$.

8 Cosmological Background and Linear Perturbations

From (2.1), the Friedmann (FRW) equations in coordinate time are

$$\Big(\frac{\dot a}{a\,\Theta}\Big)^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \qquad \frac{1}{\Theta}\frac{d}{dt}\Big(\frac{\dot a}{a\,\Theta}\Big) = -\,4\pi G\Big(\rho + \frac{p}{c^2}\Big) + \frac{k}{a^2}\,. \tag{8.1}$$

In proper time $d\tau_{\text{prop}} = \Theta\,dt$ (gauge $\Theta \equiv 1$), these reduce to the standard Friedmann equations. In conformal time $d\eta = dt/a$, the lapse density rescales as

$$\Theta_\eta := \frac{d\tau_{\text{prop}}}{d\eta} = a\,\Theta, \tag{8.2}$$

so $\Theta_\eta = a$ holds only if $\Theta \equiv 1$. For linear perturbations, scalar modes satisfy $v'' + (k^2 - z''/z)\,v = 0$, with $z = a\,\varphi_0'/\mathcal{H}$. Tensor modes satisfy $u'' + (k^2 - a''/a)\,u = 0$, where $H = a'/a$ is the conformal Hubble parameter and $'$ denotes $d/d\eta$. Midpoint discretization preserves a discrete Wronskian and yields leading spectral corrections

$$\frac{\Delta\mathcal{P}_{S,T}}{\mathcal{P}_{S,T}}(k) = \alpha_{S,T}(k)\,(\tau\,\omega_k)^2 + O\big((\tau\,\omega_k)^4\big), \tag{8.3}$$

with $\alpha_{S,T}(k) = O(1)$ governed by z''/z and a''/a. Thus the relative power-spectrum error vanishes: $\Delta\mathcal{P}/\mathcal{P} \to 0$ as $\tau \to 0$.

9 Compactness and Convergence to Continuum Solutions

Theorem 9.1. *Let $(\gamma_{ij}^{(\tau)}, \pi^{ij,(\tau)};\ \psi^{(\tau)}, \pi_\psi^{(\tau)})$ be discrete solutions of (3.2)–(3.4) from uniformly constraint-compatible initial data, with $\omega \ll \omega_{\max} = \pi/(\Theta\tau)$. On finite proper-time intervals, the sequence is precompact in $H^s \times H^{s-1}$; every limit point as $\tau \to 0$ solves the continuum GR equations with the same initial data (weak/strong depending on s).*

Sketch. Uniform energy estimates stem from symplecticity and the modified Hamiltonian; discrete Aubin–Lions (or compensated compactness) gives precompactness. Consistency at $O(\tau^2)$ and constraint closure imply the continuum PDEs in the limit. □

10 Discussion: Retranslation Posture

The posture here is not to *align* with an external theory by fiat. Rather, the approach starts from the Core Axioms as laid out in [1], and then shows that these axioms *retranslate* GR in a discrete proper-time syntax: closure arises smoothly at the discrete level, and all statements return to standard GR as $\tau \to 0$. Terminology such as “unitary one-step map” and “tempo lapse” comes from the starting axioms and does not constitute new nomenclature.

References

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