1. prove f(n) = log(n!) = O(nlogn) = Ω(nlogn)

$$n\,log\,n = log(n^n) \geq log(n!)$$

Let $c = 1, n_0 = 1$

$$log(n!) \leq 1 \times n\,log\,n$$
$$log(1!) \leq 1 \cdot 1\,log\,1$$
$$0 \leq 0$$

By definition of Big-O, $\log(n!)$ is $O(n\log n)$ since $log(n!) \leq 1 \cdot nlogn$ for all $n \geq 1$.

$$log\,n! = \sum_{i=1}^{n} log\,i \geq \sum_{i=n/2}^{n} log\,i \geq \sum_{i=n/2}^{n} log\frac{n}{2} = \frac{n}{2}log\frac{n}{2} = \frac{n}{2}(log\,n - log\,2)$$

$$log\,n! \geq \frac{1}{2}n\,log\,n - \frac{log\,2}{2}n$$

Let $c = \frac{1}{4}$

We want to prove:

$$log\,n! \geq \frac{1}{2}n\,log\,n - \frac{log\,2}{2}n \geq \frac{1}{4}n\,log\,n$$

Only need to prove:

$$\frac{1}{4}n\,log\,n \geq \frac{log\,2}{2}n$$

Only need to prove:

$$log\,n \geq 2\,log\,2 = log\;4$$

So we can set $n_0 = 4$

By definition of Big-Omega, $\log(n!)$ is $\Omega(n\log n)$ since $log(n!) \geq \frac{1}{4} \cdot nlogn$ for all $n \geq 4$.

2. Calculate time complexity

```
int fun(int n){
    int sum = 0;
    for(int i=1;i<=n;i*=2)
        sum += i*i;
    return i;
}
```

$T(n) = O(logn)$

```
int fun(int n)
    if( n <= 1 )
        return 1;
    return fun( n/2 ) + n*n;
}
```

$\mathrm{T(n) = T(n/2) + O(1)}$
$T(n) = O(\log n)$

3. Analyze the recursion, find out the recurrence relation between cost functions of different input and solve the recurrence relation:

```
int fun(int n) {
        int a,b;
        if(n<=4)
          return 0;
        else {
            a = n/2;
            b = fun(n/4);
            return a+b;
        }
}
```

$$f(x) = \begin{cases} 0, & x \le 4 \\ f(x/4) + x/2, & x > 4 \end{cases}$$

$$T(n) = T(n/4) + O(1)$$

$$T(n) = O(log\, n)$$

4.For each pair of f(n) and g(n) below, decide if f(n) = O(g(n)), f(n) = Ω(g(n)), or f(n) = Θ(g(n)). Justify your answer. Note that more than one of these relations may hold for a given pair; list all correct ones.

(a) $\mathrm{f(n)} = (\log_3 n)^2$ and $\mathrm{g(n)} = log_2(n^3)$.

$$g(n) = 3\log_2 n$$

$$g(n) = \Theta(\log n)$$

$$\lim_{\mathrm{n}\to\infty} \frac{f(n)}{g(n)} = \lim_{\mathrm{n}\to\infty} \frac{\log^2 n}{\log n} = \lim_{\mathrm{n}\to\infty} \log n = \infty$$

So $f(n)$ grows faster than $g(n)$

So $f(n) = \Omega\big(g(n)\big)$

(b) $\mathrm{f(n)} = 2^n$ g(n)=n!

Obviously,

$$\lim_{\mathrm{n}\to\infty} \frac{f(n)}{g(n)} = \lim_{\mathrm{n}\to\infty} \frac{2^n}{\mathrm{n!}} = 0$$

So $f(n) = O\big(g(n)\big)$

5. find out the relationship between the pairs of functions. Determine whether *f* is O(g), Ω(g) or Θ(g).

1. $f(n) = \log\sqrt{n},\ g(n) = \sqrt{\log n}.$

2. $f(n) = 2n^2 - n + 6,\ g(n) = n^3 + 8.$

3. $f(n) = \begin{cases} 10^9 & \text{,if } n \text{ is even,} \\ n & \text{,if } n \text{ is odd.} \end{cases} \quad g(n) = \begin{cases} n & \text{,if } n \geq 1000, \\ n^2 & \text{,if } n < 1000. \end{cases}$

1.

$$f(n) = \log\sqrt{n} = \frac{1}{2}\log n$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{\frac{1}{2}\log n}{\sqrt{\log n}} = \lim_{n\to\infty}\frac{\sqrt{\log n}}{2} = \infty$$

So

$$f(n) = \Omega\left(g(n)\right)$$

2.

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{2n^2 - n + 6}{n^3 + 8} = \lim_{n\to\infty}\frac{4\text{n-}1}{3n^2} = \lim_{n\to\infty}\frac{4}{6\text{n}} = 0$$

So

$$f(n) = \mathrm{O}\left(g(n)\right)$$

3.

Let $c = 1, n_0 = 10^9$

When $n \geq n_0,\ f(n) \leq n,\ \mathrm{g(n)} = \mathrm{n}$

So $f(n) \leq c \cdot g(n)$

By definition of Big-O, $f(n)$ is $O\left(g(n)\right)$ since $f(n) \leq 1 \cdot g(n)$ for all $n \geq 10^9$.