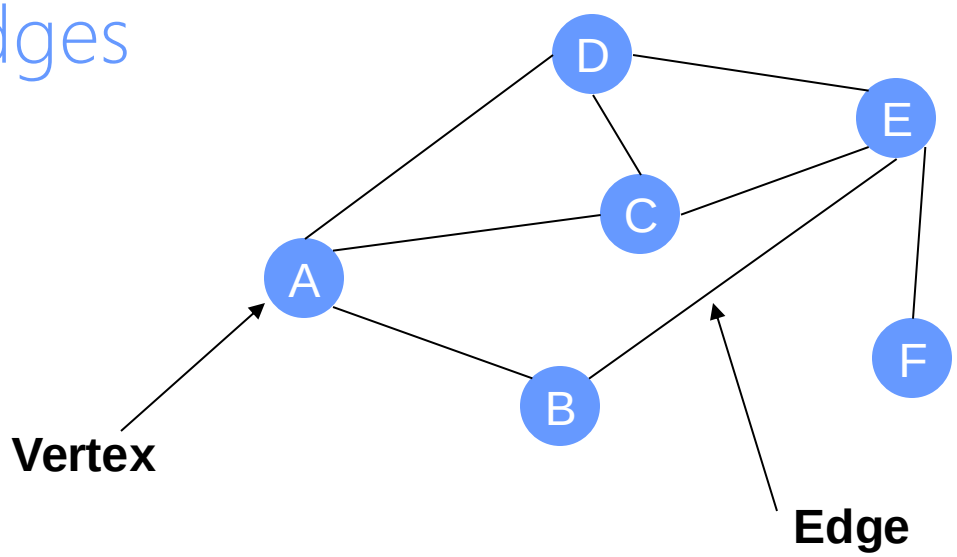
Data Structures and Algorithms

Lecture 14: Graphs and Breadth First Search

Department of Computer Science & Technology
United International College

Graphs

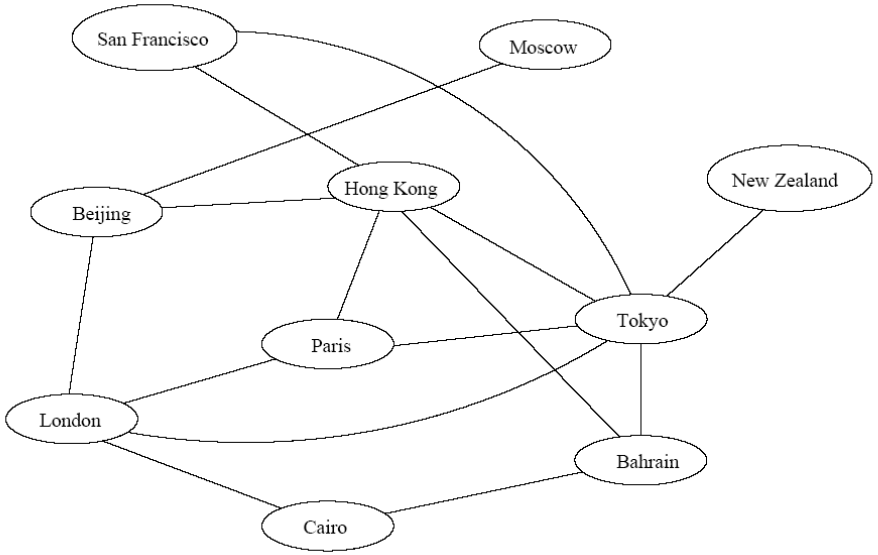
- Extremely useful tool in modeling problems
- Consist of:
  - Vertices
  - Edges



Vertices can be considered “sites” or locations.

Edges represent connections.

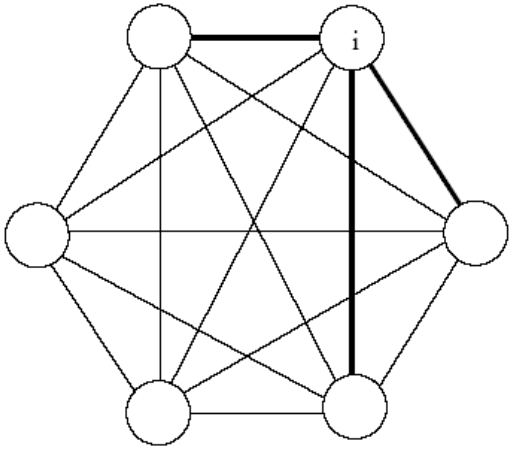
Application 1

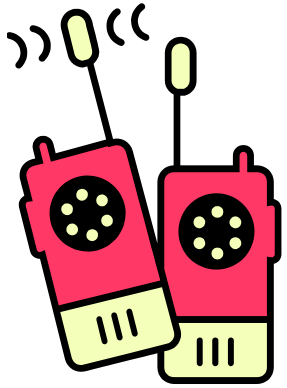


Air flight system

- Each vertex represents a city
- Each edge represents a direct flight between two cities
- A query on direct flights = a query on whether an edge exists
- A query on how to get to a location = does a path exist from A to B?
- We can even associate costs to edges (weighted graphs), then ask “what is the cheapest path from A to B?”

Application 2

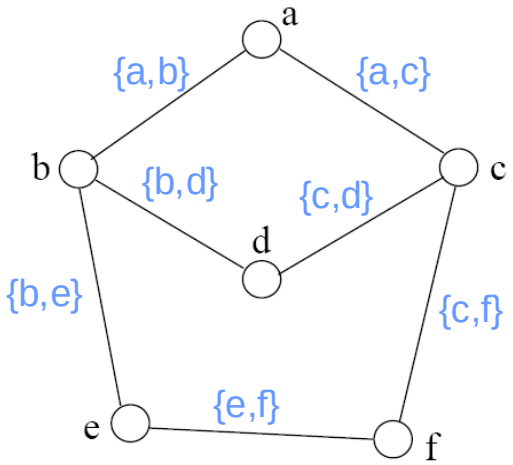




- Represented by a weighted complete graph (every two vertices are connected by an edge)
- Each edge represents the Euclidean distance d_{ij} between two radio stations
- Each station uses a certain power i to transmit messages. Given this power i, only a few nodes can be reached (bold edges). A station reachable by i then uses its own power to relay the message to other stations not reachable by i.
- A typical wireless communication problem is: how to broadcast between *all* stations such that they are all connected and the power consumption is minimized.

Definition

- A graph G=(V, E) consists a set of vertices V and a set of edges E.
- Each edge is a pair of *(v, w)* where v and w belong to V
- If the pair is unordered, the graph is undirected; otherwise it is directed



$$V = \{a, b, c, d, e, f\}$$

$$E = \{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}, \{b, e\}, \{c, f\}, \{e, f\}\}$$

An undirected graph

Terminology

1. If v_1 and v_2 are connected, they are said to be adjacent vertices
   - v_1 and v_2 are endpoints of the edge $\{v_1, v_2\}$
2. If an edge e is connected to v then v is said to be incident on e. Also, the edge e is said to be incident on v.
3. $\{v_1, v_2\} = \{v_2, v_1\}$

If we are talking about directed graphs, where edges have direction, then $\{v_1,v_2\} \neq \{v_2,v_1\}$
Directed graphs are drawn with arrows (called arcs) between edges.

This means {A,B} only, not {B,A}

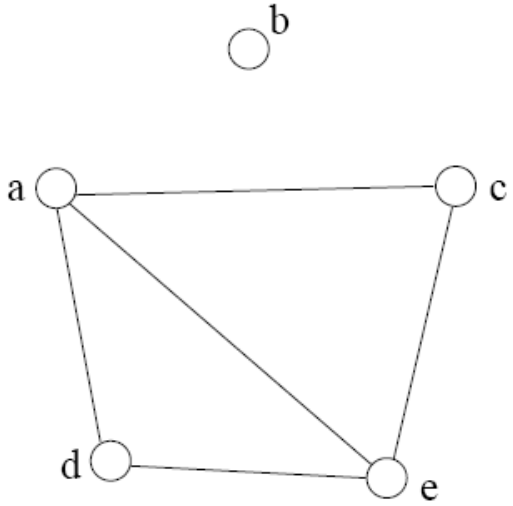
Graph Representation

- Two popular computer representations of a graph. Both represent the vertex set and the edge set, but in different ways.
  1. Adjacency Matrix

     Use a 2D matrix to represent the graph
  2. Adjacency List

     Use a 1D array of linked lists

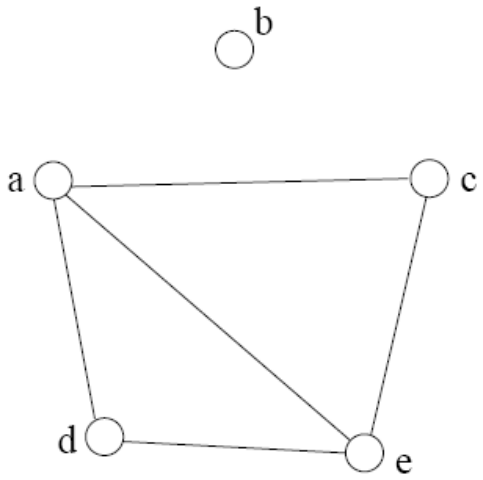
Adjacency Matrix

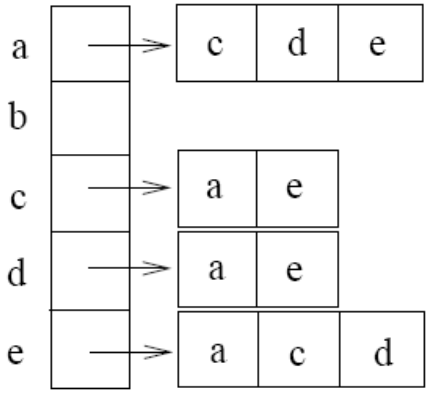


	a	b	c	d	e
a	0	0	1	1	1
b	0	0	0	0	0
c	1	0	0	0	1
d	1	0	0	0	1
e	1	0	1	1	0

- 2D array A[0..n-1, 0..n-1], where *n* is the number of vertices in the graph.
- Each row and column is indexed by the vertex id
  - e,g a=0, b=1, c=2, d=3, e=4
- *A[i][j]=1* if there is an edge connecting vertices *i* and *j*
  otherwise *A[i][j]=0.*
- The storage requirement is $\Theta(n^2)$. It is not efficient if the graph has few edges.
  An adjacency matrix is an appropriate representation if the graph is dense:
  $|E|=\Theta(|V|^2)$
- We can detect in O(1) time whether two vertices are connected.

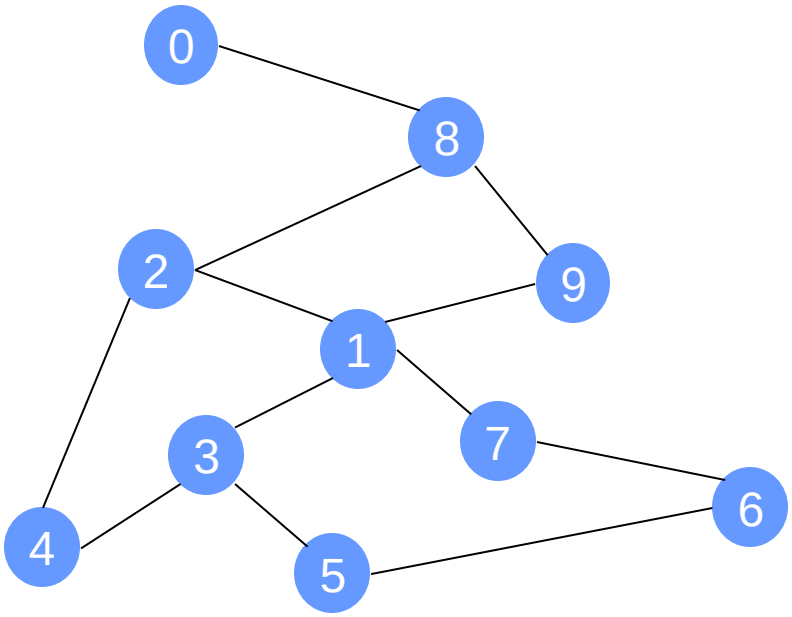
Adjacency List





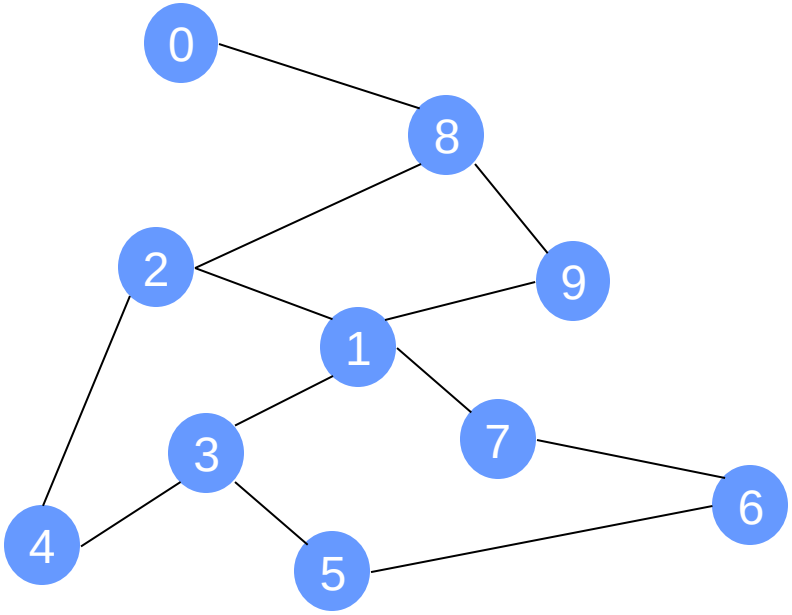
- If the graph is not dense, in other words, sparse, a better solution is an adjacency list
- The adjacency list is an array A[0..n-1] of lists, where n is the number of vertices in the graph.
- Each array entry is indexed by the vertex id
- Each list *A[i]* stores the ids of the vertices adjacent to vertex i

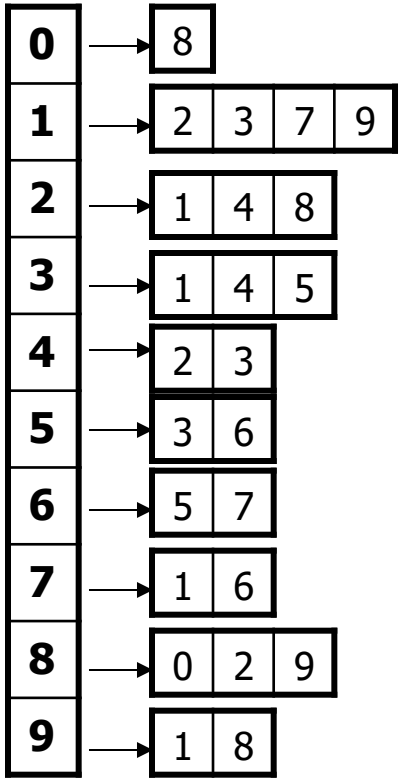
Adjacency Matrix Example



	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	1	0
1	0	0	1	1	0	0	0	1	0	1
2	0	1	0	0	1	0	0	0	1	0
3	0	1	0	0	1	1	0	0	0	0
4	0	0	1	1	0	0	0	0	0	0
5	0	0	0	1	0	0	1	0	0	0
6	0	0	0	0	0	1	0	1	0	0
7	0	1	0	0	0	0	1	0	0	0
8	1	0	1	0	0	0	0	0	0	1
9	0	1	0	0	0	0	0	0	1	0

Adjacency List Example





Storage of Adjacency List

- The array takes up $\Theta(n)$ space
- Define degree of v, $\deg(v)$, to be the number of edges incident to v. Then, the total space to store the lists is proportional to:

$$\sum_{\text{vertex } v} \deg(v)$$

- An edge $e=\{u,v\}$ of the graph contributes a count of 1 to $\deg(u)$ and contributes a count of 1 to $\deg(v)$
- Therefore $\Sigma_{\text{vertex } v} \deg(v) = 2m$ where m is the total number of edges
- In all, the adjacency list takes up $\Theta(n+m)$ space
  - If $m = O(n^2)$ (i.e. dense graphs), both the adjacency matrix and the adjacency list use $\Theta(n^2)$ space.
  - If $m = O(n)$, the adjacency list outperforms the adjacency matrix

- However, with an adjacency list one cannot tell in O(1) time whether two vertices are connected! A linked list search is required.

Adjacency List vs. Matrix

- Adjacency List
  - More compact than adjacency matrices if graph has few edges
  - Requires more time to find if an edge exists

- Adjacency Matrix
  - Always require n^2 space
    - This can waste a lot of space if the graph is sparse
  - Can quickly find if an edge exists

Path between Vertices

- A path is a sequence of vertices (v_0, v_1, v_2,… v_k) such that:
  - For $0 \leq i < k$, $\{v_i, v_{i+1}\}$ is an edge

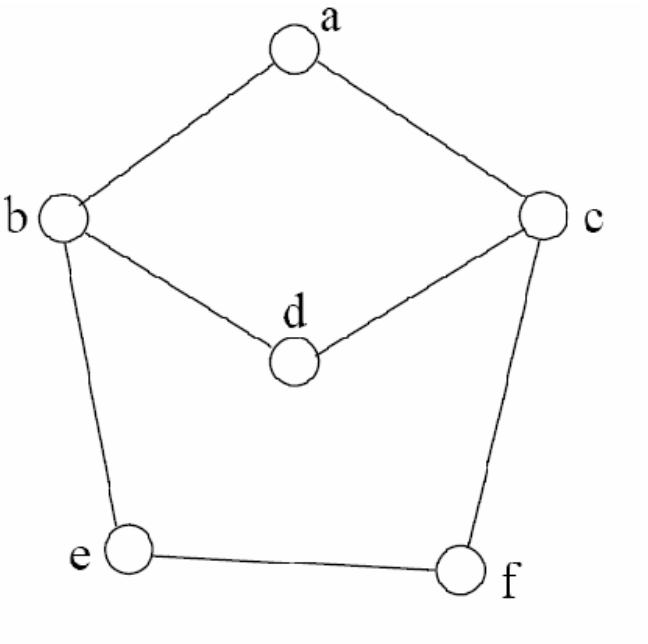
Note: a path is allowed to go through the same vertex or the same edge any number of times!

- The length of a path is the number of edges on the path

Types of paths

- A path is simple if and only if it does not contain a vertex more than once.
- A path is a cycle if and only if $v_0 = v_k$
  - The beginning and end are the same vertex!
- A path contains a cycle as its sub-path if some vertex appears twice or more

Path Examples



Are these paths?

Any cycles?

What is the path's length?

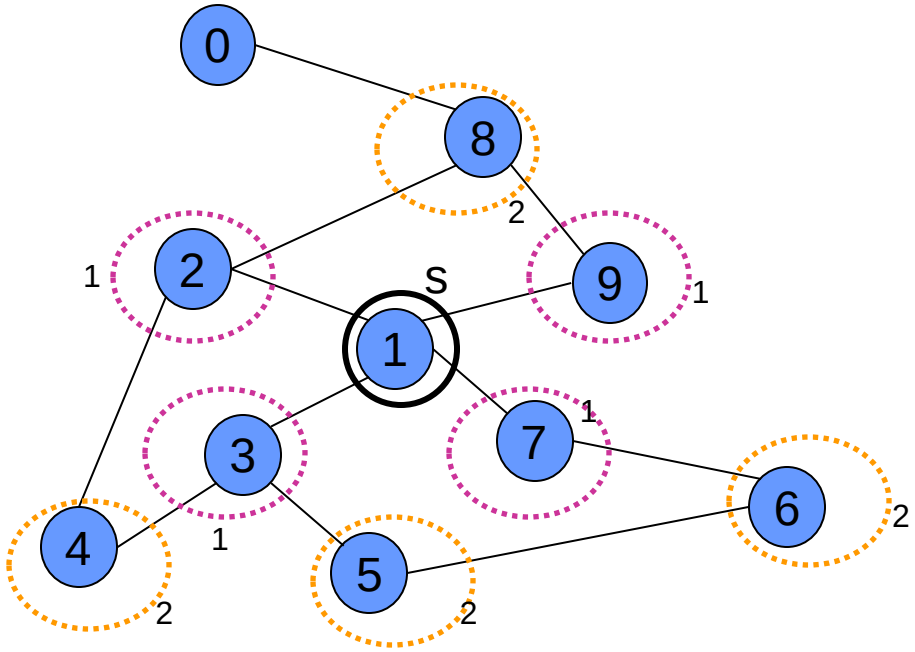
1. {a,c,f,e}
2. {a,b,d,c,f,e}
3. {a, c, d, b, d, c, f, e}
4. {a,c,d,b,a}
5. {a,c,f,e,b,d,c,a}

Graph Traversal

- Application example
  - Given a graph representation and a vertex **s** in the graph
  - Find all paths from **s** to other vertices
- Two common graph traversal algorithms
  - Breadth-First Search (BFS)
    - Find the shortest paths in an unweighted graph
  - Depth-First Search (DFS)
    - Topological sort
    - Find strongly connected components

BFS and Shortest Path Problem

- Given any source vertex *s*, BFS visits the other vertices at increasing distances away from s. In doing so, BFS discovers paths from s to other vertices
- What do we mean by "distance"? The number of edges on a path from s



Example

Consider s = vertex 1

Nodes at distance 1?
2, 3, 7, 9

Nodes at distance 2?
8, 6, 5, 4

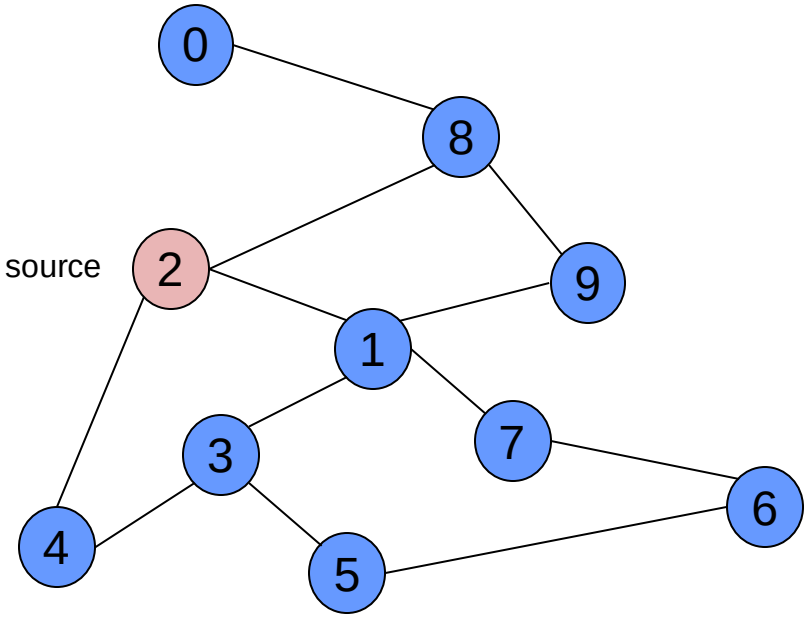
Nodes at distance 3?
0

BFS Algorithm

Algorithm $BFS(s)$
Input: s is the source vertex
Output: Mark all vertices that can be visited from s.

```
1.    for each vertex v
2.        do flag[v] := false;  // flag[ ]: visited table
3.    Q = empty queue;          Why use queue? Need FIFO
4.    flag[s] := true;
5.    enqueue(Q, s);
6.    while Q is not empty
7.       do v := dequeue(Q);
8.          for each w adjacent to v
9.              do if flag[w] = false
10.                    then flag[w] := true;
11.                         enqueue(Q, w)
```

BFS Example



Adjacency List

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

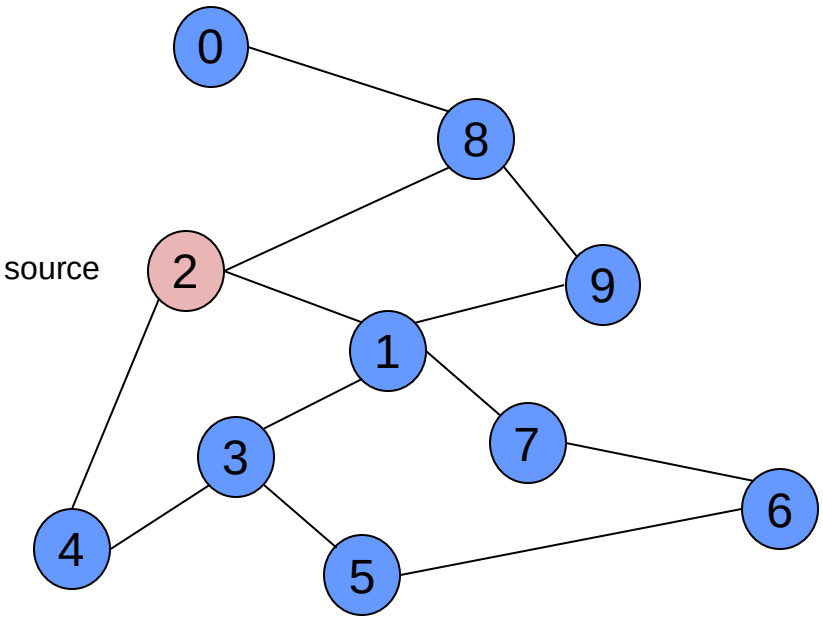
0	F
1	F
2	F
3	F
4	F
5	F
6	F
7	F
8	F
9	F

Initialize visited table (all False)

Q = { }

Initialize **Q** to be empty

BFS Example (Cont'd)



Adjacency List

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

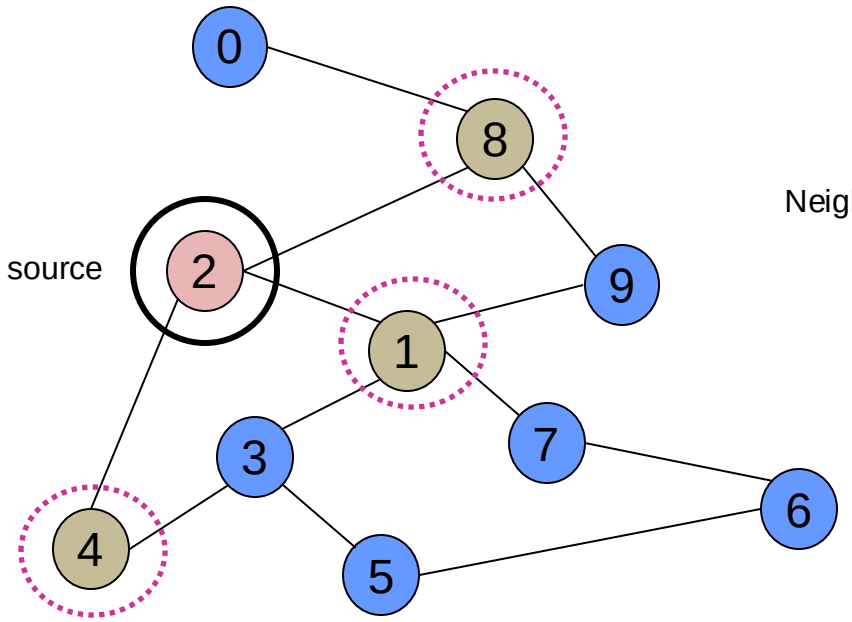
0	F
1	F
2	T
3	F
4	F
5	F
6	F
7	F
8	F
9	F

Q = { 2 }

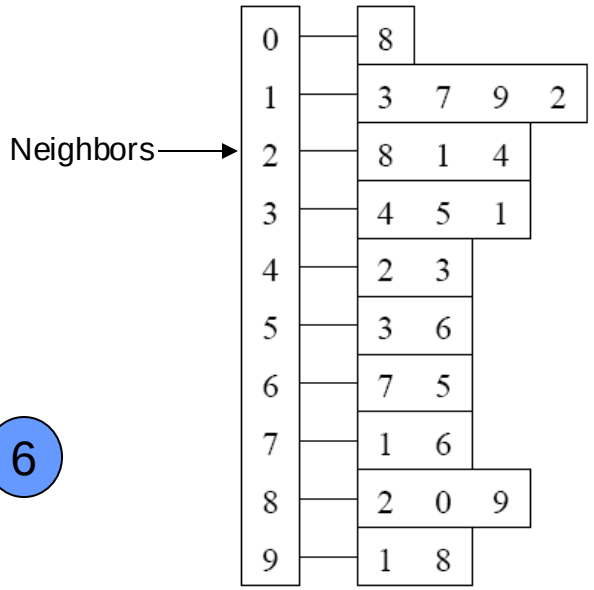
Place source 2 on the queue

Flag that 2 has been visited

BFS Example (Cont'd)



Adjacency List



Visited Table (T/F)

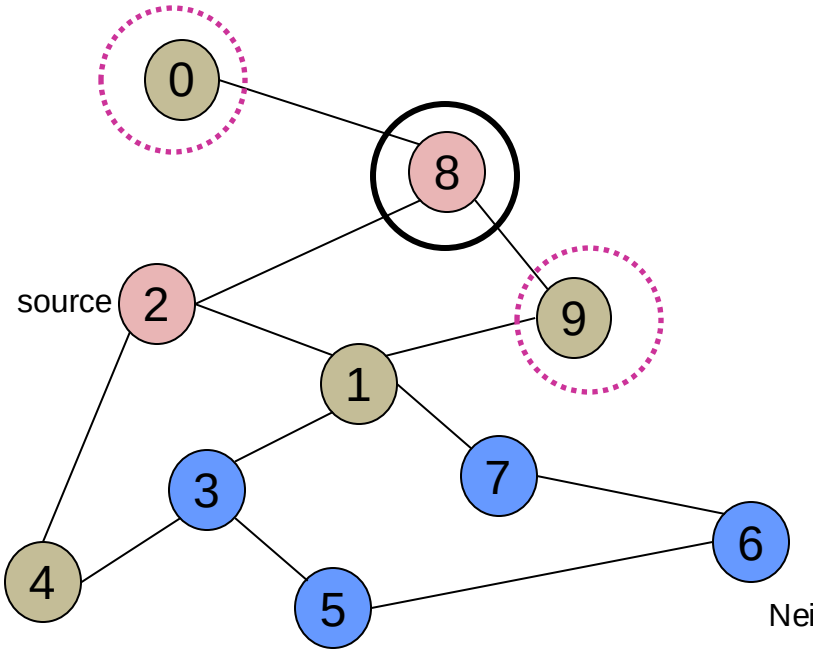
0	F
1	T
2	T
3	F
4	T
5	F
6	F
7	F
8	T
9	F

Mark neighbors as visited 1, 4, 8

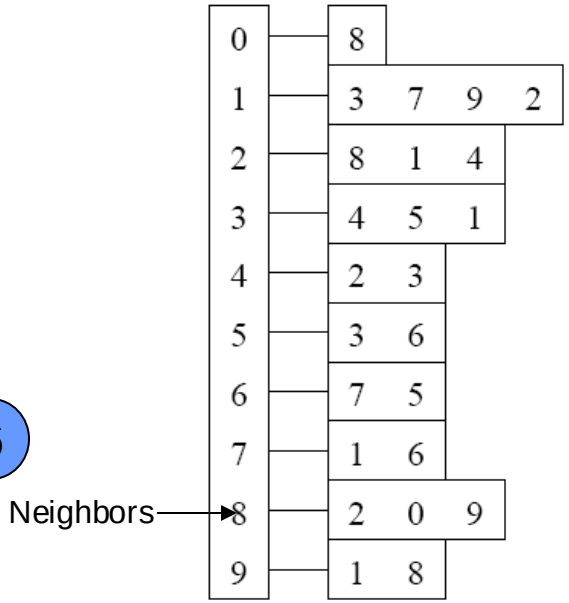
Q = {2} → { 8, 1, 4 }

Dequeue 2.
Place all unvisited neighbors of 2 on the queue

BFS Example (Cont'd)



Adjacency List



Visited Table (T/F)

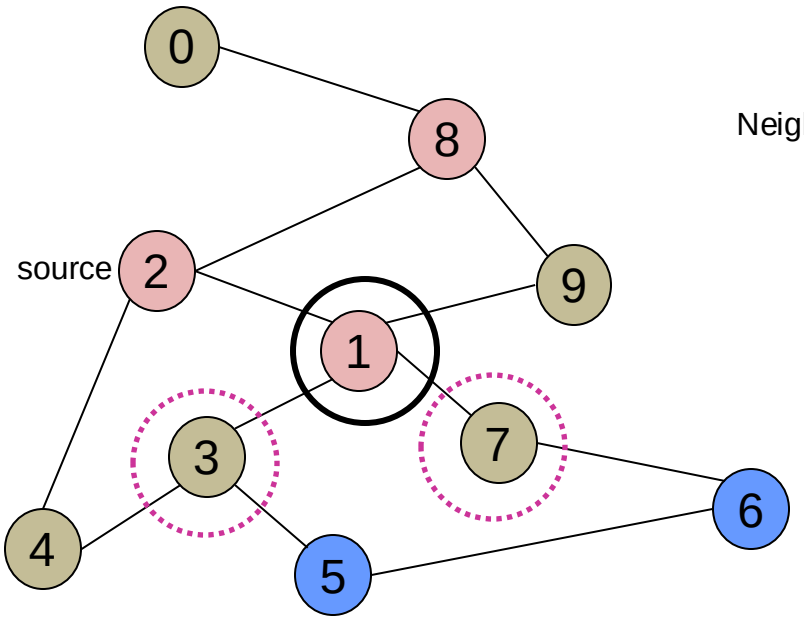
0	T
1	T
2	T
3	F
4	T
5	F
6	F
7	F
8	T
9	T

Mark new visited
Neighbors 0, 9

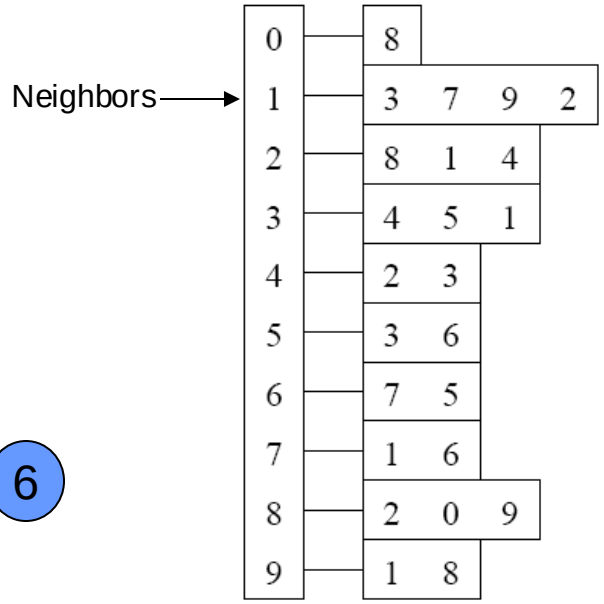
Q = { 8, 1, 4 } → { 1, 4, 0, 9 }

Dequeue 8.
-- Place all unvisited neighbors of 8 on the queue.
-- Notice that 2 is not placed on the queue again, it has been visited!

BFS Example (Cont'd)



Adjacency List



Visited Table (T/F)

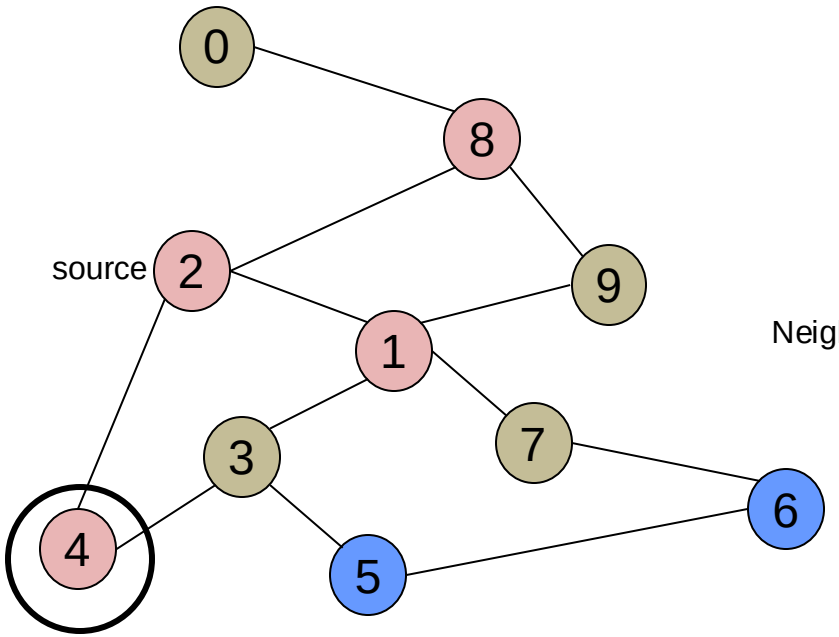
0	T
1	T
2	T
3	T
4	T
5	F
6	F
7	T
8	T
9	T

Mark new visited
Neighbors 3, 7

Q = { 1, 4, 0, 9 } → { 4, 0, 9, 3, 7 }

Dequeue 1.
-- Place all unvisited neighbors of 1 on the queue.
-- Only nodes 3 and 7 haven't been visited yet.

BFS Example (Cont'd)



Adjacency List

Neighbors →

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

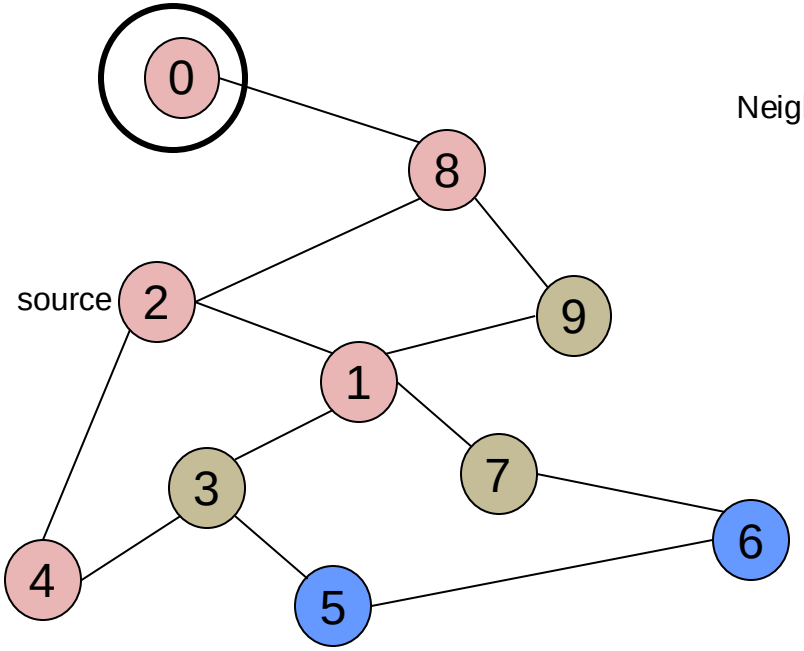
Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	F
6	F
7	T
8	T
9	T

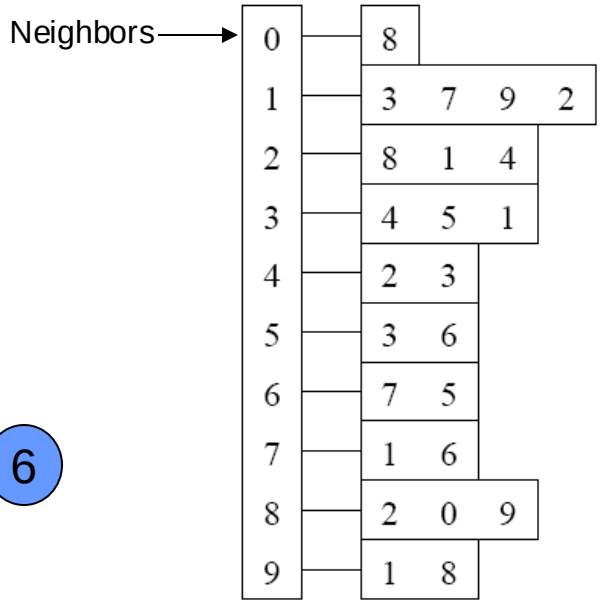
Q = { 4, 0, 9, 3, 7 } → { 0, 9, 3, 7 }

Dequeue 4.
-- 4 has no unvisited neighbors!

BFS Example (Cont'd)



Adjacency List



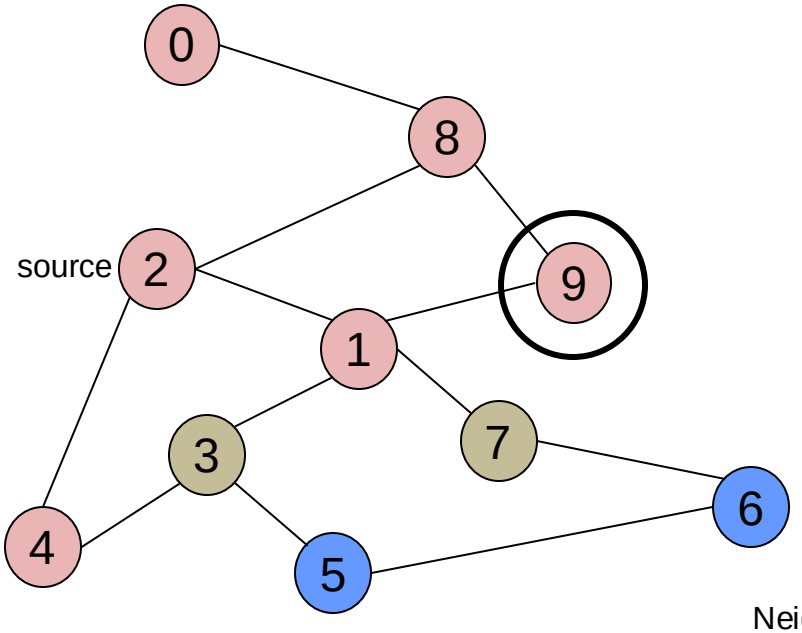
Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	F
6	F
7	T
8	T
9	T

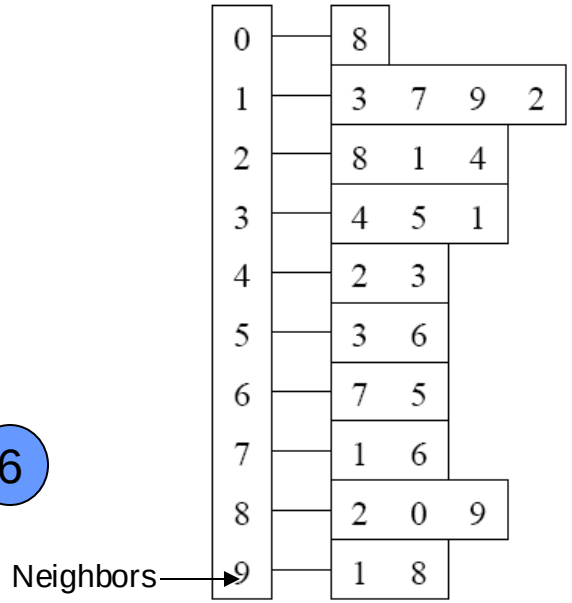
Q = { 0, 9, 3, 7 } → { 9, 3, 7 }

Dequeue 0.
-- 0 has no unvisited neighbors!

BFS Example (Cont'd)



Adjacency List



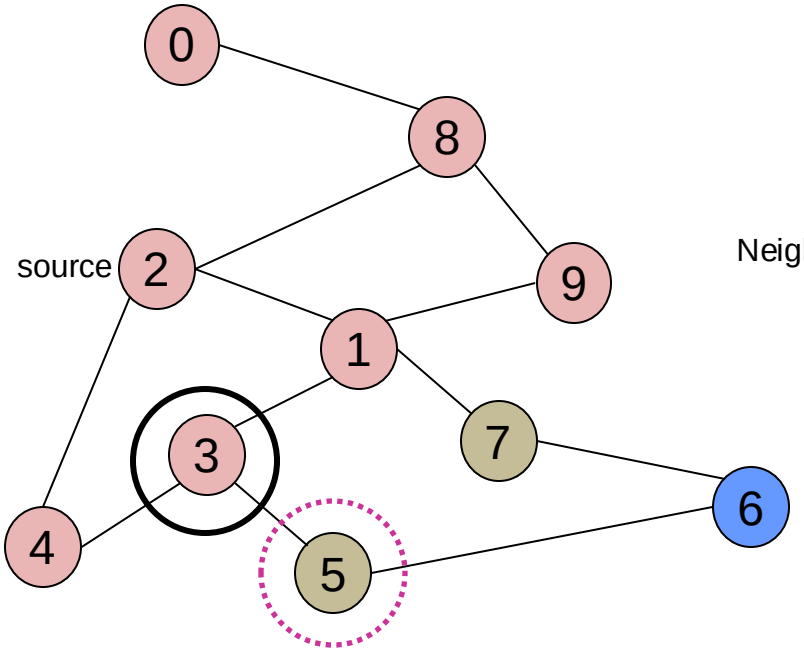
Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	F
6	F
7	T
8	T
9	T

Q = { 9, 3, 7 } → { 3, 7 }

Dequeue 9.
-- 9 has no unvisited neighbors!

BFS Example (Cont'd)



Adjacency List

Neighbors →

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

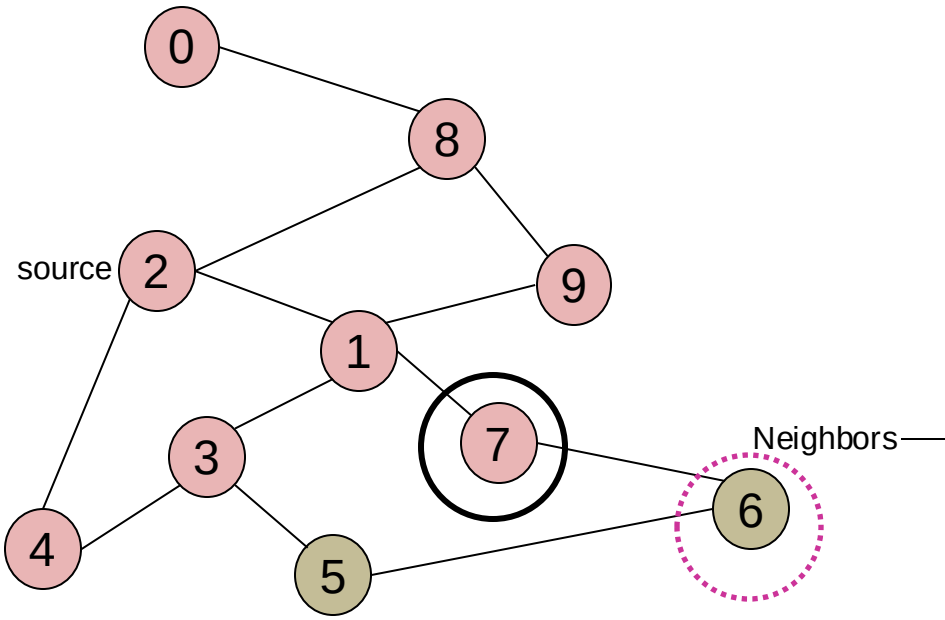
0	T
1	T
2	T
3	T
4	T
5	T
6	F
7	T
8	T
9	T

Mark new visited Vertex 5

Q = { 3, 7 } → { 7, 5 }

Dequeue 3.
-- place neighbor 5 on the queue.

BFS Example (Cont'd)



Adjacency List

Neighbors

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

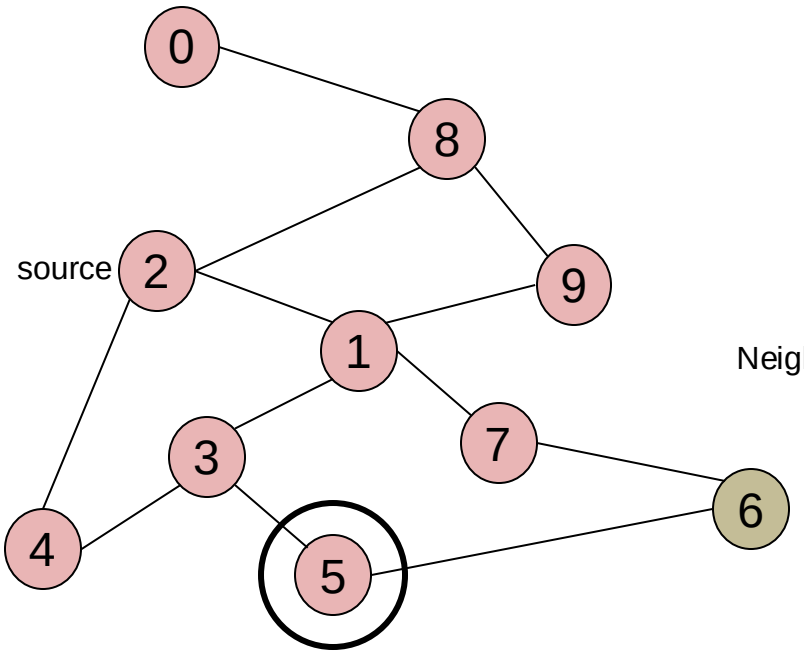
0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T

Mark new visited Vertex 6

Q = { 7, 5 } → { 5, 6 }

Dequeue 7.
-- place neighbor 6 on the queue

BFS Example (Cont'd)



Adjacency List

Neighbors →

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

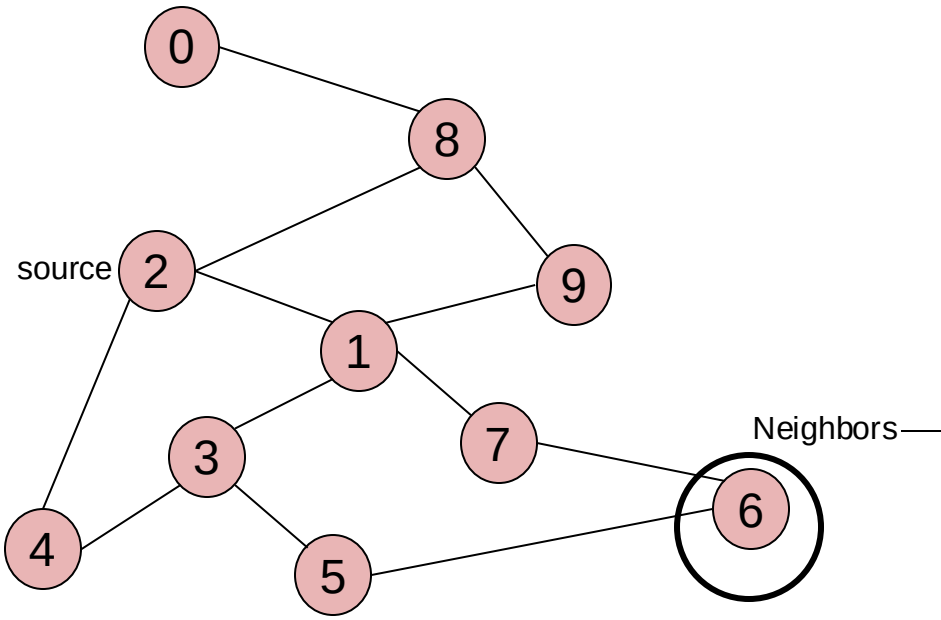
Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T

Q = { 5, 6} → { 6 }

Dequeue 5.
-- no unvisited neighbors of 5

BFS Example (Cont'd)



Adjacency List

Neighbors →

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

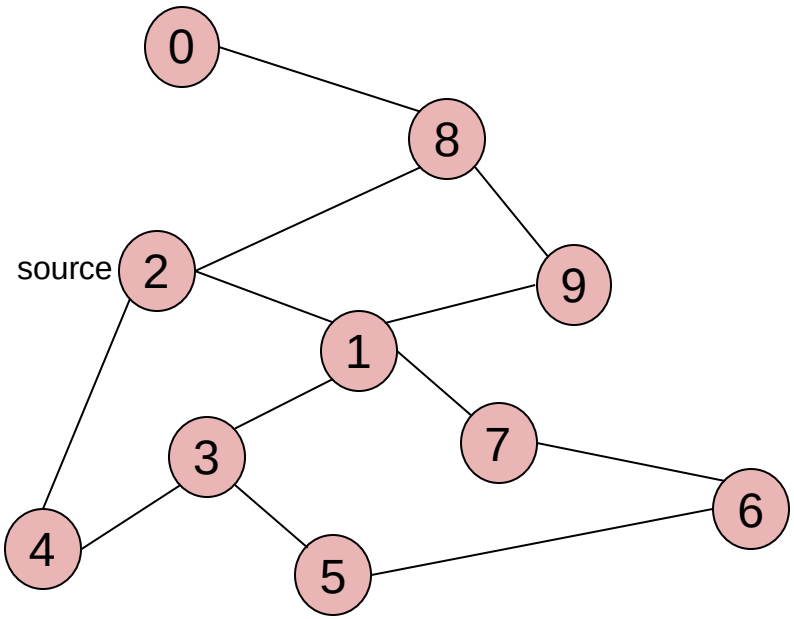
Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T

Q = { 6 } → { }

Dequeue 6.
-- no unvisited neighbors of 6

BFS Example (Cont'd)



Adjacency List

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T

Q = { } STOP!!! Q is empty!!!

What did we discover?

Look at "visited" tables.

There exists a path from source vertex 2 to all vertices in the graph

Time Complexity of BFS
(Using Adjacency List)

- Assume adjacency list
  - n = number of vertices m = number of edges

Algorithm *BFS*(s)
Input: s is the source vertex
Output: Mark all vertices that can be visited from s.

```
1.  for each vertex v
2.      do flag[v] := false;
3.  Q = empty queue;
4.  flag[s] := true;
5.  enqueue(Q, s);
6.  while Q is not empty
7.      do v := dequeue(Q);
8.          for each w adjacent to v
9.              do if flag[w] = false
10.                 then flag[w] := true;
11.                     enqueue(Q, w)
```

O(n + m)

Each vertex will enter Q at most once.

Each iteration takes time proportional to deg(v) + 1 (the number 1 is to account for the case where deg(v) = 0 --- the work required is 1, not 0).

Running Time

- Total time for the enqueue and deqeue operations: O(n)
- Recall: Given a graph with m edges, what is the total degree?

$$\Sigma_{\text{vertex } v} \deg(v) = 2m$$

- Total time for processing adjacent vertices:

$$O(\Sigma_{\text{vertex } v} (\deg(v) + 1)) = O(m)$$

This is summing over all the iterations in the while loop!

- The total running time is then: O(n + m)

Time Complexity of BFS
(Using Adjacency Matrix)

- Assume adjacency list
  - n = number of vertices m = number of edges

Algorithm *BFS*(s)
Input: s is the source vertex
Output: Mark all vertices that can be visited from s.

```
1.  for each vertex v
2.      do flag[v] := false;
3.  Q = empty queue;
4.  flag[s] := true;
5.  enqueue(Q, s);
6.  while Q is not empty
7.      do v := dequeue(Q);
8.          for each w adjacent to v
9.              do if flag[w] = false
10.                 then flag[w] := true;
11.                     enqueue(Q, w)
```

$O(n^2)$

Finding the adjacent vertices of v requires checking all elements in the matrix row. This takes linear time O(n).

Summing over all the n iterations, the total running time is $O(n^2)$.

Time Complexity of BFS

- When using an adjacency matrix, BFS is $O(n^2)$, independent of the number of edges m.
- When using an adjacency list, BFS is $O(n + m)$.
- If the graph is dense then $m = O(n^2)$ and then $O(n + m) = O(n^2)$.