Advanced Counting Techniques

1) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 1s. What are the initial conditions?
2) Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 1s. What are the initial conditions?
3) Find a recurrence relation for the number of bit strings of length n that contain the string 10. What are the initial conditions?
4) Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 1s. What are the initial conditions?
5) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes.
6) A vending machine dispensing books of stamps accepts only \$3 bills and \$5 bills. Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the bills are deposited does not matter.
7) Solve these recurrence relations together with the initial conditions given.
   a) $a_n = a_{n-1} + 12a_{n-2}$ for $n \geq 2$, and $a_0 = 5, a_1 = 7$
   b) $a_n = 4a_{n-2}$ for $n \geq 2$, and $a_0 = 1, a_1 = -1$
8) *Solve the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ with the initial conditions $a_0 = 1$ and $a_1 = 3$.
9) *Solve the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ with the initial conditions $a_0 = 1, a_1 = 2$ and $a_2 = 5$.
10) *Solve the recurrence relation $a_n = 2a_{n-1} + 3n$ with the initial condition $a_1 = 5$.

	8	4	2	1
	Q3	Q2	Q1	Q0
0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1
12	1	1	0	0
13	1	1	0	1
14	1	1	1	0
15	1	1	1	1

1) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 1s. What are the initial conditions?

if 1... also if → $10\overbrace{\dots}^{n-2}$ then $a_n = a_{n-2}$

→ $11\overbrace{\dots}^{n-2}$ then $a_n = 2^{n-2}$

if 0... also if → 01... then $a_n = a_{n-1}$ because of 1 having options.

→ 00... then nothing changes

Thus, $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$ & $a_0 = 0, a_1 = 0$.

2) Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 1s. What are the initial conditions?

$\overbrace{1\dots}^{n}$ => a_{n-1} string with no 111s

$10\dots$ => a_{n-2} string with no 111s

$110\dots$ => a_{n-3} string with no 111s

~~111~~...

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

$a_0 = 1$ $a_1 = 2$ $a_2 = 4$

3) Find a recurrence relation for the number of bit strings of length n that contain the string 10. What are the initial conditions?

	8	4	2	1
	Q3	Q2	Q1	Q0
0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1
12	1	1	0	0
13	1	1	0	1
14	1	1	1	0
15	1	1	1	1

$2^1 - 1 \{ 10,$

$2^2 - 1 \{ 110, \; 100, \; 101$

$0\underbrace{10}_{a_{n-1}}$

$\overbrace{111}^{k}\overbrace{0}^{n-k}, \ldots$

1100

$\underbrace{11}_{k}\underbrace{01}_{n-k-1}$

$2^3 - 1 \{ 1000, \; 1001, \; 1010, \; 1011$

$0\underbrace{\ldots\ldots}_{a_{n-1}}$

$2^{n-1} - 1 \{ \overbrace{11\ldots1\ldots1}^{k}\overbrace{0}^{n-k}$

$\overbrace{1\ldots1}^{k}\,0\overbrace{\ldots0}^{n-k-1 \text{ strings}}$

$\ldots$

$10\ldots1\ldots1$

$0\underbrace{\ldots\ldots\ldots}_{a_{n-1}}$

Answer: $a_n = a_{n-1} + 2^{n-1} - 1$

Initial conditions: $a_0 = a_1 = 0$

4) Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 1s. What are the initial conditions?

binary string – 0 & 1s string

ternary string – 0, 1, 2s string

$0 \to 0, 1, 2 \quad a_{n-1}$

$1 \to 0, \cancel{1}, 2 \quad 2a_{n-2}$

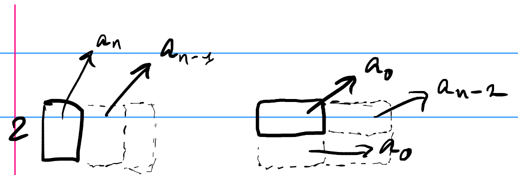
$2 \to 0, 1, 2 \quad a_{n-1}$

$a_n = 2a_{n-1} + 2a_{n-2}$

$a_0 = 1$

$a_1 = 3$

5) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes.



$a_1 = 1 \quad a = 2$

$a_n = a_{n-1} + a_{n-2}$

6) A vending machine dispensing books of stamps accepts only \$3 bills and \$5 bills. Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the bills are deposited does not matter.

$$a_n = a_{n-3} + a_{n-5}$$

a_0(\$1), a_1(\$2), a_3(\$4), and bills b such that $3 \nmid b$ or $5 \nmid b$ couldn't be deposited, however they'll be 0, $a_0 = a_1 = a_3 = 0$. But $a_2 = 1$ and $a_4 = 1$. Therefore, $a_5 = 1$, $a_6 = 0$, $a_7 = 1$, and so on...

7) Solve these recurrence relations together with the initial conditions given.

a) $a_n = a_{n-1} + 12a_{n-2}$ for $n \geq 2$, and $a_0 = 5, a_1 = 7$

b) $a_n = 4a_{n-2}$ for $n \geq 2$, and $a_0 = 1, a_1 = -1$

a) $r^2 - r - 12 = (r-4)(r+3) \Rightarrow r_1 = 4$ & $r_2 = -3$

$a_0 = 5 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1(4) + \alpha_2(-3)$ $\Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 5 \\ 4\alpha_1 - 3\alpha_2 = 7 \end{cases} \Rightarrow$ $\alpha_1 = 5 - \alpha_2$ $\rightarrow \alpha_1 = 5 - \frac{13}{7} = \frac{35-13}{7} = \frac{22}{7} = \alpha_1$

$4(5-\alpha_2) - 3\alpha_2 = 7$

$20 - 4\alpha_2 - 3\alpha_2 = 7$

$20 - 7 = 7\alpha_2$

$13 = 7\alpha_2$

$\frac{13}{7} = \alpha_2$

$\frac{22}{7}(4)^n + \frac{13}{7}(-3)^n = a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

b) $r^2 - 4 = (r+2)(r-2) \Rightarrow r_1 = -2$ & $r_2 = 2$

$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_1(-2) + \alpha_2(2) = -1 \end{cases} \Rightarrow$ $\alpha_1 = 1 - \alpha_2$ $\rightarrow \alpha_1 = 1 - \frac{1}{4} = \frac{3}{4}$

$(1-\alpha_2)(-2) + 2\alpha_2 = -1$

$-2 + 2\alpha_2 + 2\alpha_2 = -1$

$4\alpha_2 = 1$

$\alpha_2 = \frac{1}{4}$

$a_n = \frac{3}{4}(-2)^n + \frac{1}{4}(2)^n$

8) *Solve the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ with the initial conditions $a_0 = 1$ and $a_1 = 3$.

Try it!

9) *Solve the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ with the initial conditions $a_0 = 1, a_1 = 2$ and $a_2 = 5$.

As previously but $r^3 - 2r^2 - r + 2$. By fundamental theorem of algebra, if the roots are in the form of $\frac{a}{b}$ then any divisor of $\frac{a}{b}$, where $a = 2$ and $b = 1$ (coef. of r^3) is a candidate, ± 1 and ± 2 are the only ones. If $r = 1$ then $1^3 - 2(1)^2 - 1 + 2 = 0$. Let's do long division:

$$\begin{array}{r} r^2 - r - 2 \\ r-1 \overline{) r^3 - 2r^2 - r + 2} \\ \underline{r^3 - r^2} \qquad\qquad \\ -r^2 - r \qquad \\ \underline{-r^2 + r} \qquad \\ -2r + 2 \end{array} \qquad \begin{array}{r} -2r+2 \\ \underline{-2r+2} \\ 0 \end{array}$$

$$r^3-2r^2-r+2=(r-1)(r^2-r-2)=(r-1)(r-2)(r+1) \Rightarrow \begin{array}{l} r_1=1 \\ r_2=2 \\ r_3=-1 \end{array}$$

$$\begin{aligned} a_0 &= 1 = \alpha_1+\alpha_2+\alpha_3 \\ a_1 &= 2 = \alpha_1+2\alpha_2+(-1)\alpha_3 \\ a_2 &= 5 = \alpha_1+4\alpha_2+\alpha_3 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \Rightarrow Ax=B \Rightarrow A^{-1}Ax=A^{-1}B$$

$$x=A^{-1}B$$

or by Gauss

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & 4 & 1 & 5 \end{array}\right] \xrightarrow{\substack{R_2-R_1 \\ R_3-R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 3 & 0 & 5 \end{array}\right] \xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 6 & -1 \end{array}\right] \rightarrow$$

$$\xrightarrow{R_3\left(\frac{1}{6}\right)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -\frac{1}{6} \end{array}\right] \xrightarrow{R_2+2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2-\frac{2}{6}=\frac{5}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \end{array}\right] \xrightarrow{R_1-R_2-R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1-\frac{5}{3}+\frac{1}{6}=\frac{6-10+1}{6}=-\frac{3}{6}=-\frac{1}{2} \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -\frac{1}{6} \end{array}\right]$$

$$a_n=-\frac{1}{2}(1)^n+\frac{5}{4}(2)^n-\frac{1}{6}(-1)^n$$

10) *Solve the recurrence relation $\overbrace{a_n = 2a_{n-1} + 3n}^{\text{RR}}$ with the initial condition $a_1 = 5$.

$$\overbrace{\underbrace{a_n = 2a_{n-1}}_{\text{homogenous}} + \underbrace{3n}_{\text{particular}}}^{\text{nonhomogeneous (NH)}}$$

Theorem 5: If $a_n^{(p)}$ is particular solution for NHRR then every solution is of the form $a_n^{(p)}+a_n^{(h)}$ where $a_n^{(h)}$ is a solution for homogenous (H) RR

Solution for HRR $a_n=2a_{n-1}$ is of the form $a_n^{(h)}=\alpha 2^n$ $(r^2-2r=r(r-2) \Rightarrow a_n=\alpha_1 0^n+\alpha_2 2^n)$.

Notice, α hasn't been evaluated.

Because $3n$ is a polynomial of the form $p_n=cn+d$, let's use p_n in $a_n=2a_{n-1}+3n$

Then, $cn+d=2(c(n-1)+d)+3n=2cn-2c+2d+3n \Rightarrow 2cn-2c+2d+3n-cn-d=0$

which is $cn+3n+d=0 \Rightarrow n(c+3)+d=0$. $cn+d$ is solution only if $c+3=0$ and $d=0$.

It's seen $c=-3$. Thus, $a_n^{(p)}=-3n+0=-3n$. Now, $a_n=a_n^{(p)}+a_n^{(h)}=-3n+\alpha 2^n$, almost done.

$a_1=5$, it means $a_1=a_1^{(p)}+a_1^{(h)}=-3(1)+\alpha 2^1=a_1=5 \Rightarrow -3+2\alpha=5$

$$2\alpha=8$$

$$\alpha=4 \Rightarrow a_n=-3n+4(2)^n.$$