Residues and Duality

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Preface

In the spring of 1963 I suggested to Grothendieck the possibility of my running a seminar at Harvard on his theory of duality for coherent sheaves — a theory which had been hinted at in his talk to the Séminaire Bourbaki in 1957 [8], and in his talk to the International Congress of Mathematicians in 1958 [9], but had never been developed systematically. He agreed, saying that he would provide an outline of the material, if I would fill in the details and write up lecture notes of the seminar. During the summer of 1963, he wrote a series of "prénotes" [10] which were to be the basis for the seminar.

I quote from the preface of the prénotes: "Les presentes notes donnent une esquisse assez detaillee d'une theorie cohomologique de la dualite des Modules coherents sur les preschemas. Les idees principales de la theorie m'etaient connues des 1959, mais Ie manque de fondements

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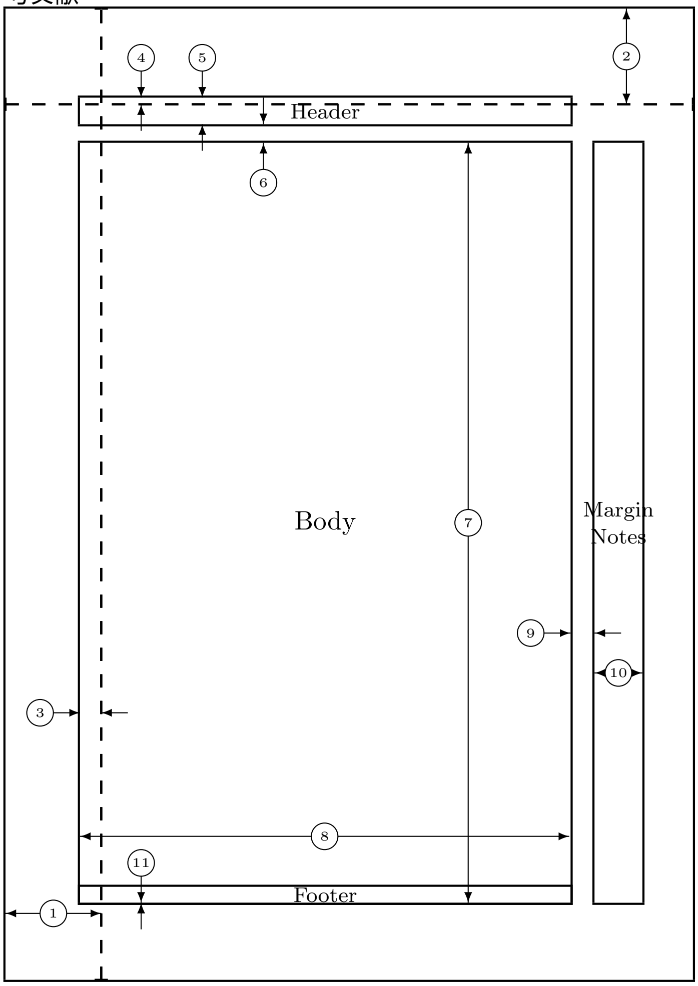

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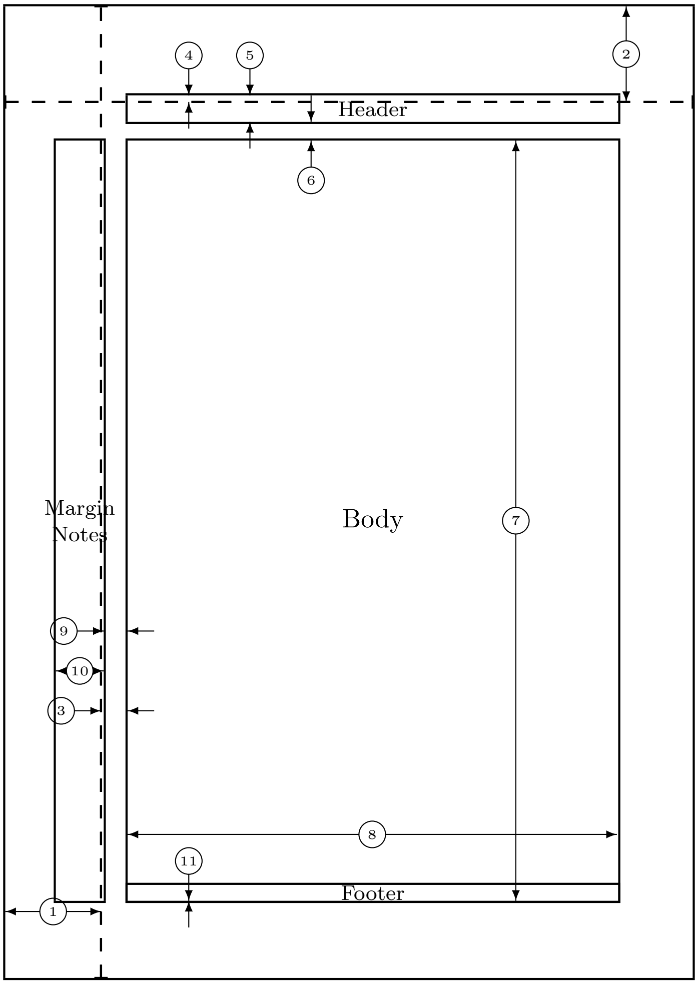

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Introduction

The main purpose of these notes is to prove a duality theorem for cohomology of quasi-coherent sheaves, with respect to a proper morphism of locally noetherian preschemes. Various such theorems are already known. Typical is the duality theorem for a non-singular complete curve X over an algebraically closed field k, which says that

$$h^0(D) = h^1(K - D),$$

where D is a divisor, K is the canonical divisor, and

$$h^i(D) = \dim_k H^i(X, L(D)),$$

for any i, and any divisor D. (See e.g. [16,Ch. II] for a proof.)

Various attempts were made to generalize this theorem to varieties of higher dimension, and as Zariski points out in his report [20], his generalization of a lemma of Enriques-Severi [19] is equivalent to the statement that for a normal projective variety X of dimension n over k,

$$h^0(D) = h^n(K - D)$$

for any divisor D. This is also equivalent to a theorem of Serre [FAC § 76 Thm. 4] on the vanishing of the cohomology group $H^1(X, L(-m))$ for m large and L locally free. Using a related theorem [FAC § 75 Thm. 3], Zariski shows how one can deduce on a non-singular projective variety the formula

$$h^i(D) = h^{n-i}(K - D)$$

for $0 < i < n$. In terms of sheaves, this result corresponds to the fact that the k-vector spaces

$$H^i(X, F) = H^{n-i}(X, F^\vee \otimes \omega)$$

are dual to each other, where F is a locally free sheaf, $F^\vee$ is the dual sheaf $\mathrm{Hom}(F, \mathcal{O}_X)$ and ω is the sheaf $\omega = \Omega^n_{X/k}$ of n-differentials on X. Serre [15] gives a proof of this same theorem by analytic methods for a compact complex analytic manifold X.

Grothendieck [8] gave some generalizations of these theorems for non-singular projective varieties, and then in [9] announced the general theorem for schemes proper over a field, with arbitrary singularities, which is the subject of the present lecture notes.

To motivate the statement of our main theorem, let us consider the case of projective space $X = \mathbf{P}_k^n$ over an algebraically closed field k. Then there is a canonical isomorphism

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