Microlocal Study of Sheaves

Contents

Introduction

Let X be a real manifold, F a sheaf on X, or better an objet of $\mathsf{D}^+(X)$, the derived category of the category of complexes bounded from below of sheaves on X. Let TX be the cotangent bundle to X. We associate to F a closed conic subset of TX, denoted $SS(F)$, the "micro-support of F", as follows:

Definition 0.0.1 . Let $p = (x_0, \xi_0) \in T^*X$. Then $p \notin SS(F)$ if and only if there exists an open neighborhood U of p in TX such that for any $(x_1, \xi_1) \in U$, any real C^1-function ϕ on X, with $\phi(x_1) = 0$, $d\phi(x_1) = \xi_1$, we have: $(\mathbf{R}\Gamma_{x;\phi(x)\geqq 0}(F))_{x_1} = 0$.

In other words the micro-support of F describes the set of codirections of X where F, and its cohomology, "do not propagate". This definition is motivated by the following situation.

Assume X is a complex manifold, and let $\mathcal{M}$ be a coherent module over the Ring $\mathcal{D}_X$ of (holomorphic, finire order) differential operators. Let $\mathrm{char}(\mathcal{M})$ be the characteristic variety of $\mathcal{M}$ in T^*X. Then we can interpret a well-known result of Zerner [1], Bony-Schapira [1], Kashiwara [5], through the formula:

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \subset \mathrm{char}(\mathcal{M}). \tag{0.0.1}$$

A natural problem then arising in the theory of (micro-)differential equations, is to evaluate the set of codirections of propagation for the sheaf of hyperfunction or microfunction solutions of $\mathcal{M}$ (or more generally of a system of micro-differential equations). To be more precise, let M be a real analytic manifold of dimension n, X a complexification of M. Recall that the sheaf $\mathcal{B}_M$ (resp. $\mathcal{C}_M$) of Sato's hyperfunctions on M (resp. Sato's microfunctions on $T^*_M X$, the conormal bundle to M in X) is defined by:

$$\mathcal{B}_M = \mathrm{R}\Gamma_M(\mathcal{O}_X) \otimes \omega_M[n]$$
$$(\text{resp. } \mathcal{C}_M = \mu_M(\mathcal{O}_X) \otimes \omega_M[n])$$

where ω_M is the orientation sheaf on M, $[n]$ means the n-shift in $\mathsf{D}^+(X)$, and $\mu(\cdot)$ is the functor of Sato's microlocalization along M (cf. Chapter 2).

Then the problem is: evaluate $\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M))$ or $\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_M))$. Taking $F = \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)$ this is a particular case of the following problem: given $F \in \mathsf{Ob}(\mathsf{D}^+(X))$, and M a real submanifold of X, calculate $\mathrm{SS}(\mathrm{R}\Gamma_M(F))$ or $\mathrm{SS}(\mu_M(F))$.

As we see, in this new formulation, we may forget that X is a complex-manifold, and we do not study separately the $\mathcal{D}_X$-module $\mathcal{M}$ from one side and the sheaf $\mathcal{O}_X$ on the other side. On the contrary we work with the whole complex of solutions of $\mathcal{M}$ in $\mathcal{O}_X$. The only information that we keep is the geometrical data of the characteristic variety, which is interpreted in terms of micro-support (in fact we shall prove in Chapter 10 that the inclusion in (0.1) is an equality).

Now let us come back to the subject of this paper.

We study in Chapter 4 and 5 the functorial properties of the micro-support: behavior under direct or inverse images, functors $\mathrm{R}\mathscr{H}om(\cdot,\cdot)$, $\cdot \otimes^L \cdot$, specialization, Fourier-Sato transformation, microlocalization (the construction of these functors are recalled in Chapter 1 and 2). But in order to manipulate micro-supports, the definition (0.1) given above is too much of a local one and one has to replace it by a more global criterium. This is achieved in Chapter 3, using a Mittag-Leffler procedure for sheaves (Theorem 1.4.3).

The calculations of Chapters 4 and 5 are all essentially based on the computation of the micro-support of the direct image of a sheaf by an open immersion. In this case the procedure, and the result, are very similar to those encountered in the theory of micro-hyperbolic systems (cf. our work [2]), and the set we obtain is defined as a "normal cone", (Theorem 4.3.1.). The preliminaries concerning such normal cones are presented in Chapter 1, §2.

The notion of micro-support allows us to work with sheaves "micro-locally", that is, locally in T^*X. In fact for a subset Ω of T^*X, we introduce the triangulated category $\mathsf{D}^+(X;\Omega)$ obtained from $\mathsf{D}^+(X)$ by localization on Ω, that is, by regarding as the zero object the sheaves whose micro-support do not meet Ω. A useful tool in the microlocal study of sheaves, is the "G-topology". The idea of the G-topology is the following: in order to work microlocally, let us say on $X \times U$ where X is open in a real vector space E and U is an open cone in the dual space E^*, the usual topology on X is too strong, and may be weakened by introducing a closed convex proper cone G in E whose polar set G° is contained in $(-U) \cup \{0\}$, and by considering only those open sets Ω of X such that:

$$\Omega = (\Omega + G) \cap X. \tag{0.2}$$

Let X_G be the space X endowed with the G-topology (i.e.: the open subsets of X_G satisfy (0.2)) and let be the continuous map $X \to X_G$. Let Ω_0 and Ω_1 be two G-open subsets of E such that $\Omega_0 \subset \Omega_1$, $\Omega_1 \setminus \Omega_0 \Subset X$.

Then one proves (cf. Theorem 3.2.2.) that for $F \in \mathsf{Ob}(\mathsf{D}^+(X))$ one has the isomorphism:

$$\Phi_G^{-1}\mathbf{R}\Phi_{G^*}\mathbf{R}\Gamma_{\Omega_1\setminus\Omega_0}(F) \xrightarrow{\sim} F \quad \text{in} \quad \mathsf{D}^+(X : \mathrm{Int}(\Omega_1 \setminus \Omega_0) \times \mathrm{Int}(-G^\circ))$$

and moreover:

$$\mathrm{SS}(\Phi_G^{-1}\mathbf{R}\Phi_{G^*}\mathbf{R}\Gamma_{\Omega_1\setminus\Omega_0}(F)) \subset X \times \mathrm{Int}(-G^\circ)$$

Chapter 1

Preliminaries

1.1 Notations and conventions

1.2 Normal cones

1.3 Sheaves

1.4 An extension theorem for sheaves

1.5 G-toplogy

Bibliography

[Arn67] Vladimir I. Arnold, *On a characteristic class entering into conditions of quantization*, Funkcional. Anal. i Prilozen (1967), 1–14. in Russian.

[Gab81] Ofer Gabber, *The integrability of the characteristic variety*, Amer. Journ. Math. 103 (1981), 445-–468.

[GKS12] Stéphane Guillermou, Masaki Kashiwara and Pierre Schapira, *Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems*, Duke Math. J. 161(2): 201–245 (2012).

[KS90] Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer-Verlag, Berlin, 1990.

[Ler76] Jean Leray, *Analyse Lagrangienne et mécanique quantique*, Collège de France, 1976.

[Mas65] Viktor P. Maslov, *Theory of perturbations and asymptotic methods*, Moskow Gos. Univ., 1965. [邦訳] マスロフ, 摂動論と漸近的方法, 岩波書店, 1976 年.

[Sch21] Pierre Schapira, *Microlocal analysis and beyond*, New spaces in Mathematics, edited by Mathieu Anel and Gabriel Catren, Cambridge University Press, 2021, pp. 117–152.

[SKK73] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau), Springer, Berlin, 1973, pp. 265—529. Lecture Notes in Math., Vol. 287.