# Normal Deformation and Normal Cones 本多研 院生ゼミ

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- X: a manifold of dim M = n
- $M \subset X$ : a closed submanifold of  $\operatorname{codim} M = l$
- $T_MX$ : the normal bundle to M in X

We defined the **normal deformation** of M in X:

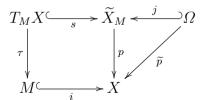
- $\bullet$   $\widetilde{X}_M$
- $p \colon \widetilde{X}_M \to X$
- $t \colon \widetilde{X}_M \to \mathbf{R}$

p and t satisfy the following conditions:

(4.1.3) 
$$\begin{cases} p^{-1}(X - M) \cong (X - M) \times (\mathbf{R} - \{0\}), \\ t^{-1}(\mathbf{R} - \{0\}) \cong X \times (\mathbf{R} - \{0\}), \\ t^{-1}(0) \cong T_M X. \end{cases}$$

- $\Omega := t^{-1}(]0, +\infty[)$
- $j \colon \Omega \hookrightarrow \widetilde{X}_M$
- $\bullet \ \widetilde{p} \coloneqq p \circ j$

(4.1.5)



#### Claim

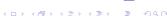
 $\widetilde{p}$  is smooth and  $\Omega$  is isomorphic to  $X \times \mathbf{R}^+$  by the map  $(\widetilde{p}, t)$ .

**Proof.** We have  $\widetilde{p}^{-1}(X) = j^{-1}p^{-1}(X) = \Omega$  by the definition of  $\widetilde{p}$  and the surjectivity of p. The condition about tangent maps is a local property, and the claim follows.

We have  $t^{-1}(\mathbf{R}^+) \cong \Omega$ . Therfore

$$(\widetilde{p}, t) (\Omega) \cong \widetilde{p}(\Omega) \times t(\Omega)$$
  
 $\cong X \times \mathbf{R}^+.$ 

The inverse morphism is induced by (4.1.3).



#### **Claim**

 $p^{-1}(M)$  is the union of  $T_MX$  and  $M \times \mathbf{R}$ .

## Proof. We can see locally

$$p^{-1}(M) = \left\{ (x,t) \in \widetilde{X}_M; \ (tx',x'') \in M \right\}$$
$$= \left\{ (x,t) \in \widetilde{X}_M; \ tx' = 0 \right\}$$
$$= \left\{ (x,t) \in \widetilde{X}_M; \ t = 0, \text{ or } \ x' = 0 \right\}$$
$$= T_M X \cup (M \times \mathbf{R}).$$

#### Claim

 $T_MX \cap (M \times \mathbf{R}) = M \times \{0\}$  coincides with the zero-section of  $T_MX$ .

**Proof.** As how we consider above,

$$T_M X \cap (M \times \mathbf{R}) = t^{-1}(0) \cap (M \times \mathbf{R})$$
$$= \left\{ (x, t) \in \widetilde{X}_M; \ t = 0, \text{ and } x' = 0 \right\}$$
$$= M \times \{0\},$$

and  $M \times \{0\} \cong M \subset T_M X$ .

# Normal Cones

# Definition ([KS90, Def.4.1.1])

(i) For  $S \subset X$ , the normal cone to S along M is

$$C_M(S) = T_M X \cap \overline{\widetilde{p}^{-1}(S)}.$$

(ii) 
$$S_1, S_2 \subset X$$

## References I

[KS90] Kashiwara, Schapira Sheaves on Manifolds, Springer, 1990.