Normal Deformation and Normal Cones 本多研 院生ゼミ

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- X: a manifold of $\dim M = n$
- $M \subset X$: a closed submanifold of $\operatorname{codim} M = l$
- T_MX : the normal bundle to M in X

We defined the **normal deformation** of M in X:

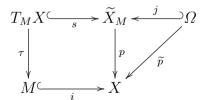
- \bullet \widetilde{X}_M
- $p \colon \widetilde{X}_M \to X$
- $t: \widetilde{X}_M \to \mathbf{R}$

p and t satisfy the following conditions:

(4.1.3)
$$\begin{cases} p^{-1}(X - M) \cong (X - M) \times (\mathbf{R} - \{0\}), \\ t^{-1}(\mathbf{R} - \{0\}) \cong X \times (\mathbf{R} - \{0\}), \\ t^{-1}(0) \cong T_M X. \end{cases}$$

- $\Omega := t^{-1}(]0, +\infty[)$
- $j \colon \Omega \hookrightarrow \widetilde{X}_M$
- $\bullet \ \widetilde{p} \coloneqq p \circ j$

(4.1.5)



Claim

 \widetilde{p} is smooth and Ω is isomorphic to $X \times \mathbf{R}^+$ by the map (\widetilde{p}, t) .

Proof. We have $\widetilde{p}^{-1}(X) = j^{-1}p^{-1}(X) = \Omega$ by the definition of \widetilde{p} and the surjectivity of p. The condition about tangent maps is a local property, and the claim follows.

We have $t^{-1}(\mathbf{R}^+) \cong \Omega$. Therfore

$$(\widetilde{p}, t) (\Omega) \cong \widetilde{p}(\Omega) \times t(\Omega)$$

 $\cong X \times \mathbf{R}^+.$

The inverse morphism is induced by (4.1.3).





Claim

 $p^{-1}(M)$ is the union of T_MX and $M\times \mathbf{R}$.

Proof. We can see locally

$$p^{-1}(M) = \left\{ (x,t) \in \widetilde{X}_M; \ (tx',x'') \in M \right\}$$
$$= \left\{ (x,t) \in \widetilde{X}_M; \ tx' = 0 \right\}$$
$$= \left\{ (x,t) \in \widetilde{X}_M; \ t = 0, \text{ or } \ x' = 0 \right\}$$
$$= T_M X \cup (M \times \mathbf{R}).$$

Claim

 $T_MX \cap (M \times \mathbf{R}) = M \times \{0\}$ coincides with the zero-section of T_MX .

Proof. As how we consider above,

$$T_M X \cap (M \times \mathbf{R}) = t^{-1}(0) \cap (M \times \mathbf{R})$$
$$= \left\{ (x, t) \in \widetilde{X}_M; \ t = 0, \text{ and } x' = 0 \right\}$$
$$= M \times \{0\},$$

and $M \times \{0\} \cong M \subset T_M X$.

Definition ([KS90, Def.4.1.1])

(i) For $S \subset X$, the **normal cone to** S **along** M is

$$C_M(S) = T_M X \cap \overline{\widetilde{p}^{-1}(S)}.$$

(ii) For $S_1, S_2 \subset X$, we define $C(S_1, S_2) := C_{\Delta_X}(S_1 \times S_2) \subset TX$.

$$\begin{array}{ccc} T_{\Delta_X}(X \times X) & \longrightarrow & TX \\ & & & & & \\ ((x,x),(\xi,0)) & \longmapsto & (x,\xi) \end{array}$$

We can write an element of $T_{\Delta_X}(X \times X)$ as

$$((x,x),(\xi,0))$$
.

Indeed, for $(\xi, \eta) \in T_{(x,x)}(X \times X)$, we have

$$(\xi, \eta) + T_{(x,x)} \Delta_X = \{ (\xi + \zeta, \eta + \zeta); \zeta \in T_{(x,x)} \Delta_X \}$$

$$= \{ (\xi + (\zeta - \eta), \eta + (\zeta - \eta)); \zeta \in T_{(x,x)} \Delta_X \}$$

$$= \{ (\xi - \eta, 0) + \zeta; \zeta \in T_{(x,x)} \Delta_X \}$$

$$= (\xi - \eta, 0) + T_{(x,x)} \Delta_X.$$

That is to say, we can make any vectors (ξ, η) in $T_{\Delta_X}(X \times X)$ in the form $(\zeta, 0)$ preserving the difference between the first and second entries.

Claim

- 1. $C_M(S)$ is a closed conic subset of T_MX .
- 2. its projection onto M is the set $M \cap \overline{S}$.

Proof. 1. The claim follows from

$$C_M(S) = T_M X \cap \overline{\widetilde{p}^{-1}(S)}$$

$$= T_M X \cap \overline{\{(x'/t, x'') \in \Omega; x \in S, t \in \mathbf{R}^+\}}$$

$$= T_M X \cap \overline{\mathbf{R}^+ \{(x', x'') \in \Omega; x \in S\}}.$$

Proof. 2. ????

$$\tau(C_M(S)) \stackrel{?}{=} M \cap \overline{S}$$

$$\tau(C_M(S)) = \tau(T_M X \cap \overline{\widetilde{p}^{-1}(S)})$$

$$= \tau\left(s^{-1}\left(\overline{\widetilde{p}^{-1}(S)}\right)\right)$$

$$= \tau\left(s^{-1}\left(\overline{j^{-1}(p^{-1}(S))}\right)\right)$$

$$= \tau\left(\overline{s^{-1}\left(\overline{j^{-1}(p^{-1}(S))}\right)}\right)$$

As a consequence of the claim,

$$C(S_1, S_2) \subset TX$$
: closed conic.

If M is a closed submanifold, then C(S,M) is the inverse image of $C_M(S)$ by the projection $M\times_X TX\to T_MX$ by the next proposition.

References I

[KS90] Kashiwara, Schapira Sheaves on Manifolds, Springer, 1990.