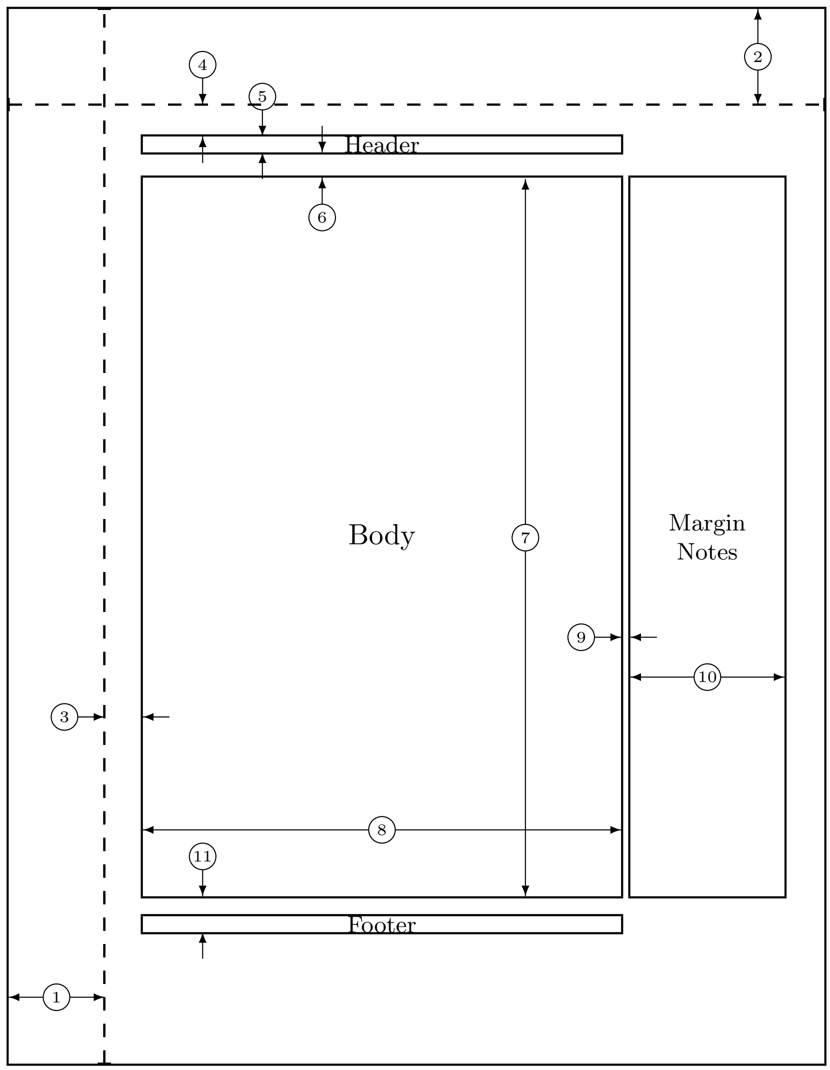
Systems of Microdifferential Equations

# Contents

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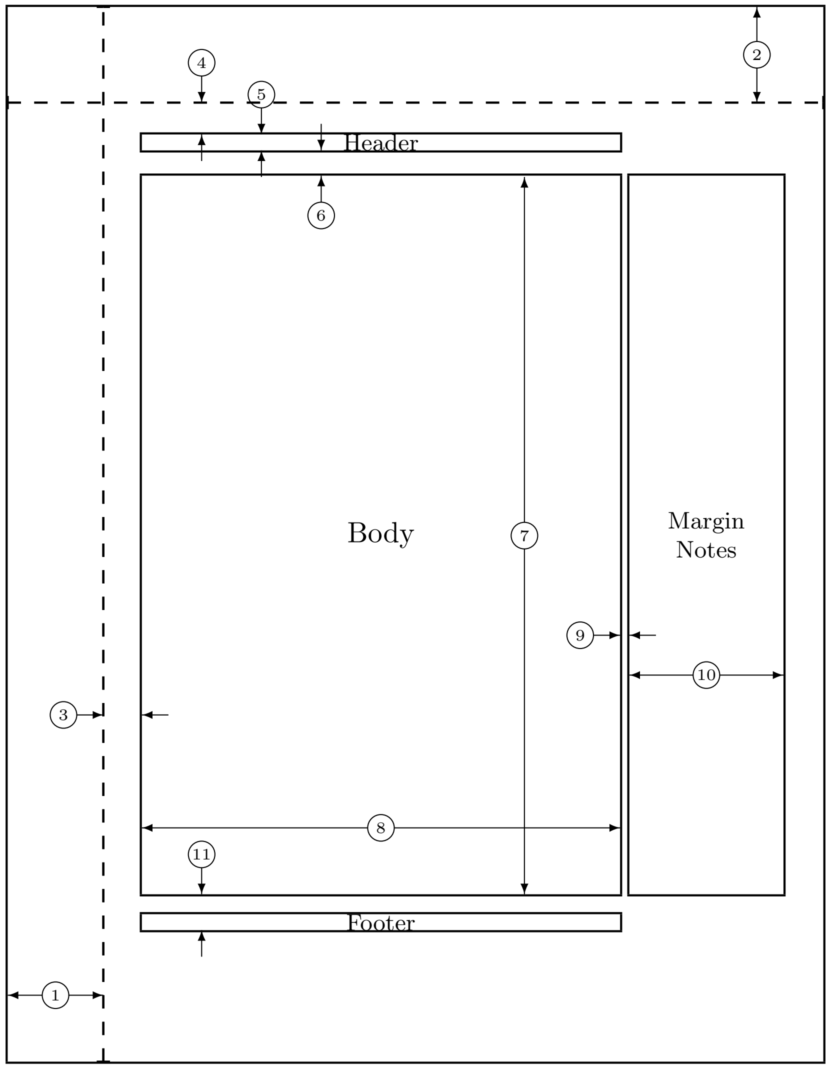


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# Introduction

This book is grew out from a course that Masaki Kashiwara gave at the "Université Paris-Nord" during the first term of the academic year 1976–77. Teresa Monterio Fernandes worked out lecture notes for this course, which were preprinted in 1979 by Paris-Nord under the title "Systèms d'équations microdifférentielles" and distributed to a happy few. On the grounds that such a basic textbook should be made available to a wider mathematical audience, the Birkhäuser Publishing Company proposed to make it a new volume of its series "Progress in Mathematics". Kashiwara and the publisher agreed not to change the overall structure of the text (which at first was supposed to be provisional); T. Monterio Fernandes then was kind enough to make care of the translation into English and of the necessary minor corrections.

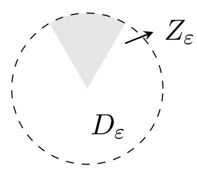
It was decided that an introduction might be written, outlining the purposes and main features of the text, and mentioning recent developments connected with the index fomula in Chapter 6. Kashiwara suggested that I would write this introduction, and I accepted to do so with pleasure. I thought it might be a way of thanking Kashiwara for all he has taught me since I benefited so much from talking to him and reading his works.

In short, this book is an introduction to systems of microdifferential equations and to some of the tools of micro-local analysis. Here a remark on the terminology; after [Boutet de Monvel-Kree], pseudo-differential operators on a complex manifold $X$ were considered in [44]: They used the notation $\mathcal{P}_X$ for the sheaf of rings of pseudo-differential operators (this was a sheaf on the projectivised cotangent bundles of $X$). In recent years, instead $\mathcal{P}_X$, one has been using $\mathcal{E}_X$ (now viewed as a sheaf of rings on the cotangent bundle itself); $\mathcal{E}_X$ is called the sheaf of *micro-differential operators*: micro-differential operators are defined and studied locally in the cotangent bundle, i.e. *microlocally.*

As is well-known, say for $X$ of complex dimension one with variable $t$, both $\frac{d}{dt}$ and $\left(\frac{d}{dt}\right)^{-1}$ are micro-differential operators (more precisely, as a section of $\mathcal{E}_X$, $\left(\frac{d}{dt}\right)^{-1}$ is defined outside the zero-section of the cotangent bundle). An infinite series $\sum_{n\geqq 0} a_n(t)\left(\frac{d}{dt}\right)^{-n}$ also defines a micro-differential operator, providing the holomorphic functions $a_n(t)$ satisfy, on any compact $K$ of $X$, an estimate of the type

$$\sup_{t\in K}|a_n(t)| < (n!)C_K^n, \quad \text{for some } C_K > 0.$$

There is no way to make such operators act on the space of holomorphic functions. However, let $X = \mathbf{C}$ and take a closed angular sector $Z_\varepsilon$ inside the disk $D_\varepsilon = \{t \in \mathbf{C}; |t| < \varepsilon\}$.




Let $\mathcal{O}(D_\varepsilon)$ resp. $\mathcal{O}(D_\varepsilon - Z_\varepsilon)$ be the space of holomorphic functions on $D_\varepsilon$ resp. $D_\varepsilon - Z_\varepsilon$; then $\left(\frac{d}{dt}\right)^{-1}$ operates on $\mathcal{O}(D_\varepsilon - Z_\varepsilon)|\mathcal{O}(D_\varepsilon)$, to get an operation of microdifferential operators, it suffices to make $\varepsilon$ smaller and smaller, i.e. to consider $\varinjlim_{\varepsilon\to 0} \mathcal{O}(D_\varepsilon - Z_\varepsilon) \mid \mathcal{O}(D_\varepsilon)$.

The reader will find two definitions

# Chapter 1

# Microfunctions

# Chapter 2

# Microdifferential Systems

# Chapter 3

# Structure of Coherent $\mathcal{E}_X$-modules

Chapter 4

# Holomorphic Solutions of Systems of Partial Differential Equations

# Chapter 5

# Solutions of Holonomic Systems

# Chapter 6

# Index Theorems

# Appendix A

# Derived Categories and Functors

# Appendix B

# Whitney Stratification and Constructible Sheaves

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