Normal Deformation and Normal Cones

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Normal Deformation

- X: a manifold of $\dim M = n$
- $M \subset X$: a closed submanifold of$\operatorname{codim} M = l$
- $T_M X$: the normal bundle to M in X

We defined the **normal deformation** of M in X:

- $\widetilde{X}_M$
- $p\colon \widetilde{X}_M \to X$
- $t\colon \widetilde{X}_M \to \mathbf{R}$

Normal Deformation

p and t satisfy the following conditions:

$$
(4.1.3) \qquad \begin{cases} p^{-1}(X - M) \cong (X - M) \times (\mathbf{R} - \{0\}), \\ t^{-1}(\mathbf{R} - \{0\}) \cong X \times (\mathbf{R} - \{0\}), \\ t^{-1}(0) \cong T_M X. \end{cases}
$$

Normal Deformation

- $\Omega := t^{-1}\big(]0, +\infty[\big)$
- $j \colon \Omega \hookrightarrow \widetilde{X}_M$
- $\widetilde{p} := p \circ j$

(4.1.5)
$$
\begin{tikzcd}
T_M X \arrow[r, hook, "s"'] \arrow[d, "\tau"'] & \widetilde{X}_M \arrow[d, "p"] & \Omega \arrow[l, hook', "j"'] \arrow[dl, "\widetilde{p}"] \\
M \arrow[r, hook, "i"'] & X &
\end{tikzcd}
$$

Normal Deformation

Claim

$\widetilde{p}$ is smooth and Ω is isomorphic to $X \times \mathbf{R}^+$ by the map $(\widetilde{p}, t)$.

Proof. We have $\widetilde{p}^{-1}(X) = j^{-1}p^{-1}(X) = \Omega$ by the definition of $\widetilde{p}$ and the surjectivity of p. The condition about tangent maps is a local property, and the claim follows.
We have $t^{-1}(\mathbf{R}^+) \cong \Omega$. Therfore

$$\begin{aligned}(\widetilde{p}, t)(\Omega) &\cong \widetilde{p}(\Omega) \times t(\Omega) \\ &\cong X \times \mathbf{R}^+.\end{aligned}$$

The inverse morphism is induced by (4.1.3). □

Normal Deformation

Claim

$p^{-1}(M)$ is the union of $T_M X$ and $M \times \mathbf{R}$.

Proof. We can see locally

$$
\begin{aligned}
p^{-1}(M) &= \left\{(x,t) \in \widetilde{X}_M;\ (tx', x'') \in M\right\} \\
&= \left\{(x,t) \in \widetilde{X}_M;\ tx' = 0\right\} \\
&= \left\{(x,t) \in \widetilde{X}_M;\ t = 0,\ \text{or}\ \ x' = 0\right\} \\
&= T_M X \cup (M \times \mathbf{R}).
\end{aligned}
$$

$\square$

Normal Deformation

Claim

$T_M X \cap (M \times \mathbf{R}) = M \times \{0\}$ coincides with the zero-section of $T_M X$.

Proof. As how we consider above,

$$\begin{aligned} T_M X \cap (M \times \mathbf{R}) &= t^{-1}(0) \cap (M \times \mathbf{R}) \\ &= \left\{ (x, t) \in \widetilde{X}_M ;\ t = 0, \text{ and } x' = 0 \right\} \\ &= M \times \{0\}, \end{aligned}$$

and $M \times \{0\} \cong M \subset T_M X$. □

Normal Cones

Definition ([KS90, Def.4.1.1])

(i) For $S \subset X$, the **normal cone to** S **along** M is

$$C_M(S) = T_M X \cap \overline{\widetilde{p}^{-1}(S)}.$$

(ii) $S_1, S_2 \subset X$

References I

[KS90] Kashiwara, Schapira *Sheaves on Manifolds*, Springer, 1990.