Hyperfunctions and Pseudo-differential Equations

March 21, 2024

Contents

Preface for Part II

This is the last of two parts of the Proceedings of the conference on Hyperfunctions and Pseudo-Differential Equations held at Katata on October 12–14, 1971.

This part consists of a paper by M. Sato, T. Kawai and M. Kashiwara which is an enlarged version of four lectures by them delivered at the conference.

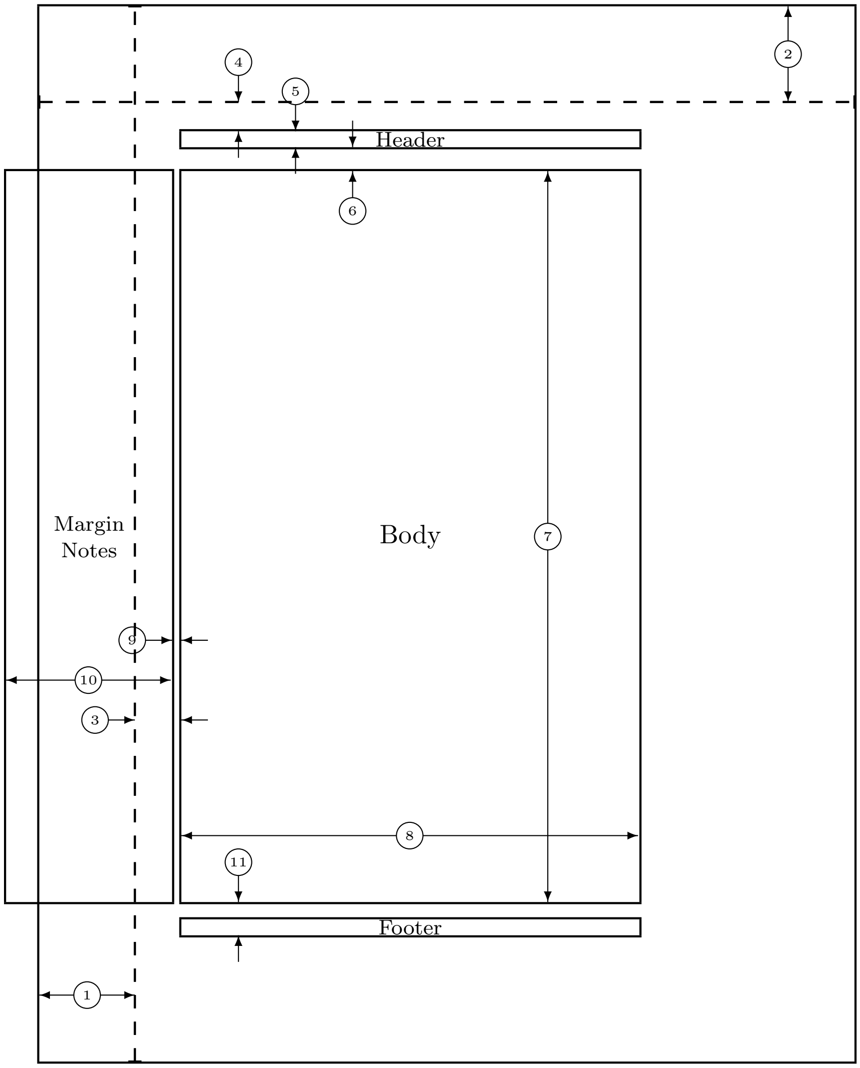
We received the final manuscript in June, 1971 but have postponed the publication because the authors had the intention of adding an introduction to the paper. Since we do not think it appropriate to wait for it forever, we have decided to publish this part in the present form.

In place of the introduction, we advise the reader to read the lectures by the authors at different occasions, the Nice Congress, 1970, the A. M. S. Symposium on Partial Differential Equations at Berkeley, 1971, and the Colloque C. N. R. S. Equations aux Dérivées Partielles Linéaires at Orsay, 1972.

We thank Miss C. Sagawa for typing and Mr. T. Miwa and Mr. T. Oshima for proof-reading.

December 28, 1972

Hikosaburo Komatsu



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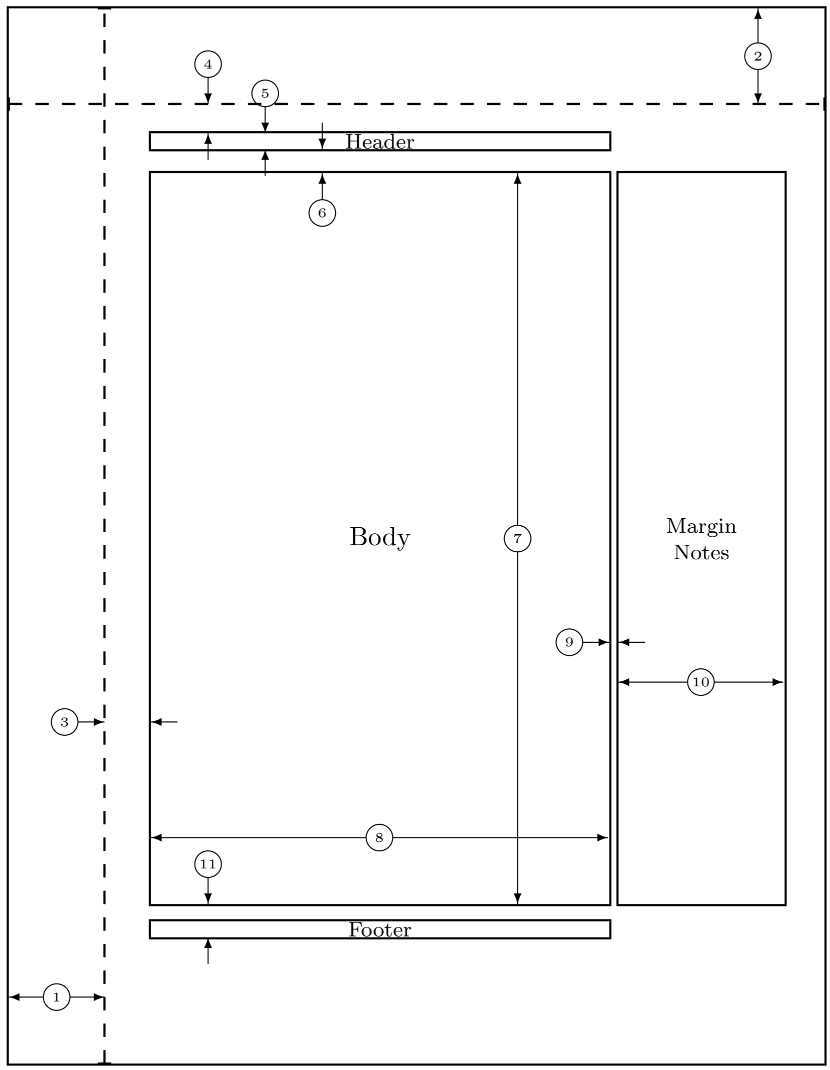

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Chapter 1

Theory of Microfunctions

1.1 Construction of the sheaf of microfunctions

1.1.1 Hyperfunctions

Let M be an n-dimentional real analytic manifold and X be a complex neighborhood of M. X is uniquely determined by M if we pay attention only to a neighborhood of M. We denote by $\mathscr{O}_X$ the sheaf of holomorphic functions on X and by $\mathscr{A}_M$ the sheaf of real analytic functions on M, that is, $\mathscr{A}_M = \iota^{-1}\mathscr{O}_X$ by definition, where $\iota\colon M \hookrightarrow X$ is the canonical injection. We denote by ω_M the sheaf of orientation of M. ω_M is isomorphic to $\mathscr{H}^n_M(\mathbb{Z}_M)$. ω_M is locally isomorphic to $\mathbb{Z}_M$, and giving an isomorphism $\omega_M|_U \simeq \mathbb{Z}_M|_U$ on an open subset U of M is equivalent to giving an orientation of U.

As in [Sato1], we define the sheaf of hyperfunctions on M:

Definition 1.1.1. The sheaf $\mathscr{B}_M$ is by definition

$$\mathscr{B}_M = \mathscr{H}^n_M(\mathscr{O}_X) \otimes_{\mathbb{Z}_M} \omega_M. \tag{1.1.1.1}$$

A section of $\mathscr{B}_M$ is called a *hyperfunction*.

As stated in [Sato1], $H^i_M(\mathscr{O}_X) = 0$ for $i \neq n$ and $\mathscr{B}_M$ constitutes a flabby sheaf on M.

We first recall the fallowing general lemma:

Lemma 1.1.2. Let Y be a d-codimensional submanifold of a topological manifold X of dimension n. Then, for any sheaf (or complex of sheaves) $\mathscr{F}$ on X, we can define the following homomorphism

$$\mathscr{F}|_Y \longrightarrow \mathbb{R}\Gamma_Y(\mathscr{F})[d] \otimes \omega_{Y/X}, \tag{1.1.1.2}$$

where $\omega_{Y/X} = \mathscr{H}^d_Y(\mathbb{Z}_X)$ is the orientation sheaf of $Y \subset X$ and $\mathbb{R}$ and Γ_Y denote respectively the derived functor in the derived category and the functor of taking the sub sheaf with support in Y of [Hatshorne1].

Proof. Since $\mathbb{R}\Gamma_Y(\mathbb{Z}_X) = \omega_{Y/X}[-d]$, we obtain the desired homomorphism as the composite of the following:

$$\begin{aligned}\mathscr{F}|_Y \simeq \mathscr{F} \otimes_{\mathbb{Z}_X} \mathbb{Z}_Y \simeq \mathscr{F} \otimes_{\mathbb{Z}_X} \mathbb{R}\Gamma_Y(\mathbb{Z}_X) \otimes \omega_{Y/X}[d] \\ \longrightarrow \mathbb{R}\Gamma_Y(\mathscr{F})[d] \otimes \omega_{Y/X}[d].\end{aligned}$$

q.e.d.

We apply this lemma to our case where $\mathscr{F}, X, Y$ correspond to $\mathscr{O}_X, X$ and M respectively. Then we obtain the sheaf homomorphism

$$\mathscr{A}_M \longrightarrow \mathscr{B}_M, \tag{1.1.1.3}$$

which will be proved to be injective later. This injection allows us to consider hyperfunctions as a generalization of functions. The purpose of this section is to analyse the structure of the quotient sheaf $\mathscr{B}_M/\mathscr{A}_M$ from a very new point of view.

1.1.2 Real monoidal transformation and real comonoidal transformation

Now consider the following situation, although we apply it to a special case in this section.

Let N and M be real analytic manifolds and $f\colon M \to N$ be a real analytic map. We denote by TN (resp. TM) the tangent vector bundle of N (resp. M) and by T^*N (resp. T^*M) the cotangent vector bundle over N (resp. M). We can define the following canonical homomorphisms:

$$\begin{aligned}&0 \to TM \to TN \times_N M \to T_M N \to 0 \quad \text{(when } f \text{ is an embedding)} \\ &T^*M \leftarrow T^*N \times_N M \leftarrow T^*_M N \leftarrow 0\end{aligned} \tag{1.1.2.1}$$

where $T_M N$ (resp. $T^*_M N$) is the normal (resp. conormal) fiber space. We denote by SM (resp. S^*M, SN, S^*N, $S_M N$, $S^*_M N$) the spherical bundle $(TM - M)/\mathbb{R}^+$ (resp. $(T^*M - M)/\mathbb{R}^+$,…), where $\mathbb{R}^+$ is the multiplicative group of strictly positive real numbers. $S^*_M N$ is not necessarily a fiber bundle.

Then,

$$S^*_M N \hookrightarrow S^*N \times_N M$$

and we have a projection

$$\rho\colon S^*N \times_N M - S^*_M N \longrightarrow S^*M. \tag{1.1.2.2}$$

Suppose moreover that $\iota\colon M \to N$ is an embedding. Then we can provide the disjoint union $\widetilde{M_N} = (N - M) \sqcup S_M N$ with a structure of real analytic manifold with boundary $S_M N$. Since this is constructed in the same way as monoidal transforms of complex manifolds, we call $\widetilde{M_N}$ the *real monoidal transform* of

N with center M. Let $\{U_j\}$ be a set of coordinate patches of N with a local coordinate $x_j = (x_j^1, \dots, x_j^n)$ such that

$$M \cap U_j = \{x_j \in U_j;\ x_j^1 = \cdots = x_j^m = 0\}.$$

Let

$$x_j^\nu = f_{jk}^\nu(x_k) \quad \nu = m+1, \dots, n, \tag{1.1.2.3}$$

$$x_j^\nu = \sum_{\mu=1}^{m} x_k^\mu g_{jk,\mu}^\nu(x_k) \quad \nu = 1, \dots, m \tag{1.1.2.4}$$

be a coordinate transformation. We put

$$U_j' = \Big\{(x_j, \xi_j);\quad x_j = (x_j^1, \dots, x_j^n) \in U_j,\ \xi_j = (\xi_j^1, \dots, \xi_j^m) \in \mathbb{R}^m - \{0\}$$
$$\text{such that}\quad x_j^\nu \xi_j^\mu = x_j^\mu \xi_j^\nu \quad \text{for} \quad \nu, \mu = 1, \dots, m, \quad x_j^\nu \xi_j^\nu \geqq 0\Big\}.$$

The multiplicative group $\mathbb{R}^+$ of positive numbers operates on U_j' by

$$((x_j, \xi_j), t) \mapsto (x_j, t\xi_j).$$

We denote by $\widetilde{U_j}$ the quotient $U_j'/\mathbb{R}^+$. We glue together $\widetilde{U_j}$ in the following manner: $(x_j, \xi_j) \in U_j$ and $(x_k, \xi_k) \in U_k$ are identified if x_j and x_k satisfy (1.1.2.3) and (1.1.2.4) and

$$\xi_j^\nu = \sum_{\mu=1}^{m} \xi_k^\mu g_{jk,\mu}^\nu(x_k) \quad \nu = 1, \dots, m.$$

We denote by $\widetilde{M_N}$ the real analytic manifold with boundary obtained by gluing $\widetilde{U_j}$. Then, $\tau\colon \widetilde{M_N} \to N$ is the projection defined by $\widetilde{U_j} \ni (x_j, \xi_j) \overset{\tau}{\mapsto} x_j \in U_j$. Then, $\tau^{-1}(M)$ is isomorphic to the normal spherical bundle $S_M N$, and seen to be the boundary of $\widetilde{M_N}$. Moreover, τ gives an isomorphism $\widetilde{M_N} - S_M N \to N - M$. For a tangent vector $\xi \in T_M N_X - \{0\}$, we denote by $x + \xi 0$ the corresponding point of $S_M N \subset \widetilde{M_N}$.

$D_M N$ is a subset of the fiber product $S_M N \times_M S_M^* N$ defined by $\{(\xi, \eta) \in S_M N \times_M S_M^* N; \langle \xi, \eta \rangle \geqq 0\}$. We define the toplogy on the set $\widetilde{M_{N^+}} = (N - M) \sqcup D_M N$ as follows: $N^M \subset \widetilde{M_{N^+}}$ is an open set and the toplogy of $N - M$ induced from $\widetilde{M_{N^+}}$ is usual one, and for a point $x \in D_M N \subset \widetilde{M_{N^+}}$, a neighborhood of x is a subset U such that $U \cap D_M N$ is a neighborhood of x with respect to the usual toplogy of $D_M N$ ant that the image of U under the projection $\pi\colon \widetilde{M_{N^+}} \to \widetilde{M_N}$ is a neighborhood of $\pi(x)$. We note that the topology of $\widetilde{M_{N^+}}$ is not Hausdorff.

Let $\widetilde{M_N^*}$ be disjoint union of $(N - M)$ and $S_M^* N$, $\tau\colon \widetilde{M_{N^*}}$, $\pi\colon \widetilde{M_{N^*}} \to N$ be the canonical projections. $\widetilde{M_{N^*}}$ will be equipped with the quotient toplogy of $\widetilde{M_{N^*}}$ under τ.

In this way we obtain a diagram of maps of topological spaces:

$$
\begin{tikzcd}
 & \widetilde{M_{N^+}} \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & D_M N \arrow[l, hook'] \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & \\
\widetilde{M_N} \arrow[dr, "\tau"'] & S_M N \arrow[l, hook'] \arrow[dr, "\tau"] & \widetilde{M_{N^*}} \arrow[dl, "\pi"'] & S_M^* N \arrow[l, hook'] \arrow[dl, "\pi"] \\
 & N & M \arrow[l, hook'] &
\end{tikzcd} \quad . \tag{1.1.2.5}
$$

Note that

1) all horizontal inclusionas are closed embeddings;

2) $\widetilde{M_{N^+}}$ can be considered as a closed subspace of $\widetilde{M_N} \underset{N}{\times} \widetilde{M_{N^*}}$;

3) $\widetilde{M_N} \to N$ and $\widetilde{M_{N^+}} \to \widetilde{M_{N^*}}$ are proper and separated.

Remark 1.1.3. The map $f\colon X \to Y$ of topological spaces is said to be *separated* if X is closed in $X \times_Y X$. f is said to be *proper* if every fibre of f is compact and f is closed (that is, the image of a closed set in X by f is closed in Y). The following lemma is used frequetly in this note.

Lemma 1.1.4. Let $f\colon X \to Y$ be separated and proper, $\mathscr{F}$ be a sheaf on X. Then, for every point y of Y, the homomorphism

$$\mathrm{R}^k f_*(\mathscr{F})_y \to H^k\left(f^{-1}(y); \mathscr{F}|f^{-1}(y)\right)$$

is isomorphic for every integer k.

For the proof, we refer to Bredon.

In the sequel, the notion of derived category will be of constant use. We refer to Hartshorne as to derived category. We will not distinguishe the sheaf, the complex of sheaves and the corresponding object of the derived category.

Proposition 1.1.5. *Let $\mathscr{F}$ be a complex of sheves on N (or more precisely an object of the derived category of sheaves on N). Then we have an isomorphism*

$$\mathrm{R}\tau_*\pi^{-1}\,\mathrm{R}\Gamma_{S_M N}\left(\tau^{-1}\mathscr{F}\right) \xrightarrow{\sim} \mathrm{R}\Gamma_{S_M^* N}\left(\pi^{-1}\mathscr{F}\right).$$

Proof. At first note that

$$\pi^{-1}\,\mathrm{R}\Gamma_{S_M N}\left(\tau^{-1}\mathscr{F}\right) \xrightarrow{\sim} \mathrm{R}\Gamma_{D_M N}\left(\pi^{-1}\tau^{-1}\mathscr{F}\right)$$

is an isomorphism. This follows from the fact that for every point x in $D_M N$, the family $\{U - S_M N\}$ where U runs through the neighborhoods of $\pi(x)$ is equivalent to the family $V - D_M N$ where V runs through the neighborhoods of x.

Now we have a triangle:

$$\begin{tikzcd}[column sep=small] & \mathrm{R}\Gamma_{S_M^* N}\,\mathrm{R}\mathscr{D}ist_\tau\left(\pi^{-1}\mathscr{F}\right) \arrow[dr] & \\ \mathrm{R}\tau_*\,\mathrm{R}\Gamma_{D_M N}\left(\tau^{-1}\pi^{-1}\mathscr{F}\right) \arrow[ur, "+1"] & & \mathrm{R}\Gamma_{S_M^* N}\left(\pi^{-1}\mathscr{F}\right). \arrow[ll] \end{tikzcd}$$

(See Hartshorne for the notion of triangle.) Since $\tau\colon \widetilde{M_{N^+}} \to \widetilde{M_{N^*}}$ is proper and separated with contractible fiber,

$$\mathrm{R}\mathscr{D}ist_\tau\left(\pi^{-1}\mathscr{F}\right) = 0.$$

This proves the isomorphism

$$\mathrm{R}\tau_*\,\mathrm{R}\Gamma_{D_M N}\left(\tau^{-1}\pi^{-1}\mathscr{F}\right) \xrightarrow{\sim} \mathrm{R}\Gamma_{S_M^* N}\left(\pi^{-1}\mathscr{F}\right).$$

q.e.d.

Remark 1.1.6. Let $f\colon X \to Y$ be a continuous map, and $\mathscr{F}$ be a sheaf on Y. Let $\mathscr{F} \to \mathscr{L}^\bullet$, $f^{-1}\mathscr{F} \to \mathscr{M}^\bullet$ be flabby resolutions of $\mathscr{F}$ and $f^{-1}\mathscr{F}$ with $\mathscr{L}^\bullet \to f_*\mathscr{M}^\bullet$.

1.1.3 Definition of microfunctions

Now we will come back to the original situation.

1.1.4 Sheaves on sphere bundle and on cosphere bundle

We consider the following situation.

1.1.5 Fundamental diagram on $\mathscr{C}$

We will apply the arguements in the preceding section to a special case.

1.2 Several oprations on hyperfunctions and microfunctions

1.2.1 Linear differential operators

1.2.2 Substitution

1.2.3 Integration along fibers

1.2.4 Products

1.2.5 Micro-local operators

1.2.6 Complex conjugation

1.3 Techniques for construction of hyperfunctions and microfunctions

1.3.1 Real analytic functions of positive type

1.3.2 Boundary values of hyperfunctions with holomorphic parameters and examples

Chapter 2

Foundation of the Theory of Pseudo-differential Equations

2.1 Definition of pseudo-differential operators

Is a

2.2 Fundamental properties of pseudo-differential operators

2.2.1 Theorems on ellipticity and the equivalence of pseudo-differential operators

2.2.2 Theorems on division of pseudo-differential operators

2.3 Algebraic properties of the sheaf of pseudo-differential operators

2.3.1 Pseudo-differential operators with holomorphic parameters

2.3.2 Properties of the ring of formal pseudo-differential operators

2.3.3 Contact structure and quantized contact transforms

2.3.4 Faithful flatness

Remark 2.3.1. Let X be a complex manifold. We denote by $\mathscr{D}_X$ (resp. $\mathscr{D}_X^{\mathrm{f}}$) the sheaf of differential operators on X (resp. differential operators of finite order on X). $\mathscr{P}_X$ (resp. $\mathscr{P}_X^{\mathrm{f}}$) is a $\pi^{-1}\mathscr{D}_X$-Algebra (resp. $\pi^{-1}\mathscr{D}_X^{\mathrm{f}}$-Algebra). By using the method

2.3.5 Operations on systems of pseudo-differential equations

2.4 Maximally overdetermined systems

2.4.1 Definition of maximally overdetermined systems

2.4.2 Invariants of maximally overdetermined systems

2.4.3 Quantized contact transform — general case —

2.5 Structure theorem for systems of pseudo-differential equations in the complex domain

In this section we establish the fundamental theorem concerning the structure of a system of pseudo-differential equations of finite order in complex domain at generic points, i.e., we will firstly prove in theorem 2.5.1 as the simplest case that any system $\mathscr{M}$ of pseudo-differential equations of finite order with one unknown function and simple characteristics can be transformed micro-locally into the partial de Rham systems

$$\mathscr{N} : \frac{\partial}{\partial x_i'} u = 0, \quad i = 1, \dots, d$$

by a suitable "quantized" contact transformation. Here "micro-locally" means "locally on P^*X, not on X". In the sequal we use the word "micro-locally" in this sense (and sometimes in the sense that "locally on S^*M, not on M" when we consider the problems in the real domain). Later we extend Theorem2.5.1 to more general systems by the aid of pseudo-differential operators of infinite order.

2.5.1 Structure theorem for systems of pseudo-differential equations with simple characteristics

Theorem 2.5.1. *a*

2.5.2 Equivalence of pseudo-differential operators with constant multiple characteristics

Remark 2.5.2. This example shows that the structure of the hyperfunction solution sheaf, not merely the microfunction solution sheaf, of the equation of $P_1(D)u = 0$ and that of $P_2(D)u = 0$ are the same, because the operators $A_j(x, D)$ are differential operators, not merely pseudo-differential operators. Note that, more generally, if $P(x, D)$ is a linear differential operator of order m defined in a neighborfood of the origin of $\mathbb{C}^n$ whose principal symbol is ι_1^m, then the differential equation $P(x, D)u = 0$ and $D_1^m u = 0$ are equivalent as left $\mathscr{D}$-modules.

2.5.3 Structure theorem for regular systems of pseudo-differential equations

Chapter 3

Structure of Systems of Pseudo-differential Equations

3.1 Realification of holomorphic microfunctions

3.1.1 Realification of holomorphic hyperfunctions

3.1.2 Realification of holomorphic microfunctions

3.1.3 Real "quantized" contact transforms

3.2 Structure theorems for systems of pseudo-differential equations in the real domain

3.2.1 Structure theorem I — partial de Rham type —

3.2.2 Structure theorem II — partial Cauchy Riemann type —

3.2.3 Structure theorem III — Lewy-Mizohata type —

3.2.4 Structure theorem IV — general case —

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