

## Microlocal Study of Sheaves



# Introduction

Let  $X$  be a real manifold,  $F$  a sheaf on  $X$ , or better an object of  $D^+(X)$ , the derived category of the category of complexes bounded from below of sheaves on  $X$ . Let  $TX$  be the cotangent bundle to  $X$ . We associate to  $F$  a closed conic subset of  $TX$ , denoted  $SS(F)$ , the “micro-support of  $F$ ”, as follows:

**Definition 0.0.1 .** Let  $p = (x_0, \xi_0) \in T^*X$ . Then  $p \notin SS(F)$  if and only if there exists an open neighborhood  $U$  of  $p$  in  $TX$  such that for any  $(x_1, \xi_1) \in U$ , any real  $C^1$ -function  $\phi$  on  $X$ , with  $\phi(x_1) = 0$ ,  $d\phi(x_1) = \xi_1$ , we have:  $(\mathbf{R}\Gamma_{x; \phi(x) \geq 0}(F))_{x_1} = 0$ .

In other words the micro-support of  $F$  describes the set of codirections of  $X$  where  $F$ , and its cohomology, “do not propagate”. This definition is motivated by the following situation.

Assume  $X$  is a complex manifold, and let  $\mathcal{M}$  be a coherent module over the Ring  $\mathcal{D}_X$  of (holomorphic, finire order) differential operators. Let  $\text{char}(\mathcal{M})$  be the characteristic variety of  $\mathcal{M}$  in  $T^*X$ . Then we can interpret a well-known result of Zerner [1], Bony-Schapira [1], Kashiwara [5], through the formula:

$$SS(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \subset \text{char}(\mathcal{M}). \quad (0.0.1)$$

A natural problem then arising in the theory of (micro-)differential equations, is to evaluate the set of codirections of propagation for the sheaf of hyperfunction or microfunction solutions of  $\mathcal{M}$  (or more generally of a system of micro-differential equations). To be more precise, let  $M$  be a real analytic manifold of dimension  $n$ ,  $X$  a complexification of  $M$ . Recall that the sheaf  $\mathcal{B}_M$  (resp.  $\mathcal{C}_M$ ) of Sato’s hyperfunctions on  $M$  (resp. Sato’s microfunctions on  $T_M^*X$ , the conormal bundle to  $M$  in  $X$ ) is defined by:

$$\begin{aligned} \mathcal{B}_M &= \mathbf{R}\Gamma_M(\mathcal{O}_X) \otimes \omega_M[n] \\ (\text{resp. } \mathcal{C}_M &= \mu_M(\mathcal{O}_X) \otimes \omega_M[n]) \end{aligned}$$

where  $\omega_M$  is the orientation sheaf on  $M$ ,  $[n]$  means the  $n$ -shift in  $D^+(X)$ , and  $\mu(\bullet)$  is the functor of Sato's microlocalization along  $M$  (cf. Chapter 2).

Then the problem is: evaluate  $\text{SS}(\text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M))$  or  $\text{SS}(\text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M))$ .  
 Taking  $F = \text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)$  this is a particular case of the following problem: given  $F \in \text{Ob}(D^+(X))$ , and  $M$  a real submanifold of  $X$ , calculate  $\text{SS}(\text{R}\Gamma_M(F))$  or  $\text{SS}(\mu_M(F))$ .

As we see, in this new formulation, we may forget that  $X$  is a complex-manifold, and we do not study separately the  $\mathcal{D}_X$ -module  $\mathcal{M}$  from one side and the sheaf  $\mathcal{O}_X$  on the other side. On the contrary we work with the whole complex of solutions of  $\mathcal{M}$  in  $\mathcal{O}_X$ . The only information that we keep is the geometrical data of the characteristic variety, which is interpreted in terms of micro-support (in fact we shall prove in Chapter 10 that the inclusion in (0.1) is an equality).

Now let us come back to the subject of this paper.

We study in Chapter 4 and 5 the functorial properties of the micro-support: behavior under direct or inverse images, functors  $\text{R}\mathcal{H}om(\bullet, \bullet)$ ,  $\bullet \otimes^L \bullet$ , specialization, Fourier-Sato transformation, microlocalization (the construction of these functors are recalled in Chapter 1 and 2). But in order to manipulate micro-supports, the definition (0.1) given above is too much of a local one and one has to replace it by a more global criterium. This is achieved in Chapter 3, using a Mittag-Leffler procedure for sheaves (Theorem 1.4.3).

# Bibliography

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