Normal Deformation and Normal Cones

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sect.4.1

Normal Deformation

- X: a manifold of $\dim M = n$
- $M \subset X$: a closed submanifold of$\operatorname{codim} M = l$
- $T_M X$: the normal bundle to M in X

We defined the **normal deformation** of M in X:

- $\widetilde{X}_M$
- $p\colon \widetilde{X}_M \to X$
- $t\colon \widetilde{X}_M \to \mathbf{R}$

Normal Deformation

p and t satisfy the following conditions:

$$(4.1.3) \qquad \begin{cases} p^{-1}(X - M) \cong (X - M) \times (\mathbf{R} - \{0\}), \\ t^{-1}(\mathbf{R} - \{0\}) \cong X \times (\mathbf{R} - \{0\}), \\ t^{-1}(0) \cong T_M X. \end{cases}$$

Normal Deformation

- $\Omega \coloneqq t^{-1}\big(]0,+\infty[\big)$
- $j\colon \Omega \hookrightarrow \widetilde{X}_M$
- $\widetilde{p} \coloneqq p \circ j$

(4.1.5)
$$\begin{tikzcd}
T_M X \arrow[r, hook, "s"'] \arrow[d, "\tau"'] & \widetilde{X}_M \arrow[d, "p"] & \Omega \arrow[l, hook', "j"'] \arrow[dl, "\widetilde{p}"] \\
M \arrow[r, hook, "i"'] & X &
\end{tikzcd}$$

Normal Deformation

Claim

$\widetilde{p}$ is smooth and Ω is isomorphic to $X \times \mathbf{R}^{+}$ by the map $(\widetilde{p}, t)$.

Proof. We have $\widetilde{p}^{-1}(X) = j^{-1}p^{-1}(X) = \Omega$ by the definition of $\widetilde{p}$ and the surjectivity of p. The condition about tangent maps is a local property, and the claim follows.

We have $t^{-1}(\mathbf{R}^{+}) \cong \Omega$. Therfore

$$\begin{aligned}(\widetilde{p}, t)\,(\Omega) &\cong \widetilde{p}(\Omega) \times t(\Omega) \\ &\cong X \times \mathbf{R}^{+}.\end{aligned}$$

The inverse morphism is induced by (4.1.3). □

Normal Deformation

Claim

$p^{-1}(M)$ is the union of $T_M X$ and $M \times \mathbf{R}$.

Proof. We can see locally

$$\begin{aligned} p^{-1}(M) &= \left\{(x,t) \in \widetilde{X}_M;\ (tx', x'') \in M\right\} \\ &= \left\{(x,t) \in \widetilde{X}_M;\ tx' = 0\right\} \\ &= \left\{(x,t) \in \widetilde{X}_M;\ t = 0, \text{ or } \ x' = 0\right\} \\ &= T_M X \cup (M \times \mathbf{R}). \end{aligned}$$

$\square$

Normal Deformation

Claim

$T_M X \cap (M \times \mathbf{R}) = M \times \{0\}$ coincides with the zero-section of $T_M X$.

Proof. As how we consider above,

$$\begin{aligned} T_M X \cap (M \times \mathbf{R}) &= t^{-1}(0) \cap (M \times \mathbf{R}) \\ &= \left\{ (x, t) \in \widetilde{X}_M;\ t = 0,\ \text{and } x' = 0 \right\} \\ &= M \times \{0\}, \end{aligned}$$

and $M \times \{0\} \cong M \subset T_M X$. □

Normal Cones

Definition ([KS90, Def.4.1.1])

(i) For $S \subset X$, the **normal cone to S along M** is

$$C_M(S) = T_M X \cap \overline{\widetilde{p}^{-1}(S)}.$$

(ii) For $S_1, S_2 \subset X$, we define $C(S_1, S_2) := C_{\Delta_X}(S_1 \times S_2) \subset TX$.

$$\begin{array}{ccc} T_{\Delta_X}(X \times X) & \longrightarrow & TX \\ \Cup & & \Cup \\ ((x,x),(\xi,0)) & \longmapsto & (x,\xi) \end{array}$$

Normal Cones

We can write an element of $T_{\Delta_X}(X \times X)$ as

$$((x, x), (\xi, 0)).$$

Indeed, for $(\xi, \eta) \in T_{(x,x)}(X \times X)$, we have

$$\begin{aligned}
(\xi, \eta) + T_{(x,x)}\Delta_X &= \{(\xi + \zeta, \eta + \zeta); \zeta \in T_{(x,x)}\Delta_X\} \\
&= \{(\xi + (\zeta - \eta), \eta + (\zeta - \eta)); \zeta \in T_{(x,x)}\Delta_X\} \\
&= \{(\xi - \eta, 0) + \zeta; \zeta \in T_{(x,x)}\Delta_X\} \\
&= (\xi - \eta, 0) + T_{(x,x)}\Delta_X.
\end{aligned}$$

That is to say, we can make any vectors (ξ, η) in $T_{\Delta_X}(X \times X)$ in the form $(\zeta, 0)$ preserving the difference between the first and second entries.

Normal Cones

Claim

1. $C_M(S)$ is a closed conic subset of $T_M X$.
2. its projection onto M is the set $M \cap \overline{S}$.

Proof. 1. The claim follows from

$$
\begin{aligned}
C_M(S) &= T_M X \cap \overline{\widetilde{p}^{-1}(S)} \\
&= T_M X \cap \overline{\{(x'/t, x'') \in \Omega; x \in S, t \in \mathbf{R}^+\}} \\
&= T_M X \cap \overline{\mathbf{R}^+ \{(x', x'') \in \Omega; x \in S\}}.
\end{aligned}
$$

$\square$

Normal Cones

Proof. 2. ????

$$\tau(C_M(S)) \stackrel{?}{=} M \cap \overline{S}$$

$$\begin{aligned}
\tau(C_M(S)) &= \tau(T_M X \cap \overline{\widetilde{p}^{-1}(S)}) \\
&= \tau\left(s^{-1}\left(\overline{\widetilde{p}^{-1}(S)}\right)\right) \\
&= \tau\left(s^{-1}\left(\overline{j^{-1}(p^{-1}(S))}\right)\right) \\
&= \tau\left(\overline{s^{-1}\left(j^{-1}(p^{-1}(S))\right)}\right)
\end{aligned}$$

$\square$

Normal Cones

As a consequence of the claim,

$$C(S_1, S_2) \subset TX\text{: closed conic.}$$

If M is a closed submanifold, then $C(S, M)$ is the inverse image of $C_M(S)$ by the projection $M \times_X TX \to T_M X$ by the next proposition.

References

References i

[KS90] Kashiwara, Schapira *Sheaves on Manifolds*, Springer, 1990.