Hulten's Theorem

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This note provides an exposition of Hulten's (1978) Theorem, mostly following Baqaee and Farhi's (2019) Appendix A.

Hulten's theorem considers a multi-sector economy, where one sector can use other sectors' products as intermediate inputs. It shows that, denoting the total factor productivity (TFP) of sector i as A_i, the % change of the entire economy's TFP can be approximated

$$\frac{dTFP}{TFP} = \sum_{i=1}^{N} \delta_i \frac{dA_i}{A_i} \tag{1}$$

in the first-order approximation, where

$$\delta_i = \frac{p_i y_i}{\sum_{i=1}^{N} p_i c_i}$$

is called the Domar weight. Here, p_i is the price of good i, y_i is the production of good i and c_i is consumption (and investment, if there is capital stock) of good i. Note that, because the denominator and the numerator do not necessarily match when the products are used as intermediate inputs, the Domar weight may not sum up to one.

Below we show Hulten's theorem in a relatively simple static economy. The only input is labor and labor is inelastically supplied. Thus the % change in the aggregate TFP in this economy is the same as the % change in GDP (later denoted by Y). Suppose that there are N goods. Assume that all markets are competitive. The representative consumer's utility takes the form

$$U(c_1, ..., c_N)$$

and the budget constraint is

$$\sum_{i=1}^{N} p_i c_i = w\bar{\ell} + \sum_{i=1}^{N} \pi_i,$$

where w is the wage rate and $\bar{\ell}$ is fixed amount of labor supply, which is the only input for production. π_i is profit from sector i. Assume that the preferences are homothetic so that $U(c_1, ..., c_N)$ is linearly homogeneous.

The production function for sector i is

$$y_i = A_i F_i(\ell_i, x_{i1}, x_{i2}, ..., x_{iN}),$$

where ℓ_i is labor input at sector i and x_{ij} is the quantity of product j used in sector i. The profit is

$$\pi_i = p_i y_i - w\ell_i - \sum_{j=1}^{N} p_j x_{ij}.$$

The market clearing conditions are

$$y_i = \sum_{j=1}^{N} x_{ji} + c_i$$

for all i and

$$\bar{\ell} = \sum_{i=1}^{N} \ell_i.$$

First, consider the consumer's expenditure minimization problem for a given utility level.

$$\min_{c_1,\ldots c_N} \sum_{i=1}^{N} p_i c_i$$

subject to

$$U(c_1, ..., c_N) = u.$$

The Lagrangian is

$$L = \sum_{i=1}^{N} p_i c_i - \lambda(U(c_1, ..., c_N) - u).$$

The first-order condition for this problem is

$$p_i = \lambda \frac{\partial U(c_1, ..., c_N)}{\partial c_i}. \tag{2}$$

Let us normalize the prices (i.e. choose the numeraire) p_i so that $\lambda = 1$ in equilibrium. Let

$$Y \equiv \sum_{i=1}^{N} p_i c_i$$

be the GDP (and TFP) of this economy. From (2) and linear homogeneity, it can be rewritten as

$$Y = \sum_{i=1}^{N} \frac{\partial U(c_1, ..., c_N)}{\partial c_i} c_i = U(c_1, ..., c_N). \tag{3}$$

Therefore, in equilibrium, the level of utility also represents GDP.

Because the first welfare theorem holds, the competitive equilibrium is Pareto optimal and it solves the social planner's problem

$$\max_{c_i, x_{ij}, \ell_i} U(c_1, ..., c_N)$$

subject to

$$c_i + \sum_{j=1}^{N} x_{ji} = A_i F_i(\ell_i, x_{i1}, x_{i2}, ..., x_{iN})$$

and

$$\sum_{i=1}^{N} \ell_i = \bar{\ell}.$$

The Lagrangian for the social planner is

$$L = U(c_1, ..., c_N) + \sum_{i=1}^{N} \mu_i \left(A_i F_i(\ell_i, x_{i1}, x_{i2}, ..., x_{iN}) - c_i - \sum_{j=1}^{N} x_{ji} \right) + \nu \left(\bar{\ell} - \sum_{i=1}^{N} \ell_i \right).$$

From the first-order condition,

$$\frac{\partial U(c_1, ..., c_N)}{\partial c_i} = \mu_i. \tag{4}$$

The envelope theorem imply

$$\frac{dU}{dA_i} = \mu_i F_i(\ell_i, x_{i1}, x_{i2}, ..., x_{iN}) = \mu_i y_i \frac{1}{A_i}.$$

From (3), this equation implies

$$\frac{dY}{dA_i} = \mu_i y_i \frac{1}{A_i}. \tag{5}$$

From (2) and (4), together with our normalization of $\lambda = 1$,

$$\mu_i = p_i$$

holds. Using this relationship and dividing the both sides of (5) by $Y = \sum_{i=1}^{N} p_i c_i$ yields

$$\frac{dY}{Y} = \frac{p_i y_i}{\sum_{i=1}^{N} p_i c_i} \frac{dA_i}{A_i}.$$

Repeating the same procedure for all A_i and noting that $TFP = Y/\bar{\ell}$ in this economy, we obtain the equation (1).

References

Baqaee, D. R. and E. Farhi (2019). The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten's Theorem. *Econometrica 87*, 1155–1203.

Hulten, C. R. (1978). Growth Accounting with Intermediate Inputs. *Review of Economic Studies 45*, 511–518.