Incomplete Market Models in Macroeconomics: Basics

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The lectures

- In the two lectures, I will talk about the incomplete-market models in macroeconomics.
- The first lecture lays out the basics: (i) what makes the incomplete-market models different (from complete-market/representative-agent models); (ii) two basic models of incomplete markers in macroeconomics.
- The second lecture talks about computation and applications.
- Lecture slides will be on my lecture notes page: `https://sites.google.com/view/toshimukoyama/notes` (you can google my name to find my webpage)

Why incomplete market models?

- To think about the benefit of incorporating market incompleteness into the macroeconomic analysis, we will contrast complete-market models and incomplete-market models.
- **Comparison #1**: When (i) the utility function is in a certain class (including the CRRA utility that is often used in macroeconomics) and (ii) the markets are complete,
  - Distribution of wealth does not matter for the macroeconomic outcome.
  - We can analyze the macro questions with the representative consumer (and only with aggregate shocks, ignoring idiosyncratic shocks that washes out in macro).
  - When my consumption goes down, the consumption of Jeff Bezos also goes down, even if it is the time I lose a job and Amazon makes more profit.

Gorman aggregation (Gorman, 1961; Mukoyama, 2010)

- Consider consumer i who maximizes the expected utility

$$E\left[\sum_{t=0}^{\infty}\beta^t\frac{c_{it}(s^t)^{1-\nu}-1}{1-\nu}\right]=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\frac{c_{it}(s^t)^{1-\nu}-1}{1-\nu}\pi(s^t)$$

  where s^t is the state (including all history up to t), $c_{it}(s^t)$ is the consumption of i at t, $\pi(s^t)$ is the probability of the realization of state s^t from the viewpoint of time 0.
- $\nu > 0$ and $\nu \neq 1$ and $\nu \to 1$ corresponds to the log utility.
- Assume that the asset market is complete, that is, at time 0, the market for contingency claims (**Arrow securities**) for **all** states s^t opens. Let $p_t(s^t)$ be the time-0 price of an Arrow security that promises one unit of consumption good at time t if state s^t realizes.

Gorman aggregation

- The budget constraint is

$$\sum_{t=0}^{\infty}\sum_{s^t} p_t(s^t)c_{it}(s^t) \leq W_{i0}(\mathbf{p}) \equiv \sum_{t=0}^{\infty}\sum_{s^t} p_t(s^t)\omega_{it}(s^t),$$

  where $W_{i0}(\mathbf{p})$ is the time-0 wealth of consumer i, which can depend on the vector of prices $\mathbf{p}$. $\omega_{it}(s^t)$ is the endowment (income) of i at time t if the state is s^t.

- For example, the deterministic flow budget constraint

$$a_{i,t+1} \leq (1+r)a_{it} + y_{it} - c_{it}$$

  (a_0 given and $\lim_{T\to\infty} a_T/(1+r)^T = 0$) can be rewritten as

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} c_{it} \leq \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \tilde{y}_{it},$$

  where $\tilde{y}_{10} \equiv y_{i0} + (1+r)a_0$ and $\tilde{y}_{it} = y_{it}$ for $t = 1, 2,$...

Gorman aggregation

- From the Euler equation

$$\frac{1}{p_t(s^t)}\pi(s^t)c_{it}(s^t)^{-\nu} = \beta\frac{1}{p_{t+1}(s^{t+1})}\pi(s^{t+1})c_{i,t+1}(s^{t+1})^{-\nu}$$

c_{it} can be rewritten as a function of c_{i0}. Then c_{i0} can be solved using the budget constraint. Consumption at each period can be expressed as

$$c_{it}(s^t) = g(s^t, \mathbf{p})W_{i0}(\mathbf{p}).$$

Everything in $g(s^t, \mathbf{p})$ function is common across consumers. Thus the consumption for consumers i and j satisfies

$$\frac{c_{it}(s^t)}{c_{jt}(s^t)} = \frac{W_{i0}(\mathbf{p})}{W_{j0}(\mathbf{p})}.$$

Thus I behave like "mini Jeff Bezos"—my consumption is a fixed fraction of his, no matter what the state is.

Gorman aggregation

- This result also imply the aggregate consumption $C_t(s^t)$ is

$$C_t(s^t) = \sum_i c_{it}(s^t) = \sum_i W_{i0}(\mathbf{p}) = p_t(s^t) \sum_i \omega_{it}(s^t),$$

  so that for a given total endowment $\sum_i \omega_{it}(s^t)$, the distribution of endowment doesn't matter for macroeconomic outcome.
- Normatively, the welfare effect of a policy change is in the same direction for everyone unless the policy treats poor and rich differently (e.g. tax the rich and redistribute to the poor) AND there are no Arrow securities for policy changes.

Then maybe we can just use non-CRRA utility?

- In macroeconomics, we use CRRA utility all the time, and the reason is not unrelated to the above argument.
- CRRA utilities belong to a class of **balanced-growth preferences** (King et al., 1988). These preferences have a property that people behave "similarly" (working similar hours, for example) despite that the income per capita is a lot larger. This time-series property requires "poor behaving similarly to rich" as in the case of cross-section aggregation.
- The balanced-growth preferences also suits the analysis of recurring situations ("business cycles are all alike") because we can simply detrend the growth component in the model.
- There are some subtleties—there is a fairly large literature on both aggregation. Here I'd just like to point out the "homotheticity" requirement is similar for the cross-sectional aggregation and the balanced-growth property.

Why incomplete market models?

- **Comparison #2**: With CRRA utility, the wealth distribution is "indeterminate": it is not pinned down by endogenous forces.
- Often the stationary wealth distribution is unique, and it is shaped by economic forces.
- Even if people are ex ante homogeneous, they can be ex post heterogeneous (compare this with the overlapping-generations model), because there are "lucky" people and "unlucky " people.
- Policy effects (positive and normative) are heterogeneous. Room for politics, even with ex ante homogeneous consumers, as policy opinions differ (see, for example, Mukoyama, 2013). More on this next week.

Existence and uniqueness of stationary distribution

- Consider a model with idiosyncratic shock ϵ_{it} and asset a_{it} for consumer i. The decision rule for $a_{i,t+1}$ is
$$a_{i,t+1} = a'(\epsilon_{it}, a_{it}).$$
Assuming that ϵ_{it} is a Markov process, the probability of moving to (a', ϵ') tomorrow given (a, ϵ) today $\pi_{\epsilon a \epsilon' a'}$ can be constructed. Let $\mathbf{p}$ be the vector listing the population (total is normalized to one) for each state (a, ϵ) and $\mathbf{A}$ be the matrix of $\pi_{\epsilon a \epsilon' a'}$. (Suppose that a is discrete.) Then
$$\mathbf{p}' = \mathbf{A}\mathbf{p}.$$
- A stationary distribution is $\bar{\mathbf{p}}$ that satisfies $\bar{\mathbf{p}} = \mathbf{A}\bar{\mathbf{p}}$.
- With a fairly weak condition on $\pi_{\epsilon a \epsilon' a'}$ (e.g. there exists a state (a, ϵ) that is visited with strictly positive probability starting from anywhere), the stationary distribution exists, is unique, and can be computed by an iterative procedure $\mathbf{A}^N \mathbf{p}_0$, $N \to \infty$ for any $\mathbf{p}_0$ (Stokey et al., 1989, Theorem 11.4). An application of the contraction mapping theorem.

Existence and uniqueness of stationary distribution

- An example: Two points in $\epsilon \in \{b, g\}$ with transition probabilities $\nu_{\epsilon\epsilon'}$ and three points in $a \in \{\ell, m, h\}$. $a'(b,\ell) = \ell$, $a'(b,m) = \ell$, $a'(b,h) = m$, $a'(g,\ell) = m$, $a'(g,m) = h$, $a'(g,h) = h$. Then

$$\begin{bmatrix} p'_{b\ell} \\ p'_{bm} \\ p'_{bh} \\ p'_{g\ell} \\ p'_{gm} \\ p'_{gh} \end{bmatrix} = \begin{bmatrix} \nu_{bb} & \nu_{bb} & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_{bb} & \nu_{gb} & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu_{gb} & \nu_{gb} \\ \nu_{bg} & \nu_{bg} & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_{bg} & \nu_{gg} & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu_{gg} & \nu_{gg} \end{bmatrix} \begin{bmatrix} p_{b\ell} \\ p_{bm} \\ p_{bh} \\ p_{g\ell} \\ p_{gm} \\ p_{gh} \end{bmatrix}.$$

- if $\nu_{\epsilon\epsilon'} > 0$ for all (ϵ, ϵ'), this matrix satisfies the condition for the Stokey et al. (1989) theorem, and the stationary distribution exists and is unique.

Why incomplete market models?

- **Comparison #3**: In the complete market model, we cannot talk about insurance policy on idiosyncratic risks.
- Various government policies (even macro models) are about insuring against idiosyncratic risks.
- Examples: unemployment insurance, welfare, and government-sponsored health insurance.
- It is reasonable to think about a situation where consumers are partially (and endogenously) insured.
- For example, many incomplete-market models consider a situation consumers have an access only to self-insurance (accumulating only some types of assets).

Asset pricing (Huggett, 1993; Krusell et al., 2011)

A review of the complete-market asset pricing model (Lucas, 1978):

- The problem for the representative agent:

$$V(a, x, z) = \max_{a'_g, a'_b, x'} \quad \frac{c^{1-\sigma}}{1-\sigma} + \beta \sum_{z'=g,b} \phi_{zz'} V(a'_{z'}, x', z')$$

subject to

$$c = a + xY_z - Q_g a'_g - Q_b a'_b - p_z(x' - 1).$$

The aggregate state z has two realizations: $z = g$ and $z = b$. The consumer has an access to three assets: two Arrow securities (price Q_z and quantity a_z) and a "tree" (price p_z and quantity x). Assume (for simplicity) that the tree lasts only for one period and the consumer receives one unit of a new tree every period as endowment.

Asset pricing

Continuing the Lucas asset pricing model:

- The consumer's FOCs:

$$-Q_{zz'}c_z^{-\sigma} + \beta\phi_{zz'}V_1(a'_{z'}, x', z') = 0$$

for $z' = g, b$ and

$$-p_z c_z^{-\sigma} + \beta \sum_{z'=g,b} \phi_{zz'}V_2(a'_{z'}, x', z') = 0.$$

- The envelope conditions

$$V_1(a, x, z) = c_z^{-\sigma}$$

$$V_2(a, x, z) = Y_z c_z^{-\sigma}.$$

- Combining,

$$Q_{zz'} = \beta\phi_{zz'}\left(\frac{c_{z'}}{c_z}\right)^{-\sigma}$$

$$p_z = \beta \sum_{z'=g,b} \phi_{zz'}\left(\frac{c_{z'}}{c_z}\right)^{-\sigma} Y_{z'}$$

Asset pricing

Continuing the Lucas asset pricing model:

- Because everyone is identical, $a_{z'} = 0$ and $x = 1$ in equilibrium. Therefore $c_z = Y_z$. The pricing formula:

$$Q_{zz'} = \beta \phi_{zz'} \left(\frac{Y_{z'}}{Y_z} \right)^{-\sigma}$$

$$p_z = \beta \sum_{z'=g,b} \phi_{zz'} \left(\frac{Y_{z'}}{Y_z} \right)^{-\sigma} Y_{z'}.$$

- An observation: From above, $p_z = \sum_{z'} Q_{zz'} Y_{z'}$ holds ($Q_{zz'}$ is the "state price"). This relationship can also be derived from no arbitrage. Thus this relationship holds even with an incomplete-market setting as long as all these securities exist.

Asset pricing

Continuing the Lucas asset pricing model:

- Another observation: when Y_z is constant (no aggregate shocks), a bond (promises one unit of consumption good regardless of the state) price is

$$q = \beta \sum_{z'=g,b} \phi_{zz'} = \beta.$$

- When there is a growth in consumption $c_{z'}/c_z = 1+g$, where $g > 0$,

$$q = \beta \sum_{z'=g,b} \phi_{zz'} = \beta(1+g)^{-\sigma}.$$

  when σ is large, q is small (the safe rate is high). The "safe rate puzzle."

Asset pricing

Now, consider Huggett (1993).

- First consider the bond economy without aggregate shock (but idiosyncratic shock ϵ, which can take ϵ_h or ϵ_ℓ randomly).

$$V(a,\epsilon) = \max_{a'} \quad \frac{c^{1-\sigma}}{1-\sigma} + \beta \sum_{\epsilon=\epsilon_h,\epsilon_\ell} \pi_{\epsilon\epsilon'} V(a',\epsilon')$$

subject to

$$c = a + \epsilon - qa'$$

and the borrowing constraint

$$a' \geq \underline{a}.$$

Note that the borrowing constraint can be at the "natural borrowing constraint" level (which ensures repayment) or a tighter one.

Asset pricing

Huggett (1993).

- Following similar steps, one can show

$$-qc_s^{-\sigma} + \beta[\pi_{sh}c_h'^{-\sigma} + (1-\pi_{sh})c_\ell'^{-\sigma}] + \lambda_s = 0$$

  holds for $s = h, \ell$, where $\lambda_s \geq 0$ is the Lagrange multiplier on the constraint $a' \geq \underline{a}$.

- From here, consider $\underline{a} = 0$. Then, no one can borrow, which means no one can lend, and thus the equilibrium is **autarky**.

$$-q\epsilon_s^{-\sigma} + \beta[\pi_{sh}\epsilon_h^{-\sigma} + (1-\pi_{sh})\epsilon_\ell^{-\sigma}] + \lambda_s = 0$$

- One can show that (see Krusell et al., 2011), in this case, $\lambda_\ell > \lambda_h \geq 0$ and therefore the constraint is always binding for the low-endowment consumer.

Asset pricing

Huggett (1993).

- From

$$-q\epsilon_s^{-\sigma} + \beta[\pi_{sh}\epsilon_h^{-\sigma} + (1-\pi_{sh})\epsilon_\ell^{-\sigma}] + \lambda_s = 0,$$

  Any value of q that is large can satisfy $\lambda_\ell > \lambda_h \geq 0$.
- **Proposition:** Any q that satisfies

$$q \geq q^* = \beta\left[\pi_{hh} + (1-\pi_{hh})\left(\frac{\epsilon_h}{\epsilon_\ell}\right)^\sigma\right].$$

  is an equilibrium bond price.
- One can show that q^* is the limit of the bond price as $\underline{a} \to 0$ from below. Note that $q^* > \beta$.
- Recall that $q = \beta$ in the complete market case. The bond price is higher (the safe rate is lower) in the incomplete-market case. Why? The ϵ_h consumer has a strong desire to save, to prepare for the possibility of ϵ_ℓ next period (precautionary saving motive).
- A large σ can be consistent with a large q^*.

Asset pricing

Huggett (1993).

- When there are aggregate shocks, we consider a market structure where there is an **aggregate Arrow security** (a security that provides one unit of good if the next period aggregate state is z') for each aggregate state in the next period z'.
- The financial market is still incomplete because the idiosyncratic risks are not spanned by the aggregate securities.
- Various assets can be created by combining the aggregate Arrow securities.
- The price of the aggregate Arrow security for state z' when the current state is z: $Q_{zz'}$.
- The individual holding of the security is denoted $a_{z'}$. Note that $a_{z'}$ is also the total asset balance for the consumer going into z'. Thus it is reasonable to consider the borrowing constraint on $a_{z'}$: $a_{z'} \geq \underline{a}_{z'}$ for $z' \in \{g, b\}$.
- Here, once again, consider $\underline{a}_{z'} = 0$. The equilibrium is autarky.

Asset pricing

Huggett (1993).

- With similar steps (see Krusell et al., 2011), one can show **Proposition:** Any $Q_{zz'}$ that satisfies
$$Q_{zz'} \geq Q^*_{zz'} = \beta\phi_{zz'}\left[\pi_{hh|zz'} + (1-\pi_{hh|zz'})\left(\frac{\epsilon_h}{\epsilon_\ell}\right)^\sigma\right].$$
is an equilibrium aggregate Arrow security price.
- Krusell et al. (2011) investigates the implications for pricing various assets, including the equity premium puzzle.
- I like the aggregate Arrow security approach because:
    1. $Q_{zz'}$ is a state price. It is a simple way to discount the future, including the firm profit (used in Krusell et al., 2010, for example)
    2. The security demand is easier to compute, compared to other portfolio choice problems, because the corner solution is relatively unlikely (compare with Krusell and Smith, 1997)
    3. Conceptually, it makes sense to impose a borrowing constraint on the total net asset holding going into each state, rather than constraints on the holding of each asset.

Production economy

Aiyagari (1994).

- Production economy: competitive firms produce with the (aggregate) production function

$$Y_t = K_t^\alpha L_t^{1-\alpha}$$

where Y_t is output, K_t is capital, and L_t is labor.

- Consumers: maximize

$$E_0\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right]$$

subject to

$$c_t + a_{t+1} = w_t \ell_t + (1 + r_t - \delta) a_t$$

and

$$a_{t+1} \geq b.$$

ℓ_t is random (and exogenous), a_t is capital stock holding.

Production economy

Aiyagari (1994).

- The prices w_t and r_t are determined in competitive markets:

$$w_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha-1}$$

and

$$r_t = \alpha\left(\frac{K_t}{L_t}\right)^{\alpha}$$

where

$$K_t = \int a_t(i)di$$

and

$$L_t = \int \ell_t(i)di.$$

Production economy

Aiyagari (1994).

- I will talk about computation next week.
- This model looks very much like the neoclassical growth model (Ramsey), except that each individual behavior is consistent with the permanent-income (or life-cycle) hypothesis.
- This model (and its variations) has been used extensively in the policy analysis (especially fiscal policy): taxation, government debt, etc.
- Now, for these purposes, more people use finite-life (overlapping generations) versions of this model, because life-cycle elements are important for many policy questions (e.g. social security).
- Labor market policies can be analyzed, too, especially after ℓ_t process is endogenized (see, for example, Krusell et al., 2010).

Recap of today

- Why incomplete market models?
  - Distribution of wealth can have impact on aggregate dynamics
  - Wealth distribution is endogenously determined. The effects of a policy can be different across individuals.
  - Can analyze the government's insurance policy in a partial-insurance environment.
- Asset pricing implications (Huggett, 1993)
- Introducing production (Aiyagari, 1994)

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