Firm Growth through New Establishments*

Dan Cao[†]
Georgetown University

Toshihiko Mukoyama[‡]
Georgetown University and University of Virginia

Erick Sager[§]
Bureau of Labor Statistics

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[We are currently awaiting data disclosure and therefore a significant part of empirical and estimation results are not presented in this version. Please visit
https://sites.google.com/site/toshimukoyama/CMSdraft.pdf
for the latest draft with up-to-date disclosed empirical and estimation results.]

Abstract

This paper analyzes firm growth in two margins: extensive margin (adding more establishments) and intensive margin (adding more workers per establishment). Utilizing the Quarterly Census of Employment and Wages dataset, we document how both margins are related to changes in firm size in the U.S. economy. Between 1990 and 2014, the average size of firms has increased significantly. It is shown that this increase in firm size is driven primarily by an expansion along the extensive margin, particularly in large firms. We build a tractable general equilibrium model of endogenous innovation and firm growth that feature both margins. The model is quantitatively compared to the data facts. We investigate the cause of time-series changes in the data through the lens of our model, in particular the implications of changing the costs for innovations that lead to extensive margin expansions.

Keywords: firm growth, firm size distribution, establishment, innovation
JEL Classifications: E24, J21, L11, O31

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[†]Contact information: dc448@georgetown.edu
[‡]Contact information: tm1309@georgetown.edu
[§]Contact information: sager.erick@bls.gov

1 Introduction

Understanding the process of firm growth is essential in the analysis of macroeconomic performance. Firms that innovate and expand are the driving force of output and productivity growth. Recent analysis of macroeconomic productivity emphasizes the role of innovation and reallocation at the firm level, both from theoretical and empirical standpoint.

In this paper, we focus on a particular aspect of firm growth: growth through adding new establishments. In general, a firm can increase its size, measured in employment, along two margins. It can add more workers for given establishments, or build new establishments. We call the former the *intensive margin* and the latter the *extensive margin*. This distinction is important because these margins typically imply different reasons for expansion. In the context of manufacturing, a new plant often is built to produce a new product. In the service sector, building a new store or a new restaurant imply venturing into a new market. In both contexts, creating a new establishment requires a significant investment.

Given the importance of this distinction, it is surprising that very little is known about how firms grow through building establishments. With the exception of a few papers, the literature separately treats firms and establishments without linking the two. Our first goal is to establish stylized facts about how firms and establishments are linked. We then interpret the facts through the lens of a macroeconomic model of endogenous innovation and firm dynamics.

In the literature on firm dynamics, firms and establishments are often treated as interchangeable. One justification for this indifference is the fact that the majority of firms are single-establishment firms. While this assumption is justifiable in some situations, it is misleading in many macroeconomic contexts. In the U.S. economy, while 95% of firms are single-establishment firms, their share in total employment is less than half (45%).[1] Furthermore, the firm size distribution exhibits a Pareto tail, which implies that a large firm with many establishments has a disproportionately large impact on macroeconomic performance (Gabaix, 2011). Thus, understanding how these large firms are created is an important research question.

We find that the average firm size has grown in recent years, and it is largely driven by the firms in the right tail of the size distribution. This result echoes the recent emphasis on the emergence of superstar firms (Autor et al., 2017) and also consistent with the increase in concentration measured by the markup (De Loecker and Eeckhout, 2017). These studies suggest that this increase in size has had important implications on other changes in macroeconomic variables, such as the decline in labor share. What is novel about our empirical analysis is that we show that this expansion is driven by extensive margin growth.[2]

In order to investigate what changes in the economic environment have contributed to this

[1]The numbers are taken from 2015 Q1 Business Employment Dynamics (which is drawn from the Quarterly Census of Employment and Wages, explained in Section 2.1): `https://www.bls.gov/bdm/sizeclassqanda.htm`. In the manufacturing sector, single-plant firms own 72% of plants, they produce only 22% of the value added (Kehrig and Vincent, 2017).

[2]Choi and Spletzer (2012) and Hathaway and Litan (2014) also note the discrepancy in the recent trend between firm size and establishment size.

phenomenon, we build a macroeconomic model of endogenous firm growth. Our model extends previous work by Klette and Kortum (2004) and Luttmer (2011). In Klette and Kortum (2004) and Luttmer (2011), each individual firm grows by adding production units (*external innovations* in our terminology), "product lines" in Klette and Kortum's (2004) terminology and "blueprints" in Luttmer's (2011) terminology. A natural interpretation of such a production unit, as is explicit in Luttmer (2011), is an establishment. Thus, this type of framework provides a perfect vehicle for analyzing firm growth through new establishments. The major departure of our model, compared to Klette and Kortum (2004) and Luttmer (2011), is the recognition that each establishment can grow. In fact, the critique of Klette and Kortum (2004) by Acemoglu and Cao (2015) points out the lack of such growth that we observe in the data. We introduce this technological improvement at the establishment level (we call it *internal innovation*), and explicitly compare our model outcomes to the data on establishments.

The model constructed by Akcigit and Kerr (2016) has similar features to that in this paper. They also consider innovations that are internal to the establishments ("products" in their terminology) that the firm already produces and external innovations that increases the number of establishments that the firm operates. At the same time, many of their model assumptions are different from ours.[3] The differences in assumptions mainly stem from a difference in purpose for each model. Our main purpose is to map the model to the data on firm and establishment sizes, measured by employment. Akcigit and Kerr (2016) primarily use patent data and therefore consider "products" instead of "establishments." Our model is very tractable, and it allows for analytical characterizations of Pareto tails in the firm size, the establishment size, and the number of establishments per firm. Our model is tailored to the question we ask: what is behind the increase in firm size over the recent years? In particular, we focus on the role of changes in innovation costs in allowing firms to expand on the extensive margin.

Lentz and Mortensen (2008) extends Klette and Kortum's (2004) model and estimates it using Danish data. While their main focus is on productivity and reallocation, this paper mainly targets labor market facts. We directly exploit establishment-level information in analyzing firm dynamics. Furthermore, the empirical phenomenon we highlight mainly concerns large firms, and the model is tailored to fit the right tail of the firm size distribution.

The paper is organized as follows. Section 2 describes the empirical patterns of firm growth we find using our dataset. Section 3 sets up and characterizes the theoretical model. Section 4 estimates the model and conduct counterfactual experiments. Section 5 concludes.

[3]For example, in Akcigit and Kerr (2016) an external innovation improves a product that is already produced by another firm, while in our model it creates a new establishment whose quality is the same as the other establishments in the firm. Our assumption seems more appropriate especially in the context of service sector innovations. For example, when a retail firm opens a new store after researching the placement of a new location, the quality of the new store would more likely be associated with the firm itself than the existing stores in that location.

2 Empirical facts

2.1 Data

Our data set is the Quarterly Census of Employment and Wages (QCEW). The microdata in the QCEW is collected for the official administration of state unemployment insurance programs, and therefore contains a near census of establishments in the United States. The QCEW reports payroll information on the number of workers and total wage bill by establishment and month between the years 1990-2016. Each establishment in the QCEW has an employer identification number (EIN) and a 6-digit NAICS code associated with its self-reported primary industry. This paper contains calculations from a sample of 38 states (including the District of Columbia, Puerto Rico and the U.S. Virgin Islands) that allow access to their confidential data through the Bureau of Labor Statistics' visiting researcher program.[4]

We use the employer identification numbers (EINs) as the definition of the firm. This is the level at which companies file their tax returns, and is often considered as the boundary of the firm in recent studies. As is frequently discussed, EINs can be different from the ultimate ownership of a firm, especially for large firms. Song et al. (2015) discusses this at length in their study of inequality. They state that for 4,233 New York Stock Exchange publicly listed firms in the Dunn & Bradstreet database, 13,377 EINs are reported. For example, Walmart operates with separate EINs for "Walmart Stores," the Supercenter, Neighborhood Market, Sam's Club, and On-line divisions. Stanford University has separate EINs for the university, the bookstore, the main hospital, and children's hospitals. We view the EIN as a reasonable definition of a firm for our analysis, given that it is an economically meaningful unit (especially from the accounting perspective), and that typically the EINs for these large firms with multiple EINs are still substantially more aggregated than establishments.[5] Further details of the dataset are described in Appendix A.

2.2 Cross-sectional characteristics

We start from the cross-sectional properties of firms and establishments. Although publicly-available datasets such as Business Dynamics Statistics of the U.S. Census Bureau contain size distribution of establishments and firms separately, it has not been documented how these are linked.

The particular interest of this paper is how firms grow in two margins, the intensive margin and the extensive margin. It is therefore useful as a first step to describe the distributions of the firm size, the establishment size, and the number of establishment per firm.

[4]Further details of the dataset are described in Appendix A, as well as previous work that uses the QCEW such as Ratner (2013), Siemer (2014), Chodorow-Reich (2014), and Chodorow-Reich and Wieland (2017).

[5]For example, as of October 31 2017, Walmart has 402 Discount Stores, 3552 Supercenters, 702 Neighborhood Markets, 660 Sam's Clubs, and 97 Small Formats including E-Commerce Acquisition/C-stores (numbers are taken from `https://corporate.walmart.com/our-story/locations/united-states`). Sorkin (2018) (footnote 7) quotes a communication with a Census employee and state that the occurrences of firms multiple SEINs (State EINs) are rare.

[We are awaiting data disclosure for the results.]

Denoting the firm size by Z, the number of establishment per firm as X, and the average establishment size for each firm as Y, one can decompose the firm size into extensive and intensive margin:

$$\log(Z) = \log(E[X|XY = Z]) + \log(E[Y|XY = Z]) + \Omega, \tag{1}$$

where $\Omega \leq 0$ and it is equal to zero if and only if $var[X|XY = Z] = 0$.[6]

[We are awaiting data disclosure for the results.]

2.3 Firm life cycle

It is well known that the firm growth exhibits a strong life cycle pattern. Thus it is useful to document the difference of growth pattern in terms of extensive and intensive margins for young and old firms.

[We are awaiting data disclosure for the results.]

2.4 Changes of cross-sectional characteristics: 1990–2014

There have been notable changes in firm characteristics over our sample period. Figure 1 plots average firm size, measured by the number of workers within a firm. Over our sample period, the average firm size has increased from about 23 employees to over 25 employees. This fact accords well with the rise in concentration in the U.S. economy documented by, for example, Autor et al. (2017).

[6] The derivation of (1) is in Appendix B.

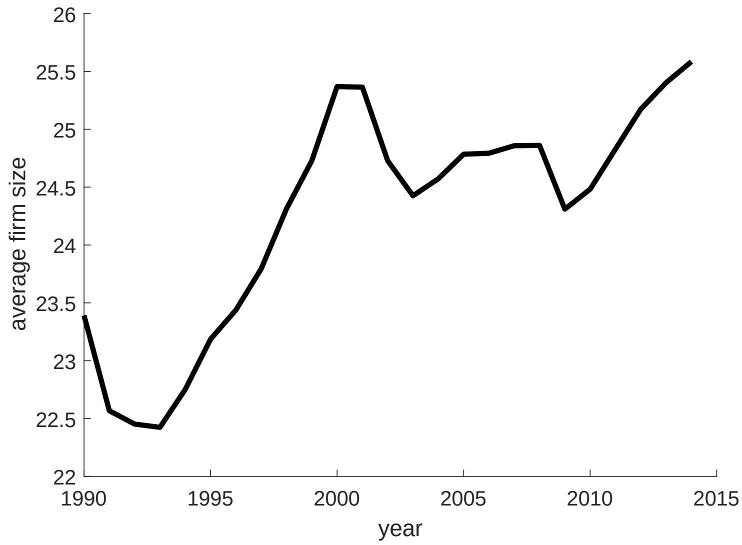

Figure 1: Average firm size (number of workers)

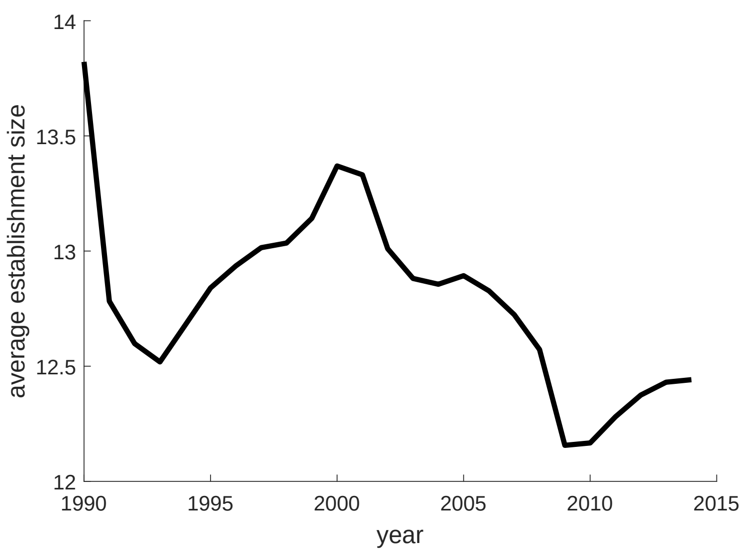

Figure 2: Average establishment size (number of workers)

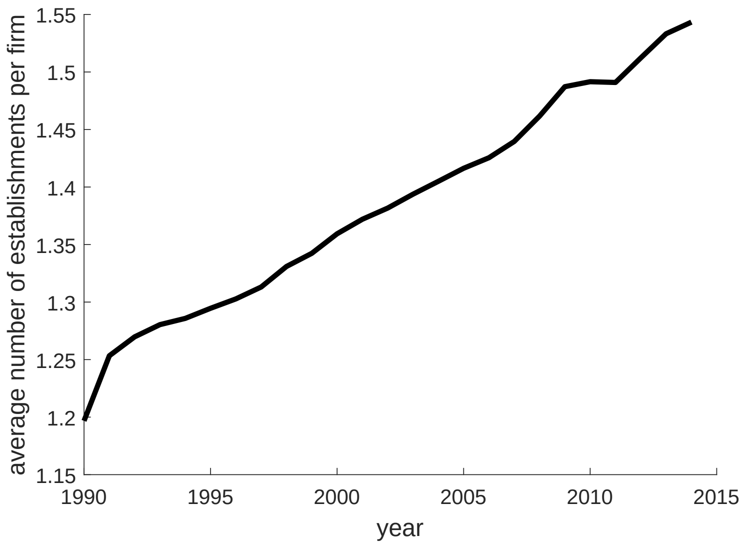

Figure 3: Average number of establishment per firm

Figures 2 and 3 present the novel fact that we focus on in this paper.[7] Figure 2 plots the average establishment size of each firm, measured by employment.[8] It shows that, despite the increase in the firm size over our sample period, the establishment size remains stationary. In contrast, the average number of establishments per firm, shown in Figure 3, exhibits a strong upward trend. The average number of establishments per firm starts from 1.2 in 1990 and increases to over 1.5 in 2014. This contrasting behavior implies that different forces are at work for these different components of firm growth.

What drives the increase in firm size, in particular along the extensive margin? To investigate this phenomenon further in the micro level, we consider three dimensions of disaggregation: different sectors, different ages, and different size bins.

First, to see how the patterns are different across different sectors, Figure 4 plots the time series of the average firm size in each sector, compared to the size in 1990. It can be seen that all sectors experienced the size increase over the period of 1991-2013. Over the whole sample period, the service sector, which employs the majority of the U.S. workers, have experienced the largest size increase.

Similarly to the overall economy, the intensive margins plotted in Figure 5 exhibit a flat profile or a slight downward trend for all sectors. In contrast, Figure 6, which plot the extensive margin

[7]Choi and Spletzer (2012) and Hathaway and Litan (2014) also document trends in firm size and establishment size, but do not analyze the number of establishments per firm, which is central to our analysis.

[8]The average establishment size of each firm, calculated here, is a different concept from the average size of establishments in the whole economy. Here we first calculate the average establishment size within each firm. After that, we average this number over all firms. This is different from simply dividing the number of all workers by the number of the establishments in the whole economy. Therefore, in calculating the numbers here, having an access to the microdata is essential— this cannot be calculated from the publicly-available information of the total numbers of employment, the total number of firms, and the total number of establishments.

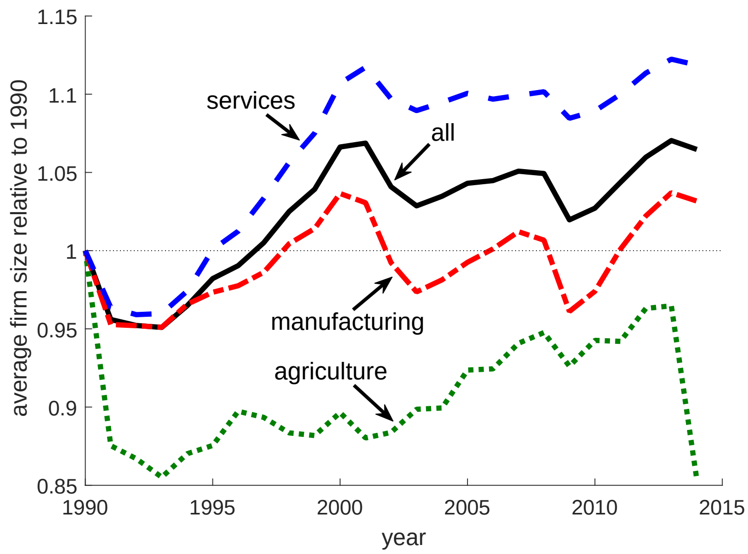

Figure 4: Average firm size (number of workers), different sectors

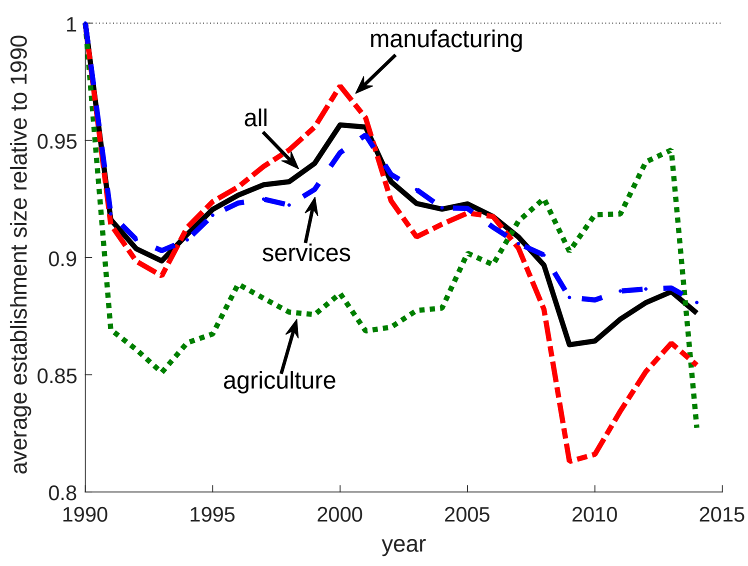

Figure 5: Average establishment size (number of workers), different sectors

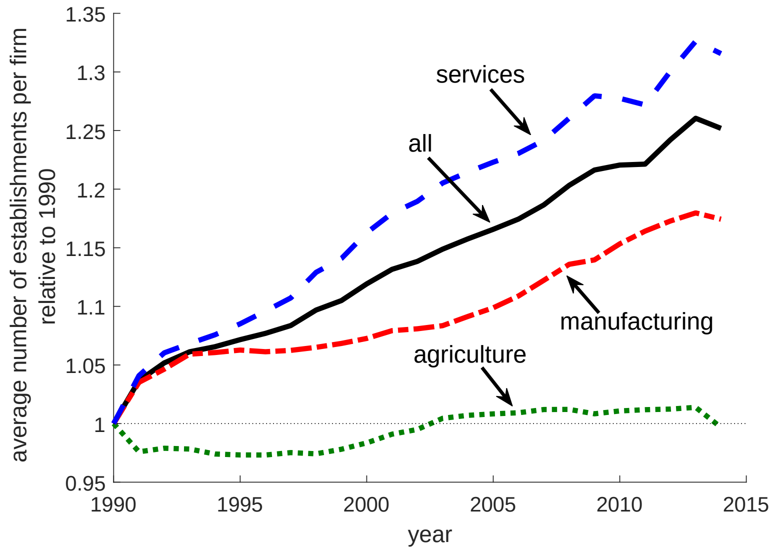

Figure 6: Average number of establishments per firm, different sectors

for different sectors, outlines the same message as the entire firm size: a significant increase in the expansion of service sector firms is the driving force of the overall firm size increase.

For the second dimension, we look at the outcomes for different ages of the firm. Note that the graphs for the age groups starts at the year 2001, so that we are able to consistently measure the age groups across different time periods. Figures 7, 8, and 9 repeat the same plots as earlier. Because the significant increase in the firm size occurs during 1990s, Figure 7 does not exhibit any obvious trend over the sample period for overall firms. However, we can see that there still is an increasing trend for the firms that are older than 11 years old. As in the case for all firms, the intensive size does not exhibit an increasing trend in Figure 8 for any age group. In contrast, Figure 9 reveals a striking contrast across different age groups: the increase in extensive margin for overall economy is driven by the older firms. This motivates our modeling choice—firms are not born with different sizes across different time period; rather, their pattern of growth has changed over time.

To see another dimension of heterogeneity, Figure 10 calculates the average size within size bins. There is a pattern of spreading out: very small firms with 1 to 4 employees have tended to become smaller, while the average size of larger firms with 100 employees has increased over time. If we examine the very right tail of firms with 5000 workers or more, firm size has been increasing over time since 1997, with a similar increase to firms that have 100 employees or more.

Figure 11 looks at this for the intensive margin, and here we do not see an obvious pattern. None of the series have an increasing trend, and in fact, the overall time-series pattern looks similar between very small firms (1 to 4 employees) and very large firms (5000 or more employees) except for a spike for very large firms in early 2000s.

The average number of establishment per firm, shown in Figure 12, reveals different trends for different firm sizes. Very small firms are predominantly single-establishment firms over the entire

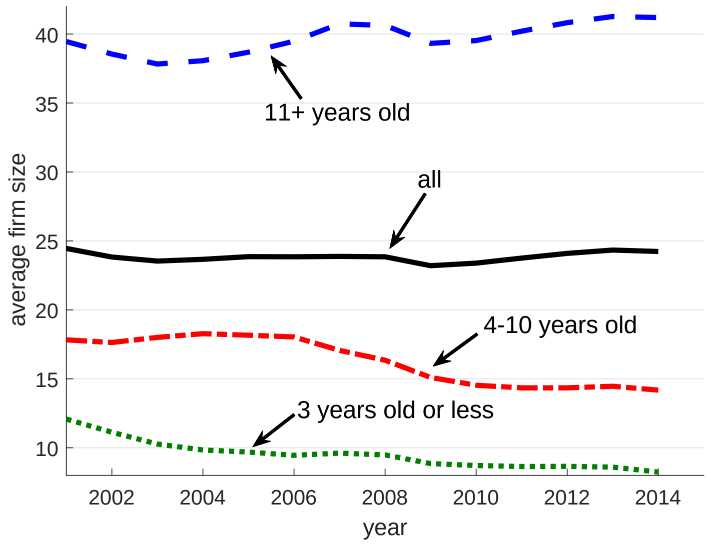

Figure 7: Average firm size (number of workers), different age groups

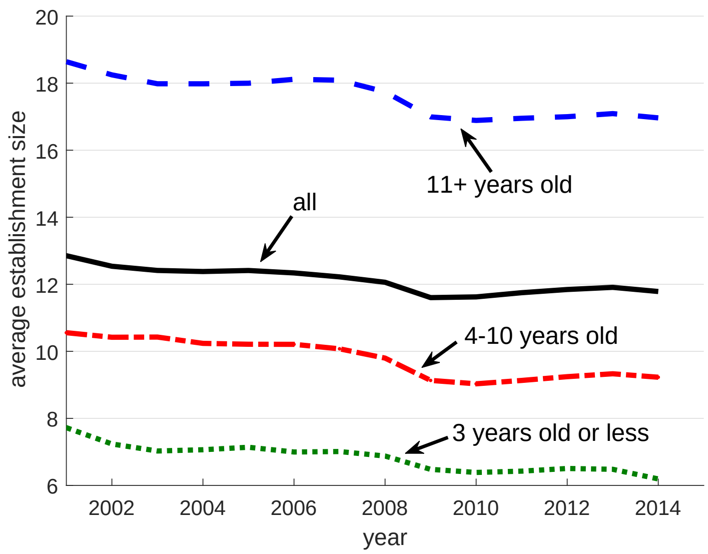

Figure 8: Average establishment size (number of workers), different age groups

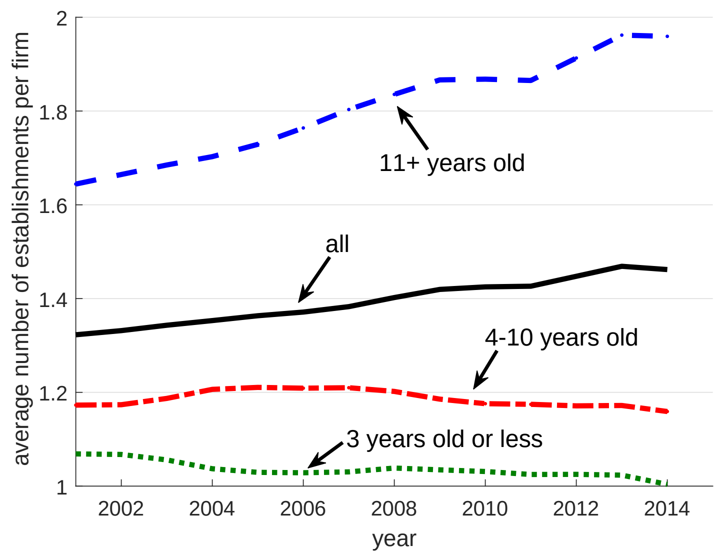

Figure 9: Average number of establishments per firm, different age groups

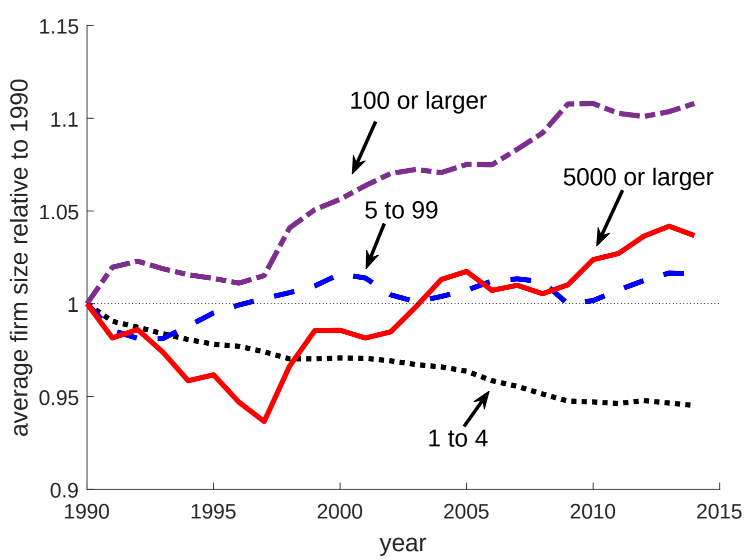

Figure 10: Average firm size (number of workers), different size bins

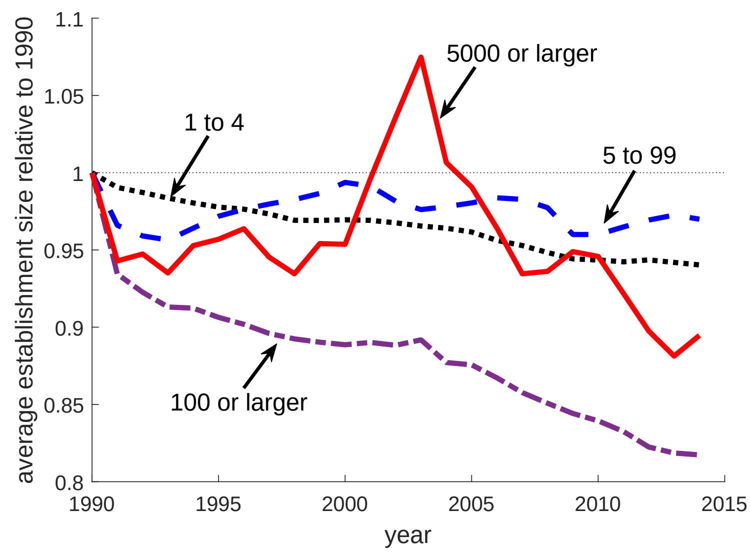

Figure 11: Average establishment size (number of workers), different size bins

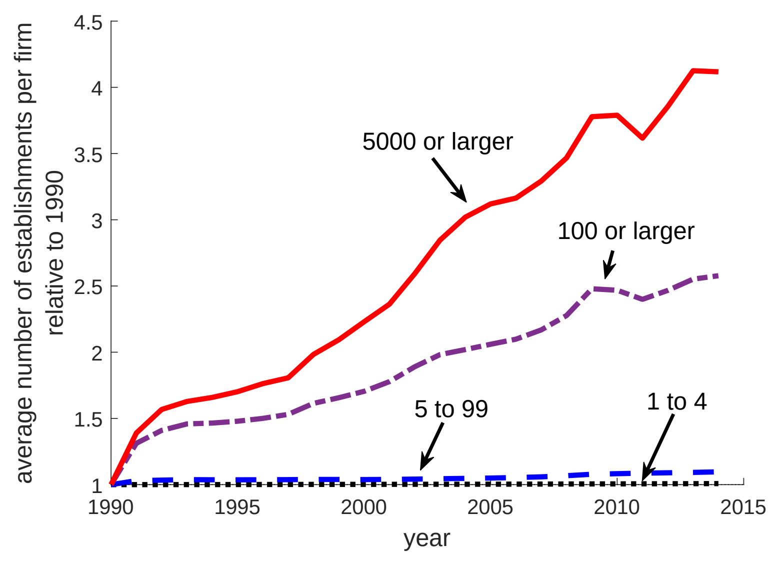

Figure 12: Average number of establishments per firm, different size bins

sample period. Medium-size firms with 5 to 99 employees have had a modest increase in the number of establishments. Larger firms have had a startling increase in the number of establishments. On average, the firms in 5000 or more employees category have about 4 times more establishments in 2014 compared to 1990. Thus, we conclude that a key mechanism that generated the increase in firm size in recent years is expansion through the number of establishments in very large firms.

3 Model

In this section, we construct a model of firm dynamics which can be mapped to our data observations. The heart of the model is the *endogenous* productivity improvements by intermediate-good firms, which allows us to analyze the fundamental causes of the recent changes in firm characteristics, which we described in Section 2 above.

3.1 Model setting

Time is continuous. The representative consumer provides labor and consumes the final good. The final good is produced from differentiated intermediate goods.

3.1.1 Consumer

The consumer side is intentionally kept simple, as we focus mainly on firm growth. The utility function of the representative households is

$$U = \int_0^\infty e^{-\rho t} L(t) \frac{(C(t)/L(t))^{1-\sigma}}{1-\sigma} dt.$$

The consumer consumes, owns firms, and supplies labor. The labor supply is given exogenously and grows at the rate $\gamma \geq 0$. Letting the real interest rate r, the Euler equation for the consumer is

$$\frac{\dot{C}(t)}{C(t)} = \frac{r-\rho}{\sigma}. \tag{2}$$

The final goods are used for consumption, R&D activities, and fixed costs for operation and entry. Thus

$$Y(t) = C(t) + R(t) + K(t),$$

where $Y(t)$ is the final goods output, $R(t)$ is total R&D of the incumbents, and $K(t)$ is the total entry cost.

3.1.2 Final good producers

There is a perfectly competitive final good sector. The final good is produced from differentiated intermediate goods. Intermediate goods have different qualities, and a high-quality intermediate good contributes more to the final good production. The production function for the final good is

$$Y(t)=\left(\int_{\mathcal{N}(t)} q_j(t)^{\beta} x_j(t)^{1-\beta} dj\right)^{\frac{1}{1-\beta}}, \tag{3}$$

where $x_j(t)$ is the quantity of intermediate good j, and $q_j(t)$ is its quality. $\mathcal{N}(t)$ is the set of actively-produced intermediate goods and $N(t)=|\mathcal{N}(t)|$ denote the number of actively-produced intermediate goods. We assume that $\beta \in (0,1)$.

With the maximization problem

$$\max_{x_j(t)} \left(\int_{\mathcal{N}(t)} q_j(t)^{\beta} x_j(t)^{1-\beta} dj\right)^{\frac{1}{1-\beta}} - \int_{\mathcal{N}(t)} p_j(t) x_j(t) dj,$$

the inverse demand function for the intermediate good j is

$$p_j(t)=Y(t)^{\beta}\left(\frac{q_j(t)}{x_j(t)}\right)^{\beta}.$$

Let

$$Q(t)\equiv \frac{1}{N(t)}\int_{\mathcal{N}(t)} q_j(t) dj$$

be the average quality of intermediate goods.

3.1.3 Intermediate good producers

The intermediate good sector is monopolistically competitive. Each intermediate good is produced by one firm. A firm can potentially produce many intermediate goods. A firm can add a new intermediate good to its portfolio by investing in R&D (external innovation). It can also increase the quality of the intermediate goods that it already produces by investing in R&D (internal innovation). A new firm can enter by coming up with its first product by innovation. Later we map one intermediate good to one establishment in the data.

We assume that the intermediate goods are produced only by labor. This is the only process in the entire economy that uses labor as an input. This allows us to map the employment dynamics of the intermediate good sector to our data analysis in Section 2.[9] The production function for intermediate good j is

$$x_j(t)=Z(t)\ell_j(t), \tag{4}$$

where $Z(t)$ is the labor productivity. Here, we allow a geometric weighted average of exogenous factor, at rate θ and endogenous factor, $Q(t)$:

$$Z(t)=(e^{\theta t})^{\alpha} Q(t)^{1-\alpha}. \tag{5}$$

[9]A similar idea of mapping the employment process to the productivity process is employed by Hopenhayn and Rogerson (1993), Garcia-Macia et al. (2016), and Mukoyama and Osotimehin (2018).

The optimization for the intermediate goods producer results in the usual mark-up rule:

$$p_j(t) = \frac{1}{1-\beta}\frac{w_t}{Z(t)}.$$

The resulting intermediate goods production is

$$x_j(t) = (1-\beta)^{\frac{1}{\beta}} \left(\frac{w(t)}{Z(t)}\right)^{-\frac{1}{\beta}} Y(t)q_j(t). \tag{6}$$

Thus (4) and (6) imply that the labor demand is proportional to $q_j(t)$ given the aggregate variables:

$$\ell_j(t) = \left(\frac{w(t)}{(1-\beta)Z(t)}\right)^{-\frac{1}{\beta}} \frac{q_j(t)Y(t)}{Z(t)}. \tag{7}$$

The firms (and establishments) are different only in $q_j(t)$ here. Thus the employment per establishment varies proportionally to $q_j(t)$ in cross section. The profit is also proportional to $q_j(t)$:

$$\pi_j(t) = \bar{\pi}(t)q_j(t),$$

where

$$\bar{\pi}(t) = \beta(1-\beta)^{\frac{1}{\beta}-1}\left(\frac{w(t)}{Z(t)}\right)^{1-\frac{1}{\beta}} Y(t) = \beta(1-\beta)^{\frac{1}{\beta}-1}\bar{w}(t)^{1-\frac{1}{\beta}}, \tag{8}$$

where

$$\bar{w}(t) \equiv \frac{w(t)}{Z(t)Y(t)^{\frac{\beta}{1-\beta}}}.$$

Innovations are carried out through R&D activity. The input for R&D is in final goods. For an existing intermediate-good firm, there are two kinds of innovations: internal innovation and external innovation.

Internal innovation raises the quality of the goods that they already produce. The total intensity of internal innovation is denoted by $Z_{I,j}(t)$. The innovation intensity per good is $z_{I,j}(t) \equiv Z_{I,j}(t)/n_j(t)$, where $n_j(t)$ is the number of goods firm j produces. Then the quality improves following

$$\frac{dq_j(t)}{dt} = z_{I,j}(t)q_j(t).$$

Here we index q only by j, because with the assumption we will make on external innovation, the qualities of all goods produced by firm j is always the same within the firm.

We assume that different firms can have different costs for innovation. In particular, we partition firms into different (finite) types, and assume that different types have different costs for innovation. We will detail later how types evolve over time. We denote the number of types by T and index the types by τ. The R&D cost for internal innovation is assumed to be $\mathbf{R}_I^\tau(Z_{I,j}(t), n_j(t), q_j(t))$. As in Klette and Kortum (2004), we assume that the R&D cost function $\mathbf{R}_I^\tau(Z_{I,j}(t), n_j(t), q_j(t))$ exhibits constant returns to scale with respect to $Z_{I,j}(t)$ and $n_j(t)$. Then the R&D cost per good can be denoted as

$$R_I^\tau(z_{I,j}(t), q_j(t)) \equiv \mathbf{R}_I^\tau(z_{I,j}(t), 1, q_j(t)) = \frac{\mathbf{R}_I^\tau(Z_{I,j}(t), n_j(t), q_j(t))}{n_j(t)}.$$

We further assume that

$$R_I^\tau(z_{I,j}(t), q_j(t)) = h_I^\tau(z_{I,j}(t))q_j(t)$$

for a strictly convex function $h_I^\tau(\cdot)$.

External innovation adds brand-new intermediate goods to the production portfolio of the firm. We assume that the new good has the same quality as the average quality of the goods produced by that firm. Thus, the quality of all products that firm j produces always have the same quality. The total intensity of external innovation is denoted by $Z_{X,j}(t)$. The innovation intensity per good is $z_{X,j}(t) \equiv Z_{X,j}(t)/n_j(t)$. The R&D cost for external innovation is assumed to be $\mathbf{R}_X^\tau(Z_{X,j}(t), n_j(t), q_j(t))$, which is assumed to be constant returns to scale with respect to $Z_{X,j}(t)$ and $n_j(t)$. Once again, we can denote the cost per good as

$$R_X^\tau(z_{X,j}(t), q_j(t)) \equiv \mathbf{R}_X^\tau(z_{X,j}(t), 1, q_j(t)) = \frac{\mathbf{R}_X^\tau(Z_{X,j}(t), n_j(t), q_j(t))}{n_j(t)},$$

and we assume that

$$R_X^\tau(z_{X,j}(t), q_j(t)) = h_X^\tau(z_{X,j}(t))q_j(t)$$

for a strictly convex function $h_X^\tau(\cdot)$.

A major departure from Luttmer (2011) is that we allow for internal innovation. Internal innovation allows us to capture the characteristics of intensive margin growth; we have shown in Section 2 that the size of establishments grow over time as firms age.

We assume that firms transition between different types from τ to τ' with Poisson transition rates $\lambda_{\tau\tau'}$. Each establishment depreciates (is forced to exit) with the Poisson rate δ_τ. We also impose an exogenous exit shock at the firm level. Let d_τ be the Poisson rate of the firm exit shock for a type-τ firm.

As in Klette and Kortum, the Hamilton-Jacobi-Bellman (HJB) equation for the firm can be written separately for each establishment. The establishment-level HJB equation of a type-τ firm is

$$rV_\tau(q) - \dot{V}_\tau(q) = \max_{z_I, z_X} \left[\begin{array}{l} \pi(q) - R_I^\tau(z_I, q) - R_X^\tau(z_X, q) + z_I \frac{\partial V_\tau(q)}{\partial q} q \\ + z_X V_\tau(q) - (\delta_\tau + d_\tau)V_\tau(q) + \sum_{\tau'} \lambda_{\tau\tau'}(V_{\tau'}(q) - V_\tau(q)) \end{array} \right],$$

where $V_\tau(q)$ is the value of type-τ establishment with quality q and $\dot{V}_\tau(q)$ is the time derivative of $V_\tau(q)$ function. We omit the time notation here, because all variables and functions here are constant over time along the balanced-growth path.

As in Mukoyama and Osotimehin (2018), $V_\tau(q)$ can be shown to be linearly homogeneous in q along the balanced-growth path. That is, $V_\tau(q) = v_\tau q$ for a constant v_τ. The HJB equation above can be normalized to

$$rv_\tau = \max_{z_I, z_X} \left[\bar{\pi} - h_I^\tau(z_I) - h_X^\tau(z_X) + (z_I + z_X - \delta_\tau - d_\tau)v_\tau + \sum_{\tau'} \lambda_{\tau\tau'}(v_{\tau'} - v_\tau) \right]. \tag{9}$$

Here, $\bar{\pi}$ is given by (8). We assume that $r - z_I - z_X + \delta_\tau + d_\tau > 0$.

The HJB equation (9) implies that the choice of innovation intensities (z_I, z_X) is a function of the firm type only. We denote the decision rules as (z_I^τ, z_X^τ). Recall that, from (8), $\bar{\pi}$ is a function of $\bar{w}$ only. Thus, given r and $\bar{w}$, the equation (9) and the first-order conditions can solve for v_τ, z_I^τ, and z_X^τ.

3.1.4 Entry

A firm can enter by creating a new product. A new firm draws its type from an exogenous distribution. Let the probability that an entrant draws the type τ be m_τ. Given the type τ, the entrant draws the relative quality $\hat{q}$, which is equal to $q(t)/Q(t)$, from a distribution $\Phi_\tau(\hat{q})$. The value for entry $V^e(t)$ is thus

$$V^e(t) = \sum_\tau m_\tau \int V_\tau(\hat{q}Q(t))d\Phi_\tau(\hat{q}).$$

Let $v^e \equiv V^e(t)/Q(t)$. Then

$$v^e = \sum_\tau v_\tau m_\tau \int \hat{q} d\Phi_\tau(\hat{q}) \tag{10}$$

holds.

We assume that anyone can enter by paying the cost $\phi Q(t)$ in final goods. Then the free-entry condition is

$$V^e(t) = \phi Q(t),$$

and therefore

$$v^e = \phi. \tag{11}$$

Note that once r is given, we can find a value of $\bar{w}$ that satisfies the free entry condition (11), where v^e is given by (10). Let the number of entrants at time t be $\mu_e N(t)$, where μ_e is a constant along the balanced-growth path.

3.1.5 Labor market

Before going into the general equilibrium, it is useful to summarize the equilibrium in the labor market. The total labor demand can be calculated from (7), and the equilibrium condition is

$$\int_{\mathcal{N}(t)} \left(\frac{w(t)}{(1-\beta)Z(t)} \right)^{-\frac{1}{\beta}} \frac{q_j(t)Y(t)}{Z(t)} dj = \left(\frac{w(t)}{(1-\beta)Z(t)} \right)^{-\frac{1}{\beta}} \frac{Y(t)N(t)Q(t)}{Z(t)} = L(t). \tag{12}$$

Thus, from the definition of $\bar{w}(t)$,

$$\bar{w}(t) = (1-\beta)\left(\frac{N(t)Q(t)}{L(t)Z(t)Y(t)^{\frac{\beta}{1-\beta}}}\right)^{\beta} \tag{13}$$

holds. Later we will see that $\bar{w}(t)$ is determined in general equilibrium. Because $N(0)$, $Q(0)$, $L(0)$, and $Z(0)$ are given by initial conditions (assuming that the economy is in a balanced-growth path at time 0), the equation (13) has to be viewed as the equation to pin down the level of $Y(0)$ when $\bar{w}(0)$ is determined in general equilibrium.

3.1.6 General equilibrium on a balanced growth path

Let us first go over the properties of a balanced growth path of this economy. Assume that the population $L(t)$ grows at an exogenous rate γ. Let the growth rate of $Q(t)$ along the balanced-growth path by ζ and the growth rate of $N(t)$ as η. Denote the growth rate of output $Y(t)$ by g. From the Euler equation (2),

$$g = \frac{r-\rho}{\sigma}$$

has to hold, because $C(t)$ grows at the same rate as $Y(t)$. This implies that once we know g, we can calculate r. From the argument in Section 3.1.4, we can characterize the firm behaviour once g is known.

Along the balanced-growth path, the normalized profit $\bar{\pi}(t)$ in (8) is constant. This implies that the growth rate of $w(t)/Z(t)$ is $\beta g/(1-\beta)$. Because the growth rate of $Z(t)$ is $\alpha\theta + (1-\alpha)\zeta$ from (5), $w(t)$ has to grow at the rate $\beta g/(1-\beta) + \alpha\theta + (1-\alpha)\zeta$. The labor income of the representative consumer $w(t)L(t)$ has to grow at the rate $\beta g/(1-\beta) + \alpha\theta + (1-\alpha)\zeta + \gamma$. This has to be equal to the growth rate of $Y(t)$, $C(t)$, $R(t)$, and $K(t)$. Thus

$$g = \gamma + \alpha\theta + (1-\alpha)\zeta + \frac{\beta}{1-\beta}g \tag{14}$$

holds. Note that in (13), $\bar{w}(t)$ has to be constant along the balanced-growth path. Therefore,

$$\eta + \zeta = \gamma + \alpha\theta + (1-\alpha)\zeta + \frac{\beta}{1-\beta}g$$

holds and thus

$$g = \eta + \zeta. \tag{15}$$

The growth of aggregate output is driven by two factors: the growth of the number of establishments $N(t)$ and the growth of the average quality of products $Q(t)$.

Rewriting (14), we can obtain an explicit formula for ζ given g:

$$\zeta = \frac{1}{1-\alpha}\left(\frac{1-2\beta}{1-\beta}g - \gamma - \alpha\theta\right).$$

Because $\eta = g - \zeta$,

$$\eta = g - \frac{1}{1-\alpha}\left(\frac{1-2\beta}{1-\beta}g - \gamma - \alpha\theta\right) \tag{16}$$

holds.

Note that the total number of establishments at each type (denote it as $N_\tau(t)$ for type τ) has to grow at the same rate, η. In other words, $N_\tau(t) = M_\tau N(t)$ has to hold for a constant M_τ that satisfies $M_\tau \in [0,1]$ for all τ and

$$\sum_\tau M_\tau = 1. \tag{17}$$

The law of motion for $N_\tau(t)$ is

$$\dot{N}_\tau(t) = z_X^\tau N_\tau(t) - (\delta_\tau + d_\tau)N_\tau(t) + \mu_e m_\tau N(t) - \sum_{\tau' \neq \tau} \lambda_{\tau\tau'} N_\tau(t) + \sum_{\tau' \neq \tau} \lambda_{\tau'\tau} N_{\tau'}(t).$$

The first term is the increase in products due to external innovation, the second term is the exit, the third term is the entry, and the fourth and fifth terms are the changes in firm types. The growth rate of $N_\tau(t)$ has to be the common value η. Thus this equation can be rewritten as

$$\eta = z_X^\tau - (\delta_\tau + d_\tau) + \mu_e \frac{m_\tau}{M_\tau} - \sum_{\tau' \neq \tau} \lambda_{\tau\tau'} + \sum_{\tau' \neq \tau} \lambda_{\tau'\tau} \frac{M_{\tau'}}{M_\tau}. \tag{18}$$

A simpler expression for η can be found by multiplying M_τ to both sides of (18) and adding up:

$$\eta = \sum_\tau M_\tau [z_X^\tau - (\delta_\tau + d_\tau)] + \mu_e.$$

Note that using (16), (18) can be rewritten as

$$g - \frac{1}{1-\alpha}\left(\frac{1-2\beta}{1-\beta}g - \gamma - \alpha\theta\right) = z_X^\tau - (\delta_\tau + d_\tau) + \mu_e \frac{m_\tau}{M_\tau} - \sum_{\tau' \neq \tau} \lambda_{\tau\tau'} + \sum_{\tau' \neq \tau} \lambda_{\tau'\tau} \frac{M_{\tau'}}{M_\tau}. \tag{19}$$

For notational convenience, define

$$Q_\tau(t) \equiv \frac{1}{N_\tau(t)} \int_{\mathcal{N}_\tau(t)} q_j(t) dj,$$

where $\mathcal{N}_\tau(t)$ is the set of actively produced goods by type-τ firms. Balanced growth implies that $Q_\tau(t)$ has to grow at the same rate as $Q(t)$. Also define

$$s_\tau \equiv M_\tau Q_\tau(t)/Q(t). \tag{20}$$

Note that s_τ is constant along the balanced-growth path and it satisfies

$$\sum_\tau s_\tau = 1. \tag{21}$$

Using these definitions, we can show that for any τ, g satisfies

$$g = z_I^\tau + z_X^\tau - (\delta_\tau + d_\tau) + \mu_e \frac{m_\tau}{s_\tau} \int \hat{q} d\Phi(\hat{q}) - \sum_{\tau' \neq \tau} \lambda_{\tau\tau'} + \sum_{\tau' \neq \tau} \lambda_{\tau'\tau} \frac{s_{\tau'}}{s_\tau}. \tag{22}$$

The derivation is in Appendix B. A simpler expression for g can be found by multiplying s_τ on both sides of (22) and summing across τ:

$$g = \sum_\tau s_\tau [z_I^\tau + z_X^\tau - (\delta_\tau + d_\tau)] + \mu_e \int \hat{q} d\Phi(\hat{q}).$$

The first term is the incumbent firms' contribution to g and the second term comes from entrants.

In sum, we can proceed with computation as follows. First, make a guess on g. Once g is known, we can compute r by $r = \rho + \sigma g$ and look for $\bar{w}$ that satisfies the free-entry condition (11) by solving (9) for given r and $\bar{w}$. Then, we compare the initial guess with the g calculated from the general equilibrium. In particular, the set of equations (17), (19), (21), and (22) can be used for pinning down μ_e, g, M_τ, and s_τ ($2T+2$ equations and $2T+2$ unknowns). If this g is different from the initial guess, adjust g until we find the fixed point.

When $\alpha = 1$, the algorithm is slightly different. Because we can solve for g from (14) as a function of parameters (exogenous growth), we can obtain r without making any guess. After solving the HJB equations and finding $\bar{w}$ that satisfies free-entry condition, we can use (21) and (22) to obtain μ_e and s_τ. Given these, η and M_τ can be solved from (17) and (18).

3.1.7 Accounting for aggregate resources

In this subsection, we decompose $Y(t)$ into $C(t)$, $R(t)$, and $K(t)$. This decomposition will be used in the estimation of the model later on.

Plugging the optimal choice for $x_j(t)$, (6), in the aggregate production function (3), we obtain

$$\begin{aligned} Y(t) &= \left(\int_{\mathcal{N}(t)} q_j(t)^\beta x_j(t)^{1-\beta} dj \right)^{\frac{1}{1-\beta}} \\ &= \left(\int_{\mathcal{N}(t)} q_j(t)^\beta \left[(1-\beta)^{\frac{1}{\beta}} \left(\frac{w(t)}{Z(t)} \right)^{-\frac{1}{\beta}} Y(t) q_j(t) \right]^{1-\beta} dj \right)^{\frac{1}{1-\beta}} \\ &= (1-\beta)^{\frac{1}{\beta}} \left(\frac{w(t)}{Z(t)} \right)^{-\frac{1}{\beta}} Y(t) \left(\int_{\mathcal{N}(t)} q_j(t) dj \right)^{\frac{1}{1-\beta}} \\ &= \left(\frac{w(t)}{(1-\beta)Z(t)} \right)^{-\frac{1}{\beta}} Y(t) \left(N(t) Q(t) \right)^{\frac{1}{1-\beta}}, \end{aligned}$$

where the last equality uses the definition of $Q(t)$. Simplifying $Y(t)$ from both sides, we arrive at:

$$1 = \left(\frac{w(t)}{(1-\beta)Z(t)} \right)^{-\frac{1}{\beta}} \left(N(t) Q(t) \right)^{\frac{1}{1-\beta}}.$$

Combining this identity with the labor market clearing condition (13) yields

$$Y(t) = L(t)Z(t)\left(N(t)Q(t)\right)^{\frac{\beta}{1-\beta}}.$$

We then use (13) to replace $L(t)Z(t)$ with $\left(\frac{\bar{w}(t)}{1-\beta}\right)^{-\frac{1}{\beta}} N(t)Q(t)Y(t)^{-\frac{\beta}{1-\beta}}$. Thus

$$Y(t) = \left(\frac{\bar{w}(t)}{1-\beta}\right)^{-\frac{1-\beta}{\beta}} N(t)Q(t).$$

On a balanced-growth path, $K(t) = \mu_e N(t)\phi Q(t)$. Therefore

$$\frac{K(t)}{Y(t)} = \mu_e \phi \left(\frac{\bar{w}(t)}{1-\beta}\right)^{\frac{1-\beta}{\beta}}.$$

The aggregate cost of intensive and extensive margin investment is given by:

$$\begin{aligned} R(t) &= \textstyle\sum_\tau \left(h_X^\tau(z_X^\tau) + h_I^\tau(z_I^\tau)\right) \int_{\mathcal{N}_\tau(t)} q_j(t)dj \\ &= \textstyle\sum_\tau \left(h_X^\tau(z_X^\tau) + h_I^\tau(z_I^\tau)\right) N_\tau(t)Q_\tau(t) \\ &= N(t)Q(t)\textstyle\sum_\tau \left(h_X^\tau(z_X^\tau) + h_I^\tau(z_I^\tau)\right) s_\tau, \end{aligned}$$

where s_τ is defined in (20). As a result

$$\frac{R(t)}{Y(t)} = \left(\sum_\tau \left(h_X^\tau(z_X^\tau) + h_I^\tau(z_I^\tau)\right) s_\tau\right)\left(\frac{\bar{w}(t)}{1-\beta}\right)^{\frac{1-\beta}{\beta}}.$$

Combining the expressions for the ratios K/Y and R/Y, we obtain the fraction of total investment over output:

$$\frac{K(t)+R(t)}{Y(t)} = \left(\mu_e\phi + \sum_\tau \left(h_X^\tau(z_X^\tau) + h_I^\tau(z_I^\tau)\right) s_\tau\right)\left(\frac{\bar{w}(t)}{1-\beta}\right)^{\frac{1-\beta}{\beta}}. \tag{23}$$

In the estimation of the model below, we use this fraction as a target.

3.2 Model properties

Before evaluating the model quantitatively, it is useful to establish some general properties of the model. First we look at the partial-equilibrium properties of the general model. Then we obtain sharper characterizations for the case with a single type.

3.2.1 Partial-equilibrium properties of the general model

First, we establish some partial-equilibrium properties of the HJB equation (9). It can be shown that v_τ moves monotonically with $\bar{\pi}$ for all types.

Lemma 1 *When r is fixed, $\partial v_\tau/\partial\bar{\pi} \geq 0$ for all τ, with strict inequality for at least one τ.*

Proof. Suppose not. It is straightforward to see that $\partial v_\tau/\partial\bar{\pi} = 0$ for all τ would not satisfy the HJB equation (9). The equation (9) can be rewritten as

$$(r - z_I^\tau - z_X^\tau + \delta_\tau + d_\tau)v_\tau = \bar{\pi} - h_I^\tau(z_I^\tau) - h_X^\tau(z_X^\tau) + \sum_{\tau'} \lambda_{\tau\tau'}(v_{\tau'} - v_\tau). \tag{24}$$

Choose τ that provides the minimum value of $\partial v_\tau/\partial\bar{\pi}$. Then, from the supposition, $\partial v_\tau/\partial\bar{\pi} < 0$. For this τ, $\partial(v_{\tau'} - v_\tau)/\partial\bar{\pi} \geq 0$ has to hold for all τ'. We can ignore the effect through endogenous z_I^τ and z_X^τ because of the envelope theorem. Then, when $\bar{\pi}$ increases, the left-hand side of (24) strictly decreases while the right-hand side has to increase. Contradiction. ■

A similar logic can be used for analyzing the change in r.

Lemma 2 *When $\bar{\pi}$ is fixed, $\partial v_\tau/\partial r \leq 0$ for all τ, with strict inequality for at least one τ.*

Proof. Suppose not. Choose τ that provides the maximum value of $\partial v_\tau/\partial r$. Then, from the supposition, $\partial v_\tau/\partial r > 0$. For this τ, $\partial(v_{\tau'} - v_\tau)/\partial\bar{\pi} \leq 0$ has to hold for all τ'. We can ignore the effect through endogenous z_I^τ and z_X^τ because of the envelope theorem. When r increases, for this τ, the left-hand side of (24) strictly increases while the right-hand side has to decrease. Contradiction. ■

Lemma 1 and Lemma 2 imply that $\bar{\pi}$ and r have opposite influences on v_τ. Since the first-order conditions for the innovation decision

$$h_I^{\tau\prime}(z_I^\tau) = v_\tau$$

and

$$h_X^{\tau\prime}(z_X^\tau) = v_\tau$$

crucially depend on the value of v_τ, these results provide information on how innovations react to the external environment.

In general equilibrium, the fact that the free-entry condition (11) has to hold imposes how equilibrium objects react to the external changes.

Proposition 1 *Suppose that ϕ, m_τ, $\Phi_\tau(\cdot)$, δ_τ, d_τ, $\lambda_{\tau\tau'}$, $h_I^\tau(\cdot)$ and $h_X^\tau(\cdot)$ are kept constant. Then a change in a parameter moves r and $\bar{\pi}$ in the same direction.*

Proof. Suppose not, and consider the case where r goes up and $\bar{\pi}$ goes down. Because δ_τ, d_τ, $\lambda_{\tau\tau'}$, $h_I^\tau(\cdot)$ and $h_X^\tau(\cdot)$ are kept constant, the HJB equation (9) remains the same. Therefore, from Lemma 1 and Lemma 2, v_τ goes down for all τ. This contradicts to the free-entry condition (11) because (10) implies that v_e has to go down. The opposite case leads to a contradiction with the same steps. ■

The result of this proposition applies to the comparative statics on the parameters such as β, α, and θ. Because $\bar{\pi}$ is negatively linked to $\bar{w}$ through (8) and r is positively linked to g through $r = \rho + \sigma g$, this proposition imply a negative relationship between $\bar{w}$ and g.

3.2.2 Properties of the model with one type

Although the quantitative exercise below features multiple types, the case with one type allows us sharper characterizations of the general equilibrium and help us develop intuitions. We omit the subscripts and superscripts τ as there is only one type.

Recall the output growth rate g can be decomposed into the growth of $Q(t)$ and the growth of $N(t)$ with the formula (15). The separate formula for ζ and η are particularly simple for the single-type case:

$$\zeta = z_I + \mu_e \left(\int \hat{q} d\Phi(\hat{q}) - 1 \right) \tag{25}$$

and

$$\eta = z_X - \delta - d + \mu_e \tag{26}$$

The sum, g, is

$$g = z_I + z_X - \delta - d + \mu_e \int \hat{q} d\Phi(\hat{q}). \tag{27}$$

The single-type case is particularly convenient as it has the recursive structure in solution. The free-entry condition (11) and (10) imply that

$$v = v_e = \phi.$$

From the first-order condition for innovations, this imply that z_I and z_X are determined only by ϕ and the innovation cost functions $h_I(\cdot)$ and $h_X(\cdot)$. Then the only endogenous variable to determine ζ and g in (25) and (27) is μ_e. Plugging (25) and (27) into (14) yields an equation with one unknown μ_e. With some algebra, this equation becomes

$$\left[\left(\frac{1-2\beta}{1-\beta} - (1-\alpha) \right) \int \hat{q} d\Phi(\hat{q}) + 1 - \alpha \right] \mu_e + \left[\frac{1-2\beta}{1-\beta} - (1-\alpha) \right] z_I + \frac{1-2\beta}{1-\beta} (z_X - \delta - d) = \gamma + \alpha\theta.$$

We assume that β and α are sufficiently small so that the coefficients of μ_e, z_I, and $(z_X - \delta - d)$ on the left-hand side are all positive. Then the above steps solve out μ_e, ζ, g, η, z_I, z_X, and v as functions of parameters. The only equilibrium object left here is $\bar{w}$, which can be solved by the HJB equation

$$(\rho + \sigma g) v = \bar{\pi} - h_I(z_I) - h_X(z_X) + (z_I + z_X - \delta - d) v$$

and the relationship (8).

The following propositions are straightforward and therefore presented without proof.

Proposition 2 *An increase in the entry cost, ϕ, increases z_I and z_X, while reducing μ_e. When $\alpha = 1$ (exogenous growth), these effects exactly offset and g stays constant. When $\alpha < 1$ and $\int \hat{q} d\Phi(\hat{q}) \leq 1$, g increases.*

Proposition 3 *Suppose that the innovation cost functions take the form*

$$h_i(z_i) = \chi_i z_i^{\psi},$$

where $i = I, X$. The parameters satisfy $\chi_i > 0$ and $\psi > 1$. Then a decrease in χ_I increases z_I but keeps z_X the same. The entry rate μ_e decreases, and when $\alpha < 1$, the overall growth rate increases. A decrease in χ_X increases z_X but keeps z_I the same. The entry rate μ_e decreases, and when $\alpha < 1$ and $\int \hat{q} d\Phi(\hat{q}) \leq 1$, g increases.

In above Propositions, the direct effect through the changes in z_I and z_X have stronger impact on growth than the (offsetting) effect of μ_e, unless $\int \hat{q} d\Phi(\hat{q}) > 1$.

Proposition 4 *An increase in δ, d, γ, or θ keeps z_I and z_X constant, while increasing g through an increase in μ_e. Changes in α and β do not affect z_I and z_X either, although they influence g through the change in μ_e.*

Proposition 5 *Changes in the preference parameters σ and ρ do not have any effects on z_I, z_X, and g.*

From above analysis, we can suspect that the increase in the number of establishments per firm, which is likely be driven by an increase in z_X, can be explained by the decrease in χ_X. Other changes in parameters either move z_I and z_X together or move only μ_e without affecting z_X. The slowdown in overall growth cannot be accounted for by the change χ_X. Additional change in other technology parameters, such as δ, d, γ, or θ, can account for the overall growth slowdown.

4 Distributions of firm sizes and establishment sizes

This section characterizes the firm size distribution in our model analytically. The analytical results will be useful for our estimation later on.

The properties of our model allow us to analyze the firm size distribution from two different margins. In one margin, the number of establishments per firm evolves through the external innovation and exit shock. In the other margin, the size of each establishment evolves through the internal innovation. Note that the existence of these two margins is the major departure from Klette and Kortum (2004) and Luttmer (2011). In these papers, establishments are homogeneous and each establishment does not grow, so that the only relevant innovation is the external one.

Before analyzing the details, first note one general property of the model: the model assumptions imply that the establishments are homogeneous within a firm. A firm starts with one establishment, and whenever it expands the number of establishments, a new establishment inherits the same quality as the existing establishments. The intensity of the internal innovation, z_I^{τ}, is common across establishments within a firm, while it may change over time. These two facts mean that establishments share common quality within a firm at any point in time. This also imply that the establishment sizes are also common within a firm. Although in reality establishment sizes are not the same within a firm, we view this as a useful simplification.

In the following analysis, we focus on the balanced-growth equilibrium. Along a balanced-growth equilibrium, the number of firms grow at the same rate as the number of establishments $N(t)$. The distribution of the number of establishments per firm, $n(t)$, is stationary, and the average quality of each establishment, $q(t)$, grows at the same rate as $Q(t)$.

4.1 General characterization

Let $\mathcal{M}_\tau(n, \hat{q})$ be the normalized measure of type-τ firms with n establishments and $q(t) \geq \hat{q}Q(t)$. We start from looking at the two separate margins of the firm size. First, we denote the normalized measure of type-τ firms with $n(t) = n$ as $\bar{\mathcal{M}}_\tau(n)$ Second, we denote the fraction of type-τ establishments with $q(t) \geq \hat{q}Q(t)$ as $\bar{\mathcal{H}}_\tau(\hat{q})$.

The first margin, $\bar{\mathcal{M}}_\tau(n)$, can be characterized by the following set of difference equations. For each $\tau \in \Gamma$,

$$\begin{aligned} 0 = \quad & -(z_X^\tau + \delta_\tau + d_\tau + \eta)\bar{\mathcal{M}}_\tau(1) + 2\delta_\tau \bar{\mathcal{M}}_\tau(2) + \mu_e m_\tau \\ & - \textstyle\sum_{\tau' \neq \tau} \lambda_{\tau\tau'} \bar{\mathcal{M}}_\tau(1) + \textstyle\sum_{\tau' \neq \tau} \lambda_{\tau'\tau} \bar{\mathcal{M}}_{\tau'}(1) \end{aligned} \tag{28a}$$

has to hold and

$$\begin{aligned} 0 = \quad & -(n(z_X^\tau + \delta_\tau) + d_\tau + \eta)\bar{\mathcal{M}}_\tau(n) + (n+1)\delta_\tau \bar{\mathcal{M}}_\tau(n+1) + (n-1) z_X^\tau \bar{\mathcal{M}}_\tau(n-1) \\ & - \sum_{\tau' \neq \tau} \lambda_{\tau\tau'} \bar{\mathcal{M}}_\tau(n) + \sum_{\tau' \neq \tau} \lambda_{\tau'\tau} \bar{\mathcal{M}}_{\tau'}(n) \end{aligned} \tag{28b}$$

holds for $n > 1$.

The second margin, $\bar{\mathcal{H}}_\tau(\hat{q})$, is governed by the following Kolmogorov forward equation:

$$\begin{aligned} (z_I^\tau - \zeta)\hat{q} \frac{d\bar{\mathcal{H}}_\tau(\hat{q})}{d\hat{q}} = \quad & -(\delta_\tau + d_\tau + \eta - z_X^\tau)\bar{\mathcal{H}}_\tau(\hat{q}) + \mu_e \tfrac{m_\tau}{M_\tau}(1 - \Phi(\hat{q})) \\ & - \textstyle\sum_{\tau' \neq \tau} \lambda_{\tau\tau'} \bar{\mathcal{H}}_\tau(\hat{q}) + \textstyle\sum_{\tau' \neq \tau} \lambda_{\tau'\tau} \tfrac{M_{\tau'}}{M_\tau} \bar{\mathcal{H}}_{\tau'}(\hat{q}). \end{aligned} \tag{29}$$

The distribution of firm sizes $\mathcal{M}_\tau(n, \hat{q})$ is some "convolution" of the distribution of the number of establishments per firm and the distribution of establishment sizes. This can be derived as follows. For each $\tau \in \Gamma$, we have

$$\begin{aligned} (z_I^\tau - \zeta)\hat{q} \frac{d\mathcal{M}_\tau(1, \hat{q})}{d\hat{q}} &= -(z_X^\tau + \delta_\tau + d_\tau + \eta)\mathcal{M}_\tau(1, \hat{q}) \\ &\quad + 2\delta_\tau \mathcal{M}_\tau(2, \hat{q}) + \mu_e m_\tau (1 - \Phi(\hat{q})) \\ &\quad + \sum_{\tau' \neq \tau} \lambda_{\tau'\tau} \mathcal{M}_{\tau'}(1, \hat{q}) - \sum_{\tau' \neq \tau} \lambda_{\tau\tau'} \mathcal{M}_\tau(1, \hat{q}). \end{aligned} \tag{30a}$$

and for $n > 1$:

$$(z_I^\tau - \zeta)\hat{q}\frac{d\mathcal{M}_\tau(n,\hat{q})}{d\hat{q}} = -(n(z_X^\tau + \delta_\tau) + d_\tau + \eta)\mathcal{M}_\tau(n,\hat{q}) + (n+1)\delta_\tau\mathcal{M}_\tau(n+1,\hat{q};t) + (n-1)z_X^\tau\mathcal{M}_\tau(n-1,\hat{q}) + \sum_{\tau'\neq\tau}\lambda_{\tau'\tau}\mathcal{M}_{\tau'}(n,\hat{q}) - \sum_{\tau'\neq\tau}\lambda_{\tau\tau'}\mathcal{M}_\tau(n,\hat{q}). \tag{30b}$$

It is not straightforward to make a further progress with the general formulation here. Below, we look at the case of one firm type more closely.

4.2 Distributions for one-type economy

First we characterize the distribution of establishment sizes and the distribution of number of establishment per firm. Then we characterize the distribution of firm sizes.

4.2.1 Establish size distribution (intensive margin)

When there is only one firm type, firm growth is governed by three endogenous numbers: z_I, z_X, and μ_e. Note that, in this case, our model assumptions imply that for a given firm, the quality (and therefore the size) of each establishment grows at a deterministic rate z_I that is common across all firms. The average quality $Q(t)$ grows at the rate ζ given in (25). Thus, the quality of the establishments in a firm who started at time t_0 and whose initial draw of the normalized quality is $\hat{q}Q(t_0)$ can be represented as (denoting it by $q_{t_0}(t)$)

$$q_{t_0}(t) = \hat{q}Q(t_0)e^{z_I(t-t_0)} = \hat{q}Q(t)e^{(z_I-\zeta)(t-t_0)}.$$

From the labor demand (7) and the labor market equilibrium condition (12), it is straightforward to show that along the balanced-growth equilibrium, the relative labor demand $\ell(t)/L(t)$ of a particular establishment with quality $q_{t_0}(t)$ is equal to $q_{t_0}(t)/(N(t)Q(t))$. Therefore, the cross-sectional distribution of establishment size at a given time t is the same as the distribution of $\hat{q}e^{(z_I-\zeta)(t-t_0)}$. Denoting the time-$t$ number of establishments for a firm that starts at time t_0 as $n_{t_0}(t)$ (note that $n_{t_0}(t)$ is stochastic as the external innovation is random), the (relative) firm size distribution follows the distribution of $n_{t_0}(t)\hat{q}e^{(z_I-\zeta)(t-t_0)}/N(t)$.

For one-type economy, equation (29) becomes

$$(z_I - \zeta)\hat{q}\frac{d\bar{\mathcal{H}}(\hat{q})}{d\hat{q}} = -(\delta + d + \eta - z_X)\bar{\mathcal{H}}(\hat{q}) + \mu_e(1 - \Phi(\hat{q})).$$

Let us use the following change of variable $p = \log(\hat{q})$ and $\tilde{\mathcal{H}}(p) = \bar{\mathcal{H}}(\exp(p))$. Then this equation can be rewritten as:

$$(z_I - \zeta)\frac{d\tilde{\mathcal{H}}(p)}{dp} = -(\delta + d + \eta - z_X)\tilde{\mathcal{H}}(p) + \mu_e(1 - \Phi(\exp(p))).$$

This is a first-order ordinary differential equation, which has a general solution of

$$\tilde{\mathcal{H}}(p) = e^{\frac{\delta+d+\eta-z_X}{z_I-\zeta}(\underline{p}-p)}\tilde{\mathcal{H}}(\underline{p}) + \int_{\underline{p}}^{p} e^{\frac{\delta+d+\eta-z_X}{z_I-\zeta}(\tilde{p}-p)} \frac{\mu_e}{z_I-\zeta}(1-\Phi(\exp(\tilde{p})))d\tilde{p},$$

for each $\underline{p}$. Taking the limit $\underline{p} \to -\infty$, and using (26) to replace μ_e with $\delta + d + \eta - z_X$, we arrive at:

$$\tilde{\mathcal{H}}(p) = \int_{-\infty}^{p} e^{\frac{\delta+d+\eta-z_X}{z_I-\zeta}(\tilde{p}-p)} \frac{\delta+d+\eta-z_X}{z_I-\zeta}(1-\Phi(\exp(\tilde{p})))d\tilde{p}. \tag{31}$$

This expression shows that $\tilde{\mathcal{H}}(\log x)$ is the CCDF of a random variable X defined by a convolution between a Pareto distribution with scale parameter 1 and tail index $(\delta+d+\eta-z_X)/(z_I-\zeta)$ and a distribution with CDF Φ: $X = YZ$, $Y \sim Pareto\,(1,(\delta+d+\eta-z_X)/(z_I-\zeta))$ and $Z \sim \Phi$.

Notice also that, when Φ is a log-normal distribution, $\tilde{\mathcal{H}}$ is a convolution of a Pareto distribution and a log-normal distribution analyzed in Reed (2001), and more recently, Cao and Luo (2017) and Sager and Timoshenko (2018). Therefore, we offer an alternative micro-foundation of this convolution distribution with endogenous establishment growth rate, relative to the micro-foundation in Reed (2001) with exogenous growth rate. Our micro-foundation is also more general because it allows for any distribution of Φ, while Reed (2001) by using Brownian motions only allows Φ to be a log-normal distribution.

Using this explicit solution, it is easy to show that when Φ has thin right tail, for example when Φ is a (left-truncated) log-normal distribution, and $z_I > \zeta$, $\bar{\mathcal{H}}(p)$ has a Pareto tail with the index given by:[10]

$$\frac{\delta+d+\eta-z_X}{z_I-\zeta}. \tag{32}$$

4.2.2 Distribution of the number of establishments per firm (extensive margin)

For the distribution of the number of establishments per firm, (28) becomes

$$0 = -(z_X+\delta+d+\eta)\bar{\mathcal{M}}(1) + 2\delta\bar{\mathcal{M}}(2) + \mu_e$$

and for $n > 1$:

$$0 = -(n(z_X+\delta)+d+\eta)\bar{\mathcal{M}}(n) + (n+1)\delta\bar{\mathcal{M}}(n+1) + (n-1)z_X\bar{\mathcal{M}}(n-1).$$

Luttmer (2011) provides a closed form solution for $\{\bar{\mathcal{M}}(n)\}_{n=1}^{\infty}$:

$$\bar{\mathcal{M}}(n) = \frac{1}{n}\frac{\mu_e}{z_X}\sum_{k=0}^{\infty}\frac{1}{\beta_{n+k}}\left(\prod_{m=n}^{n+k}\beta_m\right)\prod_{m=1}^{n+k}\frac{z_X\beta_m}{\delta}, \tag{33}$$

[10] Using (25) and (26), this tail index is also equal to $1/(1-\int \hat{q}d\Phi(\hat{q}))$.

where the sequence $\{\beta_n\}_{n=0}^{\infty}$ is defined recursively by $\beta_0 = 0$ and

$$\frac{1}{\beta_{n+1}} = 1 - \frac{z_X \beta_n}{\delta} + \frac{\eta + d + z_X n}{\delta n}.$$

Luttmer (2011) also shows that when $z_X > \delta$, $\bar{\mathcal{M}}(n)$ has Pareto tail with the index given by

$$\frac{\eta + d}{z_X - \delta}. \tag{34}$$

The following proposition summarizes the last two results.

Proposition 6 *In a balanced growth equilibrium with $z_I > \zeta$ and $z_X > \delta$, the stationary distribution of establishment sizes and the stationary distribution of the number of establishments per firm have Pareto right tail and the tail indices are given by* (32) *and* (34) *respectively.*

4.2.3 Firm size distribution

Now, we analyze the distribution of firm sizes, which is the combination of the previous two margins. First, we consider a special case where the initial draw satisfies $\int \hat{q} d\Phi(\hat{q}) = 1$. In this case, (25) implies that $z_I = \zeta$ holds, and thus the establishment size distribution is identical to the distribution of $\hat{q}$. Furthermore, in this case, the distribution of $\hat{q}$ for a given $n(t)$ follows the identical distribution $\Phi(\hat{q})$. The distribution of $n(t)$ evolves with the standard birth-death process, as in Klette and Kortum (2004) and Luttmer (2011). Let the number of firms with $n(t) = n$ in the balanced-growth path be $\bar{\mathcal{M}}(n)N(t)$.[11]

The $N(t)$-normalized measure of firms with $n(t) = n$ and $q(t) \geq \hat{q}Q(t)$, $\mathcal{M}(n, \hat{q})$, is

$$\mathcal{M}(n, \hat{q}) = \bar{\mathcal{M}}(n)(1 - \Phi(\hat{q})),$$

where $\bar{\mathcal{M}}(n)$ is given by (33). Once we have $\mathcal{M}(n, \hat{q})$, the firm size distribution can be computed easily. The fraction of firms with size $\mathbf{l}(t) \geq \hat{\mathbf{l}}L(t)$, denoted by $\mathbf{M}(\hat{\mathbf{l}})$, can be computed as

$$\mathbf{M}(\hat{\mathbf{l}}) = \frac{\sum_n \mathcal{M}(n, \hat{\mathbf{l}}/n)}{\sum_n \bar{\mathcal{M}}(n)}.$$

To determine the tail index of $\mathbf{M}(.)$, we consider the Laplace transformation:[12]

$$\varphi(s) = \int_0^{\infty} \hat{l}^s \left(-d\mathbf{M}(\hat{l}) \right). \tag{35}$$

[11] Note that below we normalize all measures of firms, such as $\bar{\mathcal{M}}(n)$, by $N(t)$. This is because the number of firms grows at the same rate as $N(t)$ does. Normalizing by $N(t)$ implies that $\sum_n n\bar{\mathcal{M}}(n) = 1$ and the fraction of firms with $n(t) = n$ can be obtained by calculating $\bar{\mathcal{M}}(n)/\sum_n \bar{\mathcal{M}}(n)$.

[12] After a change of variable $\hat{l} = \exp(p)$, the transformation can be re-written in its more familiar form: $\int_{-\infty}^{\infty} e^{sp} \left(-d\mathbf{M}(\exp(p))\right)$.

Using the expression for $\mathbf{M}$ above, we rewrite φ as:

$$\begin{aligned}\varphi(s) &= \int_0^\infty \hat{l}^s \frac{-d\sum_n \bar{\mathcal{M}}(n)(1-\Phi(\hat{l}/n))}{\sum_n \bar{\mathcal{M}}(n)} \\ &= \int_0^\infty \hat{l}^s \frac{\sum_n \bar{\mathcal{M}}(n) d\Phi(\hat{l}/n)}{\sum_n \bar{\mathcal{M}}(n)} \\ &= \int_0^\infty \frac{\sum_n \bar{\mathcal{M}}(n) n^s (\hat{l}/n)^s d\Phi(\hat{l}/n)}{\sum_n \bar{\mathcal{M}}(n)} \\ &= \left\{\int_0^\infty \hat{l}^s d\Phi(\hat{l})\right\}\left\{\frac{\sum_n \bar{\mathcal{M}}(n) n^s}{\sum_n \bar{\mathcal{M}}(n)}\right\}\end{aligned}$$

Assume that the entry distribution Φ has thin right tail and using the closed form solution in Luttmer (2011), we can show that

$$\varphi(s) \sim \frac{c}{\frac{\eta+d}{z_X-\delta} - s}$$

as $s \uparrow (\eta + d)/(z_X - \delta)$. Therefore, by the Tauberian theorem in Mimica (2016, Theorem 1.2), $\mathbf{M}$ has right Pareto tail with the tail index given by $(\eta + d)/(z_X - \delta)$. This tail index is identical to the index for the number of establishment per firm, (34).

The following proposition summarizes the result for this case.

Proposition 7 *In a balanced growth equilibrium with $z_I = \zeta$ and $z_X > \delta$, and the distribution of entry sizes Φ has thin right tail, firm size distribution has Pareto right tail with the tail index equals to the tail index of the distribution of the number of establishments per firm given by* (34).

Now we consider the case where $z_I > \zeta$, i.e. $\int \hat{q} d\Phi(\hat{q}) < 1$. This case is more challenging than the previous case. The system of differential equations (30) for $\mathcal{M}(n, \hat{q})$ simplifies to:

$$(z_I - \zeta)\hat{q}\frac{d\mathcal{M}(1,\hat{q})}{dq} = -(z_X + \delta + d + \eta)\mathcal{M}(1,\hat{q}) + 2\delta\mathcal{M}(2,\hat{q}) + \mu_e(1-\Phi(\hat{q}))$$

and

$$(z_I - \zeta)\hat{q}\frac{d\mathcal{M}(n,\hat{q})}{dq} = -(n(z_X + \delta) + d + \eta)\mathcal{M}(n,\hat{q}) + (n+1)\delta\mathcal{M}(n+1,\hat{q}) + (n-1)z_X\mathcal{M}(n-1,\hat{q})$$

for $n > 1$. Multiplying both sides of these equations with $\hat{q}^{s-1}$ and integrate from 0 to ∞ then integrating by parts

$$\int_0^\infty \hat{q}^{s-1}\mathcal{M}(n,\hat{q})d\hat{q} = -\frac{1}{s}\int_0^\infty \hat{q}^s d\mathcal{M}(n,\hat{q}),$$

we obtain:

$$-(z_I - \zeta)s\hat{\varphi}(1,s) = -(z_X + \delta + d + \eta)\hat{\varphi}(1,s) - 2\delta\hat{\varphi}(1,s) - \int_0^\infty \hat{q}^{s-1}\mu_e(1-\Phi(\hat{q}))$$

and

$$-(z_I - \zeta)s\hat{\varphi}(n,s) = -(n(z_X+\delta)+d+\eta)\hat{\varphi}(n,s) + (n+1)\delta\hat{\varphi}(n+1,s) + (n-1)z_X\hat{\varphi}(n-1,s)$$

for $n > 1$, where

$$\hat{\varphi}(n,s) \equiv \int_0^\infty \hat{q}^s(-d\mathcal{M}(n,\hat{q})).$$

For each $s \geq 0$, the equations form a system of difference equations and allow us to solve for $\hat{\varphi}(n,s)$ for all $n \geq 1$ using the closed form solution from Luttmer (2011) (with η being replaced by $\eta - (z_I - \zeta)s$). We can also show that $\{\hat{\varphi}(n,s)\}_{n=1}^\infty$ has a Pareto tail with the tail index given by

$$\frac{d+\eta-(z_I-\zeta)s}{z_X-\delta}.$$

Now, with the solution for $\hat{\varphi}(n,s)$, we can calculate the Laplace transform (35) as follows:

$$\begin{aligned}
\varphi(s) &= \int_0^\infty \hat{l}^s \frac{-d\sum_n \mathcal{M}(n,\hat{l}/n)}{\sum_n \bar{\mathcal{M}}(n)} \\
&= \frac{1}{\sum_n \bar{\mathcal{M}}(n)} \sum_n n^s \int_0^\infty (\hat{l}/n)^s(-d\mathcal{M}(n,\hat{l}/n)) \\
&= \frac{\sum_n n^s \hat{\varphi}(n,s)}{\sum_n \bar{\mathcal{M}}(n)}.
\end{aligned}$$

Using the tail property of $\hat{\varphi}(n,s)$, it is easy to show that $\varphi(s)$ is finite up to s^* determined by

$$\frac{d+\eta-(z_I-\zeta)s}{z_X-\delta} = s,$$

or equivalently

$$s^* = \frac{\eta+d}{z_X-\delta+z_I-\zeta}. \tag{36}$$

Therefore $\mathbf{M}$ has a Pareto tail with the tail index given by s^* (again using the Tauberian theorem in Mimica (2016, Theorem 1.2)). We summarize these derivations in the following proposition.

Proposition 8 *In a balanced-growth equilibrium with $z_I > \zeta$ and $z_X > \delta$, and the distribution of establishments per firm has fatter tail than the distribution of establishment sizes, firm size distribution has Pareto tail with the tail index given by* (36)*.*

Compared to the previous case, for the same values of $\eta + d$ and $z_X - \delta$, the tail is thicker. This is natural, as in this case the relative size of the incumbent establishments also grow.

5 Estimation and quantitative experiments

We first present the estimation procedure and the estimates. Then we use the estimated model to carry out counter-factual exercises. An important difference from the previous studies that estimate the model of innovation, such as Lentz and Mortensen (2008) and Akcigit and Kerr (2016), is that we exploit establishment-level information in analyzing firm dynamics.

5.1 Estimation of the model

We first estimate a simple version of the model with two-types: $\Gamma = \{H, L\}$. H-type firms have a lower cost of external investment and expand their number of establishments faster. H-type firms transition to L-type firms at the rate $\lambda_{HL} > 0$ while L-type firms do not transition to H-type firms, i.e. $\lambda_{LH} = 0$. These are the assumptions made in Luttmer (2011) (but without differentiating the extensive versus intensive margins as we do here). The assumptions give closed-form solutions for the distribution of the number of establishments per firm and we use the closed-form solutions to estimate the model. To simplify the estimation, we fix d_H, d_L at $0\%, 2\%$ respectively and δ_H, δ_L at 10% because these parameters can potentially be directly estimated from the data.

We estimate the model in two steps. In Step 1, we estimate (Step 1a)

$$z_X^H, z_X^L, \lambda_{HL}, \mu_e, m_H, m_L,$$

using the moments related to the number of establishments per firm; and (Step 1b)

$$z_I^H, z_I^L, \Phi(.)$$

using the moments related to the number of employees per establishments. In Step 2, we assume functional forms for the cost functions h_X, h_I and estimate the parameters of these functions using the estimates from Step 1.

Step 1a In this step, we target distribution of establishments per firm, in addition to the growth rate of the number of the establishments.

[We are awaiting data disclosure for the inputs of the estimation.]

Step 1b In this step, we target the distribution of the number of employees per establishments as well as the average growth rate of establishment sizes.

[We are awaiting data disclosure for the inputs of the estimation.]

The estimates in Step 1a and Step 1b imply the stationary distribution of firm sizes.

Step 2 In this step we use the estimates from Step 1a and Step 1b, including $z_i^\tau, \tau \in \{H, L\}, i \in \{X, I\}$, to estimate the original model. To simplify the estimation, we assume that $\alpha = 1$, and fix several parameters at standard values:[13]

$$\beta = 1 - \frac{1}{1.30}; \sigma = 1; \gamma = 0.01; \rho = 0.01.$$

In addition, we assume that the innovation cost functions are iso-elastic:

$$h_i^\tau(z) \equiv \chi_i^\tau z^\psi,$$

for $\tau \in \Gamma$ and $i \in \{X, I\}$, with $\psi = 2$.

The first order condition in (9) implies $\psi \chi_i^\tau (z_i^\tau)^{\psi-1} = v_\tau$, and hence

$$-h_i^\tau(z_i^\tau) + z_i^\tau v_\tau = \left(1 - \frac{1}{\psi}\right) z_i^\tau v_\tau.$$

Substituting this expression in (9) and re-arranging, we arrive at:

$$A \begin{bmatrix} v_H \\ v_L \end{bmatrix} = \bar{\pi} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} r - (1 - \frac{1}{\psi})(z_X^H + z_I^H) + \delta_H + d_H + \lambda_{HL} & -\lambda_{HL} \\ 0 & r - (1 - \frac{1}{\psi})(z_X^L + z_I^L) + \delta_L + d_L + \lambda_{HL} \end{bmatrix}$$

From the estimates in Step 1a and Step 1b, all the elements of matrix A are known, including

$$r = \rho + \sigma g = \rho + \sigma(\eta + \zeta).$$

We can then solve for v_H, v_L as functions of $\bar{\pi}$:

$$\begin{bmatrix} v_H \\ v_L \end{bmatrix} = \bar{\pi} A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

[13] The value of β implies a markup of 30% in light of the estimate for average markup in 1990 in De Loecker and Eeckhout (2017).

Now, combining this result with equations (10) and (11), we obtain

$$\phi = \bar{\pi} \begin{bmatrix} m_H \\ m_L \end{bmatrix}' A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(-\varrho + \frac{\varsigma^2}{2}).$$

In other words, $\bar{\pi}$ is uniquely determined as a function of ϕ. Lastly, we use (23) to choose ϕ such that the total investment is 10% of total output.

[We are awaiting data disclosure for the inputs of the estimation.]

5.2 Quantitative experiments

In this section, we carry out quantitative experiments utilizing the estimated model in the previous section. In particular, our interest is to explain the mechanism of the increase in firm sizes, documented in Section 2. First, we analyze an effect of the change in a cost parameter for external innovation, χ_X. the upper panel in Figure 13 shows how the Pareto tail index of the distribution of number of establishments per firm vary if we change the cost of external innovation from χ_X^H to $\tilde{\chi}_X^H$: as the cost of external innovation decreases slightly (by less than 5%), the tail of the distribution of number of establishments per firm becomes significantly fatter (lower tail index corresponds to fatter tail, with tail index of 1 corresponds to Zipf's law distribution which has infinite mean). The lower panel shows how the tail index of the establishment size distribution changes: the tail of the distribution becomes thinner. These features are broadly consistent with the facts documented in Figure 11 and Figure 12.

[We are awaiting data disclosure for the inputs of the estimation.]

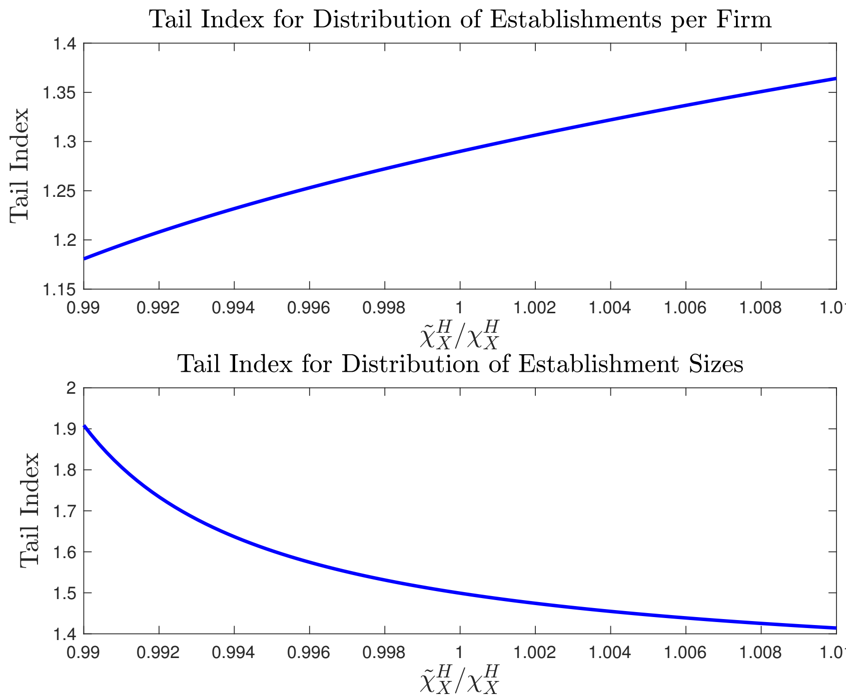

Figure 13: Distributions of number of establishment per firm and establishment size as χ_X^H changes

6 Conclusion

[To be concluded, once the data disclosure goes through.]

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Appendix

A Data

This data appendix describes the Quarterly Census of Employment and Wages (QCEW) and draws heavily from the BLS Handbook of Methods.[14]

A.1 Definitions

The Quarterly Census of Employment and Wages (QCEW) is a count of employment and wages obtained from quarterly reports filed by almost every employer in the U.S., Puerto Rico and the U.S. Virgin Islands, for the purpose of administering state unemployment insurance programs. These reports are compiled by the Bureau of Labor Statistics (BLS) and supplemented with the Annual Refiling Survey and the Multiple Worksite Report for the purpose of validation and accuracy. The reports include an establishment's monthly employment level upon the twelfth of each month and counts any employed worker, whether their position is full time, part time, permanent or temporary. Counted employees include most corporate officials, all executives, all supervisory personnel, all professionals, all clerical workers, many farmworkers, all wage earners and all piece workers. Employees are counted if on paid sick leave, paid holiday or paid vacation. Employees are not counted if they did not earn wages during the pay period covering the 12th of the month, because of work stoppages, temporary layoffs, illness, or unpaid vacations. The QCEW does not count proprietors, the unincorporated self-employed, unpaid family members, certain farm and domestic workers that are exempt from reporting employment data, railroad workers covered by the railroad unemployment insurance system, all members of the Armed Forces, and most student workers at schools. If a worker holds multiple jobs across multiple firms, then that worker may be counted more than once in the QCEW.

A.2 Sample

Our sample includes each month from 1990 to 2016 and covers 38 states: Alaska, Alabama, Arkansas, Arizona, California, Colorado, Connecticut, Delaware, Georgia, Hawaii, Iowa, Idaho, Indiana, Kansas, Louisiana, Maryland, Maine, Minnesota, Montana, North Dakota, New Jersey, New Mexico, Nevada, Ohio, Oklahoma, Rhode Island, South Carolina, South Dakota, Tennessee, Texas, Utah, Virginia, Vermont, Washington, West Virginia, as well as the District of Columbia, Puerto Rico and the U.S. Virgin Islands.

A.3 Data Cleaning and Variable Construction

To conform to official statistics, we clean the data in accordance with BLS procedure. First, while the QCEW contains monthly data as of the 12th of each month, we follow BLS convention by

[14] See `https://www.bls.gov/opub/hom/cew/home.htm` for the complete BLS Handbook of Methods.

only using data from the final month within a quarter. As a result, our sample does not capture establishments that enter and exit within the same quarter. We additionally exclude firms from calculations in a given quarter if the absolute change in employment from the previous quarter exceeds 10 times the average employment between the two quarters. Statistics within this paper are not sensitive to the choice of multiple being 10.

We construct firms as the summation over employment identification numbers (EINs). Firm-level employment is the sum of all employment in establishments associated with the same EIN and the number of establishments within a firm as the number of establishments that report using a common EIN. To classify a firm's industry, we assign to a firm the average self-reported, 6-digit NAICS code of its establishments so that the firm is classified in the same way as its establishments are on average.

A firm's entry date is measured as the date at which the QCEW records a non-zero number of workers associated with a particular EIN after four consecutive quarters of recording zero workers. A firm's exit date is measured as the last date at which the QCEW records a non-zero number of workers associated with a particular EIN prior to four consecutive quarters of recording zero workers. A firm's age is measured by tracking firms after entering. Upon entry, the firm is assigned an age of 1 quarter and the firm's age is incremented by 1 quarter for each period that it does not exit.

B Some derivations

B.1 Derivation of (1)

$$\begin{aligned}\log(Z) &= \log(E[XY|XY=Z]) \\ &= \log(E[X|XY=Z]) + \log(E[Y|XY=Z]) + \log\left(\frac{E[XY|XY=Z]}{E[X|XY=Z]E[Y|XY=Z]}\right).\end{aligned}$$

Call the final term as Ω. Because $E[XY|XY=Z]=Z$ and $E[X|XY=Z]E[Y|XY=Z] = E[X|XY=Z]E[Z/X|XY=Z] = E[X|XY=Z]E[1/X|XY=Z]Z$. From Jensen's inequality,

$$E[X|XY=Z]E\left[\frac{1}{X}\middle| XY=Z\right] \geq E[X|XY=Z]\frac{1}{E[X|XY=Z]} = 1,$$

and thus $\Omega \leq 0$ and the equality holds when $var[X|XY=Z]=0$.

B.2 Derivation of (22)

From the definition of $Q_\tau(t)$,

$$\frac{\dot{Q}_\tau(t)}{Q_\tau(t)} = -\frac{\dot{N}_\tau(t)}{N_\tau(t)} + \frac{d\int_{\mathcal{N}_\tau(t)} q_j(t)dj/dt}{\int_{\mathcal{N}_\tau(t)} q_j(t)dj}. \tag{37}$$

The first term of the right-hand side is $-\eta$. To compute the second term, consider a discrete time interval $\Delta t > 0$, compute $(\int_{\mathcal{N}_\tau(t+\Delta t)} q_j(t+\Delta t)dj - \int_{\mathcal{N}_\tau(t)} q_j(t)dj)/\Delta t$ and set $\Delta_t \to 0$. Note that the denominator of the second term is equal to $Q(t)N(t)$. Because

$$\begin{aligned}
&\int_{\mathcal{N}_\tau(t+\Delta t)} q_j(t+\Delta t)dj - \int_{\mathcal{N}_\tau(t)} q_j(t)dj \\
&\qquad = [z_I^\tau + z_X^\tau]\Delta t Q_\tau(t)N_\tau(t) - (\delta_\tau + d_\tau)\Delta t Q_\tau(t)N_\tau(t) + \mu_e \Delta t m_\tau Q(t)N(t)\int \hat{q}d\Phi(\hat{q}) \\
&\qquad\quad - \sum_{\tau'\neq\tau} \lambda_{\tau\tau'}\Delta t Q_\tau(t)N_\tau(t) + \sum_{\tau'\neq\tau} \lambda_{\tau'\tau}\Delta t Q_{\tau'}(t)N_{\tau'}(t) + o(\Delta t),
\end{aligned}$$

where the first term is the additional quality by internal and external innovation, the second term is the lost quality by exit, the third term is the gain from entry, and the fourth and the fifth terms are the loss and gain from the transitions of firm types. (The higher-order terms are omitted as $o(\Delta t)$.) Dividing by Δt and taking $\Delta t \to 0$,

$$\begin{aligned}
&\frac{d\int_{\mathcal{N}_\tau(t)} q_j(t)dj}{dt} \\
&\quad = Q_\tau(t)N_\tau(t)\left(z_I^\tau + z_X^\tau - (\delta_\tau + d_\tau) + \mu_e m_\tau \frac{Q(t)N(t)}{Q_\tau(t)N_\tau(t)}\int \hat{q}d\Phi(\hat{q}) - \sum_{\tau'\neq\tau}\lambda_{\tau\tau'} + \sum_{\tau'\neq\tau}\lambda_{\tau'\tau}\frac{Q_{\tau'}(t)N_{\tau'}(t)}{Q_\tau(t)N_\tau(t)} \right).
\end{aligned}$$

Therefore, (37) can be rewritten as

$$\zeta = -\eta + z_I^\tau + z_X^\tau - (\delta_\tau + d_\tau) + \mu_e \frac{m_\tau}{M_\tau}\frac{Q(t)}{Q_\tau(t)}\int \hat{q}d\Phi(\hat{q}) - \sum_{\tau'\neq\tau}\lambda_{\tau\tau'} + \sum_{\tau'\neq\tau}\lambda_{\tau'\tau}\frac{Q_{\tau'}(t)M_{\tau'}}{Q_\tau(t)M_\tau}$$

Using the definition of s_τ and $g = \eta + \zeta$, we can obtain (22).