Diamond Overlapping Generations Model

Toshihiko Mukoyama

April 2020

This note explains the Diamond (1965) overlapping generations (OLG) model. A part of the exposition follows Blanchard and Fischer (1989).

1 Setting

Consider each generation living two periods. Call each period young and old. When young, a consumer work (supply inelastic labor of one unit), save (in capital stock), and consume. The capital stock will become productive in the next period. When old, the consumer receives income from renting out the capital stock that has been accumulated when young, and also consume. A generation-t consumer (a consumer who is born in the beginning of period t) is

$$u(c_{1t}) + \beta u(c_{2.t+1}),$$

where $\beta \in (0, 1)$ is the discount factor, c_{1t} is the consumption when young ("1" in the subscript) at time t ("t" in the subscript), and $c_{2,t+1}$ is the consumption when old ("2" in the subscript) at time $t + 1$ ("$t + 1$" in the subscript). The period utility $u(\cdot)$ satisfies $u'(\cdot) > 0$ and $u''(\cdot) < 0$.

The population of generation t N_t grows at rate n:

$$N_t = N_0(1 + n)^t.$$

The production function is constant-returns-to-scale:

$$Y_t = F(K_t, N_t),$$

which can be transformed to

$$y_t = f(k_t),$$

where $y \equiv Y/N$, $k \equiv K/N$, and $f(k) \equiv F(k, 1)$. We assume that $f(\cdot)$ satisfies the usual regularity conditions (monotonicity, concavity, differentiability, and Inada conditions).

2 Equilibrium

The consumer's optimization

$$\max_{c_{1t}, c_{2,t+1}, s_t} u(c_{1t}) + \beta u(c_{2.t+1}),$$

subject to

$$c_{1t} + s_t = w_t$$

and

$$c_{2.t+1} = (1 + r_{t+1} - \delta)s_t.$$

The first-order condition is

$$u'(w_t - s_t) = \beta(1 + r_{t+1} - \delta)u'((1 + r_{t+1} - \delta)s_t). \tag{1}$$

From this equation, we can derive the saving function

$$s_t = s(w_t, r_{t+1}).$$

One can show that it is always the case that $s_w \in (0, 1)$, while s_r can be positive or negative. (s_w means ds/dw and s_r means ds/dr.) For s_w, for example, differentiating (1) (omitting time subscripts)

$$u_1'' dw - u_1'' ds - \beta(1 + r - \delta)^2 u_2'' ds = 0,$$

where u_1 is the first-period utility and u_2 is the second-period utility and double-prime ($''$) indicates the second derivative. Thus

$$\frac{ds}{dw} = \frac{u_1''}{u_1'' + \beta(1 + r - \delta)^2 u_2''},$$

which is between 0 and 1 from our assumption.

The firm's optimization results in:

$$w_t = f(k_t) - k_t f'(k_t) \tag{2}$$

and

$$r_t = f'(k_t). \tag{3}$$

Denote above as $w(k_t)$ and $r(k_t)$. Note that this implies

$$w'(k_t) = -f''(k_t)k_t \tag{4}$$

and

$$r'(k) = f''(k). \tag{5}$$

Because the aggregate capital is the sum of all consumers' saving from the previous period,

$$K_{t+1} = N_t s(w_t, r_{t+1}).$$

Note that, implicitly, the capital depreciation occurs after it is used. Note also that the previous period (undepreciated) capital is eaten by the old who owns the capital. This implies (dividing by N_t)

$$(1 + n)k_{t+1} = s(w_t, r_{t+1}). \tag{6}$$

With (2) and (3),

$$k_{t+1} = \frac{s(f(k_t) - k_t f'(k_t), f'(k_{t+1}))}{1+n}. \tag{7}$$

The right-hand-side can take many different shapes, and various dynamics are possible in general. With our assumptions, Below, we impose two conditions to ensure a unique and stable steady-state. First, *global monotonicity*:

$$\frac{dk_{t+1}}{dk_t} > 0$$

evaluated at all k_t. Second, the *local stability*:

$$\frac{dk_{t+1}}{dk_t} < 1$$

at all steady states, that is, whenever $k_t = k_{t+1}$. Differentiating (7),

$$(1+n)dk_{t+1} = s_w w'(k_t)dk_t + s_r r'(k_{t+1})dk_{t+1}.$$

Thus the global monotonicity means that

$$\frac{s_w w'(k_t)}{1+n-s_r r'(k_{t+1})} > 0$$

for all k_t (and k_{t+1} from (7)). Because the numerator is always positive, this assumption is equivalent to (using (5))

$$1+n-s_r f''(k_{t+1}) > 0 \tag{8}$$

always holds (for any k_t and k_{t+1} that satisfies (7)). The local stability means that

$$\frac{s_w w'(k_t)}{1+n-s_r r'(k_{t+1})} < 1,$$

evaluated at a steady state (a point where $k_{t+1} = k_t$ is satisfied). Call the unique steady-state capital $\bar{k}$. Given (8), this condition is equivalent to (using (4) and (5))

$$s_w f''(\bar{k})\bar{k} + 1 + n - s_r f''(\bar{k}) > 0. \tag{9}$$

3 The Golden Rule and dynamic inefficiency

The *Golden Rule* allocation is a steady-state allocation which maximizes the total consumption in the steady-state. The total resource constraint in the economy is

$$N_t c_{1t} + N_{t-1}c_{2t} + K_{t+1} - (1-\delta)K_t = F(K_t, N_t).$$

Dividing both sides by N_t,

$$c_t + (1+n)k_{t+1} - (1-\delta)k_t = f(k_t), \tag{10}$$

where

$$c_t \equiv c_{1t} + \frac{1}{1+n} c_{2t}$$

represents the total consumption. In the steady state,

$$\bar{c} = f(\bar{k}) - (n+\delta)\bar{k}$$

holds, and the Golden Rule level of capital (capital-labor ratio to be exact, but here labor is exogenous) satisfies

$$f'(k_G) = n + \delta.$$

A steady-state allocation with capital level $\bar{k}$ larger than k_G (implying $f'(k_G) < n+\delta$) is called dynamically inefficient. This is because, by reducing $\bar{k}$ by Δ units every period, from (10), c_t (at the impact) increases by $(1+n)\Delta$ and c_{t+i} $(i = 1, 2, ...)$ increases by $(n + \delta - f'(\bar{k}))\Delta$, implying that a Pareto improvement is feasible regardless of the shape of the utility function.

4 Social security

We consider two system of social security. For each policy below, we look at four things.

1. Starting from a steady-state, the impact of the policy on the next-period capital stock.
2. The effect of the policy on the steady-state capital stock.
3. The impact of the policy on the current generations (both old and young) welfare.
4. The impact of the policy on the steady-state welfare.

4.1 Fully funded

The fully-funded social security takes away $\mathbf{d}_t$ units of goods from a young t generation, invest in capital, and give back $b_{t+1} = (1 + r_{t+1} - \delta)\mathbf{d}_t$ to the same person. The first-order condition for saving is

$$u'(w_t - (s_t + \mathbf{d}_t)) = \beta(1 + r_{t+1} - \delta)u'((1 + r_{t+1} - \delta)(s_t + \mathbf{d}_t)).$$

Because both s_t and $\mathbf{d}_t$ are invested in capital,

$$(1+n)k_{t+1} = s_t + \mathbf{d}_t$$

holds. Comparing these two equations with (1) and (6), it is clear that, given $\mathbf{d}_t$, the allocation is equivalent by setting the $s_t^b = s_t^{FF} + \mathbf{d}_t$, where s_t^b is the baseline saving from (1) and s_t^{FF} is the saving with the social security. This equivalence holds as long as $\mathbf{d}_t \leq s_t^b$. In this situation, the policy has no effect on capital stock and welfare. However, if $\mathbf{d}_t > s_t^b$, the forced saving by $\mathbf{d}_t$ exceeds the amount that the young agent wants to save. In this case with constant $\mathbf{d}_t$, $s_t^{FF} = 0$ and

$$(1+n)k_{t+1} = \mathbf{d}.$$

If $\mathbf{d}$ goes up, first,

$$\frac{dk_{t+1}}{d\mathbf{d}} = \frac{1}{1+n} > 0$$

and

$$\frac{d\bar{k}}{d\mathbf{d}} = \frac{1}{1+n} > 0.$$

For welfare, at the impact, because the old agent's utility is

$$W_O = u((1 + r(k_t) - \delta)\mathbf{d}_{-1})$$

and $\mathbf{d}_{-1}$ is last period's $\mathbf{d}_{-1}$ (and thus it does not change) and k_t is also determined in the last period,

$$\frac{dW_O}{d\mathbf{d}} = 0.$$

The current young agent's utility is

$$W_Y = u(w(k_t) - \mathbf{d}) + \beta u((1 + r(k_{t+1}) - \delta)\mathbf{d}),$$

and k_t is predetermined but k_{t+1} changes with d.

$$\frac{dW_Y}{d\mathbf{d}} = -u_1' + \beta(1 + r(k_{t+1}) - \delta)u_2' + \beta r'(k_{t+1})\mathbf{d}u_2' \frac{dk_{t+1}}{d\mathbf{d}}.$$

Given that the starting $\mathbf{d}$ was already oversaving, the first two terms $-u_1' + \beta(1 + r(k_{t+1}) - \delta)u_2' < 0$. The third term is the general equilibrium effect. Because $r'(k_{t+1}) < 0$, $u_2' > 0$, and $dk_{t+1}d\mathbf{d} = 1/(1+n) > 0$, the third term is also negative. Thus increasing the value of $\mathbf{d}$ makes the current young generation worse off. In the future steady state,

$$\bar{W} = u(w(\bar{k}) - \mathbf{d}) + \beta u((1 + r(\bar{k}) - \delta)\mathbf{d}),$$

and thus

$$\frac{d\bar{W}}{d\mathbf{d}} = -u_1' + \beta(1 + r(\bar{k}) - \delta)u_2' + [w'(\bar{k})u_1' + \beta r'(\bar{k})\mathbf{d}u_2'] \frac{d\bar{k}}{d\mathbf{d}}.$$

The first two terms is the direct effect, and in the same logic as above, $-u_1' + \beta(1 + r(\bar{k}) - \delta)u_2' < 0$. To evaluate the final term (the general equilibrium effect), rewrite the final term using (4) and (5):

$$[w'(\bar{k})u_1' + \beta r'(\bar{k})\mathbf{d}u_2'] \frac{d\bar{k}}{d\mathbf{d}} = f''(\bar{k})\bar{k}[-u_1' + \beta(1+n)u_2'] \frac{d\bar{k}}{d\mathbf{d}}.$$

Assuming that $r(\bar{k}) - \delta \geq n$ (dynamic efficiency),

$$-u_1' + \beta(1+n)u_2' \leq -u_1' + \beta(1 + r(\bar{k}) - \delta)u_2' < 0$$

and thus this final term would be positive. Somewhat paradoxically, because there is already a clear oversaving, for the general-equilibrium effect, an even larger value of $\bar{k}$ benefits agents by moving resources from period 2 to period 1, via increasing w and decreasing r. In this case, depending on the strengths of the direct effect (the first two terms) and the general equilibrium effect (the final term), the welfare effect can be positive or negative. If $r(\bar{k})-\delta < n$ (dynamic inefficiency), we cannot determine the sign of the general equilibrium effect. again, it is possible that the welfare effect can be positive or negative.

4.2 Pay as you go

A pay-as-you-go (PAYG) social security system takes $\mathbf{d}_t$ units of goods from the current young (generation t) and transfers it to the current old (generation $t-1$). Each old generation consumer receives $b_t = (1+n)\mathbf{d}_t$ units.

With this system, the first-order condition and the dynamics of capital is

$$u'(w_t - (s_t + \mathbf{d}_t)) = \beta(1 + r_{t+1} - \delta)u'((1 + r_{t+1} - \delta)s_t + (1+n)\mathbf{d}_{t+1}) \tag{11}$$

and

$$(1+n)k_{t+1} = s_t. \tag{12}$$

Now it is clear that the equivalence result in the FF case doesn't hold. There are two immediate insights from these two equations. First, the social security acts as a saving vehicle, as in the FF case, but now the "return" to the $\mathbf{d}_t$ portion is n and what the generation t receives comes from generation $t+1$. When the population grows rapidly, each generation receives large returns to the social security. Also note that now the policy change impact the old generation consumers because what the generation $t-1$ receives depends on the period-t policy. Second, because $\mathbf{d}_t$ now doesn't go into investment, there is a crowding-out effect on capital accumulation.

When $\mathbf{d}$ goes up from time t on, from

$$(1+n)k_{t+1} = s(w(k_t), r(k_{t+1}), \mathbf{d}_t, \mathbf{d}_{t+1}),$$

$$(1+n)dk_{t+1} = s_r r'(\bar{k})dk_{t+1} + s_{\mathbf{d}}d\mathbf{d} + s_{\mathbf{d}'}d\mathbf{d},$$

where $s_{\mathbf{d}} \equiv \partial s/\partial \mathbf{d}_t$ and $s_{\mathbf{d}'} \equiv \partial s/\partial \mathbf{d}_{t+1}$. Because $r'(\bar{k}) = f''(\bar{k})$,

$$\frac{dk_{t+1}}{d\mathbf{d}} = \frac{s_{\mathbf{d}} + s_{\mathbf{d}'}}{1 + n - s_r f''(\bar{k})}. \tag{13}$$

From the global monotonicity assumption, (8) holds (it is straightforward to show that the condition is the same in this setting), and the denominator is always positive. For the numerator, from the first-order condition above,

$$-u_1''ds - u_1''d\mathbf{d} = \beta(1 + r(\bar{k}) - \delta)^2 u_2''ds$$

and

$$-u_1''ds = \beta(1+r(\bar{k})-\delta)^2 u_2'' ds + \beta(1+r(\bar{k})-\delta)(1+n)u_2'' d\mathbf{d}'$$

hold, implying

$$s_{\mathbf{d}} = \frac{ds}{d\mathbf{d}} = -\frac{u_1''}{u_1'' + \beta(1+r(\bar{k})-\delta)^2 u_2''} < 0$$

and

$$s_{\mathbf{d}'} = \frac{ds}{d\mathbf{d}'} = -\frac{\beta(1+r(\bar{k})-\delta)(1+n)}{u_1'' + \beta(1+r(\bar{k})-\delta)^2 u_2''} < 0.$$

Increasing in $\mathbf{d}$ reduces incentive to save by two channels: (i) there are less resources today to save, and (ii) tomorrow there is an additional income. Therefore, from (13),

$$\frac{dk_{t+1}}{d\mathbf{d}} < 0.$$

For the steady-state capital, from

$$(1+n)\bar{k} = s(w(\bar{k}), r(\bar{k}), \mathbf{d}, \mathbf{d}),$$

$$(1+n)d\bar{k} = s_w w'(\bar{k})d\bar{k} + s_r r'(\bar{k})d\bar{k} + s_{\mathbf{d}} d\mathbf{d} + s_{\mathbf{d}'} d\mathbf{d}$$

holds, and using (4) and (5), we can conclude

$$\frac{d\bar{k}}{d\mathbf{d}} = \frac{s_{\mathbf{d}} + s_{\mathbf{d}'}}{-s_w f''(\bar{k})\bar{k} + 1 + n - s_r f''(\bar{k})} < 0.$$

The numerator is negative from the analysis above, and the denominator is positive from (9) (again, it is straightforward to show that the condition is the same in this setting). The overall implication for the capital stock is that capital accumulation is always crowded out.

For the welfare effects of an increase in $\mathbf{d}$, the old agent's utility at the impact (at time t) is

$$W_O = u((1+r(k_t)-\delta)s_{t-1} + (1+n)\mathbf{d}),$$

where, in this case, $\mathbf{d}$ is not pre-determined. Given that everything else is determined at time $t-1$, an increase in $\mathbf{d}$ always benefits the initial old.

$$\frac{dW_O}{d\mathbf{d}} = (1+n)u_2' > 0.$$

For the young generation at the impact, because the utility is

$$W_Y = u(w(k_t) - (s_t + \mathbf{d})) + \beta u((1+r(k_{t+1})-\delta)s_t + (1+n)\mathbf{d}),$$

(using the Envelope Theorem)

$$\frac{dW_Y}{d\mathbf{d}} = -u_1' + \beta(1+n)u_2 + \beta u_2' r'(\bar{k}) s(w(\bar{k}), r(\bar{k}), \mathbf{d}, \mathbf{d}) \frac{dk_{t+1}}{d\mathbf{d}}.$$

The first two terms are the direct effect and the final term is the general equilibrium effect. Because $r'(\bar{k}) = f''(\bar{k}) < 0$ and $dk_{t+1}/d\mathbf{d} < 0$, the general equilibrium effect is always positive. Given that w is unchanged, a decrease in k in future would benefit the current young, who will earn from r (which will increase with a smaller k) when old. For the direct effect, because $u'_1 = \beta(1 + r(\bar{k}) - \delta)u'_2$ from the first-order condition at the old steady state,

$$-u'_1 + \beta(1+n)u'_2 = \beta(n - (r(\bar{k}) - \delta))u'_2.$$

This term is negative when the economy is dynamically efficient ($r(\bar{k}) - \delta > n$). Thus the total effect

$$\frac{dW_Y}{d\mathbf{d}} = \beta(n - (r(\bar{k}) - \delta))u'_2 + \beta u'_2 r'(\bar{k}) s(w(\bar{k}), r(\bar{k}), \mathbf{d}, \mathbf{d})\frac{dk_{t+1}}{d\mathbf{d}}$$

can be positive or negative depending on the relative strengths of these effects. When the steady-state is dynamically inefficient, the PAYG social security always benefits the young at the impact. Below we will see that the PAYG would always hurt the newborns in the steady-state if the economy is dynamically efficient. A remarkable finding here is that it is possible that both the initial young and old can gain from increasing **d** even when the future generations suffer from welfare loss. This is in contrast to the commonly-held belief that only the initial old can gain in such a policy change when the economy is dynamically inefficient.[1] The distinction is important, as the increase in PAYG pension may receive a larger political support, especially when $r(\bar{k}) - \delta$ is only slightly larger than n, than it was considered earlier.

In the steady state,

$$\bar{W} = u(w(\bar{k}) - (s_t + \mathbf{d})) + \beta u((1 + r(\bar{k}) - \delta)s_t + (1+n)\mathbf{d}),$$

and thus (using the Envelope Theorem)

$$\frac{d\bar{W}}{d\mathbf{d}} = -u'_1 + \beta(1+n)u'_2 + [w'(\bar{k})u'_1 + \beta r'(\bar{k})s(w(\bar{k}), r(\bar{k}), \mathbf{d}, \mathbf{d})u'_2]\frac{d\bar{k}}{d\mathbf{d}}.$$

The first two terms are the direct effects and the last term is the general equilibrium effect. Similarly to the above case, we can transform this to

$$\frac{d\bar{W}}{d\mathbf{d}} = \beta(n - (r(\bar{k}) - \delta))u'_2 + [w'(\bar{k})u'_1 + \beta r'(\bar{k})s(w(\bar{k}), r(\bar{k}), \mathbf{d}, \mathbf{d})u'_2]\frac{d\bar{k}}{d\mathbf{d}}.$$

Further, using (4), (5), (11), and (12), the last term can be simplified to

$$\frac{d\bar{W}}{d\mathbf{d}} = \beta(n - (r(\bar{k}) - \delta))u'_2 + \beta f''(\bar{k})\bar{k}(n - (r(\bar{k}) - \delta))u'_2\frac{d\bar{k}}{d\mathbf{d}}$$

and therefore

$$\frac{d\bar{W}}{d\mathbf{d}} = \beta(n - (r(\bar{k}) - \delta))u'_2\left(1 + \beta f''(\bar{k})\bar{k}\frac{d\bar{k}}{d\mathbf{d}}\right).$$

[1] For example, de la Croix and Michael (2002) state "... when the equilibrium is efficient, the introduction of pensions always benefits the first old generations at the expense of subsequent generations." (p.152)

From the above result, the term on the final parenthesis is always positive. Both the direct effect and the general equilibrium effect depend on whether the original steady state was dynamically efficient. If it was the case $((r(\bar{k}) - \delta) > n)$, the increase in social security hurts the long-run welfare. If the original steady-state was dynamically inefficient, the expansion of PAYG social security benefits the long-run welfare. There are two effects. First, when the initial steady-sate is dynamically inefficient, even without changing the capital stock (i.e. not improving the dynamic inefficiency in production), moving resources from young to old would benefit the intertemporal allocation, in the same manner as the "money" improves the efficiency in Samuelson (1958) economy. Second, as argued in Section 3, reducing capital stock in a dynamically-inefficient economy increases resources that can be consumed in the steady state.

5 Government debt

Here we consider the effect of the government debt on capital accumulation and welfare. We consider two scenarios: the debt is held internally and externally.

5.1 Internally-held debt

Suppose that the government issues a one-period bond of amount B_t at period $t-1$. Because the payoff structure of the government bond is the same as capital stock, the bond has to pay the same interest as capital: $r_t - \delta$. We assume that there are no additional government spending and the tax is levied only to young. The amount of tax is

$$T_t = (1 + r_t - \delta)B_t - B_{t+1}.$$

The tax per young worker is

$$\tau_t \equiv \frac{T_t}{N_t} = b_t(1 + r_t - \delta) - b_{t+1}(1 + n),$$

where $b_t \equiv B_t/N_t$. The consumer's budget constraints are

$$c_{1t} + s_t = w_t - (b_t(1 + r_t - \delta) - b_{t+1}(1 + n))$$

and

$$c_{2,t+1} = (1 + r_{t+1} - \delta)s_t,$$

and the transition equation for capital is

$$K_{t+1} + B_{t+1} = N_t s_t$$

and therefore

$$(1 + n)(k_{t+1} + b_{t+1}) = s_t.$$

One interesting insight to note is if we define $\tilde{s}_t \equiv s_t - (1+n)b_{t+1}$ and $\tilde{\mathbf{d}}_t = (1+r_t-\delta)b_t$, the above equations look equivalent to the PAYG social security. The effect of increasing b by one unit in the steady state would be equivalent to increasing d by $(1+r-\delta) + bdr/d\tilde{\mathbf{d}}$ units. Of course, given that r_t is an endogenous variable, there won't be an equivalence during the transition. But intuitively the equivalence provides another interpretation of the PAYG system. The PAYG social security is equivalent to the government issuing a bond at period 1 (and hand the revenue to the initial old) and roll it over to the young generation every period.

Let us consider the effect on capital and welfare explicitly. The first-order condition and the capital accumulation equation are

$$u'(w(k_t)-(b_t(1+r(k_t)-\delta)-b_{t+1}(1+n))-s_t) = \beta(1+r(k_{t+1})-\delta)u'((1+r(k_{t+1})-\delta)s_t)) \quad (14)$$

and

$$(1+n)(k_{t+1}+b_{t+1}) = s_t. \quad (15)$$

Note that the conditions for the global monotonicity and the local stability would change here. Denoting the saving function as

$$s_t = s(w(k_t) - (b_t(1+r(k_t)-\delta) - b_{t+1}(1+n)), r_{t+1}),$$

The transition equation for capital is

$$(1+n)(k_{t+1}+b_{t+1}) = s(w(k_t) - (b_t(1+r(k_t)-\delta) - b_{t+1}(1+n)), r(k_{t+1})), \quad (16)$$

and when b_t is held at a constant level b is the local dynamics is

$$(1+n)dk_{t+1} = s_w(w'(k_t) - br'(k_t))dk_t + s_r r'(k_{t+1})dk_{t+1}$$

and therefore the condition for global monotonicity is

$$\frac{dk_{t+1}}{dk_t} = \frac{s_w(w'(k_t) - br'(k_t))}{1+n-s_r r'(k_{t+1})} > 0.$$

Note that $w'(k_t) - br'(k_t) = -kf''(k_t) - bf''(k_t) > 0$. Thus this is equivalent to

$$1+n-s_r f''(k_{t+1}) > 0, \quad (17)$$

which ends up being the same as (8). The condition for local stability is

$$\frac{dk_{t+1}}{dk_t} = \frac{s_w(w'(k_t) - br'(k_t))}{1+n-s_r r'(k_{t+1})} < 1$$

evaluated at the steady state. Thus the new condition is

$$-s_w f''(\bar{k})(\bar{k}+b) + 1 + n - s_r f''(\bar{k}) > 0. \quad (18)$$

Now, consider the change in k_{t+1} when b increases. Suppose that at time t, the economy is in the steady state. From (16), at the period when b changes (note that b_t is predetermined at period $t-1$),

$$(1+n)(dk_{t+1}+db) = s_w(1+n)db + s_r r'(k_{t+1})dk_{t+1}$$

holds and thus at the impact,

$$\frac{dk_{t+1}}{db} = \frac{-(1+n)(1-s_w)}{1+n-s_r f''(\bar{k})} < 0.$$

The sign is negative, because (i) $s_w \in (0,1)$, the numerator is negative and (ii) the denominator is positive because of (17). This negative effect is due to the crowding out of capital accumulation. For the steady-state capital,

$$(1+n)(d\bar{k}+db) = s_w(w'(\bar{k})-br'(\bar{k}))d\bar{k} + s_w(n-(r(\bar{k})-\delta))db + s_r r'(\bar{k})d\bar{k}.$$

Thus

$$\frac{d\bar{k}}{db} = \frac{-(1+n)+s_w(n-(r(\bar{k})-\delta))}{-s_w f''(\bar{k})(\bar{k}+b)+1+n-s_r f''(\bar{k})} < 0.$$

The sign is determined as the following. The denominator is positive due to (18). For the numerator,

$$-(1+n)+s_w(n-(r(\bar{k})-\delta)) = -(1+n)(1-s_w)-(1+r(\bar{k})-\delta)s_w < 0.$$

Therefore, we can conclude that the steady-state response is also negative.

Let us consider the welfare effects of increasing b on various consumers. Starting from the steady state, the initial old's utility is

$$W_O = u((1+r(k_t)-\delta)s_{t-1}).$$

Given that this is unaffected by b (both k_t and s_{t-1} are predetermined),

$$\frac{dW_O}{db} = 0.$$

For the initial young,

$$W_Y = u(w(k_t)-(b_t(1+r(k_t)-\delta)-b_{t+1}(1+n))-s_t)+\beta u((1+r(k_{t+1})-\delta)s_t).$$

Because k_t and b_t are predetermined but k_{t+1} changes with b (again, using the Envelope Theorem),

$$\frac{dW_Y}{db} = (1+n)u_1' + \beta r'(\bar{k})su_2'\frac{dk_{t+1}}{db} > 0.$$

The sign is positive because both terms are positive. In the second term, $r(\bar{k}) = f''(\bar{k})$ is negative and $dk_{t+1}/db < 0$ in the analysis above. The young prefers to increase b because (i) their tax burden is decreased and (ii) a reduced k_{t+1} implies that the return from capital is larger. When $n > 0$, the young is the majority of the population at time t. This implies that a proposal to increase b is likely to pass with a majority voting. This is more so if the young can subsidize the saved tax to initial old. Even a small amount of transfer to the initial old is sufficient to change their opinion, given that they are indifferent.

Despite the "equivalence" highlighted above, the results on initial young and old are somewhat different from the PAYG case. In the PAYG case, the dynamic-efficiency property of the economy can matter for the young's welfare, while here the young always gains from the reform to increase b. This is somewhat an artifact of our timing assumption: the social security transfer at time t can be changed at time t policy change, while the government debt from the last period have to be honored so that b can change only from time $t+1$. If the PAYG reform have to honor the transfer to the initial old before the reform, the welfare outcome in the PAYG reform is qualitatively similar to the government debt increase. In general, though, the political-economy prediction is robust if there are transfers available between the initial old and initial young. There always can be an arrangement where both are persuaded to go for the reform to increase b (or $\mathbf{d}$), regardless of the dynamic efficiency of the original steady-state, while possibly hurting the future generations.

The steady-state welfare is

$$\bar{W} = u(w(\bar{k}) + (n - (r(\bar{k}) - \delta))b - s_t) + \beta u((1 + r(\bar{k}) - \delta)s_t),$$

and thus

$$\frac{d\bar{W}}{db} = (n - (r(\bar{k}) - \delta))u_1' + [(w'(\bar{k}) - r'(\bar{k})b)u_1' + \beta r'(\bar{k})su_2']\frac{d\bar{k}}{db}.$$

For the second term, rewriting using (4), (5), (14), (15) yields

$$(w'(\bar{k}) - r'(\bar{k})b)u_1' + \beta r'(\bar{k})su_2' = \beta f''(\bar{k})(\bar{k} + b)(n - (r(\bar{k}) - \delta))u_2'.$$

Thus

$$\frac{d\bar{W}}{db} = (n - (r(\bar{k}) - \delta))\left(u_1' + \beta f''(\bar{k})(\bar{k} + b)u_2'\frac{d\bar{k}}{db}\right).$$

The term in the large parenthesis is always positive. Thus the sign of $d\bar{W}/db$ entirely depends on the dynamic-efficiency property of the economy. If the economy is dynamically inefficient at the old steady state ($n > r(\bar{k}) - \delta$), increasing b raises the steady-state welfare, while if it is dynamically efficient, it harms the steady-state welfare. When n is larger than $r(\bar{k}) - \delta$, each generation (i) receives the "population bonus" from future generation in excess of interest payments and (ii) the production efficiency is enhanced because of the reduction in $\bar{k}$.

Often it is argued that internally-held government debt does not hurt the economy because it is like "borrowing from yourself." In this framework, it is not correct because the government debt crowds out the capital accumulation, which can be good or bad, depending on the dynamic efficiency of the initial situation. There is also an additional consideration of the fact that the ones who borrow and the ones who repay can be different people.

5.2 Externally-held debt

Now instead we assume that the economy is a small open economy and can issue a government bond at the interest rate $\bar{r} - \delta$. The amount of tax is now

$$T_t = (1 + \bar{r} - \delta)B_t - B_{t+1}.$$

The tax per young worker is

$$\tau_t \equiv \frac{T_t}{N_t} = b_t(1 + \bar{r} - \delta) - b_{t+1}(1 + n),$$

where $b_t \equiv B_t/N_t$. The consumers and the firms don't have an access to the international market and the domestic capital stock pays the rental rate r_t. The budget constraints are

$$c_{1t} + s_t = w_t - (b_t(1 + \bar{r} - \delta) - b_{t+1}(1 + n))$$

and

$$c_{2,t+1} = (1 + r_{t+1} - \delta)s_t,$$

and the transition equation for capital is

$$K_{t+1} = N_t s_t$$

and therefore

$$(1 + n)k_{t+1} = s_t.$$

Now, let us consider the effect on capital and welfare. The first-order condition and the capital accumulation equation are

$$u'(w(k_t) - (b_t(1 + \bar{r} - \delta) - b_{t+1}(1 + n)) - s_t) = \beta(1 + r(k_{t+1}) - \delta)u'((1 + r(k_{t+1}) - \delta)s_t)) \quad (19)$$

and

$$(1 + n)k_{t+1} = s_t. \quad (20)$$

The global monotonicity and the local stability conditions are derived as the following.

$$s_t = s(w(k_t) - (b_t(1 + \bar{r} - \delta) - b_{t+1}(1 + n)), r_{t+1}),$$

The transition equation for capital is

$$(1 + n)k_{t+1} = s(w(k_t) - (b_t(1 + \bar{r} - \delta) - b_{t+1}(1 + n)), r(k_{t+1})), \quad (21)$$

and when b_t is held at a constant level b is the local dynamics is

$$(1 + n)dk_{t+1} = s_w w'(k_t)dk_t + s_r r'(k_{t+1})dk_{t+1}$$

and therefore the condition for global monotonicity is

$$\frac{dk_{t+1}}{dk_t} = \frac{s_w w'(k_t)}{1+n-s_r r'(k_{t+1})} > 0.$$

This is equivalent to

$$1+n-s_r f''(k_{t+1}) > 0, \tag{22}$$

which is the same as (8) and (17). The condition for local stability is

$$\frac{dk_{t+1}}{dk_t} = \frac{s_w w'(k_t)}{1+n-s_r r'(k_{t+1})} < 1$$

evaluated at the steady state. Thus the new condition is

$$-s_w f''(\bar{k})\bar{k} + 1 + n - s_r f''(\bar{k}) > 0. \tag{23}$$

Now, consider the change in k_{t+1} when b increases. Suppose that at time t, the economy is in the steady state. From (21), at the period when b changes (note that b_t is predetermined at period $t-1$),

$$(1+n)dk_{t+1} = s_w(1+n)db + s_r r'(k_{t+1})dk_{t+1}$$

holds and thus at the impact,

$$\frac{dk_{t+1}}{db} = \frac{(1+n)s_w}{1+n-s_r f''(\bar{k})} > 0.$$

In contrast to the domestically-held case, the sign is positive, because (i) $s_w \in (0,1)$ and (ii) the denominator is positive because of (22). This is because the issuance of b reduces the tax burden for the initial young and there is no crowding-out effect.

For the steady-state capital,

$$(1+n)d\bar{k} = s_w w'(\bar{k})d\bar{k} + s_w(n-(\bar{r}-\delta))db + s_r r'(\bar{k})d\bar{k}.$$

Thus

$$\frac{d\bar{k}}{db} = \frac{s_w(n-(\bar{r}-\delta))}{-s_w f''(\bar{k})(\bar{k}+b) + 1 + n - s_r f''(\bar{k})}.$$

The sign is determined as the following. The denominator is positive due to (23). The numerator depends on the sign of $n-(\bar{r}-\delta)$. Thus the sign of $d\bar{k}/db$ is the same as the sign of $n-(\bar{r}-\delta)$. This is in contrast to the domestically-held case; if $n-(\bar{r}-\delta) > 0$, the domestic capital can increase because of the income gain due to the population bonus. There is no crowding out, as all spending is financed abroad.

Let us consider the welfare effects of increasing b on various consumers. Starting from the steady state, the initial old's utility is

$$W_O = u((1+\bar{r}-\delta)s_{t-1}).$$

Given that this is unaffected by b (s_{t-1} is predetermined),

$$\frac{dW_O}{db} = 0.$$

For the initial young,

$$W_Y = u(w(k_t) - (b_t(1+\bar{r}-\delta) - b_{t+1}(1+n)) - s_t) + \beta u((1+r(k_{t+1})-\delta)s_t).$$

Because k_t and b_t are predetermined but k_{t+1} changes with b,

$$\frac{dW_Y}{db} = (1+n)u_1' + \beta r'(\bar{k})su_2'\frac{dk_{t+1}}{db}.$$

Now the sign can be positive or negative because the first term is positive and the second term is negative. In contrast to the domestically-held case, the initial young doesn't necessarily have a strong incentive to increase b, because that would negatively affect the future income.

The steady-state welfare is

$$\bar{W} = u(w(\bar{k}) + (n-(\bar{r}-\delta))b - s_t) + \beta u((1+r(\bar{k})-\delta)s_t),$$

and thus

$$\frac{d\bar{W}}{db} = (n-(\bar{r}-\delta))u_1' + [w'(\bar{k})u_1' + \beta r'(\bar{k})su_2']\frac{d\bar{k}}{db}.$$

For the second term, rewriting using (4), (5), (19), (20) yields

$$w'(\bar{k})u_1' + \beta r'(\bar{k})su_2' = \beta f''(\bar{k})\bar{k}(n-(r(\bar{k})-\delta))u_2'.$$

Thus

$$\frac{d\bar{W}}{db} = (n-(\bar{r}-\delta))u_1' + \beta f''(\bar{k})\bar{k}(n-(r(\bar{k})-\delta))u_2'\frac{d\bar{k}}{db}.$$

The first term has the same sign as $n-(\bar{r}-\delta)$. The second term depends on both $n-(r(\bar{k})-\delta)$ and $n-(\bar{r}-\delta)$ (because of the $d\bar{k}/db$ component). The second term has the same sign as $-[n-(r(\bar{k})-\delta)] \times [n-(\bar{r}-\delta)]$. Suppose that the economy is dynamically efficient, that is, $n-(r(\bar{k})-\delta) < 0$. In this case, there are two scenarios. First, if $\bar{r}$ is sufficiently high so that $n-(\bar{r}-\delta) < 0$, the first term is negative and the second term is also negative because $\bar{k}$ decreases. Thus the welfare decreases. Second, if $n-(\bar{r}-\delta) > 0$, the first term is positive and the second term is also positive because $\bar{k}$ now increases. Thus an increase in b is welfare-enhancing. This is in contrast to the domestically-held case where the steady-state welfare always decreases with b when the original steady-state is dynamically efficient. This would be the opposite if $\bar{r}$ is sufficiently low compared to the domestic population growth. Combined with the above result, the political economy of current versus future generations can be completely opposite depending on whether the government debt is held domestically or internationally.

References

Blanchard, O. and S. Fischer (1989). *Lectures on Macroeconomics*. MIT Press.

de la Croix, D. and P. Michael (2002). *A Theory of Economic Growth: Dynamics and Policy in Overlapping Generations*. Cambridge University Press.

Diamond, P. A. (1965). National Debt in a Neoclassical Growth Model. *American Economic Review 55*, 1126–1150.

Samuelson, P. A. (1958). An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money. *Journal of Political Economy 66*, 467–482.