1 Integrability from extra dimensions

Throughout this paper, we discuss correspondences between a certain class of supersymmetric gauge theories and integrable lattice models:

$$\mathcal{I}_{\mathsf{T}_{4d}[\mathsf{L}_{2d}]} = Z_{\mathsf{L}_{2d}[\mathsf{T}_{4d}]}, \tag{1.1}$$

where the left-hand side is the supersymmetric index of a four-dimensional gauge theory T_{4d} and the right-hand side denotes the statistical partition function of the corresponding integrable lattice model L_{2d} specified by the four-dimensional theory T_{4d}. As it turns out, the correspondence emerges from TQFT with extra dimensions [1]. The aim of this section is to give the step-by-step explanation of the above correspondence from an elementary level. Besides, to explain these, we will start by clarifying in order what the terms TQFT, lattice model, and integrability mean.

1.1 Preliminaries

In the next subsection, we would like to introduce lattice model as discrete quantum field theory (QFT). So let us start from the general settings of QFT. We first briefly review one-dimensional QFT, which is nothing but quantum mechanics, and then extend the discussion to general quantum field theory in $(d+1)$-dimensions. Though we do not have a complete definition of general QFT, a special class of QFT which is called *topological* QFT is axiomatized by Atiyah [2].

1.1.1 What is quantum field theory?

Let us now begin with one-dimensional QFT, as known as quantum mechanics (QM). Suppose we have a one-dimensional manifold[1], which may be thought of as "time" M^1 = interval or S^1 (or $\mathbb{R}$), and data to specify a QM:

- $\mathcal{H}$: a vector space (Hilbert space/state space),
- $\mathcal{O}$: a set of self-adjoint operators (observables, acting on $\mathcal{H}$),
- H : an operator called *Hamiltonian*,

where $H \in \mathcal{O}$ and the state space $\mathcal{H}$ is a finite or infinite dimensional $\mathbb{C}$-vector space. We sometimes need to consider a set of operators $\mathcal{O}$ including not self-adjoint operators, but for simplicity we do not care about that at this moment. In undergraduate, we have learnt that one should consider the Schrödinger equation and find a good basis in $\mathcal{H}$ diagonalizing the Hamiltonian.

[1] Throughout the paper, we assume all the manifold is smooth and oriented, unless otherwise stated.

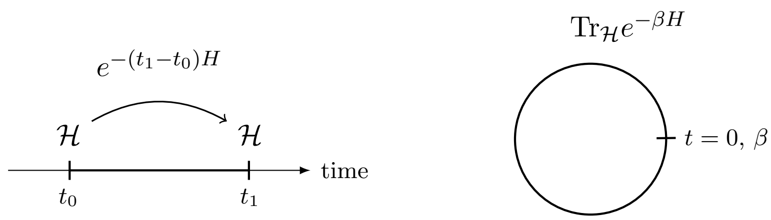
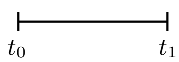


fig:qm_interval (a) fig:qm_circle (b)

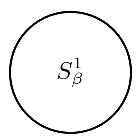
_two_properties Figure 1: Two most crucial properties of quantum mechanics.

Formally, the procedure of solving the Schrödinger equation leads to an expression of time evolution of states by a linear map[2] e^{-tH} in quantum mechanics, and the partition function in many-body statistical mechanics. In turn, we have two most crucial properties of QM: Given an interval $M^1 = [t_0, t_1]$, we have state vectors at the endpoints of the interval and a time evolution between the states:



$$\rightsquigarrow \quad e^{-(t_1-t_0)H} : \mathcal{H} \longrightarrow \mathcal{H}\,, \tag{1.2}$$

and given a circle $M^1 = S^1_\beta$, we get a number called *partition function*:



$$\rightsquigarrow \quad \mathrm{Tr}_{\mathcal{H}} e^{-\beta H}\,. \tag{1.3}$$

In addition, from the physical facts it is quite reasonable to assume that the time evolution and the partition function are compatible with cutting and gluing intervals and S^1s. This simply means that the time evolution from time t_0 to t_1 followed by another time evolution from t_1 to t_2 is equal to the single time evolution from t_0 to t_2, etc. Summarizing, the properties that quantum mechanics or one-dimensional QFT has are rephrased by the language of given one-dimensional manifold M^1 (see figure 1):

- Given an interval, produces state vectors at the endpoints and a linear map between them.
- Given an S^1, produces a number.
- Compatible with cutting and gluing intervals and S^1s.

Thus, in the abstract we conclude that one-dimensional QFT, or quantum mechanics, is a gadget satisfying the above properties for each given one-dimensional manifold M^1.

From these observations of quantum mechanics, we wish to extend the discussion to general $(d+1)$-dimensional QFT. The starting point of defining a $(d+1)$-dimensional QFT

[2] We here do not get into the argument of Euclideanization.

is the choice of a $(d+1)$-dimensional manifold M^{d+1}, which has d spatial directions and one "time" direction. For most QFTs the manifold M^{d+1} is viewed as a Riemannian manifold with a smooth metric on it. As already noticed, we will mostly consider a positive definite Riemannian metric, QFT on which is usually referred to as an Euclidean QFT, and hence precisely there is no notion of "time" in such a theory, at least globally. The manifold M^{d+1} may or may not have boundaries. In case it does have boundaries some additional information is needed at the boundaries to define the QFT. In addition to a Riemannian metric, depending on the situation one wants to consider, one often needs some more structures on the manifold M^{d+1}, e.g. smooth structure, conformal structure, spin structure, etc.

To obtain the data to specify a QFT, we now would like to extend the observations seen in QM. Suppose we have a $(d+1)$-dimensional manifold M^{d+1}, then we wish to "define" a QFT by a gadget Z, which should produce a vector when M^{d+1} has a boundary

$$Z\left(M^{d+1}\right) \in \mathcal{H}_{bdy}, \tag{1.4}$$

and should produce a number when M^{d+1} has no boundary

$$Z\left(M^{d+1}\right) \in \mathbb{C}. \tag{1.5}$$

In the case of M^{d+1} with a boundary, the vector space defined on the boundary is called the space of states or just Hilbert space in the physics literature. On the one hand, if M^{d+1} does not have boundary, the number $Z\left(M^{d+1}\right)$ is called the partition function. If the boundary of M^{d+1} has several disconnected components, namely the boundaries are given by the disjoint union of simply connected d-dimensional closed manifolds $\left\{N_i^d\right\}$: $\partial M^{d+1} = \sqcup_i N_i^d$, Z should define a linear map among the vector spaces defined on the boundaries. In particular, in the case $M^{d+1} = N^d \times I$, where I is an interval of length T, M^{d+1} has two boundaries N^d and Z now gives rise to a linear map $Z\left(N^d \times I\right) =: U\left(T\right)$,

$$U\left(T\right) : \mathcal{H}_N \longrightarrow \mathcal{H}_N. \tag{1.6}$$

From the physical facts, Z also should be compatible with cutting and gluing of $(d+1)$-dimensional manifolds, and thus we learn that the linear map given above satisfies $U\left(T_1\right)U\left(T_2\right) = U\left(T_1+T_2\right)$. This in turn defines an operator H as the generator of U,

$$U\left(T\right) = \exp\left(-TH\right). \tag{1.7}$$

H is called the Hamiltonian of the system, acting on the space of states $\mathcal{H}_N$. If one considers a manifold with Lorentzian metric, $U\left(T\right)$ is represented as

$$U\left(T\right) = \exp\left(-iTH\right), \tag{1.8}$$

and then the interval I is regarded as "physical time."

1.1.2 Construction of Z

So far we have discussed only in an abstract way what quantum field theory is, or in other words what the gadget Z should satisfy. Let us now see how we specify Z to define a QFT. Broadly speaking, there are three kinds of constructions of Z. They are not totally independent and have many aspects of applicabilities. In the rest of this section, M denotes a $(d+1)$-dimensional manifold with or without boundary, and N denotes a d-dimensional closed manifold without boundary.

Construct from the axiom

The first construction is in a sense the simplest one; we write down appropriate properties that a QFT must satisfy, and axiomatize them. Construction of Z is to give a mathematical formulation which satisfy the axioms as the data to specify the theory. This is actually the only way that one can define a QFT by mathematically rigorous procedures. Normally, such a theory is first studied by physicists as an ideal or a toy model from the physical motivation, and then refined as rigorous mathematics by mathematicians.[3] There are several kinds of such theories. We now introduce typical three examples.

The first one is *free theories* in any dimensions, which are toy models of field theories and play important roles as a probe of more complicated QFTs. In any spacetime dimensions, for the Riemannian manifolds with or without spin structure, we can rigorously define the free field theories. One of them is the free scalar field theory. Besides a $(d+1)$-dimensional closed manifold M, pick a G-bundle $P \to M$ with connection A and a G-vector space V. Then we construct the associated vector bundle $P \times_G V$, whose covariant derivative is denoted by D_A. We now have the Laplacian Δ_A given by the covariant derivative D_A, and then the free scalar field theory is defined by the partition function

$$Z_{\text{scalar}}\,(M; A) := 1/\det \Delta_A. \tag{1.9}$$

If the determinant of the Laplacian involves a divergence, it must be properly removed.

Another free theory is the free fermion theory. In this case M also needs a spin structure and the spin representations S, S'. Then we construct the Dirac operator $\mathcal{D}_A$ on the spin bundles and the partition function of the free fermion theory is given by

$$Z_{\text{fermion}}\,(M; A) := \det \mathcal{D}_A, \tag{1.10}$$

again the determinant is taken appropriately.[4]

[3] The fact that a QFT can be treated in a mathematically rigorous way implies that the theory may have an enormous amount of symmetry, and the difficulties of the QFT are completely controlled by them. Even better, these theories can often be exactly solved in an appropriate sense.

[4] Fermion theories may have anomalies. If it is the case, the axiom needs to be somehow modified. In particular the partition function for a closed manifold is given by the eta invariant [3].

Next, in two dimensions, we have another axiomatic quantum field theory, that is *conformal field theory* (CFT). Conformal field theory in two dimensions was originally formulated by Belavin, Polyakov, and Zamolodchikov [4] as a model of physical systems at critical points. They established the renowned Virasoro algebra as an infinite dimensional symmetry of the system and fully investigated the minimal model. The holomorphic part of the Virasoro algebra is captured by vertex operator algebras, and since then there are many mathematically rigorous discussions on them. Their kinematic behaviors on Riemann surfaces are governed by the conformal blocks, and its geometric meaning has been studied in [5]. In turn, the study of irrational CFTs has led to the AGT correspondence [6], which has brought to us large amount of developments in both physics and mathematics.

The final example is *topological quantum field theories* (TQFTs). TQFT is one of the main focuses in this paper. These theories are originating from Witten's proposals of topological field theories [7, 8, 9]. Inspired by Witten's proposals, Atiyah and Segal axiomatized the topological QFT,[5] and happily this also has brought to us a numerous amount of applications both to physics and to pure mathematics. For example, $(1+1)$-dimensional topological QFTs are known to be categorically equivalent to commutative Frobenius algebras. $(2+1)$-dimensional Chern-Simons theory for a compact Lie group G is rigorously constructed by the Turaev-Viro and Reshetikhin-Turaev construction using quantum groups [11, 12], and has applications to knot or link invariants and 3-manifold invariants. Moreover, both 2d and 3d TQFTs have applications to the mirror symmetry, or topological string theory (see e.g. [13]), those have led to fruitful interactions between physics and mathematics. We will take more time for TQFT and introduce the Atiyah's axioms in some detail in the next subsection.

Path integrate the Boltzmann weight

Before going to Atiyah's TQFT axioms, let us see two more constructions of Z. These are no longer mathematically rigorous, but rather familiar constructions for physicists. The second construction is to define a partition function by *path integral*. The general prescription is given as follows: One first introduces an action functional over the classical field configurations, which deduces the classical equation of motion by the variational principle. Then one exponentiates the action and integrate it over the space of fields.

The partition function for the free fields given above can also be defined by the path integral expression. For example, for the free complex scalar field theory consider a section ϕ of the vector bundle $P \times_G V$ over M, where V is a representation of a compact Lie group

[5] Precisely speaking, what Segal axiomatized is the definition of conformal field theories [10]. However, at any rate the definition of CFT by Segal is quite similar to the definition of Atiyah's TQFT, namely it is a categorical one. For the geometric definition of CFT, see [5].

G. Then define an action functional

$$S : \Gamma\left(P \times_G V\right) \longrightarrow \mathbb{R}, \tag{1.11}$$

such that

$$S\left(\phi\right) = \int_M \frac{1}{2} D_A\phi \wedge * D_A\phi, \tag{1.12}$$

where $*$ is the Hodge star on M. D_A is again the covariant derivative given by the connection A. Using this action functional, physicists "define" its partition function by

$$Z_{\text{scalar}}\left(M; A\right) := \int_{\Gamma(P\times_G V)} \mathcal{D}\phi \, e^{-S(\phi)}. \tag{1.13}$$

The integrand e^{-S} of path integral is generically called the *Boltzmann weight*. For the free field theories, the path integral can be expressed as an infinite product of the Gaussian integral. Introducing an appropriate regularization, one can compute the exact partition function and it leads to the same result as mentioned above.

As another example, let us consider Yang-Mills theory. We now introduce the kinetic term of the connection A, and define the action functional

$$S\left(A\right) = \int_M \frac{1}{4g^2} F_A \wedge * F_A, \tag{1.14}$$

where F_A is the curvature of the G-connection A. Define the partition function of Yang-Mills theory by

$$Z_{\text{YM}}\left(M\right) = \int_{\mathcal{A}_M/\mathcal{G}} \mathcal{D}A \, e^{-S(A)}, \tag{1.15}$$

where the integral is taken over the space of connections on M modulo gauge transformations. Unlike the free scalar theory, making this path integral mathematically precise is an extremely difficult problem. Although it is still ill-defined, physicists have been working on this expression to understand many properties of quantum gauge theories. Experimentally, physicists discretize the manifold M to a $(d+1)$-dimensional lattice and put it on a supercomputer. At least, numerical calculations show the above construction may be a mathematically meaningful and reproduce many experimental results to reasonable accuracy.

Deduce from string/M-theory

The final construction of Z is to use string or M-theory. This is also our main tool to construct a QFT to realize the correspondence (the first equation of this section). Nonetheless, string or M-theory is less rigorous than path-integral expression of the construction of Z, and thus it is more hopeless to give a mathematical precise meaning to the construction. We here would like to show just two examples which "define" a class of quantum field theories through string/M-theory.

The first example is the AdS/CFT correspondence.

The second example is the so-called six-dimensional $\mathcal{N} = (2,0)$ theories [14]. We start from a 10-dimensional string theory called the type IIB string theory, which roughly speaking assigns the partition function $Z_{\text{IIB}}(\tilde{M})$ to a 10-dimensional manifold $\tilde{M}$. Pick a finite subgroup Γ_G of SU(2) of type $G = A_n$, D_n or $E_{6,7,8}$. We define a six-dimensional QFT Q_G by its partition function for a six-dimensional manifold M,

$$Z_{Q_G}(M) = Z_{\text{IIB}}\left(M \times \mathbb{C}^2/\Gamma_G\right). \tag{1.16}$$

They are examples of six-dimensional $\mathcal{N} = (2,0)$ superconformal QFTs. These theories are known not to have a Lagrangian description, namely their partition function cannot be obtained from the path integral formalism. They have another description via M-theory, as the low-energy dynamics of M5-branes. The construction from M5-branes leads to the AGT correspondence and the notion of class $\mathcal{S}$ theories, which we will explain in later sections.

1.1.3 Atiyah's topological QFT

As mentioned some times, in this paper TQFT in extra dimensions will play a crucial role to realize the correspondence between supersymmetric gauge theories and integrable lattice models. So now let us pause here and introduce the Atiyah's axioms of TQFT [2]. We first list the axioms, and right then give their physical meanings. Readers will notice that most of these axioms are physically quite natural and actually mathematical rephrasing of the properties that Z should satisfy, which we have already seen. In addition, we would like to define lattice model as a discrete version of QFT in the next subsection. Reviewing the definitions of TQFT here will be really a good help of the argument of lattice model.

Axiom (Atiyah's $(d+1)$-dimensional TQFT)

$(d+1)$-dimensional TQFT is defined by Z consisting of the following two data and five assignments:

- For each oriented d-dimensional closed manifold N, assigns a finite dimensional $\mathbb{C}$-vector space $\mathcal{H}_N$:
$$Z(N) = \mathcal{H}_N. \tag{1.17}$$

This corresponds to (a half of) canonical quantization, or geometric quantization, known in physics literature. Why we say it is "a half of" will be explained in a moment. In physics, especially for a field theory which has a Lagrangian description, we consider the phase space of fields, take a constant-time surface, and then perform canonical quantization by imposing canonical commutation relations on fields and their conjugates. The constant-time surface is

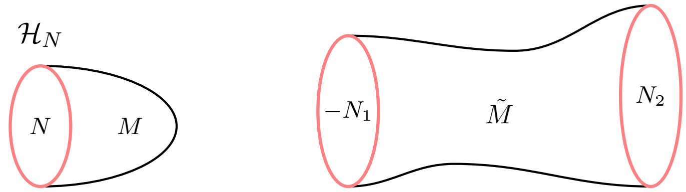
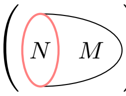


fig:one_bdy (a) fig:two_bdy (b)

fig:mfd_w_bdys Figure 2: one-boundary and two-boundary cases.

a codimension-1 hypersurface in $(d+1)$-dimensional manifold M, which in this case is nothing but the d-dimensional closed manifold N. So, the manifold N is viewed as a collection of d spatial directions. One can think of the assigned vector space $\mathcal{H}_N$ as "the space of functionals on the classical fields on N," usually called the space of states or just Hilbert space of the system. The only major difference here is that for topological theory the Hilbert space is of finite dimension.[6]

- For each oriented $(d+1)$-dimensional manifold M with a boundary $\partial M = N$, assigns a vector



$$Z\,(M) = Z\left(\quad \right) \in \mathcal{H}_N. \tag{1.18}$$

This expresses the path integral quantization on M with boundary. Recall that if one has a spacetime manifold with boundary, one needs to impose a boundary condition on fields on the boundary, and it leads to the state vector associated to the boundary condition by path integral expression, namely let φ be a fixed field configuration on N,

$$Z\,(M;\varphi) = \int_{X|_N=\varphi} \mathcal{D}X\, e^{-S(X)} \in \mathcal{H}_N. \tag{1.19}$$

In other words, the quantum state at a generic time t is given by the path integral of the Boltzmann weight over all the classical fields with time $< t$.

These data are subject to the following assignments:

1. (involutory) Let $-N$ denote a manifold N with the opposite orientation, then

$$\mathcal{H}_{-N} = \mathcal{H}_N^*, \tag{1.20}$$

where $\mathcal{H}_N^*$ is the dual to $\mathcal{H}_N$.

2. (multiplicative) If N is a disjoint union of two d-dimensional closed manifolds N_1 and N_2, then the vector space associated on it is factorized to a tensor product:

$$\mathcal{H}_{N_1 \sqcup N_2} = \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}. \tag{1.21}$$

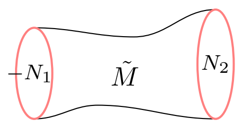
[6] One can remove the finiteness condition from the axiom. In fact, in TQFT we can define a non-degenerate bilinear form from the other axioms, and from that as a corollary we can deduce the space of states is finite-dimensional.

This is really natural in the physical point of view. This condition means that if we have two physical systems defined on spatially disjoint union $N_1 \sqcup N_2$, then the space of states is represented as the tensor product of each state space. Also, from the conditions so far one finds that if $(d+1)$-dimensional manifold M' has two boundaries such that

$$\partial M' = -N_1 \sqcup N_2, \tag{1.22}$$

then Z defines a linear map

$$Z\left(M'\right) = Z\left(\right.$$



$$\left. \right) \in \mathrm{Hom}_{\mathbb{C}}\left(\mathcal{H}_{N_1}, \mathcal{H}_{N_2}\right), \tag{1.23}$$

namely, for a cobordism between N_1 and N_2, we have a linear map between $\mathcal{H}_{N_1}$ and $\mathcal{H}_{N_2}$. This defines a time evolution, or transition amplitude, between the states in $\mathcal{H}_{N_1}$ and $\mathcal{H}_{N_2}$. When the field theory has a Hamiltonian, $Z\left(M'\right)$ may correspond to the time evolution operator

$$Z\left(M'\right) := e^{-tH} : \mathcal{H}_{N_1} \longrightarrow \mathcal{H}_{N_2}. \tag{1.24}$$

Since canonical quantization is a procedure to make an assignment of Hilbert spaces and the time evolution on them, as we saw in QM, this association of linear maps gives the other half of canonical quantization. One can also explicitly express $Z\left(M'\right)$ in the path integral expression by the Feynman kernel (also as known as propagator or Green's function). Suppose we have an initial state $\Psi_0 \in \mathcal{H}_{N_1}$, then the state $\Psi_t \in \mathcal{H}_{N_2}$ at time t is expressed by

$$\begin{aligned} \Psi_t\left(\varphi_t\right) &= \left(e^{-tH}\Psi_0\right)\left(\varphi_t\right) \\ &= \int K\left(\varphi_t, \varphi_0\right)\Psi_0\left(\varphi_0\right)\mathcal{D}\varphi_0, \end{aligned} \tag{1.25}$$

where

$$K\left(\varphi_t, \varphi_0\right) = \int_{X|_{N_1}=\varphi_0}^{X|_{N_2}=\varphi_t} \mathcal{D}X\, e^{-S(X)}. \tag{1.26}$$

3. For two cobordisms such that

$$\partial M_1 = -N_1 \sqcup N_2, \quad \partial M_2 = -N_2 \sqcup N_3, \tag{1.27}$$

it follows that

$$Z\left(M_1 \cup_{N_2} M_2\right) = Z\left(M_2\right) Z\left(M_1\right), \tag{1.28}$$

where the cobordisms M_1 and M_2 are glued along N_2 by a certain diffeomorphism on N_2. This asserts that the linear maps are transitive when we compose cobordisms. This is nothing but the physical requirement that the time evolution is compatible with the cutting and gluing the manifolds.

4. Given $N = \emptyset$ as an empty d-dimensional manifold, then the associated vector space is one-dimensional:

$$Z(\emptyset) = \mathbb{C}. \tag{1.29}$$

This is a non-triviality condition. Therefore, for each $(d+1)$-dimensional manifold M without boundary, $\partial M = \emptyset$, Z assigns a number:

$$Z(M) \in \mathbb{C}. \tag{1.30}$$

This number associated to a closed $(d+1)$-dimensional manifold is called the partition function. Not only is it an important quantity physically, but it also has mathematical applications, such as giving a topological invariant of the closed manifold M.

5. Let I be an interval, then for each cylinder $M = N \times I$, the linear map on $\mathcal{H}_N$ is trivial:

$$Z(N \times I) = \mathrm{id}_{\mathcal{H}_N}. \tag{1.31}$$

Since in a topological theory manifolds diffeomorphic to each other should be considered as the identical, each cobordism class defines a linear map. This means that the addition of a cylinder is a trivial operation, the associated linear map is the identity. This condition is another major difference from ordinary QFTs. For example, for $(2+1)$-dimensional Chern-Simons theory one can explicitly see its Hamiltonian is identically 0 and hence the time evolution of Chern-Simons theory is trivial.

This is all the assignments for Atiyah's TQFT. Although we gave the path integral expressions at the explanation of the physical meanings for some conditions, the axioms themselves are really the basis for the rigorous mathematical definition of Z. One can actually take an equivalent definition of TQFT in slightly different axioms. For more details, see the original paper by Atiyah.

At first sight, this might look a little bit complicated and too abstract. This definition, however, is really natural and works quite well in a sense that it gives a kind of "homomorphism" between geometry and algebra:

$$Z : \text{'geometry'} \longrightarrow \text{'algebra'}.$$

In fact, all in all the axioms of Atiyah's TQFT define a functor from the category of $(d+1)$-dimensional cobordisms to the category of finite-dimensional $\mathbb{C}$-vector spaces:[7]

$$Z : \mathrm{Bord}_{d+1} \longrightarrow \mathrm{Vect}_{\mathbb{C}}. \tag{1.32}$$

[7] What is more, the category of cobordisms and the category of vector spaces are endowed with the product structures, that is, taking disjoint union and taking tensor product, which are really symmetric operations. As such, the functor between them is keeping such structures and then called symmetric monoidal functor.

TQFT luckily has a mathematical precise definition. For a general QFT, i.e. not TQFT, in many cases there is no precise definition as we saw the examples of the construction of Z. However, a QFT is generically expected to be characterized by some functor from the geometric structure of spacetime manifolds to the algebraic description of physical states and observables.

1.2 Lattice model as discrete QFT

Now let us move on to the discussion on lattice model. First of all, let us begin with the question of "what is lattice model?" Probably there is no unique definition of lattice model. We here would like to characterize lattice model by a discrete version of QFT. What we have learnt in the previous subsection is that a $(d+1)$-dimensional QFT may be defined by an appropriate functor Z_{Q}^{d+1}, which produces a number for each $(d+1)$-dimensional closed manifold M:

$$Z_{\mathrm{Q}}^{d+1}(M) \in \mathbb{C}, \tag{1.33}$$

which is called the partition function of the model, and Z_{Q}^{d+1} satisfies additional some reasonable conditions for each QFT of interest.

We introduce lattice model in the same way. For a $(d+1)$-dimensional closed lattice L, a $(d+1)$-dimensional lattice model is defined by Z_{L}^{d+1}, which produces a number

$$Z_{\mathrm{L}}^{d+1}(L) \in \mathbb{C}, \tag{1.34}$$

which is again called the partition function, and Z_{L}^{d+1} satisfies additional some reasonable conditions. The most typical example of lattice model in particle physics is lattice gauge theory or lattice QCD.[8] For such theories, it is really clear that the model is defined by a discrete version of QFTs; One compactifies the spacetime manifold $\mathbb{R}^4$ to four-torus T^4, and then discretize the theory and put it on a lattice on the torus.

Throughout the paper, we will consider only $d = 0$ or 1 case of lattice model, and we call each case as 1d or 2d lattice model. To prepare for the discussion of integrability, we take the prominent example of statistical lattice model called the Ising model. We first briefly review the generality of the Ising model and compare with field theory, and then introduce the transfer matrix which is essentially a time evolution operator in a discrete quantum system.

1.2.1 Prominent example – the Ising model

The Ising model is a very good introduction to integrable lattice model. The 1d and 2d Ising model is known to be exactly solvable, whose meaning we will clarify in a moment, and it is

[8] For sure, for any field theories when one computes a physical quantity on a computer, one needs to discretize the theory and put it on a lattice. In this sense, numerical analysis of field theory is always regarded as a lattice model.

called integrable. To see the generality of the Ising model, we first define 2d lattice and spin configurations on it.

Define a 2d periodic lattice on two-torus T^2 by

$$\begin{aligned} L &:= \mathbb{Z}_M \times \mathbb{Z}_N \\ &= \{1, \ldots, M\} \times \{1, \ldots, N\} . \end{aligned} \tag{1.35}$$

A spin configuration on the lattice L is defined by a map

$$\mathbf{s} : L \longrightarrow \{+1, -1\} , \tag{1.36}$$

where usually $+1$ is called up spin and -1 down spin. The map defines up or down spin at each site of the lattice L. So it can also be thought of as an assignment of $+1$ or -1 on all the sites of the lattice L. We often denote the spin at each site by its image $s_I := \mathbf{s}(I)$, $I \in L$. Let $S(L)$ be the set of all the spin configurations. Since spin configurations are defined on the $M \times N$ periodic lattice, the number of elements in $S(L)$ is 2^{MN}. In other words, the number of all the allowed configurations of spins on the lattice L is 2^{MN}.

For each spin configuration $\mathbf{s} \in S(L)$, define the energy functional of the Ising model by

$$\begin{aligned} E_{\text{Ising}}(\mathbf{s}) &= -J \sum_{\langle I, I' \rangle} s_I s_{I'} \\ &:= -J \left(\sum_{i,j=1}^{M,N} s_{ij} s_{i,j+1} + \sum_{i,j} s_{ij} s_{i+1,j} \right) , \end{aligned} \tag{1.37}$$

where the $s_{ij} = s_{i+M,j} = s_{i,j+N}$, and $J \in \mathbb{R}_{>0}$ is a constant parameter. This is one of the simplest spin systems, in which only the nearest neighbor spins have interactions. From this energy functional, the partition function of the Ising model is defined by

$$Z(L, E_{\text{Ising}}; \beta) := \sum_{\mathbf{s} \in S(L)} e^{-\beta E_{\text{Ising}}(\mathbf{s})}, \tag{1.38}$$

where $\beta \in \mathbb{R}_{\geq 0}$ is an inverse temperature. The summand is generically called the Boltzmann weight as well as in field theory. In this expression, one recognizes that the right-hand side is a discrete version of path integral expression in field theory. The energy functional is corresponding to the action functional of a field theory, β^{-1} is the Planck's constant, and the sum over all the spin configurations is the path integral over all the field configurations. Given a periodic lattice L, the partition function returns a number, which is a discrete version of the partition function of QFT which returns a number as well if a closed manifold M is given. For later use, define another quantity called the free energy,

$$f(\beta) := -\frac{1}{\beta} \frac{1}{MN} \log Z(\beta) , \tag{1.39}$$

where $Z(\beta)$ is the partition function defined above.

According to the general theory of statistical mechanics, the probability that a configuration $\mathbf{s}$ with energy $E(\mathbf{s})$ will be realized is given by the canonical ensemble[9]

$$p(\mathbf{s}) = \frac{1}{Z} e^{-\beta E(\mathbf{s})}, \tag{1.40}$$

where Z is the partition function introduced above. In lattice model, a physical observable is in general given by a functional on the space of spin configurations:

$$\mathcal{O} : S(L) \longrightarrow \mathbb{R}, \tag{1.41}$$

and its expectation value is given by the canonical ensemble as

$$\langle \mathcal{O} \rangle_p := \sum_{\mathbf{s} \in S(L)} \mathcal{O}(\mathbf{s})\, p(\mathbf{s}) = \frac{1}{Z} \sum_{\mathbf{s} \in S(L)} \mathcal{O}(\mathbf{s})\, e^{-\beta E(\mathbf{s})}. \tag{1.42}$$

In particular, an energy functional is an example of physical observable,

$$\frac{1}{MN} \langle E \rangle_p = \frac{1}{MN} \frac{1}{Z} \sum_{\mathbf{s} \in S(L)} E(\mathbf{s})\, e^{-\beta E(\mathbf{s})}, \tag{1.43}$$

which is obtained from the derivative of the free energy,

$$\frac{1}{MN} \langle E \rangle_p = \frac{\partial}{\partial \beta} (\beta f(\beta)). \tag{1.44}$$

Generally speaking, the free energy provides us all the information of the system. For example, the state most likely to happen is given by the critical point of the free energy, and we can in principle compute physical quantities such as expectation value of energy, fluctuation, specific heat, and so on. To make a lattice model as a physically meaningful system, however, one needs to take the thermodynamic limit; $M, N \to \infty$. Therefore, in this sense, integrability of lattice model is characterized by the calculability of an exact free energy at the thermodynamic limit. This may be rephrased by the calculability of the partition function as well. Based on these argument, we shall see the integrability of the Ising model in the next subsection.

1.2.2 Transfer matrix and integrability

It is well known that 1d and 2d Ising model is exactly solvable in the sense that one can exactly compute its free energy in the thermodynamic limit. Both 1d and 2d cases are solved by so-called the method of transfer matrix. To see this, for simplicity in this section we consider 1d Ising model.

[9] To the end of this subsection, we argue for a general energy functional, but the reader may assume the Ising model.

fig:Ising_c (a) fig:Ising_q (b)

fig:1dIsing Figure 3: (a) Classical Ising spin chain. (b) Quantum states at each site.

The setup is almost the same as in the 2d case. A 1d lattice on torus and a spin configuration is defined by

$$L_{1d} = \mathbb{Z}_N = \{1, \ldots, N\}, \tag{1.45}$$

$$\mathbf{s} : L_{1d} \longrightarrow \{\pm 1\}. \tag{1.46}$$

Define the energy functional of 1d Ising model by

$$E_{1d\,\text{Ising}}(\mathbf{s}) = -J \sum_{i=1}^{N} s_i s_{i+1}, \quad s_{N+1} = s_1. \tag{1.47}$$

Then the partition function of 1d Ising model is given as

$$\begin{aligned} Z\left(L_{1d}, E_{\text{1d}}; \beta\right) &:= \sum_{\mathbf{s}} e^{-\beta E_{1d}(\mathbf{s})} \\ &= \sum_{s_1,\ldots,s_N=\pm 1} e^{K s_1 s_N} \cdots e^{K s_3 s_2} e^{K s_2 s_1}, \end{aligned} \tag{1.48}$$

where $K := \beta J$.

Let us now introduce the *transfer matrix*,

$$T := \left(e^{K s s'}\right)_{s,s'=\pm 1} = \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix}, \tag{1.49}$$

where the indices of row and column are specified by $+1$ and -1, respectively. Using this matrix T, the partition function is rewritten as

$$Z(\beta) = \sum_{s_1,\ldots,s_N=\pm 1} T_{s_1 s_N} \cdots T_{s_3 s_2} T_{s_2 s_1}. \tag{1.50}$$

By definition of matrix multiplication, the partition function is eventually given by a trace:

$$\begin{aligned} Z(\beta) &= \sum_{s_1=\pm 1} \left(T^N\right)_{s_1 s_1} \\ &= \mathrm{Tr}_{\mathbb{C}^2}\left(T^N\right). \end{aligned} \tag{1.51}$$

This expression tells us that we have the quantum Hilbert space $\mathbb{C}^2$ at the boundary of each 1d segment, and the transfer matrix T sends a state to the adjacent site, which is the discrete "time evolution" of this system; $\log T \propto$ Hamiltonian as we saw in 1d QFT in section (what is QFT?). This corresponds to the analogue of Hamiltonian or operator formalism in field theory. In operator formalism, spins are replaced by the Pauli matrices, and original classical up spin and down spin are replaced by the eigenvalues and eigenvectors of the Pauli matrix σ^z. Further, the non-commutativity is now manifest in a way that the spins and their time evolution is given by matrices, this is the consequence of quantization.

In the expression above of the partition function, let us consider the thermodynamic limit $N \to \infty$. The eigenvalues of T are easily obtained and let them be $\lambda_0 > \lambda_1$, then we have the free energy

$$\begin{aligned} -\beta f(\beta) &= \lim_{N\to\infty} \frac{1}{N} \log Z(\beta) \\ &= \lim_{N\to\infty} \frac{1}{N} \log \lambda_0 \left(1 + \left(\frac{\lambda_1}{\lambda_0}\right)^N\right) \\ &= \log \lambda_0. \end{aligned} \tag{1.52}$$

We conclude that the free energy in the thermodynamic limit is just given by the largest eigenvalue of the transfer matrix. Again, integrability of lattice model is rephrased as

One can exatly find the eigenvalues of transfer matrix

$\Rightarrow$ Can exactly compute the free enrgy

$\Rightarrow$ The system is integrable.

The transfer matrices of 1d and 2d Ising model are diagonalizable and one can exactly find their eigenvectors and eigenvalues using, for example, algebraic Bethe ansatz. In this sense, 1d and 2d Ising model are said to be integrable, or exactly solvable. Then, a natural question arises; when can we diagonalize the transfer matrix of a lattice model? This actually leads to the most fundamental answer to the question of "what is integrable model." A canonical answer is *Yang-Baxter equation.* When the Boltzmann weight with spectral parameters, which take values in a Riemann surface, satisfies the Yang-Baxter equation, the lattice model is integrable. We will give an explanation of this definition of integrability in next subsection in some detail, with its origin from TQFT with extra dimensions.

1.3 Integrability from TQFT in extra dimensions

In this section we provide a general discussion to relate four-dimensional supersymmetric gauge theories and integrable lattice models. We clarify that such a correspondence between gauge theories and lattice models emerges from a TQFT in extra dimensions. We first give a step-by-step explanation from two-dimensional TQFT to integrable lattice model, and then put the argument in the setup of brane tilings. Branes in string theory are powerful enough to yield the systematic method to relate a class of supersymmetric gauge theories with

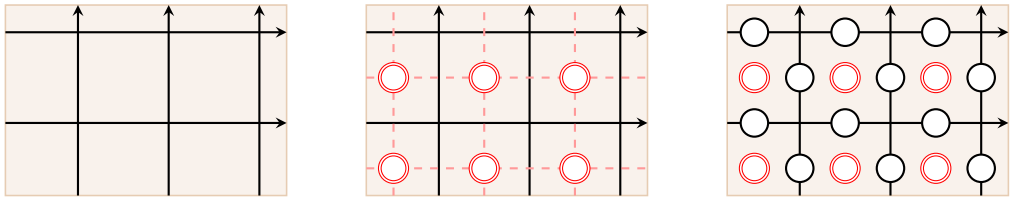


fig:line_on_torus (a) fig:line_boundary (b) fig:spin_boundary (c)

Figure 4: Construction of a lattice model from line operators. (a) A lattice of line operators on a torus. (b) The torus with holes obtained by gluing of pieces. (c) The corresponding spin model.

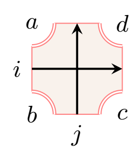
fig:TQFTandSpin

integrable lattice models, as we will see. To begin with, we present a general prescription of brane constructions at a formal level. Concrete setups and more details will be given in later sections. The discussion in this subsection is mainly based on [15].

1.3.1 Lattice models from TQFTs with line operators

Suppose that we have a two-dimensional TQFT T on a two-torus T^2 equipped with line operators $\mathcal{L}_i$, $i = 1, \ldots, l$. Wrap them around one-cycles C_i on the torus in such a way that they form an $m \times n$ lattice. We wish to consider the correlation function of this configuration of line operators. To compute it, our strategy is to break up the torus into pieces and perform in turn the path integral piecewise. Then the original correlation function is reconstructed by combining these pieces.[10]

Let us consider the following piece of surface which contains an intersection of two line operators $\mathcal{L}_i$ and $\mathcal{L}_j$:

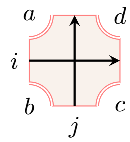


. (1.53) eq:piece_of_li

If one thinks of the above picture as the world-sheet of two scattering open strings, one can regard the above picture as follows: The endpoints of the open strings sweep out the double-lined arcs, on which D-branes are there and the boundary conditions (Chan-Paton factors) are specified by labels a, b, c, d. The line operators are the world-lines of particles associated with the open strings.

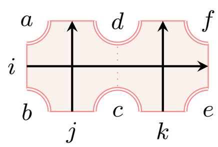
[10] In the following, we will compute the correlation function by embedding the closed TQFT with line operators into an open/closed TQFT with line operators, or defect TQFT; see e.g. [16, 17].

The path integral for the piece (1.53) produces a linear map



$$\rightsquigarrow \quad R_{ij}\begin{pmatrix} a & d \\ b & c \end{pmatrix} : V_{ab,i} \otimes V_{bc,j} \longrightarrow V_{ad,j} \otimes V_{dc,i}\,, \tag{1.54}$$

where $V_{ab,i}$ is the space of states of the open string propagating along $\mathcal{L}_i$ with the boundary conditions on the left and the right ends specified by a, b, respectively. We call this map the R-matrix (or R-operator) associated with this decorated surface. To reconstruct the original configuration of line operators, we glue pieces together. Gluing them amounts to the composition of R-matrices. For example, gluing two pieces horizontally gives

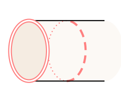
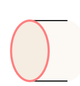
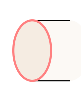
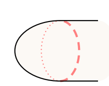


$$\rightsquigarrow \quad R_{ik}\begin{pmatrix} d & e \\ c & f \end{pmatrix} \circ_{V_{dc,i}} R_{ij}\begin{pmatrix} a & d \\ b & c \end{pmatrix}. \tag{1.55}$$

The original configuration is thus obtained by gluing all the pieces. It, however, still has holes assigned boundary conditions as in figure 4(b), which we must fill by summing over the boundary conditions. To do this, let us recall that the path integral on a finite-length cylinder with boundary condition a imposed on one end defines a closed string state $|a\rangle$ as known as a boundary state. Similarly, the path integral on a disk with no insertion of operators defines a state $|1\rangle$ on the boundary. Assume that we have chosen the set of boundary conditions B to be sufficiently large so that the state $|1\rangle$ can be written as a superposition of boundary states:

$$|1\rangle = \sum_{a\in B} c_a\,|a\rangle\,. \tag{1.56}$$

Then, the sum over the boundary conditions gives the state $|1\rangle$ on the boundary of each hole, which is replaced with a disk:

$\sum_{a\in B} c_a \Big(a$  $\Big) = \sum_{a\in B} c_a \Big(|a\rangle$  $\Big) = |1\rangle$  $=$  . (1.57)

Thus the holes are filled and the original configuration on the torus is reconstructed.

Now that we have understood how to reconstruct the correlation function of line operators in the field theoretic point of view, let us reinterpret this procedure as an operation in statistical lattice model. Rephrasing the procedures so far, it is clear that the above configuration defines a partition function of a statistical spin model. The model has spins located at two kinds of sites, ○ and ◎. They correspond to the open string states and the boundary states. A spin at ○ takes values in the chosen basis for the relevant open string states $V_{ab,i}$, while that at ◎ is valued in B. The Boltzmann weights for each spin configuration are given by the matrix elements of R-matrix and the coefficients of boundary states c_a. Thus, we conclude that the correlation function of line operators $\{\mathcal{L}_i\}$ on the torus coincides with the partition function of a statistical spin model defined on the periodic lattice formed by the line operators:

$$\left\langle \prod_{i=1}^{l} \mathcal{L}_i\left(C_i\right) \right\rangle_{\mathsf{T},T^2} = Z_{\mathsf{L}(\mathsf{T}),\{\mathcal{L}_i(C_i)\}}. \tag{1.58}$$

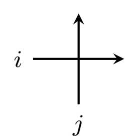
$\mathsf{L}\left(\mathsf{T}\right)$ denotes the lattice model arising from the TQFT T with the line operators and $\{\mathcal{L}_i\left(C_i\right)\}$ is the lattice formed by line operators $\mathcal{L}_i$ wrapping around C_i. By this construction, we can view typical examples of lattice models.

IRF model

If $\dim V_{ab,i} = 1$ for any a, b, i, in this case we can ignore the spins at $\bigcirc$. We just sum over the boundary conditions and such a spin system $\mathsf{L}\left(\mathsf{T}\right)$ is called *interaction-round-a-face model*, or *IRF model* for short. The spins are placed on the faces of the lattice, and interaction takes place among four spins surrounding a vertex. The 2d Ising model actually can be formulated as an IRF model by taking vertex-face correspondence.

Vertex model

If B consists of a single boundary condition, say a, we can simply write the open string state space by $V_i := V_{aa,i}$, and the R-matrix is represented only by a crossing of two lines:

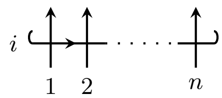
$R_{ij} =$  , (1.59)

$$R_{ij} : V_i \otimes V_j \longrightarrow V_j \otimes V_i\,. \tag{1.60}$$

The space of states V_i can also be thought of as the Hilbert space of a particle propagating on the line operator $\mathcal{L}_i$. In this case we can ignore the spins at $\circledcirc$ since there is no summation over boundary conditions. This means that the lattice model $\mathsf{L}\left(\mathsf{T}\right)$ is called a *vertex model*: Spins are living on the edges of the lattice and interact with each other at the vertices.

We can always recast our lattice model into a vertex model by setting $V_i := \bigoplus_{a,b\in B} V_{ab,i}$, at least formally, and declaring that all newly introduced R-matrix elements, which correspond to scattering processes with inconsistent Chan-Paton factors, vanish. We can also absorb the coefficients c_a into the R-matrix elements by appropriate rescaling. In what follows we implicitly perform this reformulation, and restrict ourselves to vertex model.

A remarkable aspect of this construction of lattice models is that it allows us to understand integrability from a higher-dimensional point of view. This is crucial observation by Costelllo [18]. In our lattice model, consider a horizontal line operator $\mathcal{L}_i$ intersecting vertical line operators $\mathcal{L}_j$, $j = 1, \ldots, n$. Concatenating the R-matrices in this row, we get the row-to-row transfer matrix:

$T_i =$ 

$$= \mathrm{Tr}_{V_i} \left(R_{i,n} \circ_{V_i} \cdots \circ_{V_i} R_{i,1} \right). \tag{1.61}$$

The hooks on the horizontal line are to remind us that the periodic boundary condition is imposed. This gives an endomorphism of $\bigotimes_{j=1}^{n} V_j$ which maps a state just below $\mathcal{L}_j$ to another state above it, namely the transfer matrix defines the time evolution of this spin system and $\bigotimes_{j=1}^{n} V_j$ is the quantum Hilbert space. In terms of transfer matrices, the partition function is written as a trace,

$$Z_{\mathsf{L}(\mathsf{T}),\{\mathcal{L}_i(C_i)\}} = \mathrm{Tr}_{\mathcal{H}} \left(T_{n+m} \cdots T_{n+1} \right), \tag{1.62}$$

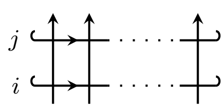
where the trace is taken over the total Hilbert space $\mathcal{H} := \bigotimes_{j=1}^{n} V_j$. An important point is that the underlying field theory is a TQFT, and hence a state evolves trivially on a cylinder unless it hits something, line operators in the present case.

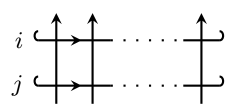
Now suppose that each line operator depends on a continuous parameter which is an element of some smooth manifold S. This parameter is called *spectral parameter* of the lattice model. We denote the spectral parameter of $\mathcal{L}_i$ by u_i, then the R-matrix and the transfer matrix are rewritten by

$$R_{ij} \longrightarrow R_{ij} \left(u_i, u_j \right), \tag{1.63}$$

$$T_i \longrightarrow T_i \left(u_i; u_1, \ldots, u_n \right). \tag{1.64}$$

To avoid clutter, we fix $u_1, \ldots, u_n$ and suppress them below. A vertex model is said to be integrable if the transfer matrices $T_i(u_i)$ are meromorphic functions of u_i and commute with each other:



$=$


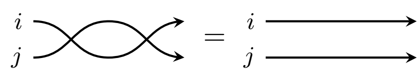
$$\Longleftrightarrow \quad \left[T_i \left(u_i \right), T_j \left(u_j \right) \right] = 0, \quad u_i \neq u_j. \tag{1.65}$$

When the transfer matrices with different spectral parameters commute, we can find a series of mutually commuting operators on the total Hilbert space $\bigotimes_{j=1}^{n} V_j$ by Laurent expansion of the transfer matrix. In particular, they commute with the transfer matrix, which means they produce an infinite number of conserved quantities. In fact, the commutativity of transfer matrix implies that one can find the exact eigenvalues and eigenvectors of the transfer matrix, as we saw in the Ising model in the previous section.

The situation considered here, namely the addition of spectral parameters and commuting transfer matrix, is naturally realized if the TQFT has "extra dimensions." In this scenario, we really start with a higher-dimensional theory $\tilde{\mathsf{T}}$ formulated on the product space $S \times T^2$, which is topological on the torus T^2 but not on S. We wrap line operators $\mathcal{L}_i$ around $\{u_i\} \times C_i$, where u_i are points on S. If one can see only the torus T^2 and is unaware of the extra dimensions S, the theory seems to be the previous two-dimensional TQFT $\mathsf{T} \cong \tilde{\mathsf{T}}[S]$ that has parameters taking values in S. One finds that line operators $\mathcal{L}_i[u_i]$ wrapping around C_i carry continuous parameters u_i in the seemingly two-dimensional theory, and the correlation function for this configuration of line operators is given by the partition function of a lattice model $\mathsf{L}\left(\tilde{\mathsf{T}}[S]\right)$ defined on the lattice $\{\mathcal{L}_i[u_i](C_i)\}$.

For a generic choice of $\{u_i\}$, the transfer matrices of the lattice model $\mathsf{L}\left(\tilde{\mathsf{T}}[S]\right)$ commute since the two horizontal line operators in the above equation can move freely and interchange their positions owing to the topological nature along T^2; no phase transition occurs when they pass each other as they do not meet in the full spacetime $S \times T^2$. Thus, integrability follows from the existence of extra dimensions, whose coordinates provide continuous spectral parameters.

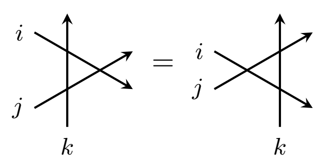
In fact, we can deduce integrability from another point of view. By the same logic, we have the unitarity relation



(1.66)

$$\Longleftrightarrow \quad R_{ji}(u_j, u_i)\, R_{ij}(u_i, u_j) = \mathrm{id}_{V_i \otimes V_j}, \tag{1.67}$$

and the Yang-Baxter equation



(1.68)

$$\Longleftrightarrow \quad R_{ij}(u_i, u_j)\, R_{ik}(u_i, u_k)\, R_{jk}(u_j, u_k) = R_{jk}(u_j, u_k)\, R_{ik}(u_i, u_k)\, R_{ij}(u_i, u_j)\ , \tag{1.69}$$

where the R-matrix R_{ij} acts on $V_i \otimes V_j$ as an intertwiner and trivial on V_k, etc. From these two relations we can reproduce the commutativity of the transfer matrix. We should emphasize that the Yang-Baxter equation is a local condition. When the Boltzmann weight locally satisfies the Yang-Baxter equation, it extends to the commutativity of the transfer matrix and hence the integrability of the model. In this sense, the Yang-Baxter equation is the fundamental condition of integrability of a lattice model.

Before going into the discussion of brane construction, we would like to generalize the above arguments further higher-dimensional situations. First of all, replace the two-torus T^2 with a general two-dimensional surface Σ, along which line operators are wrapped. We now have $S \times \Sigma$, and similarly to the above a lattice model is defined on Σ by line operators with spectral parameters. Let us consider the case that we really have more extra dimensions and a higher-dimensional theory is formulated on $S \times M \times \Sigma$, where M is some smooth manifold. The line operators we had may descend from extended operators of dimension greater than one. Let T again be the new higher-dimensional theory, and suppose that it is topological on Σ and has extended operators $\mathcal{E}_i$ whose codimension is greater than $\dim S$. Place $\mathcal{E}_i$ on submanifolds of the form $\{u_i\} \times N_i \times C_i$. Since $\mathsf{T}[S \times M]$ is again regarded as a two-dimensional TQFT defined on Σ, the correlation function of the operators $\mathcal{E}_i$ still should coincides with the partition function of an integrable lattice model $\mathsf{L}(\mathsf{T}[S \times M])$. The model is defined on the lattice formed by the line operators $\mathcal{E}_i[\{u_i\} \times N_i]$, which are the image of $\mathcal{E}_i$ in the two-dimensional theory $\mathsf{T}[S \times M]$.

As well as we can view the higher-dimensional theory as a two-dimensional TQFT, we may also view it as a theory $\mathsf{T}[\Sigma]$, which is a QFT on $S \times M$ specified by the surface Σ. In this theory, the operators $\mathcal{E}_i$ are seen as extended operators $\mathcal{E}_i[C_i]$ supported on $\{u_i\} \times N_i$. Then, we have another relation

$$\left\langle \prod_{i=1}^{l} \mathcal{E}_i[C_i](\{u_i\} \times N_i) \right\rangle_{\mathsf{T}[\Sigma], S \times M} = Z_{\mathsf{L}(\mathsf{T}[S \times M]), \{\mathcal{E}_i[\{u_i\} \times N_i](C_i)\}}. \tag{1.70}$$

Thus, we finally arrive at a correspondence between a QFT on $S \times M$ equipped with extended operators and an integrable lattice model on Σ.

1.3.2 Brane construction and correspondence

We have seen above that a lattice model is realized by a lattice of line operators in a two-dimensional TQFT, and it is integrable if the TQFT is embedded in higher dimensions and the line operators come from extended operators localized in some directions of the extra dimensions. Now we explain how to get such structures of correspondence of (previous equation) using branes in string theory. The brane construction here is still rather abstract. A little bit more concrete setups will be given in next section. The readers may skip this subsection for the first reading and jump to the next section.

Consider a type II string theory in a ten-dimensional spacetime

$$\mathbb{R}^4 \times T^*\Sigma \times \mathbb{R}^2, \tag{1.71}$$

where Σ is a two-dimensional surface embedded in $T^*\Sigma$ as a zero section. Introduce a stack of N NS5-branes supported on $\mathbb{R}^4 \times \Sigma \times \{0\}$ in this spacetime, and Dp-branes Dp_i on $\mathbb{R}^{p-1} \times \Sigma_i \times \{0\}$ ending on the NS5-branes, where $\mathbb{R}^{p-1}$ is a subspace of $\mathbb{R}^4$ (assuming $p \leq 5$) and Σ_i are surfaces in $T^*\Sigma$ such that $\Sigma_i \cap \Sigma = C_i$. See figure 5. Provided that Σ_i are suitably chosen, this brane system preserves four supercharges.

The low-energy dynamics of the NS5-branes is governed by a six-dimensional theory T_{NS5} on $\mathbb{R}^4 \times \Sigma$. The theory T_{NS5} is depending on whether IIA or IIB theory we are considering:

IIA : $\mathcal{N} = (2, 0)$ superconformal QFT of type A_{N-1},

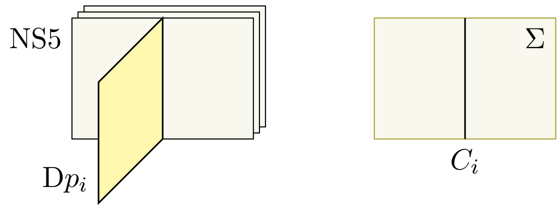


Figure 5: The Dp-brane Dp_i ending on the NS5-branes creates a defect $\mathcal{E}_{\mathrm{Dp_i}}$ along C_i. fig:Dp_on_NS5

$$\text{IIB} : \mathcal{N} = (1,1) \text{ super Yang-Mills theory with gauge group } \mathrm{SU}(N).$$

The theory $\mathsf{T}_{\mathrm{NS5}}$ is formulated on $\mathbb{R}^4 \times \Sigma$, with topological twist along Σ which breaks half of sixteen supercharges. In this twisted theory, Dp_i create p-dimensional defects $\mathcal{E}_{\mathrm{D}p_i}$ on $\mathbb{R}^{p-1} \times C_i$, reducing the number of unbroken supercharges to four. From the point of view of a four-dimensional observer, this brane configuration gives half-BPS defects $\mathcal{E}_{\mathrm{D}p_i}[C_i]$ in an $\mathcal{N} = 2$ theory $\mathsf{T}_{\mathrm{NS5}}[\Sigma]$. The total system is invariant under a U(1) R-symmetry originating from the rotational symmetry on the $\mathbb{R}^2$ factor of the ten-dimensional spacetime.

Let us take a three-manifold M and $(p-2)$-submanifolds N_i of M, and modify the above construction so that the world-volumes of the NS5-branes and the Dp-branes become $S^1 \times M \times \Sigma$ and $S^1 \times N_i \times \Sigma_i$, respectively. At low energies, we get the same theory $\mathsf{T}_{\mathrm{NS5}}$ formulated on $S^1 \times M \times \Sigma$ and defects $\mathcal{E}_{\mathrm{D}p_i}$ located on $S^1 \times N_i \times C_i$. In general, this modification completely breaks supersymmetry. For certain choices of M and N_i, however, there is a string background in which a fraction of supersymmetry is still preserved. In such a background, the path integral computes the *supersymmetric index* of $\mathsf{T}_{\mathrm{NS5}}$, defined with respect to the Hilbert space on $M \times \Sigma$ in the presence of defects $\mathcal{E}_{\mathrm{D}p_i}$ inserted on $N_i \times C_i$.

A salient feature of supersymmetric indices is that they are protected against continuous changes of various parameters of the theory. This means that the index of our theory is invariant under deformations of the geometric data of Σ and C_i, namely the metric on Σ and the shapes of C_i. In other words, the theory $\mathsf{T}_{\mathrm{NS5}}$ on $S^1 \times M \times \Sigma$ is topological on Σ, as far as the computation of the index is concerned.

To relate the present setup to the one considered in the previous subsection, we apply T-duality along S^1. It turns Dp_i into D$(p-1)$-branes $\mathrm{D}(p-1)_i$, localized at points u_i along the dual circle $\tilde{S}^1$, while sending the NS5-branes to those in the other type II string theory. The new NS5-branes produce the dual six-dimensional theory $\tilde{\mathsf{T}}_{\mathrm{NS5}}$ on $\tilde{S}^1 \times M \times \Sigma$, and in this theory $\mathrm{D}(p-1)_i$ create $(p-1)$-dimensional defects $\mathcal{E}_{\mathrm{D}(p-1)_i}$ on $\{u_i\} \times N_i \times C_i$. Thus we are in the situation studied in the last subsection, and the correlation function for this configuration coincides with the partition function of an integrable lattice model:

$$\left\langle \prod_{i=1}^{l} \mathcal{E}_{\mathrm{D}p_i}[C_i]\left(S^1 \times N_i\right) \right\rangle_{\mathsf{T}_{\mathrm{NS5}}[\Sigma], S^1 \times M} = Z_{\mathsf{L}\left(\tilde{\mathsf{T}}_{\mathrm{NS5}}\left[\tilde{S}^1 \times M\right]\right), \left\{\mathcal{E}_{\mathrm{D}(p-1)_i}[\{u_i\} \times N_i](C_i)\right\}}. \tag{1.72}$$

Here the left-hand side is expressed in the original frame; it implicitly depends on each spectral parameter u_i through the holonomy $\exp(2\pi \mathrm{i} u_i)$ around S^1 of the gauge field for the flavor symmetry $\mathrm{U}(1)_i$ supported on $\mathrm{D}p_i$. The holonomy appears in the index as a refinement parameter, or *fugacity*, associated with $\mathrm{U}(1)_i$.

1.3.3 Defects as transfer matrices

Finally, we apply the construction developed in the precious subsections to the main theme of this paper; integrable lattice models and defects as transfer matrices. Let us consider $p = 5$ case of the brane construction in the last subsection. To conform with the standard convention, take S-duality first and we still have D5- and NS5-branes

$$\begin{aligned} N\,\mathrm{D5} &\quad S^1 \times M \times \Sigma, \\ \mathrm{NS5}_i &\quad S^1 \times N_i \times \Sigma_i, \end{aligned}$$

where $\mathrm{NS5}_i$ create defects $\mathcal{E}_{\mathrm{NS5}_i}$ on $S^1 \times N_i \times C_i$.

One should notice that in this setup we necessarily have $N_i = M$ and thus the defects $\mathcal{E}_{\mathrm{NS5}_i}$ fill the whole $S^1 \times M$, which produces a four-dimensional theory

$$\mathsf{T}_{\mathrm{D5NS5}}[\Sigma]$$

with $\mathcal{N} = 1$ supersymmetry. Then we now have

$$\langle 1 \rangle_{\mathsf{T}_{\mathrm{D5NS5}}[\Sigma], S^1 \times M} = Z_{\mathsf{L}(\mathsf{T}_{\mathrm{D5NS5}}[S^1 \times M]), \{\mathcal{E}_i[S^1 \times M](C_i)\}}. \tag{1.73}$$

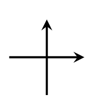
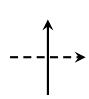
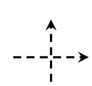
For example, when $M = S^3$, the left-hand side is given by the supersymmetric index for $\mathcal{N} = 1$ theory and the right-hand side corresponds to the partition function of Bazhanov-Sergeev integrable lattice model [19, 20, 21, 22]. When $M = L(p, 1)$, lens space, the left-hand side is computed in [23], which defines a new integrable lattice model through this correspondence.

The brane tiling construction of integrable lattice models can be enriched by introduction of additional defects. Besides the previously defined D5NS5-brane system, introduce a D3-brane such as

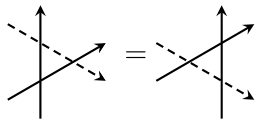
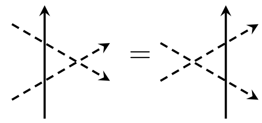
$$\mathrm{D3} \quad S^1 \times N \times C \times \mathbb{R}_+, \tag{1.74}$$

$$\mathrm{D3}' \quad S^1 \times \{0\} \times C' \times \mathbb{R}^2, \tag{1.75}$$

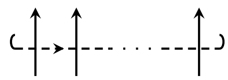
where N is a curve in M and C, C' are 1-cycles on Σ. A single D3-brane insertion corresponds to a new oriented line in the integrable lattice model, which we represent by a dashed line. Now that we have two kinds of lines, we can define three R-matrices:

$R =$  , $L =$  , $\mathcal{R} =$  . (1.76)

The middle one is usually called *L-operator*. Correspondingly, we have four Yang-Baxter equations, involving zero to three dashed lines. Those that involving one or two dashed line,

 and  (1.77)

are called *RLL relations*. The effect of the insertion of such an additional defect on the lattice model is seen in terms of the L-operator. The neighborhood of the dashed line looks like

 . (1.78)

This diagram shows that the defect acts on the lattice model by a transfer matrix constructed from L-operators. Thus, the insertion of a defect operator in four-dimensional theory is mapped into lattice model side as the action of a transfer matrix constructed from L-operators.

What we would like to discuss in detail in the subsequent sections are the introduction of a single D3-brane (equation of D3D3') to the D5NS5-brane system, and clarify that the correspondence with the additional defects. In the next two sections, we are considering the followings:

1. For the case of single D3, let $M = S^3$, $\Sigma = T^2$, and $N = S^1$. Then the D3 creates a surface defect on $S^1 \times S^1$ and it acts on supersymmetric index as a transfer matrix in the corresponding lattice model:

$$\langle 1 \rangle_{\mathsf{T}_{\mathrm{D5NS5}}[\Sigma], S^1 \times S^3} = \mathcal{I}_{S^1 \times S^3}(p,q,t)\,, \tag{1.79}$$

$$Z_{\mathsf{L}(\mathsf{T}_{\mathrm{D5NS5}}[S^1\times S^3]),\{\mathcal{E}_i[S^1\times S^3](C_i)\}} = Z_{\mathrm{Bazhanov-Sergeev}}\,, \tag{1.80}$$

   the surface defect index is represented as a difference operator acting on the original supersymmetric index [24, 25],

$$\left\langle S_{(r,s)} \right\rangle_{\mathsf{T}_{\mathrm{D5NS5}}[\Sigma]} = \mathfrak{S}_{(r,s)} \mathcal{I}_{S^1 \times S^3}(p,q,t)\,. \tag{1.81}$$

2. For the case of single D3', let $M = \mathbb{R}^3$ and $\Sigma = T^2$. D3' creates a line defect wrapping S^1 and it acts on quantum Hilbert space of a spin chain as a transfer matrix, since now we have non-compact three-manifold $\mathbb{R}^3$. The line defect in the four-dimensional theory is realized as a Wilson-'t Hooft line operator T, and its magnetic and electric charge $(\mathbf{m}, \mathbf{e})$ is specified by the one-cycle C' wrapping the torus T^2. As it turns out, the vacuum expectation values (vevs) of Wilson-'t Hooft lines realize the deformation quantization of the Hitchin moduli space, and are naturally quantized by the Weyl quantization:

$$\text{Weyl quantization of } \left\langle T_{(\mathbf{m},\mathbf{e})} \right\rangle = \text{trigonometric transfer matrix.} \tag{1.82}$$

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