1 Special functions and some formulas

sec:special_functions

1.1 Theta functions

The theta function with characteristics is defined by

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](\zeta|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi\mathrm{i}(n+a)^2\tau+2\pi\mathrm{i}(n+a)(\zeta+b)}, \tag{1.1}$$

where ζ is a complex variable and τ is a complex parameter in the upper half-plane; $\mathrm{Im}\,\tau > 0$. The Jacobi's theta functions are defined by

$$\theta_1(\zeta|\tau) = -\theta\left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right](\zeta|\tau), \tag{1.2}$$

$$\theta_2(\zeta|\tau) = \theta_1(\zeta+1/2|\tau), \tag{1.3}$$

$$\theta_3(\zeta|\tau) = e^{\pi\mathrm{i}(\zeta+\tau/4)}\theta_2(\zeta+\tau/2|\tau), \tag{1.4}$$

$$\theta_4(\zeta|\tau) = \theta_3(\zeta+1/2|\tau). \tag{1.5}$$

The first of these, θ_1, is an odd function of ζ and satisfies

$$\theta_1(\zeta+1|\tau) = -\theta_1(\zeta|\tau), \tag{1.6}$$

$$\theta_1(\zeta+\tau|\tau) = -e^{\pi\mathrm{i}(2\zeta-\tau)}\theta_1(\zeta|\tau). \tag{1.7}$$

The other three are even functions. We have

$$2\theta_1(\zeta+\zeta')\theta_1(\zeta-\zeta') = \bar\theta_4(\zeta)\bar\theta_3(\zeta') - \bar\theta_4(\zeta')\bar\theta_3(\zeta), \tag{1.8}$$

$$2\theta_2(\zeta+\zeta')\theta_2(\zeta-\zeta') = \bar\theta_3(\zeta)\bar\theta_3(\zeta') - \bar\theta_4(\zeta')\bar\theta_4(\zeta), \tag{1.9}$$

$$2\theta_3(\zeta+\zeta')\theta_3(\zeta-\zeta') = \bar\theta_3(\zeta)\bar\theta_3(\zeta') + \bar\theta_4(\zeta')\bar\theta_4(\zeta), \tag{1.10}$$

$$2\theta_4(\zeta+\zeta')\theta_4(\zeta-\zeta') = \bar\theta_4(\zeta)\bar\theta_3(\zeta') + \bar\theta_4(\zeta')\bar\theta_3(\zeta). \tag{1.11}$$

with $\theta_a(\zeta) := \theta_a(\zeta|\tau)$ and $\bar\theta_a(\zeta) := \theta_a(\zeta|\tau/2)$.

It is useful to define another kind of theta function which we call the modified theta function:

$$\theta(z;p) = (z;p)_\infty(p/z;p)_\infty, \tag{1.12}$$

$$(z;p)_\infty := \prod_{k=0}^{\infty}(1-p^k z), \qquad |p|<1. \tag{1.13}$$

It satisfies

$$\theta(z;p) = \theta(p/z;p). \tag{1.14}$$

Set $z = e^{2\pi\mathrm{i}\zeta}$, $p = e^{2\pi\mathrm{i}\tau}$ and introduce multiplicative notation

$$\theta_a(z;p) := \theta_a(\zeta|\tau). \tag{1.15}$$

Then the Jacobi's theta functions are rewritten in terms of the modified theta function as

$$\theta_1(z;p) = \mathrm{i}p^{1/8}(p;p)_\infty z^{-1/2}\theta(z;p), \tag{1.16}$$

$$\theta_2(z;p) = p^{1/8}(p;p)_\infty z^{-1/2}\theta(-z;p), \tag{1.17}$$

$$\theta_3(z;p) = (p;p)_\infty \theta(-\sqrt{p}z;p), \tag{1.18}$$

$$\theta_4(z;p) = (p;p)_\infty \theta(\sqrt{p}z;p). \tag{1.19}$$

1.2 Elliptic gamma function

The elliptic gamma function is closely related to the triple gamma function and depends on two complex parameters p and q:

$$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1-p^{j+1}q^{k+1}z^{-1}}{1-p^j q^k z}; \qquad |p|,\ |q| < 1. \tag{1.20}$$

It satisfies the identities

$$\Gamma(z;p,q)\Gamma(pq/z;p,q) = 1 \tag{1.21}$$

and

$$\Gamma(pz;p.q) = \theta(z;q)\Gamma(z;p,q), \tag{1.22}$$

$$\Gamma(qz;p.q) = \theta(z;p)\Gamma(z;p,q). \tag{1.23}$$

The function $\Gamma(z;p,q)$ has a pole at $z = p^{-j}q^{-k}$, where j, k are non-negative integers. The residue at this pole is given by

$$\mathrm{Res}_{z=p^{-j}q^{-k}}\big[\Gamma(z;p,q)\big] = \frac{(-1)^{jk+j+k}p^{(k+1)j(j+1)/2}q^{(j+1)k(k+1)/2}}{(p;p)_\infty(q;q)_\infty\theta(p,\dots,p^j;q)\theta(q,\dots,q^k;p)}, \tag{1.24}$$

eq:gamma_residue

where we have introduced the notation $\theta(z_1,\dots,z_n;q) := \theta(z_1;q)\cdots\theta(z_n;q)$.

Let t_j, $j = 1,\dots,6$ be six complex parameters such that $|t_j| < 1$ and $\prod_{j=1}^6 t_j = pq$. Then, we have the following identity proved in [1]:

$$\frac{(p;p)_\infty(q;q)_\infty}{2}\int_{\mathbb{T}}\frac{dz}{2\pi\mathrm{i}z}\frac{\prod_{j=1}^6\Gamma(t_j z^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)} = \prod_{1\le j<k\le 6}\Gamma(t_j t_k;p,q). \tag{1.25}$$

Here $\mathbb{T}$ is the unit circle with counterclockwise orientation and

$$\Gamma(z^{\pm n};p,q) := \Gamma(z^n;p,q)\Gamma(z^{-n};p,q). \tag{1.26}$$

The left-hand side of the above formula is known as the elliptic beta integral.

1.3 The function Γ_{b} and upsilon function

The function Γ_{b} is related to the double gamma function, which is another relative of the ordinary gamma function. It can be defined by an integral representation

$$\log \Gamma_{\mathsf{b}}(x) = \int_0^\infty \frac{dt}{t} \left(\frac{e^{-xt} - e^{-Qt/2}}{(1-e^{-\mathsf{b}t})(1-e^{-t/\mathsf{b}})} - \frac{(Q-2x)^2}{8e^t} - \frac{Q-2x}{t} \right), \tag{1.27}$$

where x is a complex variable and b is a complex parameter such that $\operatorname{Re} x, \operatorname{Re} \mathsf{b} > 0$, and $Q = \mathsf{b} + \mathsf{b}^{-1}$.

It is self-dual:

$$\Gamma_{\mathsf{b}}(x) = \Gamma_{1/\mathsf{b}}(x), \tag{1.28}$$

and satisfies the identity

$$\frac{\Gamma_{\mathsf{b}}(x+\mathsf{b})}{\Gamma_{\mathsf{b}}(x)} = \sqrt{2\pi} \frac{\mathsf{b}^{\mathsf{b}x - 1/2}}{\Gamma(\mathsf{b}x)}, \tag{1.29}$$

where the denominator in the right-hand side is the ordinary gamma function.

Through the function Γ_{b}, we define upsilon function as

$$\Upsilon_{\mathsf{b}}(x) = \frac{1}{\Gamma_{\mathsf{b}}(x)\Gamma_{\mathsf{b}}(Q-x)}. \tag{1.30}$$

The upsilon function has an integral representation as well, which is convergent in the strip $0 < \operatorname{Re} x < Q$:

$$\log \Upsilon_{\mathsf{b}}(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2\left((Q/2 - x)t/2\right)}{\sinh(\mathsf{b}t/2)\sinh(t/2\mathsf{b})} \right]. \tag{1.31}$$

The upsilon function is also self-dual:

$$\Upsilon_{\mathsf{b}}(x) = \Upsilon_{1/\mathsf{b}}(x), \tag{1.32}$$

and satisfies the identity

$$\frac{\Upsilon_{\mathsf{b}}(x+\mathsf{b})}{\Upsilon_{\mathsf{b}}(x)} = \frac{\Gamma(\mathsf{b}x)}{\Gamma(1-\mathsf{b}x)} \mathsf{b}^{1-2\mathsf{b}x}. \tag{1.33}$$

eq:upsilon_identity

References

[1] V. P. Spiridonov, *On the elliptic beta function*, *Uspekhi Mat. Nauk* **56** (2001) 181–182.