

Defects in Supersymmetric Gauge Theory and Integrable Lattice Models

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Abstract

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1 Introduction and Summary

2 Integrability from extra dimensions

Throughout this paper, we discuss correspondences between a certain class of supersymmetric gauge theories and integrable lattice models:

$$\mathcal{I}_{\mathsf{T}_{4d}[\mathsf{L}_{2d}]} = Z_{\mathsf{L}_{2d}[\mathsf{T}_{4d}]}, \quad (2.1)$$

where the left-hand side is the supersymmetric index of a four-dimensional gauge theory T_{4d} and the right-hand side denotes the statistical partition function of the corresponding integrable lattice model L_{2d} specified by the four-dimensional theory T_{4d} . As it turns out, the correspondence emerges from TQFT with extra dimensions [1]. The aim of this section is to give the step-by-step explanation of the above correspondence from an elementary level. Besides, to explain these, we will start by clarifying in order what the terms TQFT, lattice model, and integrability mean.

2.1 Preliminaries

In the next subsection, we would like to introduce lattice model as discrete quantum field theory (QFT). So let us start from the general settings of QFT. We first briefly review one-dimensional QFT, which is nothing but quantum mechanics, and then extend the discussion to general quantum field theory in $(d + 1)$ -dimensions. Though we do not have a complete definition of general QFT, a special class of QFT which is called *topological* QFT is axiomatized by Atiyah [2].

2.1.1 What is quantum field theory?

Let us now begin with one-dimensional QFT, as known as quantum mechanics (QM). Suppose we have a one-dimensional manifold¹, which may be thought of as “time” $M^1 = \text{interval}$ or S^1 (or \mathbb{R}), and data to specify a QM:

¹Throughout the paper, we assume all the manifold is smooth and oriented, unless otherwise stated.

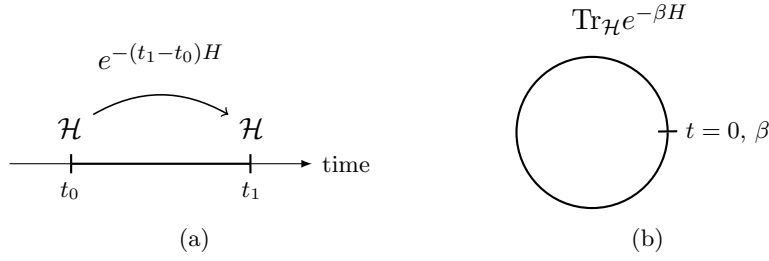


Figure 1: Two most crucial properties of quantum mechanics.

- \mathcal{H} : a vector space (Hilbert space/state space),
- \mathcal{O} : a set of self-adjoint operators (observables, acting on \mathcal{H}),
- H : an operator called *Hamiltonian*,

where $H \in \mathcal{O}$ and the state space \mathcal{H} is a finite or infinite dimensional \mathbb{C} -vector space. We sometimes need to consider a set of operators \mathcal{O} including not self-adjoint operators, but for simplicity we do not care about that at this moment. In undergraduate, we have learnt that one should consider the Schrödinger equation and find a good basis in \mathcal{H} diagonalizing the Hamiltonian.

Formally, the procedure of solving the Schrödinger equation leads to an expression of time evolution of states by a linear map² e^{-tH} in quantum mechanics, and the partition function in many-body statistical mechanics. In turn, we have two most crucial properties of QM: Given an interval $M^1 = [t_0, t_1]$, we have state vectors at the endpoints of the interval and a time evolution between the states:

$$\begin{array}{c} \text{---} \\ | \quad \quad | \\ t_0 \quad \quad t_1 \end{array} \rightsquigarrow e^{-(t_1-t_0)H} : \mathcal{H} \longrightarrow \mathcal{H}, \quad (2.2)$$

and given a circle $M^1 = S^1_\beta$, we get a number called *partition function*:

$$\begin{array}{c} \bigcirc \\ S^1_\beta \end{array} \rightsquigarrow \text{Tr}_{\mathcal{H}} e^{-\beta H}. \quad (2.3)$$

In addition, from the physical facts it is quite reasonable to assume that the time evolution and the partition function are compatible with cutting and gluing intervals and S^1 s. This simply means that the time evolution from time t_0 to t_1 followed by another time evolution from t_1 to t_2 is equal to the single time evolution from t_0 to t_2 , etc. Summarizing, the properties that quantum mechanics or one-dimensional QFT has are rephrased by the language of given one-dimensional manifold M^1 (see figure 1):

²We here do not get into the argument of Euclideanization.

- Given an interval, produces state vectors at the endpoints and a linear map between them.
- Given an S^1 , produces a number.
- Compatible with cutting and gluing intervals and S^1 s.

Thus, in the abstract we conclude that one-dimensional QFT, or quantum mechanics, is a gadget satisfying the above properties for each given one-dimensional manifold M^1 .

From these observations of quantum mechanics, we wish to extend the discussion to general $(d+1)$ -dimensional QFT. The starting point of defining a $(d+1)$ -dimensional QFT is the choice of a $(d+1)$ -dimensional manifold M^{d+1} , which has d spatial directions and one “time” direction. For most QFTs the manifold M^{d+1} is viewed as a Riemannian manifold with a smooth metric on it. As already noticed, we will mostly consider a positive definite Riemannian metric, QFT on which is usually referred to as an Euclidean QFT, and hence precisely there is no notion of “time” in such a theory, at least globally. The manifold M^{d+1} may or may not have boundaries. In case it does have boundaries some additional information is needed at the boundaries to define the QFT. In addition to a Riemannian metric, depending on the situation one wants to consider, one often needs some more structures on the manifold M^{d+1} , e.g. smooth structure, conformal structure, spin structure, etc.

To obtain the data to specify a QFT, we now would like to extend the observations seen in QM. Suppose we have a $(d+1)$ -dimensional manifold M^{d+1} , then we wish to “define” a QFT by a gadget Z , which should produce a vector when M^{d+1} has a boundary

$$Z\left(M^{d+1}\right) \in \mathcal{H}_{bdy}, \quad (2.4)$$

and should produce a number when M^{d+1} has no boundary

$$Z\left(M^{d+1}\right) \in \mathbb{C}. \quad (2.5)$$

In the case of M^{d+1} with a boundary, the vector space defined on the boundary is called the space of states or just Hilbert space in the physics literature. On the one hand, if M^{d+1} does not have boundary, the number $Z\left(M^{d+1}\right)$ is called the partition function. If the boundary of M^{d+1} has several disconnected components, namely the boundaries are given by the disjoint union of simply connected d -dimensional closed manifolds $\{N_i^d\}$: $\partial M^{d+1} = \sqcup_i N_i^d$, Z should define a linear map among the vector spaces defined on the boundaries. In particular, in the case $M^{d+1} = N^d \times I$, where I is an interval of length T , M^{d+1} has two boundaries N^d and Z now gives rise to a linear map $Z\left(N^d \times I\right) =: U(T)$,

$$U(T) : \mathcal{H}_N \longrightarrow \mathcal{H}_N. \quad (2.6)$$

From the physical facts, Z also should be compatible with cutting and gluing of $(d + 1)$ -dimensional manifolds, and thus we learn that the linear map given above satisfies $U(T_1)U(T_2) = U(T_1 + T_2)$. This in turn defines an operator H as the generator of U ,

$$U(T) = \exp(-TH). \quad (2.7)$$

H is called the Hamiltonian of the system, acting on the space of states \mathcal{H}_N . If one considers a manifold with Lorentzian metric, $U(T)$ is represented as

$$U(T) = \exp(-iTH), \quad (2.8)$$

and then the interval I is regarded as “physical time.”

2.1.2 Construction of Z

So far we have discussed only in an abstract way what quantum field theory is, or in other words what the gadget Z should satisfy. Let us now see how we specify Z to define a QFT. Broadly speaking, there are three kinds of constructions of Z . They are not totally independent and have many aspects of applicabilities. In the rest of this section, M denotes a $(d + 1)$ -dimensional manifold with or without boundary, and N denotes a d -dimensional closed manifold without boundary.

Construct from the axiom

The first construction is in a sense the simplest one; we write down appropriate properties that a QFT must satisfy, and axiomatize them. Construction of Z is to give a mathematical formulation which satisfy the axioms as the data to specify the theory. This is actually the only way that one can define a QFT by mathematically rigorous procedures. Normally, such a theory is first studied by physicists as an ideal or a toy model from the physical motivation, and then refined as rigorous mathematics by mathematicians.³ There are several kinds of such theories. We now introduce typical three examples.

The first one is *free theories* in any dimensions, which are toy models of field theories and play important roles as a probe of more complicated QFTs. In any spacetime dimensions, for the Riemannian manifolds with or without spin structure, we can rigorously define the free field theories. One of them is the free scalar field theory. Besides a $(d + 1)$ -dimensional closed manifold M , pick a G -bundle $P \rightarrow M$ with connection A and a G -vector space V . Then we construct the associated vector bundle $P \times_G V$, whose covariant derivative is denoted by

³The fact that a QFT can be treated in a mathematically rigorous way implies that the theory may have an enormous amount of symmetry, and the difficulties of the QFT are completely controlled by them. Even better, these theories can often be exactly solved in an appropriate sense.

D_A . We now have the Laplacian Δ_A given by the covariant derivative D_A , and then the free scalar field theory is defined by the partition function

$$Z_{\text{scalar}}(M; A) := 1/\det \Delta_A. \quad (2.9)$$

If the determinant of the Laplacian involves a divergence, it must be properly removed.

Another free theory is the free fermion theory. In this case M also needs a spin structure and the spin representations S, S' . Then we construct the Dirac operator \mathcal{D}_A on the spin bundles and the partition function of the free fermion theory is given by

$$Z_{\text{fermion}}(M; A) := \det \mathcal{D}_A, \quad (2.10)$$

again the determinant is taken appropriately.⁴

Next, in two dimensions, we have another axiomatic quantum field theory, that is *conformal field theory* (CFT). Conformal field theory in two dimensions was originally formulated by Belavin, Polyakov, and Zamolodchikov [4] as a model of physical systems at critical points. They established the renowned Virasoro algebra as an infinite dimensional symmetry of the system and fully investigated the minimal model. The holomorphic part of the Virasoro algebra is captured by vertex operator algebras, and since then there are many mathematically rigorous discussions on them. Their kinematic behaviors on Riemann surfaces are governed by the conformal blocks, and its geometric meaning has been studied in [5]. In turn, the study of irrational CFTs has led to the AGT correspondence [6], which has brought to us large amount of developments in both physics and mathematics.

The final example is *topological quantum field theories* (TQFTs). TQFT is one of the main focuses in this paper. These theories are originating from Witten's proposals of topological field theories [7, 8, 9]. Inspired by Witten's proposals, Atiyah and Segal axiomatized the topological QFT,⁵ and happily this also has brought to us a numerous amount of applications both to physics and to pure mathematics. For example, $(1+1)$ -dimensional topological QFTs are known to be categorically equivalent to commutative Frobenius algebras. $(2+1)$ -dimensional Chern-Simons theory for a compact Lie group G is rigorously constructed by the Turaev-Viro and Reshetikhin-Turaev construction using quantum groups [11, 12], and has applications to knot or link invariants and 3-manifold invariants. Moreover, both 2d and 3d TQFTs have applications to the mirror symmetry, or topological string theory (see e.g. [13]), those have led to fruitful interactions between physics and mathematics. We will take more time for TQFT and introduce the Atiyah's axioms in some detail in the next subsection.

⁴Fermion theories may have anomalies. If it is the case, the axiom needs to be somehow modified. In particular the partition function for a closed manifold is given by the eta invariant [3].

⁵Precisely speaking, what Segal axiomatized is the definition of conformal field theories [10]. However, at any rate the definition of CFT by Segal is quite similar to the definition of Atiyah's TQFT, namely it is a categorical one. For the geometric definition of CFT, see [5].

Path integrate the Boltzmann weight

Before going to Atiyah's TQFT axioms, let us see two more constructions of Z . These are no longer mathematically rigorous, but rather familiar constructions for physicists. The second construction is to define a partition function by *path integral*. The general prescription is given as follows: One first introduces an action functional over the classical field configurations, which deduces the classical equation of motion by the variational principle. Then one exponentiates the action and integrate it over the space of fields.

The partition function for the free fields given above can also be defined by the path integral expression. For example, for the free complex scalar field theory consider a section ϕ of the vector bundle $P \times_G V$ over M , where V is a representation of a compact Lie group G . Then define an action functional

$$S : \Gamma(P \times_G V) \longrightarrow \mathbb{R}, \quad (2.11)$$

such that

$$S(\phi) = \int_M \frac{1}{2} D_A \phi \wedge * D_A \phi, \quad (2.12)$$

where $*$ is the Hodge star on M . D_A is again the covariant derivative given by the connection A . Using this action functional, physicists “define” its partition function by

$$Z_{\text{scalar}}(M; A) := \int_{\Gamma(P \times_G V)} \mathcal{D}\phi e^{-S(\phi)}. \quad (2.13)$$

The integrand e^{-S} of path integral is generically called the *Boltzmann weight*. For the free field theories, the path integral can be expressed as an infinite product of the Gaussian integral. Introducing an appropriate regularization, one can compute the exact partition function and it leads to the same result as mentioned above.

As another example, let us consider Yang-Mills theory. We now introduce the kinetic term of the connection A , and define the action functional

$$S(A) = \int_M \frac{1}{4g^2} F_A \wedge * F_A, \quad (2.14)$$

where F_A is the curvature of the G -connection A . Define the partition function of Yang-Mills theory by

$$Z_{\text{YM}}(M) = \int_{\mathcal{A}_M/G} \mathcal{D}A e^{-S(A)}, \quad (2.15)$$

where the integral is taken over the space of connections on M modulo gauge transformations. Unlike the free scalar theory, making this path integral mathematically precise is an extremely difficult problem. Although it is still ill-defined, physicists have been working on this expression to understand many properties of quantum gauge theories. Experimentally, physicists discretize the manifold M to a $(d+1)$ -dimensional lattice and put it on

a supercomputer. At least, numerical calculations show the above construction may be a mathematically meaningful and reproduce many experimental results to reasonable accuracy.

Deduce from string/M-theory

The final construction of Z is to use string or M-theory. This is also our main tool to construct a QFT to realize the correspondence (the first equation of this section). Nonetheless, string or M-theory is less rigorous than path-integral expression of the construction of Z , and thus it is more hopeless to give a mathematical precise meaning to the construction. We here would like to show just two examples which “define” a class of quantum field theories through string/M-theory.

The first example is the AdS/CFT correspondence.

The second example is the so-called six-dimensional $\mathcal{N} = (2, 0)$ theories [14]. We start from a 10-dimensional string theory called the type IIB string theory, which roughly speaking assigns the partition function $Z_{\text{IIB}}(\tilde{M})$ to a 10-dimensional manifold \tilde{M} . Pick a finite subgroup Γ_G of $\text{SU}(2)$ of type $G = A_n, D_n$ or $E_{6,7,8}$. We define a six-dimensional QFT Q_G by its partition function for a six-dimensional manifold M ,

$$Z_{Q_G}(M) = Z_{\text{IIB}}(M \times \mathbb{C}^2/\Gamma_G). \quad (2.16)$$

They are examples of six-dimensional $\mathcal{N} = (2, 0)$ superconformal QFTs. These theories are known not to have a Lagrangian description, namely their partition function cannot be obtained from the path integral formalism. They have another description via M-theory, as the low-energy dynamics of M5-branes. The construction from M5-branes leads to the AGT correspondence and the notion of class \mathcal{S} theories, which we will explain in later sections.

2.1.3 Atiyah’s topological QFT

As mentioned some times, in this paper TQFT in extra dimensions will play a crucial role to realize the correspondence between supersymmetric gauge theories and integrable lattice models. So now let us pause here and introduce the Atiyah’s axioms of TQFT [2]. We first list the axioms, and right then give their physical meanings. Readers will notice that most of these axioms are physically quite natural and actually mathematical rephrasing of the properties that Z should satisfy, which we have already seen. In addition, we would like to define lattice model as a discrete version of QFT in the next subsection. Reviewing the definitions of TQFT here will be really a good help of the argument of lattice model.

Axiom (Atiyah’s $(d + 1)$ -dimensional TQFT)

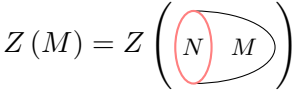
$(d + 1)$ -dimensional TQFT is defined by Z consisting of the following two data and five assignments:

- For each oriented d -dimensional closed manifold N , assigns a finite dimensional \mathbb{C} -vector space \mathcal{H}_N :

$$Z(N) = \mathcal{H}_N. \quad (2.17)$$

This corresponds to (a half of) canonical quantization, or geometric quantization, known in physics literature. Why we say it is “a half of” will be explained in a moment. In physics, especially for a field theory which has a Lagrangian description, we consider the phase space of fields, take a constant-time surface, and then perform canonical quantization by imposing canonical commutation relations on fields and their conjugates. The constant-time surface is a codimension-1 hypersurface in $(d+1)$ -dimensional manifold M , which in this case is nothing but the d -dimensional closed manifold N . So, the manifold N is viewed as a collection of d spatial directions. One can think of the assigned vector space \mathcal{H}_N as “the space of functionals on the classical fields on N ,” usually called the space of states or just Hilbert space of the system. The only major difference here is that for topological theory the Hilbert space is of finite dimension.⁶

- For each oriented $(d+1)$ -dimensional manifold M with a boundary $\partial M = N$, assigns a vector

$$Z(M) = Z \left(\text{diagram of } M \text{ with boundary } N \right) \in \mathcal{H}_N. \quad (2.18)$$


This expresses the path integral quantization on M with boundary. Recall that if one has a spacetime manifold with boundary, one needs to impose a boundary condition on fields on the boundary, and it leads to the state vector associated to the boundary condition by path integral expression, namely let φ be a fixed field configuration on N ,

$$Z(M; \varphi) = \int_{X|_N = \varphi} \mathcal{D}X e^{-S(X)} \in \mathcal{H}_N. \quad (2.19)$$

In other words, the quantum state at a generic time t is given by the path integral of the Boltzmann weight over all the classical fields with time $< t$.

These data are subject to the following assignments:

1. (involutory) Let $-N$ denote a manifold N with the opposite orientation, then

$$\mathcal{H}_{-N} = \mathcal{H}_N^*, \quad (2.20)$$

where \mathcal{H}_N^* is the dual to \mathcal{H}_N .

⁶One can remove the finiteness condition from the axiom. In fact, in TQFT we can define a non-degenerate bilinear form from the other axioms, and from that as a corollary we can deduce the space of states is finite-dimensional.

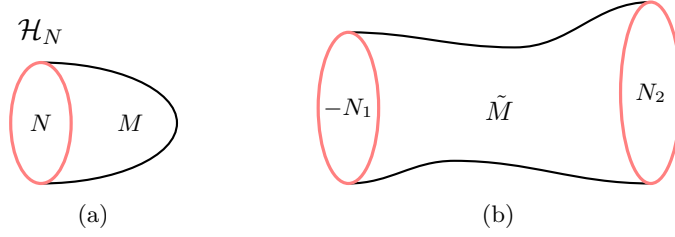


Figure 2: one-boundary and two-boundary cases.

2. (multiplicative) If N is a disjoint union of two d -dimensional closed manifolds N_1 and N_2 , then the vector space associated on it is factorized to a tensor product:

$$\mathcal{H}_{N_1 \sqcup N_2} = \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}. \quad (2.21)$$

This is really natural in the physical point of view. This condition means that if we have two physical systems defined on spatially disjoint union $N_1 \sqcup N_2$, then the space of states is represented as the tensor product of each state space. Also, from the conditions so far one finds that if $(d+1)$ -dimensional manifold M' has two boundaries such that

$$\partial M' = -N_1 \sqcup N_2, \quad (2.22)$$

then Z defines a linear map

$$Z(M') = Z \left(\begin{array}{c} \text{---} N_1 \text{---} \tilde{M} \text{---} N_2 \text{---} \end{array} \right) \in \text{Hom}_{\mathbb{C}}(\mathcal{H}_{N_1}, \mathcal{H}_{N_2}), \quad (2.23)$$

namely, for a cobordism between N_1 and N_2 , we have a linear map between \mathcal{H}_{N_1} and \mathcal{H}_{N_2} . This defines a time evolution, or transition amplitude, between the states in \mathcal{H}_{N_1} and \mathcal{H}_{N_2} . When the field theory has a Hamiltonian, $Z(M')$ may correspond to the time evolution operator

$$Z(M') := e^{-tH} : \mathcal{H}_{N_1} \longrightarrow \mathcal{H}_{N_2}. \quad (2.24)$$

Since canonical quantization is a procedure to make an assignment of Hilbert spaces and the time evolution on them, as we saw in QM, this association of linear maps gives the other half of canonical quantization. One can also explicitly express $Z(M')$ in the path integral expression by the Feynman kernel (also as known as propagator or Green's function). Suppose we have an initial state $\Psi_0 \in \mathcal{H}_{N_1}$, then the state $\Psi_t \in \mathcal{H}_{N_2}$ at time t is expressed by

$$\Psi_t(\varphi_t) = (e^{-tH} \Psi_0)(\varphi_t)$$

$$= \int K(\varphi_t, \varphi_0) \Psi_0(\varphi_0) \mathcal{D}\varphi_0, \quad (2.25)$$

where

$$K(\varphi_t, \varphi_0) = \int_{X|_{N_1}=\varphi_0}^{X|_{N_2}=\varphi_t} \mathcal{D}X e^{-S(X)}. \quad (2.26)$$

3. For two cobordisms such that

$$\partial M_1 = -N_1 \sqcup N_2, \quad \partial M_2 = -N_2 \sqcup N_3, \quad (2.27)$$

it follows that

$$Z(M_1 \cup_{N_2} M_2) = Z(M_2) Z(M_1), \quad (2.28)$$

where the cobordisms M_1 and M_2 are glued along N_2 by a certain diffeomorphism on N_2 . This asserts that the linear maps are transitive when we compose cobordisms. This is nothing but the physical requirement that the time evolution is compatible with the cutting and gluing the manifolds.

4. Given $N = \emptyset$ as an empty d -dimensional manifold, then the associated vector space is one-dimensional:

$$Z(\emptyset) = \mathbb{C}. \quad (2.29)$$

This is a non-triviality condition. Therefore, for each $(d+1)$ -dimensional manifold M without boundary, $\partial M = \emptyset$, Z assigns a number:

$$Z(M) \in \mathbb{C}. \quad (2.30)$$

This number associated to a closed $(d+1)$ -dimensional manifold is called the partition function. Not only is it an important quantity physically, but it also has mathematical applications, such as giving a topological invariant of the closed manifold M .

5. Let I be an interval, then for each cylinder $M = N \times I$, the linear map on \mathcal{H}_N is trivial:

$$Z(N \times I) = \text{id}_{\mathcal{H}_N}. \quad (2.31)$$

Since in a topological theory manifolds diffeomorphic to each other should be considered as the identical, each cobordism class defines a linear map. This means that the addition of a cylinder is a trivial operation, the associated linear map is the identity. This condition is another major difference from ordinary QFTs. For example, for $(2+1)$ -dimensional Chern-Simons theory one can explicitly see its Hamiltonian is identically 0 and hence the time evolution of Chern-Simons theory is trivial.

This is all the assignments for Atiyah's TQFT. Although we gave the path integral expressions at the explanation of the physical meanings for some conditions, the axioms themselves are really the basis for the rigorous mathematical definition of Z . One can actually take an equivalent definition of TQFT in slightly different axioms. For more details, see the original paper by Atiyah.

At first sight, this might look a little bit complicated and too abstract. This definition, however, is really natural and works quite well in a sense that it gives a kind of “homomorphism” between geometry and algebra:

$$Z : \text{‘geometry’} \longrightarrow \text{‘algebra’}.$$

In fact, all in all the axioms of Atiyah's TQFT define a functor from the category of $(d + 1)$ -dimensional cobordisms to the category of finite-dimensional \mathbb{C} -vector spaces:⁷

$$Z : \text{Bord}_{d+1} \longrightarrow \text{Vect}_{\mathbb{C}}. \quad (2.32)$$

TQFT luckily has a mathematical precise definition. For a general QFT, i.e. not TQFT, in many cases there is no precise definition as we saw the examples of the construction of Z . However, a QFT is generically expected to be characterized by some functor from the geometric structure of spacetime manifolds to the algebraic description of physical states and observables.

2.2 Lattice model as discrete QFT

Now let us move on to the discussion on lattice model. First of all, let us begin with the question of “what is lattice model?” Probably there is no unique definition of lattice model. We here would like to characterize lattice model by a discrete version of QFT. What we have learnt in the previous subsection is that a $(d + 1)$ -dimensional QFT may be defined by an appropriate functor Z_Q^{d+1} , which produces a number for each $(d + 1)$ -dimensional closed manifold M :

$$Z_Q^{d+1}(M) \in \mathbb{C}, \quad (2.33)$$

which is called the partition function of the model, and Z_Q^{d+1} satisfies additional some reasonable conditions for each QFT of interest.

We introduce lattice model in the same way. For a $(d + 1)$ -dimensional closed lattice L , a $(d + 1)$ -dimensional lattice model is defined by Z_L^{d+1} , which produces a number

$$Z_L^{d+1}(L) \in \mathbb{C}, \quad (2.34)$$

⁷What is more, the category of cobordisms and the category of vector spaces are endowed with the product structures, that is, taking disjoint union and taking tensor product, which are really symmetric operations. As such, the functor between them is keeping such structures and then called symmetric monoidal functor.

which is again called the partition function, and Z_L^{d+1} satisfies additional some reasonable conditions. The most typical example of lattice model in particle physics is lattice gauge theory or lattice QCD.⁸ For such theories, it is really clear that the model is defined by a discrete version of QFTs; One compactifies the spacetime manifold \mathbb{R}^4 to four-torus T^4 , and then discretize the theory and put it on a lattice on the torus.

Throughout the paper, we will consider only $d = 0$ or 1 case of lattice model, and we call each case as 1d or 2d lattice model. To prepare for the discussion of integrability, we take the prominent example of statistical lattice model called the Ising model. We first briefly review the generality of the Ising model and compare with field theory, and then introduce the transfer matrix which is essentially a time evolution operator in a discrete quantum system.

2.2.1 Prominent example – the Ising model

The Ising model is a very good introduction to integrable lattice model. The 1d and 2d Ising model is known to be exactly solvable, whose meaning we will clarify in a moment, and it is called integrable. To see the generality of the Ising model, we first define 2d lattice and spin configurations on it.

Define a 2d periodic lattice on two-torus T^2 by

$$\begin{aligned} L &:= \mathbb{Z}_M \times \mathbb{Z}_N \\ &= \{1, \dots, M\} \times \{1, \dots, N\}. \end{aligned} \quad (2.35)$$

A spin configuration on the lattice L is defined by a map

$$\mathbf{s} : L \longrightarrow \{+1, -1\}, \quad (2.36)$$

where usually $+1$ is called up spin and -1 down spin. The map defines up or down spin at each site of the lattice L . So it can also be thought of as an assignment of $+1$ or -1 on all the sites of the lattice L . We often denote the spin at each site by its image $s_I := \mathbf{s}(I)$, $I \in L$. Let $S(L)$ be the set of all the spin configurations. Since spin configurations are defined on the $M \times N$ periodic lattice, the number of elements in $S(L)$ is 2^{MN} . In other words, the number of all the allowed configurations of spins on the lattice L is 2^{MN} .

For each spin configuration $\mathbf{s} \in S(L)$, define the energy functional of the Ising model by

$$\begin{aligned} E_{\text{Ising}}(\mathbf{s}) &= -J \sum_{\langle I, I' \rangle} s_I s_{I'} \\ &:= -J \left(\sum_{i,j=1}^{M,N} s_{ij} s_{i,j+1} + \sum_{i,j} s_{ij} s_{i+1,j} \right), \end{aligned} \quad (2.37)$$

⁸For sure, for any field theories when one computes a physical quantity on a computer, one needs to discretize the theory and put it on a lattice. In this sense, numerical analysis of field theory is always regarded as a lattice model.

where the $s_{ij} = s_{i+M,j} = s_{i,j+N}$, and $J \in \mathbb{R}_{>0}$ is a constant parameter. This is one of the simplest spin systems, in which only the nearest neighbor spins have interactions. From this energy functional, the partition function of the Ising model is defined by

$$Z(L, E_{\text{Ising}}; \beta) := \sum_{\mathbf{s} \in S(L)} e^{-\beta E_{\text{Ising}}(\mathbf{s})}, \quad (2.38)$$

where $\beta \in \mathbb{R}_{\geq 0}$ is an inverse temperature. The summand is generically called the Boltzmann weight as well as in field theory. In this expression, one recognizes that the right-hand side is a discrete version of path integral expression in field theory. The energy functional is corresponding to the action functional of a field theory, β^{-1} is the Planck's constant, and the sum over all the spin configurations is the path integral over all the field configurations. Given a periodic lattice L , the partition function returns a number, which is a discrete version of the partition function of QFT which returns a number as well if a closed manifold M is given. For later use, define another quantity called the free energy,

$$f(\beta) := -\frac{1}{\beta} \frac{1}{MN} \log Z(\beta), \quad (2.39)$$

where $Z(\beta)$ is the partition function defined above.

According to the general theory of statistical mechanics, the probability that a configuration \mathbf{s} with energy $E(\mathbf{s})$ will be realized is given by the canonical ensemble⁹

$$p(\mathbf{s}) = \frac{1}{Z} e^{-\beta E(\mathbf{s})}, \quad (2.40)$$

where Z is the partition function introduced above. In lattice model, a physical observable is in general given by a functional on the space of spin configurations:

$$\mathcal{O} : S(L) \longrightarrow \mathbb{R}, \quad (2.41)$$

and its expectation value is given by the canonical ensemble as

$$\langle \mathcal{O} \rangle_p := \sum_{\mathbf{s} \in S(L)} \mathcal{O}(\mathbf{s}) p(\mathbf{s}) = \frac{1}{Z} \sum_{\mathbf{s} \in S(L)} \mathcal{O}(\mathbf{s}) e^{-\beta E(\mathbf{s})}. \quad (2.42)$$

In particular, an energy functional is an example of physical observable,

$$\frac{1}{MN} \langle E \rangle_p = \frac{1}{MN} \frac{1}{Z} \sum_{\mathbf{s} \in S(L)} E(\mathbf{s}) e^{-\beta E(\mathbf{s})}, \quad (2.43)$$

which is obtained from the derivative of the free energy,

$$\frac{1}{MN} \langle E \rangle_p = \frac{\partial}{\partial \beta} (\beta f(\beta)). \quad (2.44)$$

⁹To the end of this subsection, we argue for a general energy functional, but the reader may assume the Ising model.

Generally speaking, the free energy provides us all the information of the system. For example, the state most likely to happen is given by the critical point of the free energy, and we can in principle compute physical quantities such as expectation value of energy, fluctuation, specific heat, and so on. To make a lattice model as a physically meaningful system, however, one needs to take the thermodynamic limit; $M, N \rightarrow \infty$. Therefore, in this sense, integrability of lattice model is characterized by the calculability of an exact free energy at the thermodynamic limit. This may be rephrased by the calculability of the partition function as well. Based on these argument, we shall see the integrability of the Ising model in the next subsection.

2.2.2 Transfer matrix and integrability

It is well known that 1d and 2d Ising model is exactly solvable in the sense that one can exactly compute its free energy in the thermodynamic limit. Both 1d and 2d cases are solved by so-called the method of transfer matrix. To see this, for simplicity in this section we consider 1d Ising model.

The setup is almost the same as in the 2d case. A 1d lattice on torus and a spin configuration is defined by

$$L_{1d} = \mathbb{Z}_N = \{1, \dots, N\}, \quad (2.45)$$

$$\mathbf{s} : L_{1d} \longrightarrow \{\pm 1\}. \quad (2.46)$$

Define the energy functional of 1d Ising model by

$$E_{1d \text{ Ising}}(\mathbf{s}) = -J \sum_{i=1}^N s_i s_{i+1}, \quad s_{N+1} = s_1. \quad (2.47)$$

Then the partition function of 1d Ising model is given as

$$\begin{aligned} Z(L_{1d}, E_{1d}; \beta) &:= \sum_{\mathbf{s}} e^{-\beta E_{1d}(\mathbf{s})} \\ &= \sum_{s_1, \dots, s_N = \pm 1} e^{K s_1 s_N} \dots e^{K s_3 s_2} e^{K s_2 s_1}, \end{aligned} \quad (2.48)$$

where $K := \beta J$.

Let us now introduce the *transfer matrix*,

$$T := \left(e^{K s s'} \right)_{s, s' = \pm 1} = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix}, \quad (2.49)$$

where the indices of row and column are specified by $+1$ and -1 , respectively. Using this matrix T , the partition function is rewritten as

$$Z(\beta) = \sum_{s_1, \dots, s_N = \pm 1} T_{s_1 s_N} \dots T_{s_3 s_2} T_{s_2 s_1}. \quad (2.50)$$

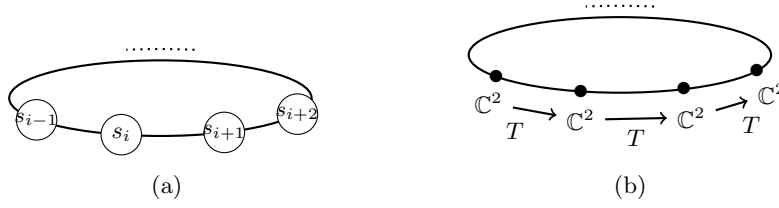


Figure 3: (a) Classical Ising spin chain. (b) Quantum states at each site.

By definition of matrix multiplication, the partition function is eventually given by a trace:

$$\begin{aligned} Z(\beta) &= \sum_{s_1 = \pm 1} (T^N)_{s_1 s_1} \\ &= \text{Tr}_{\mathbb{C}^2} (T^N). \end{aligned} \quad (2.51)$$

This expression tells us that we have the quantum Hilbert space \mathbb{C}^2 at the boundary of each 1d segment, and the transfer matrix T sends a state to the adjacent site, which is the discrete “time evolution” of this system; $\log T \propto \text{Hamiltonian}$ as we saw in 1d QFT in section (what is QFT?). This corresponds to the analogue of Hamiltonian or operator formalism in field theory. In operator formalism, spins are replaced by the Pauli matrices, and original classical up spin and down spin are replaced by the eigenvalues and eigenvectors of the Pauli matrix σ^z . Further, the non-commutativity is now manifest in a way that the spins and their time evolution is given by matrices, this is the consequence of quantization.

In the expression above of the partition function, let us consider the thermodynamic limit $N \rightarrow \infty$. The eigenvalues of T are easily obtained and let them be $\lambda_0 > \lambda_1$, then we have the free energy

$$\begin{aligned} -\beta f(\beta) &= \lim_{N \rightarrow \infty} \frac{1}{N} \log Z(\beta) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \log \lambda_0 \left(1 + \left(\frac{\lambda_1}{\lambda_0} \right)^N \right) \\ &= \log \lambda_0. \end{aligned} \quad (2.52)$$

We conclude that the free energy in the thermodynamic limit is just given by the largest eigenvalue of the transfer matrix. Again, integrability of lattice model is rephrased as

One can exactly find the eigenvalues of transfer matrix
 \Rightarrow Can exactly compute the free energy
 \Rightarrow The system is integrable.

The transfer matrices of 1d and 2d Ising model are diagonalizable and one can exactly find their eigenvectors and eigenvalues using, for example, algebraic Bethe ansatz. In this sense,

1d and 2d Ising model are said to be integrable, or exactly solvable. Then, a natural question arises; when can we diagonalize the transfer matrix of a lattice model? This actually leads to the most fundamental answer to the question of “what is integrable model.” A canonical answer is *Yang-Baxter equation*. When the Boltzmann weight with spectral parameters, which take values in a Riemann surface, satisfies the Yang-Baxter equation, the lattice model is integrable. We will give an explanation of this definition of integrability in next subsection in some detail, with its origin from TQFT with extra dimensions.

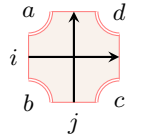
2.3 Integrability from TQFT in extra dimensions

In this section we provide a general discussion to relate four-dimensional supersymmetric gauge theories and integrable lattice models. We clarify that such a correspondence between gauge theories and lattice models emerges from a TQFT in extra dimensions. We first give a step-by-step explanation from two-dimensional TQFT to integrable lattice model, and then put the argument in the setup of brane tilings. Branes in string theory are powerful enough to yield the systematic method to relate a class of supersymmetric gauge theories with integrable lattice models, as we will see. To begin with, we present a general prescription of brane constructions at a formal level. Concrete setups and more details will be given in later sections. The discussion in this subsection is mainly based on [15].

2.3.1 Lattice models from TQFTs with line operators

Suppose that we have a two-dimensional TQFT \mathcal{T} on a two-torus T^2 equipped with line operators \mathcal{L}_i , $i = 1, \dots, l$. Wrap them around one-cycles C_i on the torus in such a way that they form an $m \times n$ lattice. We wish to consider the correlation function of this configuration of line operators. To compute it, our strategy is to break up the torus into pieces and perform in turn the path integral piecewise. Then the original correlation function is reconstructed by combining these pieces.¹⁰

Let us consider the following piece of surface which contains an intersection of two line operators \mathcal{L}_i and \mathcal{L}_j :


(2.53)

If one thinks of the above picture as the world-sheet of two scattering open strings, one can regard the above picture as follows: The endpoints of the open strings sweep out the double-lined arcs, on which D-branes are there and the boundary conditions (Chan-Paton

¹⁰In the following, we will compute the correlation function by embedding the closed TQFT with line operators into an open/closed TQFT with line operators, or defect TQFT; see e.g. [16, 17].

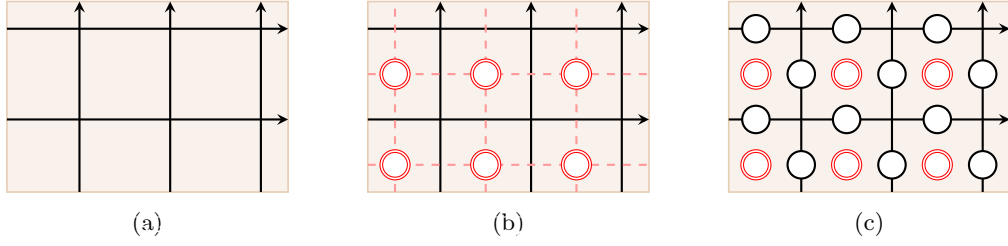


Figure 4: Construction of a lattice model from line operators. (a) A lattice of line operators on a torus. (b) The torus with holes obtained by gluing of pieces. (c) The corresponding spin model.

factors) are specified by labels a, b, c, d . The line operators are the world-lines of particles associated with the open strings.

The path integral for the piece (2.53) produces a linear map

$$\begin{array}{c} a \\ \uparrow \\ i \text{---} \text{---} d \\ \downarrow \\ b \end{array} \rightsquigarrow R_{ij} \begin{pmatrix} a & d \\ b & c \end{pmatrix} : V_{ab,i} \otimes V_{bc,j} \longrightarrow V_{ad,j} \otimes V_{dc,i}, \quad (2.54)$$

where $V_{ab,i}$ is the space of states of the open string propagating along \mathcal{L}_i with the boundary conditions on the left and the right ends specified by a, b , respectively. We call this map the R-matrix (or R-operator) associated with this decorated surface. To reconstruct the original configuration of line operators, we glue pieces together. Gluing them amounts to the composition of R-matrices. For example, gluing two pieces horizontally gives

$$\begin{array}{c} a \quad d \quad f \\ \uparrow \quad \uparrow \\ i \text{---} \text{---} \text{---} \\ \downarrow \quad \downarrow \\ b \quad c \quad e \end{array} \rightsquigarrow R_{ik} \begin{pmatrix} d & e \\ c & f \end{pmatrix} \circ_{V_{dc,i}} R_{ij} \begin{pmatrix} a & d \\ b & c \end{pmatrix}. \quad (2.55)$$

The original configuration is thus obtained by gluing all the pieces. It, however, still has holes assigned boundary conditions as in figure 4(b), which we must fill by summing over the boundary conditions. To do this, let us recall that the path integral on a finite-length cylinder with boundary condition a imposed on one end defines a closed string state $|a\rangle$ as known as a boundary state. Similarly, the path integral on a disk with no insertion of operators defines a state $|1\rangle$ on the boundary. Assume that we have chosen the set of boundary conditions B to be sufficiently large so that the state $|1\rangle$ can be written as a superposition of boundary states:

$$|1\rangle = \sum_{a \in B} c_a |a\rangle. \quad (2.56)$$

Then, the sum over the boundary conditions gives the state $|1\rangle$ on the boundary of each

hole, which is replaced with a disk:

$$\sum_{a \in B} c_a \left(a \text{ (disk with two holes)} \right) = \sum_{a \in B} c_a \left(|a\rangle \text{ (disk with one hole)} \right) = |1\rangle \text{ (disk)} = \text{(disk with one hole)} . \quad (2.57)$$

Thus the holes are filled and the original configuration on the torus is reconstructed.

Now that we have understood how to reconstruct the correlation function of line operators in the field theoretic point of view, let us reinterpret this procedure as an operation in statistical lattice model. Rephrasing the procedures so far, it is clear that the above configuration defines a partition function of a statistical spin model. The model has spins located at two kinds of sites, \bigcirc and \odot . They correspond to the open string states and the boundary states. A spin at \bigcirc takes values in the chosen basis for the relevant open string states $V_{ab,i}$, while that at \odot is valued in B . The Boltzmann weights for each spin configuration are given by the matrix elements of R-matrix and the coefficients of boundary states c_a . Thus, we conclude that the correlation function of line operators $\{\mathcal{L}_i\}$ on the torus coincides with the partition function of a statistical spin model defined on the periodic lattice formed by the line operators:

$$\left\langle \prod_{i=1}^l \mathcal{L}_i(C_i) \right\rangle_{\mathbf{T}, T^2} = Z_{\mathbf{L}(\mathbf{T}), \{\mathcal{L}_i(C_i)\}}. \quad (2.58)$$

$\mathbf{L}(\mathbf{T})$ denotes the lattice model arising from the TQFT \mathbf{T} with the line operators and $\{\mathcal{L}_i(C_i)\}$ is the lattice formed by line operators \mathcal{L}_i wrapping around C_i . By this construction, we can view typical examples of lattice models.

IRF model

If $\dim V_{ab,i} = 1$ for any a, b, i , in this case we can ignore the spins at \bigcirc . We just sum over the boundary conditions and such a spin system $\mathbf{L}(\mathbf{T})$ is called *interaction-round-a-face model*, or *IRF model* for short. The spins are placed on the faces of the lattice, and interaction takes place among four spins surrounding a vertex. The 2d Ising model actually can be formulated as an IRF model by taking vertex-face correspondence.

Vertex model

If B consists of a single boundary condition, say a , we can simply write the open string state space by $V_i := V_{aa,i}$, and the R-matrix is represented only by a crossing of two lines:

$$R_{ij} = i \begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \\ j \end{array} , \quad (2.59)$$

$$R_{ij} : V_i \otimes V_j \longrightarrow V_j \otimes V_i . \quad (2.60)$$

The space of states V_i can also be thought of as the Hilbert space of a particle propagating on the line operator \mathcal{L}_i . In this case we can ignore the spins at \odot since there is no summation over boundary conditions. This means that the lattice model $\mathbf{L}(\mathbf{T})$ is called a *vertex model*: Spins are living on the edges of the lattice and interact with each other at the vertices.

We can always recast our lattice model into a vertex model by setting $V_i := \bigoplus_{a,b \in B} V_{ab,i}$, at least formally, and declaring that all newly introduced R-matrix elements, which correspond to scattering processes with inconsistent Chan-Paton factors, vanish. We can also absorb the coefficients c_a into the R-matrix elements by appropriate rescaling. In what follows we implicitly perform this reformulation, and restrict ourselves to vertex model.

A remarkable aspect of this construction of lattice models is that it allows us to understand integrability from a higher-dimensional point of view. This is crucial observation by Costello [18]. In our lattice model, consider a horizontal line operator \mathcal{L}_i intersecting vertical line operators \mathcal{L}_j , $j = 1, \dots, n$. Concatenating the R-matrices in this row, we get the row-to-row transfer matrix:

$$T_i = \underset{1}{\overset{i}{\curvearrowright}} \underset{2}{\overset{\uparrow}{\mid}} \cdots \cdots \underset{n}{\overset{\uparrow}{\mid}} = \text{Tr}_{V_i} (R_{i,n} \circ_{V_i} \cdots \circ_{V_i} R_{i,1}). \quad (2.61)$$

The hooks on the horizontal line are to remind us that the periodic boundary condition is imposed. This gives an endomorphism of $\bigotimes_{j=1}^n V_j$ which maps a state just below \mathcal{L}_j to another state above it, namely the transfer matrix defines the time evolution of this spin system and $\bigotimes_{j=1}^n V_j$ is the quantum Hilbert space. In terms of transfer matrices, the partition function is written as a trace,

$$Z_{\mathbf{L}(\mathbf{T}), \{\mathcal{L}_i(C_i)\}} = \text{Tr}_{\mathcal{H}} (T_{n+m} \cdots T_{n+1}), \quad (2.62)$$

where the trace is taken over the total Hilbert space $\mathcal{H} := \bigotimes_{j=1}^n V_j$. An important point is that the underlying field theory is a TQFT, and hence a state evolves trivially on a cylinder unless it hits something, line operators in the present case.

Now suppose that each line operator depends on a continuous parameter which is an element of some smooth manifold S . This parameter is called *spectral parameter* of the lattice model. We denote the spectral parameter of \mathcal{L}_i by u_i , then the R-matrix and the transfer matrix are rewritten by

$$R_{ij} \longrightarrow R_{ij}(u_i, u_j), \quad (2.63)$$

$$T_i \longrightarrow T_i(u_i; u_1, \dots, u_n). \quad (2.64)$$

To avoid clutter, we fix u_1, \dots, u_n and suppress them below. A vertex model is said to be integrable if the transfer matrices $T_i(u_i)$ are meromorphic functions of u_i and commute with

each other:

$$\begin{array}{c} j \uparrow \uparrow \cdots \uparrow \\ \uparrow \downarrow \downarrow \cdots \downarrow \\ i \uparrow \uparrow \cdots \uparrow \end{array} = \begin{array}{c} i \uparrow \uparrow \cdots \uparrow \\ \uparrow \downarrow \downarrow \cdots \downarrow \\ j \uparrow \uparrow \cdots \uparrow \end{array} \iff [T_i(u_i), T_j(u_j)] = 0, \quad u_i \neq u_j. \quad (2.65)$$

When the transfer matrices with different spectral parameters commute, we can find a series of mutually commuting operators on the total Hilbert space $\bigotimes_{j=1}^n V_j$ by Laurent expansion of the transfer matrix. In particular, they commute with the transfer matrix, which means they produce an infinite number of conserved quantities. In fact, the commutativity of transfer matrix implies that one can find the exact eigenvalues and eigenvectors of the transfer matrix, as we saw in the Ising model in the previous section.

The situation considered here, namely the addition of spectral parameters and commuting transfer matrix, is naturally realized if the TQFT has “extra dimensions.” In this scenario, we really start with a higher-dimensional theory $\tilde{\mathbb{T}}$ formulated on the product space $S \times T^2$, which is topological on the torus T^2 but not on S . We wrap line operators \mathcal{L}_i around $\{u_i\} \times C_i$, where u_i are points on S . If one can see only the torus T^2 and is unaware of the extra dimensions S , the theory seems to be the previous two-dimensional TQFT $\mathbb{T} \cong \tilde{\mathbb{T}}[S]$ that has parameters taking values in S . One finds that line operators $\mathcal{L}_i[u_i]$ wrapping around C_i carry continuous parameters u_i in the seemingly two-dimensional theory, and the correlation function for this configuration of line operators is given by the partition function of a lattice model $\mathbb{L}(\tilde{\mathbb{T}}[S])$ defined on the lattice $\{\mathcal{L}_i[u_i](C_i)\}$.

For a generic choice of $\{u_i\}$, the transfer matrices of the lattice model $\mathbb{L}(\tilde{\mathbb{T}}[S])$ commute since the two horizontal line operators in the above equation can move freely and interchange their positions owing to the topological nature along T^2 ; no phase transition occurs when they pass each other as they do not meet in the full spacetime $S \times T^2$. Thus, integrability follows from the existence of extra dimensions, whose coordinates provide continuous spectral parameters.

In fact, we can deduce integrability from another point of view. By the same logic, we have the unitarity relation

$$\begin{array}{c} i \curvearrowright \\ j \curvearrowleft \end{array} = \begin{array}{c} i \longrightarrow \\ j \longrightarrow \end{array} \quad (2.66)$$

$$\iff R_{ji}(u_j, u_i) R_{ij}(u_i, u_j) = \text{id}_{V_i \otimes V_j}, \quad (2.67)$$

and the Yang-Baxter equation

$$\begin{array}{c} i \nearrow \\ j \nearrow \\ k \end{array} = \begin{array}{c} i \nearrow \\ j \nearrow \\ k \end{array} \quad (2.68)$$

$$\iff R_{ij}(u_i, u_j) R_{ik}(u_i, u_k) R_{jk}(u_j, u_k) = R_{jk}(u_j, u_k) R_{ik}(u_i, u_k) R_{ij}(u_i, u_j), \quad (2.69)$$

where the R-matrix R_{ij} acts on $V_i \otimes V_j$ as an intertwiner and trivial on V_k , etc. From these two relations we can reproduce the commutativity of the transfer matrix. We should emphasize that the Yang-Baxter equation is a local condition. When the Boltzmann weight locally satisfies the Yang-Baxter equation, it extends to the commutativity of the transfer matrix and hence the integrability of the model. In this sense, the Yang-Baxter equation is the fundamental condition of integrability of a lattice model.

Before going into the discussion of brane construction, we would like to generalize the above arguments further higher-dimensional situations. First of all, replace the two-torus T^2 with a general two-dimensional surface Σ , along which line operators are wrapped. We now have $S \times \Sigma$, and similarly to the above a lattice model is defined on Σ by line operators with spectral parameters. Let us consider the case that we really have more extra dimensions and a higher-dimensional theory is formulated on $S \times M \times \Sigma$, where M is some smooth manifold. The line operators we had may descend from extended operators of dimension greater than one. Let T again be the new higher-dimensional theory, and suppose that it is topological on Σ and has extended operators \mathcal{E}_i whose codimension is greater than $\dim S$. Place \mathcal{E}_i on submanifolds of the form $\{u_i\} \times N_i \times C_i$. Since $\mathsf{T}[S \times M]$ is again regarded as a two-dimensional TQFT defined on Σ , the correlation function of the operators \mathcal{E}_i still should coincide with the partition function of an integrable lattice model $\mathsf{L}(\mathsf{T}[S \times M])$. The model is defined on the lattice formed by the line operators $\mathcal{E}_i[\{u_i\} \times N_i]$, which are the image of \mathcal{E}_i in the two-dimensional theory $\mathsf{T}[S \times M]$.

As well as we can view the higher-dimensional theory as a two-dimensional TQFT, we may also view it as a theory $\mathsf{T}[\Sigma]$, which is a QFT on $S \times M$ specified by the surface Σ . In this theory, the operators \mathcal{E}_i are seen as extended operators $\mathcal{E}_i[C_i]$ supported on $\{u_i\} \times N_i$. Then, we have another relation

$$\left\langle \prod_{i=1}^l \mathcal{E}_i[C_i](\{u_i\} \times N_i) \right\rangle_{\mathsf{T}[\Sigma], S \times M} = Z_{\mathsf{L}(\mathsf{T}[S \times M]), \{\mathcal{E}_i[\{u_i\} \times N_i](C_i)\}}. \quad (2.70)$$

Thus, we finally arrive at a correspondence between a QFT on $S \times M$ equipped with extended operators and an integrable lattice model on Σ .

2.3.2 Brane construction and correspondence

We have seen above that a lattice model is realized by a lattice of line operators in a two-dimensional TQFT, and it is integrable if the TQFT is embedded in higher dimensions and the line operators come from extended operators localized in some directions of the extra dimensions. Now we explain how to get such structures of correspondence of (previous equation) using branes in string theory. The brane construction here is still rather abstract. A little bit more concrete setups will be given in next section. The readers may skip this

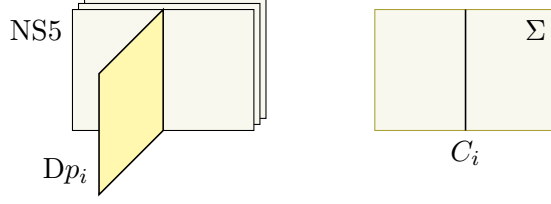


Figure 5: The Dp -brane Dp_i ending on the NS5-branes creates a defect \mathcal{E}_{Dp_i} along C_i .

subsection for the first reading and jump to the next section.

Consider a type II string theory in a ten-dimensional spacetime

$$\mathbb{R}^4 \times T^*\Sigma \times \mathbb{R}^2, \quad (2.71)$$

where Σ is a two-dimensional surface embedded in $T^*\Sigma$ as a zero section. Introduce a stack of N NS5-branes supported on $\mathbb{R}^4 \times \Sigma \times \{0\}$ in this spacetime, and Dp -branes Dp_i on $\mathbb{R}^{p-1} \times \Sigma_i \times \{0\}$ ending on the NS5-branes, where \mathbb{R}^{p-1} is a subspace of \mathbb{R}^4 (assuming $p \leq 5$) and Σ_i are surfaces in $T^*\Sigma$ such that $\Sigma_i \cap \Sigma = C_i$. See figure 5. Provided that Σ_i are suitably chosen, this brane system preserves four supercharges.

The low-energy dynamics of the NS5-branes is governed by a six-dimensional theory T_{NS5} on $\mathbb{R}^4 \times \Sigma$. The theory T_{NS5} is depending on whether IIA or IIB theory we are considering:

IIA : $\mathcal{N} = (2, 0)$ superconformal QFT of type A_{N-1} ,

IIB : $\mathcal{N} = (1, 1)$ super Yang-Mills theory with gauge group $\text{SU}(N)$.

The theory T_{NS5} is formulated on $\mathbb{R}^4 \times \Sigma$, with topological twist along Σ which breaks half of sixteen supercharges. In this twisted theory, Dp_i create p -dimensional defects \mathcal{E}_{Dp_i} on $\mathbb{R}^{p-1} \times C_i$, reducing the number of unbroken supercharges to four. From the point of view of a four-dimensional observer, this brane configuration gives half-BPS defects $\mathcal{E}_{Dp_i}[C_i]$ in an $\mathcal{N} = 2$ theory $\mathsf{T}_{\text{NS5}}[\Sigma]$. The total system is invariant under a $\text{U}(1)$ R-symmetry originating from the rotational symmetry on the \mathbb{R}^2 factor of the ten-dimensional spacetime.

Let us take a three-manifold M and $(p-2)$ -submanifolds N_i of M , and modify the above construction so that the world-volumes of the NS5-branes and the Dp -branes become $S^1 \times M \times \Sigma$ and $S^1 \times N_i \times \Sigma_i$, respectively. At low energies, we get the same theory T_{NS5} formulated on $S^1 \times M \times \Sigma$ and defects \mathcal{E}_{Dp_i} located on $S^1 \times N_i \times C_i$. In general, this modification completely breaks supersymmetry. For certain choices of M and N_i , however, there is a string background in which a fraction of supersymmetry is still preserved. In such a background, the path integral computes the *supersymmetric index* of T_{NS5} , defined with respect to the Hilbert space on $M \times \Sigma$ in the presence of defects \mathcal{E}_{Dp_i} inserted on $N_i \times C_i$.

A salient feature of supersymmetric indices is that they are protected against continuous changes of various parameters of the theory. This means that the index of our theory is

invariant under deformations of the geometric data of Σ and C_i , namely the metric on Σ and the shapes of C_i . In other words, the theory T_{NS5} on $S^1 \times M \times \Sigma$ is topological on Σ , as far as the computation of the index is concerned.

To relate the present setup to the one considered in the previous subsection, we apply T-duality along S^1 . It turns Dp_i into $D(p-1)$ -branes $D(p-1)_i$, localized at points u_i along the dual circle \tilde{S}^1 , while sending the NS5-branes to those in the other type II string theory. The new NS5-branes produce the dual six-dimensional theory $\tilde{\mathsf{T}}_{\text{NS5}}$ on $\tilde{S}^1 \times M \times \Sigma$, and in this theory $D(p-1)_i$ create $(p-1)$ -dimensional defects $\mathcal{E}_{D(p-1)_i}$ on $\{u_i\} \times N_i \times C_i$. Thus we are in the situation studied in the last subsection, and the correlation function for this configuration coincides with the partition function of an integrable lattice model:

$$\left\langle \prod_{i=1}^l \mathcal{E}_{Dp_i}[C_i](S^1 \times N_i) \right\rangle_{\mathsf{T}_{\text{NS5}}[\Sigma], S^1 \times M} = Z_{\mathsf{L}}(\tilde{\mathsf{T}}_{\text{NS5}}[\tilde{S}^1 \times M], \{\mathcal{E}_{D(p-1)_i}[\{u_i\} \times N_i](C_i)\}). \quad (2.72)$$

Here the left-hand side is expressed in the original frame; it implicitly depends on each spectral parameter u_i through the holonomy $\exp(2\pi i u_i)$ around S^1 of the gauge field for the flavor symmetry $U(1)_i$ supported on Dp_i . The holonomy appears in the index as a refinement parameter, or *fugacity*, associated with $U(1)_i$.

2.3.3 Defects as transfer matrices

Finally, we apply the construction developed in the previous subsections to the main theme of this paper; integrable lattice models and defects as transfer matrices. Let us consider $p=5$ case of the brane construction in the last subsection. To conform with the standard convention, take S-duality first and we still have D5- and NS5-branes

$$\begin{aligned} \text{ND5} \quad & S^1 \times M \times \Sigma, \\ \text{NS5}_i \quad & S^1 \times N_i \times \Sigma_i, \end{aligned}$$

where NS5_i create defects $\mathcal{E}_{\text{NS5}_i}$ on $S^1 \times N_i \times C_i$.

One should notice that in this setup we necessarily have $N_i = M$ and thus the defects $\mathcal{E}_{\text{NS5}_i}$ fill the whole $S^1 \times M$, which produces a four-dimensional theory

$$\mathsf{T}_{\text{D5NS5}}[\Sigma]$$

with $\mathcal{N} = 1$ supersymmetry. Then we now have

$$\langle 1 \rangle_{\mathsf{T}_{\text{D5NS5}}[\Sigma], S^1 \times M} = Z_{\mathsf{L}}(\mathsf{T}_{\text{D5NS5}}[S^1 \times M], \{\mathcal{E}_i[S^1 \times M](C_i)\}). \quad (2.73)$$

For example, when $M = S^3$, the left-hand side is given by the supersymmetric index for $\mathcal{N} = 1$ theory and the right-hand side corresponds to the partition function of Bazhanov-Sergeev

integrable lattice model [19, 20, 21, 22]. When $M = L(p, 1)$, lens space, the left-hand side is computed in [23], which defines a new integrable lattice model through this correspondence.

The brane tiling construction of integrable lattice models can be enriched by introduction of additional defects. Besides the previously defined D5NS5-brane system, introduce a D3-brane such as

$$\text{D3} \quad S^1 \times N \times C \times \mathbb{R}_+, \quad (2.74)$$

$$\text{D3}' \quad S^1 \times \{0\} \times C' \times \mathbb{R}^2, \quad (2.75)$$

where N is a curve in M and C, C' are 1-cycles on Σ . A single D3-brane insertion corresponds to a new oriented line in the integrable lattice model, which we represent by a dashed line. Now that we have two kinds of lines, we can define three R-matrices:

$$R = \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array}, \quad L = \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \rightarrow, \quad \mathcal{R} = \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \rightarrow. \quad (2.76)$$

The middle one is usually called *L-operator*. Correspondingly, we have four Yang-Baxter equations, involving zero to three dashed lines. Those that involving one or two dashed line,

$$\begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} = \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} = \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \quad (2.77)$$

are called *RLL relations*. The effect of the insertion of such an additional defect on the lattice model is seen in terms of the L-operator. The neighborhood of the dashed line looks like

$$\begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \dots \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ | \\ \text{---} \end{array}. \quad (2.78)$$

This diagram shows that the defect acts on the lattice model by a transfer matrix constructed from L-operators. Thus, the insertion of a defect operator in four-dimensional theory is mapped into lattice model side as the action of a transfer matrix constructed from L-operators.

What we would like to discuss in detail in the subsequent sections are the introduction of a single D3-brane (equation of D3D3') to the D5NS5-brane system, and clarify that the correspondence with the additional defects. In the next two sections, we are considering the followings:

1. For the case of single D3, let $M = S^3$, $\Sigma = T^2$, and $N = S^1$. Then the D3 creates a surface defect on $S^1 \times S^1$ and it acts on supersymmetric index as a transfer matrix in the corresponding lattice model:

$$\langle 1 \rangle_{\text{T}_{\text{D5NS5}}[\Sigma], S^1 \times S^3} = \mathcal{I}_{S^1 \times S^3}(p, q, t), \quad (2.79)$$

$$Z_{\text{L}(\text{T}_{\text{D5NS5}}[S^1 \times S^3]), \{\mathcal{E}_i[S^1 \times S^3](C_i)\}} = Z_{\text{Bazhanov-Sergeev}}, \quad (2.80)$$

the surface defect index is represented as a difference operator acting on the original supersymmetric index [24, 25],

$$\langle S_{(r,s)} \rangle_{\mathsf{T}_{\mathrm{D5NS5}}[\Sigma]} = \mathfrak{S}_{(r,s)} \mathcal{I}_{S^1 \times S^3}(p, q, t). \quad (2.81)$$

2. For the case of single D3', let $M = \mathbb{R}^3$ and $\Sigma = T^2$. D3' creates a line defect wrapping S^1 and it acts on quantum Hilbert space of a spin chain as a transfer matrix, since now we have non-compact three-manifold \mathbb{R}^3 . The line defect in the four-dimensional theory is realized as a Wilson-'t Hooft line operator T , and its magnetic and electric charge (\mathbf{m}, \mathbf{e}) is specified by the one-cycle C' wrapping the torus T^2 . As it turns out, the vacuum expectation values (vevs) of Wilson-'t Hooft lines realize the deformation quantization of the Hitchin moduli space, and are naturally quantized by the Weyl quantization:

$$\text{Weyl quantization of } \langle T_{(\mathbf{m}, \mathbf{e})} \rangle = \text{trigonometric transfer matrix.} \quad (2.82)$$

3 Surface defects as transfer matrices

In this section, we mainly review the results in [Maruyoshi-Yagi], in which the correspondence we introduce in the previous section appears.

3.1 Brane tilings and integrable lattice models

3.1.1 Supersymmetric index and integrable lattice models

3.1.2 Integrable lattice models from quiver gauge theories

3.2 Surface defects in A_1 theories of class \mathcal{S}

3.2.1 Surface defects and L-operators

3.2.2 Surface defects as transfer matrices

4 Line defects as transfer matrices

4.1 Integrable lattice models of elliptic and trigonometric type

In this section we discuss the integrable system side of the correspondence. After reviewing L-operators, transfer matrices and their relation to quantum integrable systems, we introduce an L-operator for the elliptic dynamical R-matrix. Then we define fundamental trigonometric L-operators as certain limits of the elliptic L-operator. These fundamental L-operators are building blocks of transfer matrices that correspond to Wilson-'t Hooft lines in $\mathcal{N} = 2$ supersymmetric circular quiver theories.

4.1.1 L-operator and quantum integrable system

Let \mathfrak{h} be a finite-dimensional commutative complex Lie algebra and V a finite-dimensional diagonalizable \mathfrak{h} -module. Choosing a basis $\{v_i\}$ of V that is homogeneous with respect to weight decomposition, we denote the weight of v_i by h_i and the (i, j) th entry of a matrix $M \in \text{End}(V)$ by M_j^i . We write $\mathcal{M}_{\mathfrak{h}^*}$ for the field of meromorphic functions on the dual space \mathfrak{h}^* of \mathfrak{h} .

Let $R: \mathbb{C} \times \mathfrak{h}^* \rightarrow \text{End}(V \otimes V)$ be an $\text{End}(V \otimes V)$ -valued meromorphic function on $\mathbb{C} \times \mathfrak{h}^*$ that is invertible at a generic point $(z, a) \in \mathbb{C} \times \mathfrak{h}^*$. The coordinate z is called the *spectral parameter* and a is called the *dynamical parameter*.

In the discussions that follow, fundamental roles will be played by L-operators. By an *L-operator* for R , we mean a map $L: \mathbb{C} \rightarrow \text{End}(V \otimes \mathcal{M}_{\mathfrak{h}^*} \otimes \mathcal{M}_{\mathfrak{h}^*})$, which we think of as a matrix whose entries are linear operators on meromorphic functions on $\mathfrak{h}^* \times \mathfrak{h}^*$. It must satisfy two conditions.

First, its matrix elements act on $f \in \mathcal{M}_{\mathfrak{h}^*} \otimes \mathcal{M}_{\mathfrak{h}^*}$ as

$$L(z)_i^j f(a^1, a^2) = L(z; a^1, a^2)_i^j \Delta_i^1 \Delta_j^2 f(a^1, a^2), \quad (4.1)$$

where $L(z; a^1, a^2)_i^j$ is a meromorphic function on $\mathbb{C} \times \mathfrak{h}^* \times \mathfrak{h}^*$ and Δ_i^1, Δ_j^2 are difference operators such that

$$\Delta_i^1 f(a^1, a^2) = f(a^1 - \epsilon h_i, a^2), \quad \Delta_j^2 f(a^1, a^2) = f(a^1, a^2 - \epsilon h_j). \quad (4.2)$$

Here ϵ is a fixed complex parameter.

Second, the L-operator satisfies the *RLL relation*

$$\begin{aligned} \sum_{k,l} R(z - z', a^2)_{kl}^{mn} L(z; a^1, a^2)_i^k L(z'; a^1 - \epsilon h_i, a^2 - \epsilon h_k)_j^l \\ = \sum_{k,l} L(z'; a^1, a^2)_l^n L(z; a^1 - \epsilon h_l, a^2 - \epsilon h_n)_k^m R(z - z', a^1)_{ij}^{kl}. \end{aligned} \quad (4.3)$$

Equivalently, the operator relation

$$\sum_{k,l} R(z - z', a^2)_{kl}^{mn} L(z)_i^k L(z')_j^l = \sum_{k,l} R(z - z', a^1)_{ij}^{kl} L(z')_l^n L(z)_k^m \quad (4.4)$$

holds on any meromorphic function $f(a^1, a^2)$.

It is helpful, and will turn out to be physically meaningful, to represent the L-operator graphically as two crossing oriented line segments:

$$L(z) = z \begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} \quad . \quad (4.5)$$

The solid line extending in the horizontal direction has a spectral parameter. The graphical representation of a matrix element of the L-operator is

$$L(z; a^1, a^2)_i^j = \begin{array}{c} \begin{array}{ccc} & a^1 & a^2 \\ & \circ & \circ \\ z \text{ --- } & i & j \text{ --- } \end{array} \\ \begin{array}{ccc} & a^1 - \epsilon h_i & a^2 - \epsilon h_j \end{array} \end{array} . \quad (4.6)$$

Each edge of a solid line carries a state in V , and the state may change when the line crosses another line. To each region separated by lines, a dynamical parameter is assigned. The values of dynamical parameters on the two sides of a solid line carrying state v_i differ by ϵh_i .

We also represent the operator R as two crossing solid lines:

$$R(z - z', a)_{ij}^{kl} = \begin{array}{c} \uparrow \\ a \quad \textcircled{l} \\ \textcircled{i} \quad \textcircled{k} \\ \downarrow \\ z' \quad \textcircled{j} \end{array} \quad . \quad (4.7)$$

Then, the RLL relation (4.3) simply means an equality between two configurations involving two solid and one double lines:

$$\begin{array}{c} \nearrow a^1 \\ \downarrow z \\ \nwarrow a^2 \\ \nearrow z' \end{array} = \begin{array}{c} \nwarrow a^1 \\ \downarrow z \\ \nearrow a^2 \\ \nwarrow z' \end{array} . \quad (4.8)$$

The states carried by the internal solid edges are summed over.

By comparing the values of the dynamical parameter assigned to the lower right regions of the two sides, we see that for R to satisfy the RLL relation, it must commute with $h \otimes 1 + 1 \otimes h$ for all $h \in \mathfrak{h}$; in other words, $R(z, a)_{ij}^{kl} = 0$ unless $h_i + h_j = h_k + h_l$. This is a consistency condition for the rule that determines how dynamical parameters change across solid lines.

Associated with an L-operator, there is an integrable quantum mechanical system consisting of particles moving in the space \mathfrak{h}^* . The Hilbert space of each particle is $\mathcal{M}_{\mathfrak{h}^*}$. (This is quantum mechanics in which real variables are analytically continued to complex ones.) The Hilbert space of the system is $\mathcal{M}_{\mathfrak{h}^*}^{\otimes n}$ if n is the number of particles.

To construct this system, define the *monodromy matrix* $M: \mathbb{C} \rightarrow \text{End}(V \otimes \mathcal{M}_{\mathfrak{h}^*}^{\otimes n+1})$ by the product of n copies of the L-operator: its matrix elements are given by

$$M(z)_{i^1}^{i^{n+1}} = \sum_{i^2, \dots, i^n} \prod_{r=1}^n L(z; a^r, a^{r+1})_{i^r}^{i^{r+1}} \prod_{s=1}^{n+1} \Delta_{i^s}^s, \quad (4.9)$$

acting on any meromorphic function $f(a^1, \dots, a^{n+1})$. (The superscript on Δ_i specifies the variable on which the difference operator acts.) This is a solid line crossing n double lines:

$$M(z) = z \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \text{---} \quad \text{---} \quad \text{---} \quad \rightarrow \\ \text{.....} \end{array} . \quad (4.10)$$

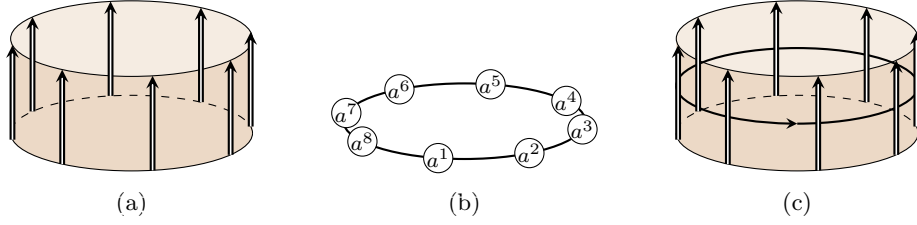


Figure 6: (a) Double lines in the longitudinal direction of a cylinder. (b) The corresponding quantum spin chain with continuous spin variables. (c) A solid line winding around the cylinder acts on the spin chain by the transfer matrix.

Identifying $a^{n+1} = a^1$ and taking the trace, one obtains the *transfer matrix* $T: \mathbb{C} \rightarrow \text{End}(\mathcal{M}_{\mathfrak{h}^*}^{\otimes n})$:

$$T(z) = \sum_{i^1, \dots, i^n} \prod_{r=1}^n L(z; a^r, a^{r+1})_{i^r}^{i^{r+1}} \prod_{s=1}^n \Delta_{i^s}^s, \quad i^{n+1} = i^1. \quad (4.11)$$

Graphically, $T(z)$ is represented by the same picture as above but with the horizontal direction made periodic.

By construction, T is an $\text{End}(\mathcal{M}_{\mathfrak{h}^*}^{\otimes n})$ -valued meromorphic function. As such, each coefficient T_m in the Laurent expansion $T(z) = \sum_{m \in \mathbb{Z}} T_m z^m$ is an operator acting on the Hilbert space $\mathcal{M}_{\mathfrak{h}^*}^{\otimes n}$. Then, one may pick a particular linear combination of these coefficients and declare that it is the Hamiltonian of the quantum mechanical system. The Hamiltonian thus obtained is a difference operator, which is typical of relativistic systems.

Alternatively, one may think of this system as a one-dimensional periodic quantum spin chain. This spin chain is constructed from n double lines extending in the longitudinal direction of a cylinder, as shown in figure 6(a). The dynamical parameter a^r resides in the region sandwiched by the r th and the $(r+1)$ th double lines. One regards the n dynamical parameters a^1, \dots, a^n as continuous spin variables; see figure 6(b). Thinking of the longitudinal direction as the time direction, the Hilbert space of the spin chain is again $\mathcal{M}_{\mathfrak{h}^*}^{\otimes n}$. An action of $T(z)$ on the Hilbert space is induced by an insertion of a solid line with spectral parameter z in the circumferential direction of the cylinder, as in figure 6(c).

The integrability of the system is a consequence of the RLL relation. By repeated use of the RLL relation, one deduces that the monodromy matrix satisfies a similar relation:

$$\sum_{k,l} R(z - z', a^{n+1})_{kl}^{mn} M(z)_i^k M(z')_j^l = \sum_{k,l} R(z - z', a^1)_{ij}^{kl} M(z')_l^n M(z)_k^m. \quad (4.12)$$

Multiplying both sides by $R^{-1}(z - z', a^1)_{mn}^{ij}$, setting $a^{n+1} = a^1$ and summing over i, j, m, n , one finds

$$T(z)T(z') = T(z')T(z). \quad (4.13)$$

In other words, transfer matrices at different values of the spectral parameter commute. It follows that the Laurent coefficients $\{T_m\}$ mutually commute and, in particular, commute with the Hamiltonian. Hence, the system has a series of commuting conserved charges.

There is a slight generalization of the above construction of commuting transfer matrices. Suppose that $g \in \text{End}(V)$ satisfies

$$(g \otimes g)R(z, a) = R(z, a)(g \otimes g) \quad (4.14)$$

and a subspace W of V is invariant under R , R^{-1} , L and g . (For instance, the invariance of W under R means that $R(z, a)(W \otimes V) \subset W \otimes V$ and $R(z, a)(V \otimes W) \subset V \otimes W$ for all z, a .) Then, the trace can be twisted by g and restricted to W :

$$T_{g,W} = \text{Tr}_W(gM). \quad (4.15)$$

If W_1, W_2 are such invariant subspaces, then

$$[T_{g,W_1}(z), T_{g,W_2}(z')] = 0. \quad (4.16)$$

Thus, we get different kinds of transfer matrices labeled by invariant subspaces, and they commute with each other. A typical situation in which this construction applies is when \mathfrak{h} is a Cartan subalgebra of a complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, V is a direct sum of irreducible representations of $\mathfrak{g}_{\mathbb{C}}$, and g is an element of $\mathfrak{g}_{\mathbb{C}}$.

Algebraically, L-operators give representations of *dynamical quantum groups* [26, 27, 28]. As an algebra, the dynamical quantum group corresponding to R is generated by the meromorphic functions on $\mathbb{C} \times \mathfrak{h}^* \times \mathfrak{h}^*$, together with additional generators $l(z)_j^i, l^{-1}(z)_j^i$. The generators $l(z)_j^i$ are to be understood as the matrix elements of an abstract L-operator and satisfy the same relations as above; $l^{-1}(z)_j^i$ are the elements of the inverse matrix. This algebra has further structures (coproduct and counit) which make it an \mathfrak{h} -bialgebroid.

4.1.2 Elliptic L-operators

An important example of an L-operator is one for the elliptic dynamical R-matrix [29, 30, 31], which is a representation of the elliptic quantum group for \mathfrak{sl}_N . In this example, \mathfrak{h} is the Cartan subalgebra of \mathfrak{sl}_N and $V = \mathbb{C}^N$ is the vector representation of \mathfrak{sl}_N .

The Lie algebra \mathfrak{sl}_N consists of the traceless complex $N \times N$ matrices and \mathfrak{h} is the subalgebra of diagonal elements. We denote by $E_{ij} \in \mathfrak{gl}_N$ the matrix that has 1 in the (i, j) th entry and 0 elsewhere, and by E_{ij}^* the element of $\mathfrak{gl}_N^* = \text{Hom}(\mathfrak{gl}_N, \mathbb{C})$ such that $\langle E_{ij}, E_{kl}^* \rangle = \delta_{ik}\delta_{jl}$. (The bilinear map $\langle -, - \rangle: \mathfrak{gl}_N \times \mathfrak{gl}_N^* \rightarrow \mathbb{C}$ is the natural pairing.) The elements of \mathfrak{h} are matrices of the form $\sum_{i=1}^N b_i E_{ii}$, with $\sum_{i=1}^N b_i = 0$. Since \mathfrak{h} is isomorphic to the quotient of the subspace of \mathfrak{gl}_N consisting of the diagonal matrices by the subspace

spanned by the identity matrix $I = \sum_{i=1}^N E_{ii}$, the dual space \mathfrak{h}^* is isomorphic to the subspace of \mathfrak{gl}_N^* consisting of elements of the form $\sum_{i=1}^N a_i E_{ii}^*$ such that $\langle I, \sum_{i=1}^N a_i E_{ii}^* \rangle = \sum_{i=1}^N a_i = 0$. Thus, \mathfrak{h}^* may also be identified with the space of traceless diagonal matrices.

The natural action of \mathfrak{sl}_N on \mathbb{C}^N defines the vector representation of \mathfrak{sl}_N . In terms of the standard basis $\{e_1, \dots, e_N\}$ of \mathbb{C}^N , we have $\sum_{j=1}^N a_j E_{jj} e_i = a_i e_i$. The weight of e_i is therefore

$$h_i = E_{ii}^* - \frac{1}{N} \sum_{j=1}^N E_{jj}^*. \quad (4.17)$$

For $a \in \mathfrak{h}^*$, we write $a_i = \langle E_{ii}, a \rangle$. Then, $\sum_{i=1}^N a_i = 0$ and $a = \sum_{i=1}^N a_i E_{ii}^* = \sum_{i=1}^N a_i h_i$.

Fix a point τ in the upper half plane, $\text{Im } \tau > 0$, and let

$$\theta_1(z) = - \sum_{j \in \mathbb{Z} + \frac{1}{2}} e^{\pi i j^2 \tau + 2\pi i j(z + \frac{1}{2})} \quad (4.18)$$

be Jacobi's first theta function. The *elliptic dynamical R-matrix* R^{ell} is defined by [26, 27, 28]

$$R^{\text{ell}}(z, a) = \sum_{i=1}^N E_{ii} \otimes E_{ii} + \sum_{i \neq j} \alpha(z, a_{ij}) E_{ii} \otimes E_{jj} + \sum_{i \neq j} \beta(z, a_{ij}) E_{ji} \otimes E_{ij}, \quad (4.19)$$

where $a_{ij} = a_i - a_j$ and

$$\alpha(z, a) = \frac{\theta_1(a + \epsilon) \theta_1(-z)}{\theta_1(a) \theta_1(\epsilon - z)}, \quad \beta(z, a) = \frac{\theta_1(a - z) \theta_1(\epsilon)}{\theta_1(a) \theta_1(\epsilon - z)}. \quad (4.20)$$

The *elliptic L-operator* L^{ell} , which satisfies the RLL relation with R^{ell} , has the matrix elements given by [32]

$$L_{w,y}^{\text{ell}}(z; a^1, a^2)_i^j = \frac{\theta_1(z - w + a_j^2 - a_i^1)}{\theta_1(z - w)} \prod_{k(\neq i)} \frac{\theta_1(a_k^1 - a_j^2 - y)}{\theta_1(a_{ki}^1)}. \quad (4.21)$$

The complex numbers w, y may be thought of as spectral parameters for the corresponding double line. The presence of the two parameters is a consequence of the fact that $R^{\text{ell}}(z, a)$ is invariant under shift of a by a multiple of the identity matrix I and in the RLL relation (4.3) the spectral parameters z, z' enter the R-matrix only through the difference $z - z'$; note also that the L-operator can be multiplied by any function of the spectral parameter.

The elliptic dynamical R-matrix and the elliptic L-operator have many more properties than just that they satisfy the RLL relation. Most importantly, the R-matrix is a solution of the dynamical Yang–Baxter equation and encodes the Boltzmann weights for a two-dimensional integrable lattice model. This model is equivalent to the eight-vertex model (or more precisely, the Belavin model [33] which is an \mathfrak{sl}_N generalization of the eight-vertex model [34, 35]) in the sense that the transfer matrices of the two models are related by a similarity transformation. The elliptic L-operator, on the other hand, satisfies the RLL relation

with another R-matrix which describes an integrable lattice model called the Bazhanov–Sergeev model [19, 20], whose spins variables take values in \mathfrak{h}^* . We will not discuss these aspects in this paper. The interested reader is referred to [36] for more details.

4.1.3 Trigonometric L-operators

The L-operators that appear in the correspondence with Wilson–’t Hooft lines are obtained from the elliptic L-operator L^{ell} via the trigonometric limit $\tau \rightarrow i\infty$. For comparison with gauge theory results, we actually need to express these L-operators in somewhat different forms.

First, we describe L-operators in a quantum mechanical language. Let us explain this description in the case in which \mathfrak{h} is the Cartan subalgebra of \mathfrak{sl}_N . Recall that \mathfrak{sl}_N has simple coroots

$$\alpha_i^\vee = E_{ii} - E_{i+1, i+1}, \quad i = 1, \dots, N-1, \quad (4.22)$$

and the fundamental weights

$$\omega_i = (\alpha_i^\vee)^* = \sum_{j=1}^i h_j. \quad (4.23)$$

Consider quantum mechanics of a particle living in $\mathfrak{h}^* \times \mathfrak{h}^*$, with Planck constant

$$\hbar = -\frac{\epsilon}{2\pi}. \quad (4.24)$$

If $(a^1, a^2) \in \mathfrak{h}^* \times \mathfrak{h}^*$ is the position of the particle, we write $a^r = \sum_{i=1}^{N-1} q_i^r \omega_i$, $r = 1, 2$. Similarly, we write the momenta $(b^1, b^2) \in \mathfrak{h} \times \mathfrak{h}$ of the particle as $b^r = \sum_{i=1}^{N-1} p_i^r \alpha_i^\vee$. The corresponding position and momentum operators \hat{q}_i^r, \hat{p}_i^s satisfy the canonical commutation relations:

$$[\hat{q}_i^r, \hat{p}_j^s] = i\hbar \delta^{rs} \delta_{ij}, \quad i, j = 1, \dots, N-1. \quad (4.25)$$

(As before, we are treating q_i^r, p_i^r as analytically continued variables.)

To rewrite the commutation relations in a form that is invariant under the action of the Weyl group, we make a change of basis

$$a^r = \sum_{i=1}^N a_i^r E_{ii}^*, \quad b^r = \sum_{i=1}^N b_i^r E_{ii}. \quad (4.26)$$

Then, the corresponding observables \hat{a}_i^r, \hat{b}_i^r obey the traceless condition, $\sum_{i=1}^N \hat{a}_i^r = \sum_{i=1}^N \hat{b}_i^r = 0$, and satisfy the commutation relations

$$[\hat{a}_i^r, \hat{b}_j^s] = i\hbar \delta^{rs} \left(\delta_{ij} - \frac{1}{N} \right), \quad i, j = 1, \dots, N. \quad (4.27)$$

Using these observables we can identify the matrix elements of an L-operator L with an operator in the Hilbert space of this quantum mechanical system:

$$L(z)_i^j = L(z; \hat{a}^1, \hat{a}^2)_i^j e^{2\pi i(\hat{b}_i^1 + \hat{b}_j^2)}. \quad (4.28)$$

In quantum mechanics, there is an invertible map from functions on the classical phase space to operators in the Hilbert space, known as the Weyl transform: if q and p are canonically conjugate variables, it maps

$$f(q, p) \mapsto \hat{f}(\hat{q}, \hat{p}) = \int_{\mathbb{R}^4} dx dy dp dq f(q, p) e^{i(x(\hat{q}-q)+y(\hat{p}-p))}. \quad (4.29)$$

The inverse map is the *Wigner transform*, which we denote by $\langle - \rangle$:

$$f(\hat{q}, \hat{p}) \mapsto \langle f(\hat{q}, \hat{p}) \rangle = \int_{\mathbb{R}} dx e^{ipx/\hbar} \left\langle q + \frac{1}{2}x \left| \hat{f}(\hat{q}, \hat{p}) \right| q - \frac{1}{2}x \right\rangle. \quad (4.30)$$

In the situation at hand, if we rewrite the expression (4.28) as

$$L(z)_i^j = e^{\pi i(\hat{b}_i^1 + \hat{b}_j^2)} \tilde{L}(z; \hat{a}^1, \hat{a}^2)_i^j e^{\pi i(\hat{b}_i^1 + \hat{b}_j^2)}, \quad (4.31)$$

then we have

$$\langle L(z)_i^j \rangle = e^{2\pi i(b_i^1 + b_j^2)} \tilde{L}(z; a^1, a^2)_i^j. \quad (4.32)$$

Next, we apply a similarity transformation to the elliptic L-operator. Assume $\text{Im } \epsilon > 0$ and let

$$\Gamma(z, \tau, \epsilon) = \prod_{m,n=0}^{\infty} \frac{1 - e^{2\pi i((m+1)\tau + (n+1)\epsilon - z)}}{1 - e^{2\pi i(m\tau + n\epsilon + z)}} \quad (4.33)$$

be the elliptic gamma function. Then, $\bar{\Gamma}(z) = e^{\pi i z^2 / 2\epsilon} \Gamma(z, \tau, \epsilon)$ has the property that $\bar{\Gamma}(z + \epsilon, \tau, \epsilon) = g(\tau, \epsilon) \theta_1(z) \bar{\Gamma}(z, \tau, \epsilon)$ for some function $g(\tau, \epsilon)$. We define the conjugated L-operator $\mathcal{L}_{w,m}^{\text{ell}}(z)$ by

$$\mathcal{L}_{w,m}^{\text{ell}}(z)_i^j = \Phi_{m-\frac{1}{2}\epsilon} L_{w,m-\frac{1}{2}\epsilon}^{\text{ell}}(z)_i^j \Phi_{m-\frac{1}{2}\epsilon}^{-1}, \quad (4.34)$$

where

$$\Phi_y = \prod_{k,l=1}^N \bar{\Gamma}(\hat{a}_k^1 - \hat{a}_l^2 - y)^{\frac{1}{2}} \prod_{k \neq l} \bar{\Gamma}(\hat{a}_{kl}^1)^{-\frac{1}{2}}. \quad (4.35)$$

It has the Wigner transform

$$\begin{aligned} \langle \mathcal{L}_{w,m}^{\text{ell}}(z)_i^j \rangle &= e^{2\pi i(b_i^1 + b_j^2)} \frac{\theta_1(z - w + a_j^2 - a_i^1)}{\theta_1(z - w)} \\ &\times \left(\frac{\prod_{k(\neq i)} \theta_1(a_k^1 - a_j^2 - m) \prod_{l(\neq j)} \theta_1(a_i^1 - a_l^2 - m)}{\prod_{k(\neq i)} \theta_1(a_{ki}^1 - \frac{1}{2}\epsilon) \theta_1(a_{ik}^1 - \frac{1}{2}\epsilon)} \right)^{\frac{1}{2}}. \end{aligned} \quad (4.36)$$

With these preparations, let us finally take the trigonometric limit to define the trigonometric L-operator:

$$\mathcal{L}_{w,m} = \lim_{\tau \rightarrow i\infty} \mathcal{L}_{w,m}^{\text{ell}}. \quad (4.37)$$

The trigonometric L-operator satisfies the RLL relation with the trigonometric limit R^{trig} of the elliptic R-matrix R^{ell} . Concretely, $\mathcal{L}_{w,m}$ and R^{trig} are obtained from $\mathcal{L}_{w,m}^{\text{ell}}$ and R^{ell} by the replacement $\theta_1(z) \rightarrow \sin(\pi z)$.

Once we are in the trigonometric setup, the quasi-periodicity in $z \rightarrow z + \tau$ is lost and we can further take the limits $w \rightarrow \pm i\infty$. This allows us to introduce more fundamental L-operators:

$$\mathcal{L}_{\pm, m} = \lim_{w \rightarrow \pm i\infty} \mathcal{L}_{w, m}. \quad (4.38)$$

These L-operators do not depend on the spectral parameters z, w , and their matrix elements have the Wigner transforms

$$\langle (\mathcal{L}_{\pm, m})_i^j \rangle = e^{2\pi i(b_i^1 + b_j^2)} e^{\pm \pi i(a_j^2 - a_i^1)} \ell_m(a^1, a^2)_i^j, \quad (4.39)$$

with

$$\ell_m(a^1, a^2)_i^j = \left(\frac{\prod_{k(\neq i)} \sin \pi(a_k^1 - a_j^2 - m) \prod_{l(\neq j)} \sin \pi(a_i^1 - a_l^2 - m)}{\prod_{k(\neq i)} \sin \pi(a_{ki}^1 - \frac{1}{2}\epsilon) \sin \pi(a_{ik}^1 - \frac{1}{2}\epsilon)} \right)^{\frac{1}{2}}. \quad (4.40)$$

The L-operator for arbitrary parameters z, w can be realized as a linear combination of $\mathcal{L}_{\pm, m}$:

$$\mathcal{L}_{w, m}(z) = \frac{e^{\pi i(z-w)} \mathcal{L}_{+, m} - e^{-\pi i(z-w)} \mathcal{L}_{-, m}}{\sin \pi(z-w)}. \quad (4.41)$$

The monodromy matrix $\mathcal{M}_{\sigma, m}$ constructed from $\mathcal{L}_{\pm, m}$ is labeled by an n -tuple of signs $\sigma = (\sigma^1, \dots, \sigma^n) \in \{\pm\}^n$ and an n -tuple of complex numbers $m = (m^1, \dots, m^n)$:

$$\langle (\mathcal{M}_{\sigma, m})_{i^1}^{i^{n+1}} \rangle = \sum_{i^2, \dots, i^n} \prod_{s=1}^{n+1} e^{2\pi i b_{i^s}^s} \prod_{r=1}^n e^{\sigma^r \pi i(a_{i^{r+1}}^{r+1} - a_{i^r}^r)} \ell_{m^r}(a^r, a^{r+1})_{i^r}^{i^{r+1}}. \quad (4.42)$$

The corresponding transfer matrix $\mathcal{T}_{\sigma, m}$ has the Wigner transform

$$\langle \mathcal{T}_{\sigma, m} \rangle = \sum_{i^1, \dots, i^n} \prod_{r=1}^n e^{2\pi i b_{i^r}^r} e^{\sigma^r \pi i(a_{i^{r+1}}^{r+1} - a_{i^r}^r)} \ell_{m^r}(a^r, a^{r+1})_{i^r}^{i^{r+1}}, \quad (4.43)$$

with $a^{n+1} = a^1$, $i^{n+1} = i^1$. Our claim is that these quantities equal the vevs of Wilson-'t Hooft lines in $\mathcal{N} = 2$ supersymmetric gauge theories.

4.2 Wilson-'t Hooft lines as transfer matrices

In the previous section we defined the fundamental trigonometric L-operators (4.38) and calculated transfer matrices constructed from them. As explained in section ??, these transfer matrices are expected to have interpretations as Wilson-'t Hooft lines in $\mathcal{N} = 2$ supersymmetric gauge theories described by a circular quiver. In this section we verify this expectation by computing the vevs of the corresponding Wilson-'t Hooft lines.

4.2.1 Wilson–’t Hooft lines in $S^1 \times_\epsilon \mathbb{R}^2 \times \mathbb{R}$

Consider a four-dimensional gauge theory whose gauge group is a compact Lie group G with Lie algebra \mathfrak{g} . Choosing a maximal torus $T \subset G$ with Lie algebra \mathfrak{t} , we let $\Lambda_r(\mathfrak{g}) \subset \mathfrak{t}^*$ and $\Lambda_{cr}(\mathfrak{g}) \subset \mathfrak{t}$ be the root lattice and the coroot lattice of \mathfrak{g} , respectively. Their duals are the coweight lattice $\Lambda_{cw}(\mathfrak{g}) = \Lambda_r(\mathfrak{g})^\vee \subset \mathfrak{t}$ and the weight lattice $\Lambda_w(\mathfrak{g}) = \Lambda_{cr}(\mathfrak{g})^\vee \subset \mathfrak{t}^*$.

An ’t Hooft line is the worldline of a very heavy monopole, that is, a nondynamical magnetically charged particle. In the presence of an ’t Hooft line, the gauge field of the theory has a singularity at the location of the monopole: in terms of the polar angle θ and the azimuthal angle ϕ of the spherical coordinates centered at the monopole, the gauge field behaves as

$$A = \frac{\mathbf{m}}{2}(1 - \cos \theta)d\phi + \dots, \quad (4.44)$$

where \dots represents less singular terms. (For simplicity we are setting the gauge theory theta angles to zero.) The coefficient \mathbf{m} is the magnetic charge of the monopole. Different singular gauge field configurations of the above form describe the same monopole if their magnetic charges are related by gauge transformation. It follows that \mathbf{m} can be chosen from \mathfrak{t} , and the choice is meaningful only up to the action of the Weyl group $W(G)$ of G .

The above expression of A is valid in a trivialization over a coordinate patch that contains the point $\theta = 0$ of a two-sphere surrounding the monopole. At $\theta = \pi$, there is a “Dirac string” which supports an unphysical magnetic flux. For the Dirac string to be invisible, we must have

$$\langle \mathbf{m}, w \rangle \in \mathbb{Z} \quad (4.45)$$

for every weight $w \in \mathfrak{t}^*$ of the representation of every field in the theory. This is simply the condition that the holonomy of A around the point $\theta = \pi$ is trivial in the bundles of which the fields are sections. The theory always contains fields in the adjoint representation, so \mathbf{m} belongs to the coweight lattice:¹¹

$$\mathbf{m} \in \Lambda_{cw}(\mathfrak{g})/W(G). \quad (4.46)$$

Equivalently, \mathbf{m} is specified by an irreducible representation of the Langlands dual ${}^L\mathfrak{g}$ of \mathfrak{g} . In general, \mathbf{m} lies in a sublattice of $\Lambda_{cw}(\mathfrak{g})/W(G)$ determined by the matter content.

We can also consider heavy particles that carry both magnetic and electric charges. The worldline of such a dyon is called a Wilson–’t Hooft line. In the path integral formalism, a Wilson–’t Hooft line is realized by an insertion of a Wilson line

$$\mathrm{Tr}_R P \exp \left(i \int_L A \right) \quad (4.47)$$

¹¹Further, \mathbf{m} belongs to the cocharacter lattice $\{v \in \mathfrak{t} \mid \exp(2\pi i v) = \mathrm{id}_G\}$, which is a sublattice of $\Lambda_{cw}(\mathfrak{g})$. If we take G to be the adjoint group, the cocharacter lattice coincides with $\Lambda_{cw}(\mathfrak{g})$.

and a singular boundary condition on the support L of the line as specified by the magnetic charge. The prescribed singularity (4.44) breaks the gauge symmetry to the stabilizer $G_{\mathbf{m}}$ of \mathbf{m} , so R is an irreducible representation of $G_{\mathbf{m}}$. (More precisely, R is an irreducible representation of the stabilizer of \mathbf{m} in the universal cover \tilde{G} of G [37].)

The data specifying such a pair (\mathbf{m}, R) is actually the same as a pair (\mathbf{m}, \mathbf{e}) of coweight \mathbf{m} and weight \mathbf{e} modulo the Weyl group action:

$$(\mathbf{m}, \mathbf{e}) \in (\Lambda_{\text{cw}}(\mathfrak{g}) \times \Lambda_{\text{w}}(\mathfrak{g})) / W(G). \quad (4.48)$$

As emphasized in [37], this data has more information than a pair of irreducible representations of \mathfrak{g} and ${}^L\mathfrak{g}$.

In [38], the vevs of Wilson–’t Hooft lines in $\mathcal{N} = 2$ supersymmetric gauge theories on $S^1 \times_{\epsilon} \mathbb{R}^2 \times \mathbb{R}$ in the Coulomb phase were computed via localization of the path integral. The geometry $S^1 \times_{\epsilon} \mathbb{R}^2$ is a twisted product of S^1 and \mathbb{R}^2 , constructed from $[0, 2\pi r] \times \mathbb{R}^2$ by the identification $(2\pi r, z) \sim (0, e^{2\pi i \epsilon} z)$, where z is the complex coordinate of $\mathbb{R}^2 \cong \mathbb{C}$. These Wilson–’t Hooft lines wind around S^1 , and are located at the origin of \mathbb{R}^2 and a point in \mathbb{R} . In order to preserve half of the eight supercharges, they require the complex scalar field ϕ in the vector multiplet to also have a singular behavior and replace the gauge field in the Wilson line (4.47) with $A + i \text{Re } \phi$. The vevs depend holomorphically on parameters

$$a \in \mathfrak{t}_{\mathbb{C}}, \quad b \in \mathfrak{t}_{\mathbb{C}}^*, \quad (4.49)$$

which are set by the values of the gauge field and the vector multiplet scalar at spatial infinity. Essentially, a is given by the holonomy around S^1 at infinity of the gauge field, while b is that of the dual gauge field. We refer the reader to [38] for the precise definitions.

The vev of a Wilson line W_R in representation R is simply given by the classical value of the holonomy:

$$\langle W_R \rangle = \text{Tr}_R e^{2\pi i a}. \quad (4.50)$$

The vevs of ’t Hooft lines are much more involved. For an ’t Hooft line $T_{\mathbf{m}}$ with magnetic charge \mathbf{m} , the vev takes the form

$$\langle T_{\mathbf{m}} \rangle = \sum_{\substack{v \in \Lambda_{\text{cr}}(\mathfrak{g}) + \mathbf{m} \\ \|v\| \leq \|\mathbf{m}\|}} e^{2\pi i \langle v, b \rangle} Z_{1\text{-loop}}(a, m, \epsilon; v) Z_{\text{mono}}(a, m, \epsilon; \mathbf{m}, v), \quad (4.51)$$

where m collectively denotes complex mass parameters. The summation over the coweights v in the shifted coroot lattice $\Lambda_{\text{cr}} + \mathbf{m}$ accounts for the so-called “monopole bubbling,” a phenomenon in which smooth monopoles are absorbed by the ’t Hooft line and screen the magnetic charge. The norm $\|v\|$ with respect to a Killing form is bounded by $\|\mathbf{m}\|$, so this is a finite sum. The first two factors in the summand are the classical action and the

one-loop determinant in the screened monopole background, respectively. The last factor is the nonperturbative contributions coming from degrees of freedom trapped on the 't Hooft line due to monopole bubbling.

Suppose that the theory under consideration consists of a vector multiplet and N_F hypermultiplets in representations R_f with mass parameters m_f , $f = 1, \dots, N_F$. The one-loop determinant $Z_{1\text{-loop}}$ is then the product of the contributions from the vector multiplet and the hypermultiplets:

$$Z_{1\text{-loop}}(a, m, \epsilon; v) = Z_{1\text{-loop}}^{\text{vm}}(a, \epsilon; v) \prod_{f=1}^{N_F} Z_{1\text{-loop}}^{\text{hm}, R_f}(a, m_f, \epsilon; v). \quad (4.52)$$

The two functions are given by

$$Z_{1\text{-loop}}^{\text{vm}}(a, \epsilon; v) = \prod_{\alpha \in \Phi(\mathfrak{g})} \prod_{k=0}^{|\langle v, \alpha \rangle| - 1} \sin^{-\frac{1}{2}} \left(\pi \langle a, \alpha \rangle + \pi \left(\frac{1}{2} |\langle v, \alpha \rangle| - k \right) \epsilon \right), \quad (4.53)$$

$$Z_{1\text{-loop}}^{\text{hm}, R}(a, m, \epsilon; v) = \prod_{w \in P(R)} \prod_{k=0}^{|\langle v, w \rangle| - 1} \sin^{\frac{1}{2}} \left(\pi \langle a, w \rangle - \pi m + \pi \left(\frac{1}{2} |\langle v, w \rangle| - \frac{1}{2} - k \right) \epsilon \right). \quad (4.54)$$

Here, $\Phi(\mathfrak{g})$ is the set of roots of \mathfrak{g} and $P(R)$ is the set of weights of R .

The factor Z_{mono} is subtle. The original computation in [38] did not give an answer that completely matches predictions from the AGT correspondence. The subtleties have been addressed in subsequent works [39, 40, 41, 42] but not resolved in full generality.

Fortunately, for Wilson-'t Hooft lines that are of interest to us, the screened magnetic charges are in the same $W(G)$ -orbit as \mathbf{m} . The corresponding contributions are therefore obtained by the $W(G)$ -action from the perturbative term, for which $v = \mathbf{m}$ and $Z_{\text{mono}} = 1$.

To our knowledge, a formula for the vevs of dyonic Wilson-'t Hooft lines generalizing the expressions (4.50) and (4.51) has not been derived. Nevertheless, for the same reason as mentioned, we can calculate the vev of a relevant Wilson-'t Hooft line by first writing down its perturbative contribution, which is simply the product of the perturbative vevs of the corresponding purely electric and purely magnetic lines, and then summing over the contributions from the nonperturbative sectors related by the $W(G)$ -action.

4.2.2 Transfer matrices from circular quiver theories

The Wilson-'t Hooft line that corresponds to the transfer matrix (4.43) is one in an $\mathcal{N} = 2$ supersymmetric gauge theory that is described by a circular quiver with n nodes:



$$. \quad (4.55)$$

Each node represents a vector multiplet for an $SU(N)$ gauge group,¹² and each edge a hypermultiplet that transforms in the bifundamental representation under the gauge groups of the nodes it connects.

Let us first consider the case in which the quiver consists of a single node and a single edge. In this case, the gauge group $G = SU(N)$ and the only hypermultiplet is in the adjoint representation. This theory is known as $\mathcal{N} = 2^*$ theory.

The roots of $\mathfrak{g} = \mathfrak{su}_N$ are $\alpha_{ij} = E_{ii}^* - E_{jj}^* = h_i - h_j$, $i \neq j$. The positive roots are α_{ij} , $i < j$, and the simple roots are $\alpha_i = \alpha_{i,i+1}$, $i = 1, \dots, N-1$. The fundamental coweights are $\omega_i^\vee = (\alpha_i^\vee)^* = \sum_{j=1}^i h_j^\vee$, with

$$h_i^\vee = E_{ii} - \frac{1}{N} \sum_{j=1}^N E_{jj}. \quad (4.56)$$

The various lattices are

$$\Lambda_r = \bigoplus_{i=1}^{N-1} \mathbb{Z}\alpha_i, \quad \Lambda_{cr} = \bigoplus_{i=1}^{N-1} \mathbb{Z}\alpha_i^\vee, \quad \Lambda_w = \bigoplus_{i=1}^{N-1} \mathbb{Z}\omega_i, \quad \Lambda_{cw} = \bigoplus_{i=1}^{N-1} \mathbb{Z}\omega_i^\vee. \quad (4.57)$$

We recall that α_i^\vee are the simple coroots and $\omega_i = (\alpha_i^\vee)^*$ are the fundamental weights.

For $\mathcal{N} = 2^*$ theory with $G = SU(N)$, minimal magnetic charges are $\mathbf{m} = \omega_1^\vee = h_1^\vee$ and $\mathbf{m} = \omega_{N-1}^\vee = -h_N^\vee$. These magnetic charges are the highest weights of the fundamental representation and the antifundamental representation of ${}^L\mathfrak{su}_N \cong \mathfrak{su}_N$, respectively.

Let us consider the 't Hooft line with $\mathbf{m} = h_1^\vee$. The vev of this 't Hooft line is expressed as a sum over the screened magnetic charges $v = h_1^\vee, h_2^\vee, \dots, h_N^\vee$. The term for $v = h_1^\vee$ is the perturbative contribution and given by

$$e^{2\pi i b_1} \prod_{j=2}^N \sin^{-\frac{1}{2}}\left(\pi a_{1j} + \frac{1}{2}\pi\epsilon\right) \sin^{-\frac{1}{2}}\left(\pi a_{j1} + \frac{1}{2}\pi\epsilon\right) \sin^{\frac{1}{2}}(\pi a_{1j} - \pi m) \sin^{\frac{1}{2}}(\pi a_{j1} - \pi m), \quad (4.58)$$

where $a_i = \langle a, h_i \rangle$, $b_i = \langle h_i^\vee, b \rangle$, $a_{ij} = a_i - a_j$ and m is the mass of the adjoint hypermultiplet. The other terms are related to this perturbative term by the Weyl group action which permutes $(h_1^\vee, \dots, h_N^\vee)$, so we find

$$\langle T_{h_1^\vee} \rangle = \sum_{i=1}^N e^{2\pi i b_i} \prod_{j(\neq i)} \left(\frac{\sin \pi(a_{ij} - m) \sin \pi(a_{ji} - m)}{\sin \pi(a_{ij} - \frac{1}{2}\epsilon) \sin \pi(a_{ji} - \frac{1}{2}\epsilon)} \right)^{\frac{1}{2}}. \quad (4.59)$$

The vev of $T_{-h_N^\vee}$ is obtained from $\langle T_{h_1^\vee} \rangle$ by the replacement $b_i \rightarrow -b_i$.

Now, let us turn to a circular quiver with n nodes. For this theory, we have $G = SU(N)^n$ and $\Lambda_{cw}(\mathfrak{g}) = \Lambda_{cw}(\mathfrak{su}_N)^{\oplus n}$. We consider the 't Hooft line with

$$\mathbf{m} = h_1^\vee \oplus \dots \oplus h_1^\vee, \quad (4.60)$$

¹²More precisely, the gauge group is a product of $PSU(N)$.

charged equally under the $SU(N)$ factors of G . This time, the summation is over all coweights of the form $v = h_{i^1}^\vee \oplus \cdots \oplus h_{i^n}^\vee$. The perturbative term, for which $i^1 = \cdots = i^n = 1$, is given by

$$\prod_{r=1}^n e^{2\pi i b_1^r} \prod_{j=2}^N \sin^{-\frac{1}{2}} \left(\pi a_{1j}^r + \frac{1}{2} \pi \epsilon \right) \sin^{-\frac{1}{2}} \left(\pi a_{j1}^r + \frac{1}{2} \pi \epsilon \right) \\ \times \sin^{\frac{1}{2}} \left(\pi (a_j^r - a_1^{r+1}) - \pi m^r \right) \sin^{\frac{1}{2}} \left(\pi (a_1^r - a_j^{r+1}) - \pi m^r \right). \quad (4.61)$$

The superscript r refers to the r th $SU(N)$ factor of G , with $a^{n+1} = a^1$. Collecting the contributions from the other coweights, we get

$$\langle T_{h_1^\vee \oplus \cdots \oplus h_1^\vee} \rangle = \sum_{i^1, \dots, i^n} \prod_{r=1}^n e^{2\pi i b_{i^r}^r} \ell_{m^r}(a^r, a^{r+1})_{i^r}^{r+1}, \quad (4.62)$$

where we have used the functions (4.40).

Comparing this expression with the Wigner transform (4.43) of the trigonometric transfer matrix $\mathcal{T}_{\sigma, m}$, we see

$$\langle T_{h_1^\vee \oplus \cdots \oplus h_1^\vee} \rangle = \langle \mathcal{T}_{(+, \dots, +), m} \rangle = \langle \mathcal{T}_{(-, \dots, -), m} \rangle \quad (4.63)$$

under the obvious identification of parameters.

In order to reproduce $\langle \mathcal{T}_{\sigma, m} \rangle$ for a general choice of the signs σ , we add to the 't Hooft line the electric charge

$$\mathbf{e} = \sum_{r=1}^n \sigma^r \frac{1}{2} (h_1^{r+1} - h_1^r) = \sum_{r=1}^n (\sigma^r 1 - \sigma^{r+1} 1) \frac{1}{2} h_1^{r+1}. \quad (4.64)$$

This electric charge is in a sense a minimal one that is compatible with the Dirac–Schwinger–Zwanziger quantization condition for locality: the charges (\mathbf{m}, \mathbf{e}) and $(\mathbf{m}', \mathbf{e}')$ of two dyons must satisfy $\langle \mathbf{m}, \mathbf{e}' \rangle - \langle \mathbf{m}', \mathbf{e} \rangle \in \mathbb{Z}$. In section 4.3, we will see the geometric meaning of this “minimality” in connection with the AGT correspondence.

The magnetic charge (4.60) breaks the gauge group to $S(U(1) \times U(N-1))^n$, and we are turning on a Wilson line that is charged under the $U(1)$ factors with charges proportional to $(\sigma^r 1 - \sigma^{r+1} 1)/2$. The Wilson line multiplies the perturbative term (4.61) by the phase factor

$$\prod_{r=1}^n e^{\sigma^r \pi i \langle a, h_1^{r+1} - h_1^r \rangle} = \prod_{r=1}^n e^{\sigma^r \pi i (a_1^{r+1} - a_1^r)}. \quad (4.65)$$

Hence, the term with $v = h_{i^1}^\vee \oplus \cdots \oplus h_{i^n}^\vee$ gets the phase factor $e^{\sigma^r \pi i (a_{i^r+1}^{r+1} - a_{i^r}^r)}$, and the vev of this Wilson–'t Hooft line matches the Wigner transform of $\mathcal{T}_{\sigma, m}$.

4.2.3 Monodromy matrices from linear quiver theories

We have considered the Wilson–t Hooft lines in the circular quiver theory and showed that their vevs match the Wigner transforms of the trigonometric transfer matrices. What correspond to the monodromy matrices then? In view of the fact that summing over the weights of the representation $V = \mathbb{C}^N$ in the integrable model amounts to summing over the different screened magnetic charges, natural candidates are Wilson–t Hooft lines in a theory described by a linear quiver with $n + 1$ nodes:

$$\boxed{N} \text{---} \bigcirc{N} \text{---} \bigcirc{N} \text{---} \cdots \text{---} \bigcirc{N} \text{---} \boxed{N} \quad . \quad (4.66)$$

The leftmost and the rightmost nodes represent $SU(N)$ flavor groups, which are not gauged.

In particular, we expect that the fundamental trigonometric L-operators (4.38) arise from the vevs of Wilson–t Hooft lines of the theory of a bifundamental hypermultiplet:

$$\boxed{N} \text{---} \boxed{N} \quad . \quad (4.67)$$

Let us see if this is the case.

We introduce nondynamical vector multiplets for the $SU(N)$ flavor groups, and consider the Wilson–t Hooft lines with magnetic charge

$$\mathbf{m} = h_i^\vee \oplus h_j^\vee \quad (4.68)$$

and electric charges

$$\mathbf{e} = \mp \frac{1}{2} h_i \oplus \pm \frac{1}{2} h_j \quad . \quad (4.69)$$

Note that the electric charges are fractional. The vevs of these Wilson–t Hooft lines are

$$e^{2\pi i(b_i^1 + b_j^2)} e^{\pm \pi i(a_j^2 - a_i^1)} \prod_{k(\neq i)} \prod_{l(\neq j)} (\sin \pi(a_k^1 - a_j^2 - m) \sin \pi(a_i^1 - a_l^2 - m))^{\frac{1}{2}} \quad . \quad (4.70)$$

The vevs do not quite match the Wigner transforms (4.39) of $(\mathcal{L}_{\pm, m})_i^j$. They differ by the factor in the denominator of the function (4.40).

This factor is the one-loop determinant associated with the first node; it would have been present had the $SU(N)$ flavor group been gauged and the vector multiplet been dynamical. From the gauge theory point of view, it is natural to think of this factor as a weight accompanying the summation over the screened magnetic charges. On the integrable system side, we could as well omit the denominator in question from the definitions of the L-operators and adopt the convention that the same weight is included when operators are multiplied within V . The L-operators would still satisfy the RLL relation.

To get the monodromy matrix (4.42), we take n L-operators and multiply them inside V . The gauge theory counterpart of this operation is to connect n copies of the

two-node quiver (4.67), in the presence of appropriate Wilson–t Hooft lines of the type considered above, by identifying and gauging flavor nodes. This produces the $n + 1$ node linear quiver (4.66) and the Wilson–t Hooft lines with magnetic charge

$$\mathbf{m} = h_{i_1}^\vee \oplus h_1^\vee \oplus h_1^\vee \oplus \cdots \oplus h_1^\vee \oplus h_{i_{n+1}}^\vee \quad (4.71)$$

and electric charge

$$\mathbf{e} = \sigma^1 \frac{1}{2} (h_1^2 - h_{i_1}^1) + \sum_{r=2}^{n-1} \sigma^r \frac{1}{2} (h_1^{r+1} - h_1^r) + \sigma^n \frac{1}{2} (h_{i_{n+1}}^{n+1} - h_1^n). \quad (4.72)$$

The vev of this Wilson–t Hooft line reproduces the Wigner transform (4.42), except that a factor corresponding to the one-loop determinant for the vector multiplet for the first node is missing.

4.2.4 Other representations

The magnetic charge (4.60) of the above Wilson–t Hooft lines is the highest weight of the representation $(\mathbb{C}^N)^{\oplus n}$ of the Langlands dual ${}^L\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{sl}_N^{\oplus n}$ of $\mathfrak{g}_{\mathbb{C}}$. The corresponding transfer matrix (4.43) is represented graphically as n double lines intersected by a single solid loop carrying the representation $V = \mathbb{C}^N$, as shown in figure 6(c). The n regions sandwiched between double lines correspond to the n copies of \mathfrak{sl}_N .

Both sides of the correspondence have a generalization in which the vector representation \mathbb{C}^N is replaced by another representation R of \mathfrak{sl}_N . On the gauge theory side, we can change the magnetic charge of the Wilson–t Hooft lines to the highest weight λ_R of $R^{\oplus n}$ while keeping the electric charges intact. On the integrable system side, the counterpart of this operation is the fusion procedure, which allows one to construct a solid line in an arbitrary finite-dimensional representation of \mathfrak{sl}_N from a collection of solid lines in the vector representation, with the spectral parameters suitably adjusted.

We naturally expect that the vev of the Wilson–t Hooft line with magnetic charge $\mathbf{m} = \lambda_R^{\oplus n}$ is equal to the Wigner transform of a transfer matrix constructed from L-operators in representation R , obtained by fusion from the L-operators (4.38) in the vector representation.

For $n = 1$ and $R = \wedge^k \mathbb{C}^N$, this equality can be verified from known results. In this case, the transfer matrix is the trigonometric limit of Ruijsenaars’ difference operator [43]

$$\sum_{\substack{I \subset \{1, \dots, N\} \\ |I|=k}} \Delta_I^{\frac{1}{2}} \prod_{\substack{i \in I \\ j \notin I}} \sqrt{\frac{\theta_1(a_{ji} - m) \theta_1(a_{ij} - m)}{\theta_1(a_{ji} - \frac{1}{2}\epsilon) \theta_1(a_{ij} - \frac{1}{2}\epsilon)}} \Delta_I^{\frac{1}{2}}, \quad \Delta_I = \prod_{i \in I} \Delta_i, \quad (4.73)$$

and is related to the Macdonald operator by a similarity transformation [32]. On the other hand, the exterior power $\wedge^k \mathbb{C}^N$ being a minuscule representation (that is, all weights are

related by the action of the Weyl group), the vev of the 't Hooft line with $\mathbf{m} = \omega_k^\vee = h_1^\vee + \dots + h_k^\vee$ in $\mathcal{N} = 2^*$ theory can be computed from the perturbative term:

$$\langle T_{\omega_k^\vee} \rangle = \sum_{\substack{I \subset \{1, \dots, N\} \\ |I|=k}} \prod_{\substack{i \in I \\ j \notin I}} e^{2\pi i b_i} \left(\frac{\sin \pi(a_{ij} - m) \sin \pi(a_{ji} - m)}{\sin \pi(a_{ij} - \frac{1}{2}\epsilon) \sin \pi(a_{ji} - \frac{1}{2}\epsilon)} \right)^{\frac{1}{2}}. \quad (4.74)$$

(For $\mathcal{N} = 2^*$ theory the choice of the signs $\sigma = \pm$ is irrelevant.) The vev matches the Wigner transform of the trigonometric Ruijsenaars operator.

In the case of symmetric powers, we find a discrepancy between this proposal and the formula (4.51). The vev of the 't Hooft line with $\mathbf{m} = kh_1^\vee$ in $\mathcal{N} = 2^*$ theory, as computed by that formula, can be expressed as the Moyal product of k copies of the vev for the vector representation [38, 44]:

$$\langle T_{kh_1^\vee} \rangle = \langle T_{h_1^\vee} \rangle \star \dots \star \langle T_{h_1^\vee} \rangle. \quad (4.75)$$

The Moyal product \star has the property that $\langle f \rangle \star \langle g \rangle = \langle fg \rangle$ with respect to the Wigner transform, so this equation means that we have

$$\langle T_{kh_1^\vee} \rangle = \langle \mathcal{T}_m^k \rangle. \quad (4.76)$$

However, \mathcal{T}_m^k is the transfer matrix in the tensor product representation $(\mathbb{C}^N)^{\otimes k}$, not the k th symmetric power of \mathbb{C}^N . The discrepancy might be ascribed to subtle monopole contributions to the vev of the 't Hooft line.

4.3 Transfer matrices from Verlinde operators

4.3.1 Review of AGT correspondence

We have computed the vevs of a class of Wilson-'t Hooft lines in $\mathcal{N} = 2$ supersymmetric gauge theories described by a circular quiver, and found that they match the Wigner transforms of transfer matrices constructed from the fundamental trigonometric L-operators. In this section, we show that these transfer matrices can also be identified with Verlinde operators in Toda theory on a punctured torus. The result is in keeping with the relation proposed in [38] based on the AGT correspondence [6] between Toda theory and $\mathcal{N} = 2$ supersymmetric field theories.

The AGT correspondence originates from six-dimensional $\mathcal{N} = (2, 0)$ supersymmetric field theory, of type A_{N-1} in our case, placed on $S_b^4 \times C_{g,n}$. Here, S_b^4 is an ellipsoid, defined as a submanifold of \mathbb{R}^5 by the equation

$$(x^1)^2 + b^{-2}((x^2)^2 + (x^3)^2) + b^2((x^4)^2 + (x^5)^2) = r^2, \quad (4.77)$$

and $C_{g,n}$ is a Riemann surface of genus g with n punctures. With partial topological twisting along $C_{g,n}$, this system preserves eight of the sixteen supercharges of $\mathcal{N} = (2, 0)$ supersymmetry in six dimensions.

In the limit in which $C_{g,n}$ shrinks to a point, the six-dimensional theory reduces to a four-dimensional $\mathcal{N} = 2$ supersymmetric field theory on $S_{\mathbf{b}}^4$, whose gauge and matter contents are determined by the choice of a pants decomposition of $C_{g,n}$ and boundary conditions at the punctures [45, 46]. The theories discussed in section 4.2 can all be obtained in this way. If one instead integrates out the modes along $S_{\mathbf{b}}^4$, one is left with A_{N-1} Toda theory on $C_{g,n}$ with central charge

$$c = 1 + 6q^2, \quad q = \mathbf{b} + \mathbf{b}^{-1}, \quad (4.78)$$

with vertex operators V_{β^r} , $r = 1, \dots, n$, inserted at the punctures. According to the AGT correspondence, the partition function of the theory on $S_{\mathbf{b}}^4$ equals the correlation function of Toda theory on $C_{g,n}$:

$$\langle 1 \rangle_{S_{\mathbf{b}}^4} = \left\langle \prod_r V_{\beta^r} \right\rangle_{C_{g,n}}. \quad (4.79)$$

Let the vertex operators at the punctures be primary fields V_{β^r} , $r = 1, \dots, n$, labeled by momenta $\beta^r \in \mathfrak{h}^*$ valued in the dual of the Cartan subalgebra \mathfrak{h} of \mathfrak{sl}_N . Given a pants decomposition of $C_{g,n}$, the Toda correlation function takes the form

$$\left\langle \prod_r V_{\beta^r} \right\rangle_{C_{g,n}} = \int [d\alpha] \mathcal{C}(\alpha; \beta) \overline{\mathcal{F}(\alpha; \beta)} \mathcal{F}(\alpha; \beta), \quad (4.80)$$

where $[d\alpha]$ is a measure of integration over the set $\alpha = \{\alpha^1, \dots, \alpha^{3g-3+n}\}$ of momenta assigned to the internal edges of the pants decomposition, $\beta = \{\beta^1, \dots, \beta^n\}$ is the set of momenta assigned to the external edges, $\mathcal{C}(\alpha; \beta)$ is the product of relevant three-point functions, and $\mathcal{F}(\alpha; \beta)$ is the corresponding conformal block which is a meromorphic function of α and β .

On the gauge theory side, $\mathcal{C}(\alpha; \beta)$ is interpreted as the product of the classical and the one-loop contributions to the partition function on $S_{\mathbf{b}}^4$, whereas $\mathcal{F}(\alpha; \beta)$ and $\overline{\mathcal{F}(\alpha; \beta)}$ represent the nonperturbative contributions from instantons localized at the two poles at $x^2 = x^3 = x^4 = x^5 = 0$. The internal momenta α are related to the zero modes \mathbf{a} of scalar fields in the vector multiplets by

$$\alpha = Q + i\mathbf{a}, \quad (4.81)$$

and the external momenta β are identified with mass parameters for matter multiplets.

4.3.2 Verlinde operators and Wilson-'t Hooft lines

To incorporate Wilson-'t Hooft lines in the gauge theory, one introduces Verlinde loop operators in the Toda theory. We will explain the construction of relevant Verlinde operators in concrete examples. For the moment, it suffices to say that they are specified by a momentum of the form $\mu = -\mathbf{b}\lambda$ and a one-cycle γ in $C_{g,n}$, where λ is the highest weight of a

representation of \mathfrak{sl}_N .¹³ In the presence of a Verlinde operator $\Phi_\mu(\gamma)$, the Toda correlation function is modified to

$$\left\langle \Phi_\mu(\gamma) \prod_r V_{\beta^r} \right\rangle_{C_{g,n}} = \int [d\alpha] \mathcal{C}(\alpha; \beta) \overline{\mathcal{F}(\alpha; \beta)} (\Phi_\mu(\gamma) \cdot \mathcal{F}(\alpha; \beta)). \quad (4.82)$$

The AGT correspondence asserts [47, 48] that this is equal to the vev of a Wilson–t Hooft line $T_{\mu,\gamma}$ winding around a circle $S^1_{\mathbf{b}}$ where $x^4 = x^5 = 0$ (at $x^1 = 0$, say):

$$\langle T_{\mu,\gamma} \rangle_{S^1_{\mathbf{b}}} = \left\langle \Phi_\mu(\gamma) \prod_r V_{\beta^r} \right\rangle_{C_{g,n}}. \quad (4.83)$$

It turns out that $\Phi_\mu(\gamma)$ acts on conformal blocks as a difference operator shifting the internal momenta α , just as Wilson–t Hooft lines in $\mathcal{N} = 2$ supersymmetric gauge theories on $S^1 \times_\epsilon \mathbb{R}^2 \times \mathbb{R}$ shift Coulomb branch parameters. Indeed, it was argued in [38] that if one defines the modified Verlinde operator

$$\mathcal{L}_\mu(\gamma) = \mathcal{C}(\alpha; \beta)^{\frac{1}{2}} \Phi_\mu(\gamma) \mathcal{C}(\alpha; \beta)^{-\frac{1}{2}}, \quad (4.84)$$

then its Wigner transform is equal to the vev of the Wilson–t Hooft line in the theory on $S^1 \times_\epsilon \mathbb{R}^2 \times \mathbb{R}$, up to an appropriate identification of parameters:

$$\langle T_{\mu,\gamma} \rangle_{S^1 \times_\epsilon \mathbb{R}^2 \times \mathbb{R}} = \langle \mathcal{L}_\mu(\gamma) \rangle. \quad (4.85)$$

Therefore, we expect that for suitable choices of $C_{g,n}$, β , μ and γ , the modified Verlinde operator $\mathcal{L}_\mu(\gamma)$ coincides with a transfer matrix constructed from the trigonometric L-operator.

4.3.3 Verlinde operators on a punctured torus

To reproduce the transfer matrix (4.43), we consider Toda theory on an n -punctured torus $C_{1,n}$ and insert vertex operators V_{β^r} with

$$\beta^r = -N \left(\frac{q}{2} + \mathrm{im}^r \right) h_N. \quad (4.86)$$

The corresponding four-dimensional theory on $S^4_{\mathbf{b}}$ is the one described by an n -node circular quiver, which we studied in section 4.2.2. The parameter \mathfrak{m}^r is the mass of the bifundamental hypermultiplet between the r th and $(r+1)$ th nodes.

To this setup we introduce the Verlinde operator $\Phi_\mu(\gamma)$ with

$$\mu = -\mathbf{b}\omega_1 = -\mathbf{b}h_1 \quad (4.87)$$

¹³More generally, the momentum takes the form $\mu = -\mathbf{b}\lambda_1 - \mathbf{b}^{-1}\lambda_2$, where λ_1, λ_2 are the highest weights of a pair of representations of \mathfrak{sl}_N . The corresponding Wilson–t Hooft line is a superposition of lines wrapping $S^1_{\mathbf{b}}$ and another circle $S^1_{\mathbf{b}^{-1}}$ where $x^2 = x^3 = 0$.

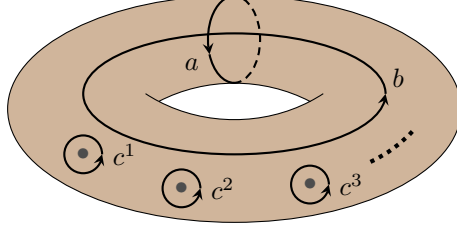


Figure 7: One-cycles on a punctured torus. The cycle c^r goes around the r th puncture.

and γ being a cycle γ_σ specified by an n -tuple of signs $\sigma \in \{\pm\}^n$. If b and c^r are the cycles shown in figure 7, then

$$\gamma_\sigma = b + \sum_r \frac{1 - \sigma^r 1}{2} c^r. \quad (4.88)$$

In other words, the curve γ_σ passes “above” or “below” the r th puncture depending on whether $\sigma^r = +$ or $-$. In the gauge theory, this operator corresponds to the Wilson–t Hooft line with magnetic charge (4.60) and electric charge (4.64).

Let us explain the construction of this Verlinde operator step by step, following the treatment in [49]. To this end, it is convenient to represent the conformal block graphically as

$$\quad (4.89)$$

The internal momenta are α^r , $r = 1, \dots, n+1$, with $\alpha^{n+1} = \alpha^1$.

The first step is to insert the identity operator between β^n and β^1 , and resolve it into the chiral vertex operators V_{-bh_1} and V_{bh_N} by fusion. This step gives the equality

$$\quad (4.90)$$

The difference operator Δ_i acts on internal momenta by

$$\Delta_i \alpha = \alpha - bh_i. \quad (4.91)$$

The function F_{i^1} is given by

$$F_{i^1} = \frac{\Gamma(Nbq)}{\Gamma(bq)} \prod_{j^1 (\neq i^1)} \frac{\Gamma(iba_{j^1 i^1}^1)}{\Gamma(bq + iba_{j^1 i^1}^1)}, \quad (4.92)$$

with

$$Q = q\rho, \quad \rho = \sum_{i=1}^{N-1} \omega_i. \quad (4.93)$$

Next, we transport V_{-bh_1} along γ_σ . Graphically, we move the external edge labeled $-bh_1$ clockwise. Every time the line passes another external edge we get a braiding factor:

$$\begin{aligned} & \begin{array}{c} \text{Diagram 1: A circle with internal labels } \alpha^1, \alpha^2, \alpha^{n+1}. \text{ External edges: } -bh_1 \text{ (top-left), } \beta^1 \text{ (top-right), } \beta^2 \text{ (right), } \beta^n \text{ (bottom-left), } bh_N \text{ (left), } \Delta_{i^1}\alpha^1 \text{ (top-left), } \Delta_{i^2}\alpha^2 \text{ (top-right), } \Delta_{i^{n+1}}\alpha^{n+1} \text{ (bottom-left).} \end{array} \\ &= \sum_{i^2} B_{i^1 i^2}^{\sigma^2} \begin{array}{c} \text{Diagram 2: Similar to Diagram 1, but } -bh_1 \text{ is moved to the top-right position, now labeled } \Delta_{i^2}\alpha^2. \end{array} \\ &= \sum_{i^2, \dots, i^{n+1}} \left(\prod_{r=1}^n B_{i^r i^{r+1}}^{\sigma^r} \right) \begin{array}{c} \text{Diagram 3: Similar to Diagram 2, but } -bh_1 \text{ is moved to the top position, now labeled } \Delta_{i^1}\alpha^1. \end{array}. \quad (4.94) \end{aligned}$$

The function $B_{i^r i^{r+1}}^{\sigma^r}$ depends on the sign σ^r , which specifies the direction of the braiding moves:

$$\begin{aligned} B_{i^r i^{r+1}}^{\sigma^r} &= e^{-\sigma^r \pi b(a_{i^r}^r - a_{i^{r+1}}^{r+1})} \prod_{j^r (\neq i^r)} \frac{\Gamma(b(q + ia_{j^r}^r))}{\Gamma(b(\frac{1}{2}q + ia_{j^r}^r - ia_{i^{r+1}}^{r+1} - im^r))} \\ &\quad \times \prod_{j^{r+1} (\neq i^{r+1})} \frac{\Gamma(iba_{j^{r+1}}^{r+1})}{\Gamma(b(\frac{1}{2}q + ia_{j^{r+1}}^{r+1} - ia_{i^r}^r + im^r))}. \quad (4.95) \end{aligned}$$

Finally, we fuse V_{-bh_1} and V_{bh_N} and project the result to the channel in which the intermediate state is the identity operator:

$$\begin{array}{c} \text{Diagram 4: Similar to Diagram 3, but } -bh_1 \text{ and } bh_N \text{ are now fused at the top position, labeled } 0. \end{array} \rightarrow \frac{\sin(\pi bq)}{\sin(\pi N bq)} F_{i^1}^{-1} \begin{array}{c} \text{Diagram 5: Similar to Diagram 4, but } -bh_1 \text{ and } bh_N \text{ are now fused at the top position, labeled } 0. \end{array}. \quad (4.96)$$

Note that the right-hand side vanishes unless $i^{n+1} = i^1$ since $\alpha^{n+1} = \alpha^1$.

Thus, dropping the overall factor $\sin(\pi bq)/\sin(\pi N bq)$, we find that the Verlinde operator is the difference operator

$$\Phi_{-bh_1}(\gamma_\sigma) = \sum_{i^1, \dots, i^n} \left(\prod_r B_{i^r i^{r+1}}^{\sigma^r} \right) \Delta_{\{i^1, \dots, i^n\}}, \quad (4.97)$$

where $i^{n+1} = i^1$ and

$$\Delta_{\{i^1, \dots, i^n\}} = \prod_r \Delta_{i^r}^r. \quad (4.98)$$

Before we compare the Verlinde operator with the transfer matrix, we must perform a change of basis and find the modified Verlinde operator (4.84). For the correlation function at hand, the product of three-point function factors is

$$\mathcal{C}(\alpha; \beta) = \prod_r \frac{\prod_{i < j} \Upsilon(\mathbf{ia}_{ij}^r) \Upsilon(-\mathbf{ia}_{ij}^{r+1})}{\prod_{i,j} \Upsilon(\frac{1}{2}q + \mathbf{im}^r - \mathbf{ia}_i^r + \mathbf{ia}_j^{r+1})}. \quad (4.99)$$

The precise definition of the function Υ is not important for us; we just need to know that it satisfies the identity

$$\frac{\Upsilon(x + \mathbf{b})}{\Upsilon(x)} = \frac{\Gamma(\mathbf{b}x)}{\Gamma(1 - \mathbf{b}x)} \mathbf{b}^{1-2\mathbf{b}x}, \quad (4.100)$$

where Γ is the gamma function.

Let us calculate $\mathcal{C}(\alpha; \beta) \Delta_{\{i^1, \dots, i^n\}} \mathcal{C}(\alpha; \beta)^{-1}$. The only nontrivial contributions come from the Υ -factors in which either of i or j (but not both) in \mathbf{a}_{ij}^r is equal to i^r :

$$\begin{aligned} & \mathcal{C}(\alpha; \beta) \Delta_{\{i^1, \dots, i^n\}} \mathcal{C}(\alpha; \beta)^{-1} \\ &= \prod_r \prod_{(i^r < j)} \frac{\Upsilon(\mathbf{ia}_{i^r j}^r)}{\Upsilon(\mathbf{ia}_{i^r j}^r - \mathbf{b})} \prod_{i(< i^r)} \frac{\Upsilon(\mathbf{ia}_{i i^r}^r)}{\Upsilon(\mathbf{ia}_{i i^r}^r + \mathbf{b})} \\ & \times \prod_{(i^{r+1} < j)} \frac{\Upsilon(-\mathbf{ia}_{i^{r+1} j}^{r+1})}{\Upsilon(-\mathbf{ia}_{i^{r+1} j}^{r+1} + \mathbf{b})} \prod_{i(< i^{r+1})} \frac{\Upsilon(-\mathbf{ia}_{i i^{r+1}}^{r+1})}{\Upsilon(-\mathbf{ia}_{i i^{r+1}}^{r+1} - \mathbf{b})} \\ & \times \prod_{i(\neq i^r)} \frac{\Upsilon(\frac{1}{2}q + \mathbf{im}^r - \mathbf{ia}_i^r + \mathbf{ia}_{i^{r+1}}^{r+1} - \mathbf{b})}{\Upsilon(\frac{1}{2}q + \mathbf{im}^r - \mathbf{ia}_i^r + \mathbf{ia}_{i^{r+1}}^{r+1})} \prod_{j(\neq i^{r+1})} \frac{\Upsilon(\frac{1}{2}q + \mathbf{im}^r - \mathbf{ia}_{i^r}^r + \mathbf{ia}_j^{r+1} + \mathbf{b})}{\Upsilon(\frac{1}{2}q + \mathbf{im}^r - \mathbf{ia}_{i^r}^r + \mathbf{ia}_j^{r+1})}. \end{aligned} \quad (4.101)$$

Combining the first two lines and using the aforementioned identity, we can rewrite this quantity as

$$\begin{aligned} & \mathcal{C}(\alpha; \beta) \Delta_{\{i^1, \dots, i^n\}} \mathcal{C}(\alpha; \beta)^{-1} \\ &= \prod_r \prod_{j^r(\neq i^r)} \frac{\Gamma(1 - \mathbf{b}(q + \mathbf{ia}_{j^r i^r}^r)) \Gamma(\mathbf{b}(\frac{1}{2}q + \mathbf{ia}_{j^r}^r - \mathbf{ia}_{i^{r+1}}^{r+1} - \mathbf{im}^r))}{\Gamma(\mathbf{b}(q + \mathbf{ia}_{j^r i^r}^r)) \Gamma(1 - \mathbf{b}(\frac{1}{2}q + \mathbf{ia}_{j^r}^r - \mathbf{ia}_{i^{r+1}}^{r+1} - \mathbf{im}^r))} \\ & \times \prod_{j^{r+1}(\neq i^{r+1})} \frac{\Gamma(1 - \mathbf{ib}a_{j^{r+1} i^{r+1}}^{r+1}) \Gamma(\mathbf{b}(\frac{1}{2}q + \mathbf{ia}_{j^{r+1}}^{r+1} - \mathbf{ia}_{i^r}^r + \mathbf{im}^r))}{\Gamma(\mathbf{ib}a_{j^{r+1} i^{r+1}}^{r+1}) \Gamma(1 - \mathbf{b}(\frac{1}{2}q + \mathbf{ia}_{j^{r+1}}^{r+1} - \mathbf{ia}_{i^r}^r + \mathbf{im}^r))}. \end{aligned} \quad (4.102)$$

Plugging this expression into the formula for the modified Verlinde operator, we see that the various factors of gamma functions combine nicely into sine functions via Euler's reflection formula

$$\Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}. \quad (4.103)$$

The final result is

$$\begin{aligned} \mathcal{L}_{-bh_1}(\gamma_\sigma) = \sum_{i^1, \dots, i^n} \left(\prod_r e^{-\sigma^r \pi \mathbf{b}(\mathbf{a}_{i^r}^r - \mathbf{a}_{i^{r+1}}^{r+1})} \prod_{j^r (\neq i^r)} \left(\frac{\sin \pi \mathbf{b}(\frac{1}{2}q + i\mathbf{a}_{j^r}^r - i\mathbf{a}_{i^{r+1}}^{r+1} - i\mathbf{m}^r)}{\sin \pi \mathbf{b}(q + i\mathbf{a}_{j^r}^r)} \right)^{\frac{1}{2}} \right. \\ \left. \times \prod_{j^{r+1} (\neq i^{r+1})} \left(\frac{\sin \pi \mathbf{b}(\frac{1}{2}q + i\mathbf{a}_{j^{r+1}}^{r+1} - i\mathbf{a}_{i^r}^r + i\mathbf{m}^r)}{\sin \pi i\mathbf{b}_{j^{r+1}i^{r+1}}^{r+1}} \right)^{\frac{1}{2}} \right) \Delta_{\{i^1, \dots, i^n\}}. \quad (4.104) \end{aligned}$$

The above expression can be written in terms of the functions (4.40) as

$$\begin{aligned} \mathcal{L}_{-bh_1}(\gamma_\sigma) \\ = \sum_{i^1, \dots, i^n} \Delta_{\{i^1, \dots, i^n\}}^{\frac{1}{2}} \left(\prod_r \ell_{i\mathbf{b}\mathbf{m}^r + \frac{1}{2}}(i\mathbf{b}\mathbf{a}^r, i\mathbf{b}\mathbf{a}^{r+1})_{i^r}^{i^{r+1}} e^{\sigma^r \pi i(i\mathbf{b}\mathbf{a}_{i^r}^r - i\mathbf{b}\mathbf{a}_{i^{r+1}}^{r+1})} \right) \Delta_{\{i^1, \dots, i^n\}}^{\frac{1}{2}}. \quad (4.105) \end{aligned}$$

Comparing this expression with the Wigner transform (4.43) of the trigonometric transfer matrix, we deduce that the modified Verlinde operator coincides with the transfer matrix,

$$\mathcal{L}_{-bh_1}(\gamma_\sigma) = \mathcal{T}_{\sigma, m}, \quad (4.106)$$

under the identification

$$\epsilon = \mathbf{b}^2, \quad a^r = i\mathbf{b}\mathbf{a}^r, \quad m^r = i\mathbf{b}\mathbf{m}^r + \frac{1}{2}. \quad (4.107)$$

It has been proposed in [38] that precisely under this identification of parameters, a modified Verlinde operator in Toda theory corresponding to a Wilson-'t Hooft line in the AGT-dual theory on $S_{\mathbf{b}}^4$ reproduces the Weyl quantization of the same Wilson-'t Hooft line in the same theory, but placed in the spacetime $S^1 \times_{\epsilon} \mathbb{R}^2 \times \mathbb{R}$. Therefore, we again reach the conclusion that the vev of the Wilson-'t Hooft line with charge (4.60) and (4.64) are equal to the Wigner transform of the trigonometric transfer matrix (4.43).

4.4 Brane realization and string dualities

The AGT correspondence between Wilson-'t Hooft lines and Verlinde operators, which we exploited in section 4.3, can be realized in terms of branes in string theory. String dualities relate the brane configuration for the AGT correspondence to another configuration that realizes four-dimensional Chern-Simons theory, and in the latter setup the emergence of quantum integrability can be seen more transparently. Another chain of dualities relate these setups to the one studied in [50, 36], which provided the initial motivation for the present work. In this last section we discuss these brane constructions.

4.4.1 M-theory setup and brane realization

As explained in section 4.3, the field theoretic origin of the AGT correspondence is six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theory, which in our context is of type A_{N-1}

and compactified on an n -punctured torus $C_{1,n}$. This theory describes the low-energy dynamics of a stack of N M5-branes (modulo the center-of-mass degrees of freedom), intersected by n M5-branes.

Consider M-theory in the eleven-dimensional spacetime

$$M_{11} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_{\epsilon} S_3^1 \times_{-\epsilon} \mathbb{R}_{45}^2 \times S_6^1 \times \mathbb{R}_7 \times \mathbb{R}_8 \times \mathbb{R}_9 \times S_{10}^1. \quad (4.108)$$

(The subscripts indicate the directions in which the spaces extend.) We put N M5-branes $M5_i$, $i = 1, \dots, N$, on

$$M_{M5_i} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_{\epsilon} S_3^1 \times \{0\} \times S_6^1 \times \{0\} \times \{0\} \times \{0\} \times S_{10}^1. \quad (4.109)$$

They realize $\mathcal{N} = (2, 0)$ superconformal field theory on $\mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_{\epsilon} S_3^1 \times C_1$, with

$$C_1 = S_6^1 \times S_{10}^1. \quad (4.110)$$

Further, we introduce n M5-branes $M5^r$, $r = 1, \dots, n$, with worldvolumes

$$M_{M5^r} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_{\epsilon} S_3^1 \times \{0\} \times \{l^r\} \times \{0\} \times \mathbb{R}_8 \times \mathbb{R}_9 \times \{\theta^r\}. \quad (4.111)$$

These M5-branes create codimension-two defects in the six-dimensional theory, located at n points (l^r, θ^r) on C_1 , making C_1 an n -punctured torus $C_{1,n}$.

The two sets of M5-branes share a four-dimensional part of the spacetime, $\mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_{\epsilon} S_3^1$, and on this four-dimensional spacetime we get an $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $G = \text{SU}(N)^n$, described by the circular quiver with n nodes. (More precisely, the gauge group is $\text{SU}(N)^n \times \text{U}(1)$ but the $\text{U}(1)$ factor is associated with the center-of-mass and decoupled from the rest of the theory.) In fact, reduction on S_{10}^1 turns $M5_i$ into D4-branes $D4_i$ on

$$M_{D4_i} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_{\epsilon} S_3^1 \times \{0\} \times S_6^1 \times \{0\} \times \{0\} \times \{0\} \quad (4.112)$$

and $M5^r$ into NS5-branes $NS5^r$ on

$$M_{NS5^r} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_{\epsilon} S_3^1 \times \{0\} \times \{l^r\} \times \{0\} \times \mathbb{R}_8 \times \mathbb{R}_9, \quad (4.113)$$

and the above brane configuration becomes the well-known D4–NS5 brane configuration for the circular quiver theory [51]. The difference $l^{r+1} - l^r$ in the x^6 -coordinate between $NS5^{r+1}$ and $NS5^r$ is inversely proportional to the square of the gauge coupling for the r th gauge group, whereas the difference $\theta^{r+1} - \theta^r$ in the x^{10} -coordinate is the θ -angle for the r th gauge group.¹⁴

¹⁴To realize nonzero values for the parameters a^r and b^r , we break each D4-brane $D4_i$ into n segments $D4_i^r$ suspended between neighboring NS5-branes and allow these segments to be located anywhere on $\mathbb{R}_8 \times \mathbb{R}_9$. Then, a_i^r is a complex linear combination of the x^9 -coordinate of $D4_i^r$ and the background holonomy of the $\text{U}(1)$ gauge field on $D4_i^r$ around S_3^1 . The definition of b_i^r is similar, but involves both the x^8 - and x^9 -coordinates as well as a chemical potential for the magnetic charge at infinity which does not have a simple interpretation in this brane system.

A Wilson–t Hooft line in this four-dimensional theory is realized by an M2-brane on

$$M_{\text{M2}} = \{t_0\} \times \{0\} \times S_3^1 \times \{0\} \times S_6^1 \times \{0\} \times \mathbb{R}_8^{\geq 0} \times \{x_0\} \times \{\theta_0\}, \quad (4.114)$$

where $\mathbb{R}_8^{\geq 0}$ is the nonnegative part of \mathbb{R}_8 . Upon reduction on S_{10}^1 , this M2-brane becomes a D2-brane on

$$M_{\text{D2}} = \{t_0\} \times \{0\} \times S_3^1 \times \{0\} \times S_6^1 \times \{0\} \times \mathbb{R}_8^{\geq 0} \times \{x_0\} \quad (4.115)$$

and creates a Wilson–t Hooft line of the type considered in section 4.2. It corresponds to a Verlinde operator in Toda theory on $C_{1,n}$, constructed from a vertex operator transported along the path $S_6^1 \times \{\theta_0\}$. We will explain in a moment how to get the other relevant Verlinde operators.

To understand the relation to quantum integrable systems, let us compactify \mathbb{R}_9 to a circle S_9^1 of radius R_9 . By doing so, we are uplifting the four-dimensional gauge theory to a five-dimensional one, compactified on a circle. Indeed, by T-duality on S_9^1 we get D5-branes $\check{\text{D}}5_i$, NS5-branes $\check{\text{NS}}5^r$ and a D3-brane $\check{\text{D}}3$ with worldvolumes

$$M_{\check{\text{D}}5_i} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_\epsilon S_3^1 \times \{0\} \times S_6^1 \times \{0\} \times \{0\} \times \check{S}_9^1, \quad (4.116)$$

$$M_{\check{\text{NS}}5^r} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_\epsilon S_3^1 \times \{0\} \times \{l^r\} \times \{0\} \times \mathbb{R}_8 \times \check{S}_9^1, \quad (4.117)$$

$$M_{\check{\text{D}}3} = \{t_0\} \times \{0\} \times S_3^1 \times \{0\} \times S_6^1 \times \{0\} \times \mathbb{R}_8^{\geq 0} \times \check{S}_9^1. \quad (4.118)$$

The D5- and NS5-branes intersect along $\mathbb{R}_0 \times \mathbb{R}_{12}^2 \times_\epsilon S_3^1 \times \check{S}_9^1$, where a five-dimensional circular quiver theory arises. Recall that the radius \check{R}_9 of the dual circle \check{S}_9^1 is inversely proportional to the original radius, $\check{R}_9 = \alpha'/R_9$.

Going back to the M-theory setup, we reduce it on S_3^1 and apply T-duality on S_9^1 . Then, $\text{M}5_i$ become D5-branes $\widetilde{\text{D}}5_i$ on

$$M_{\widetilde{\text{D}}5_i} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times \{0\} \times S_6^1 \times \{0\} \times \{0\} \times \check{S}_9^1 \times S_{10}^1, \quad (4.119)$$

$\text{M}5^r$ become D3-branes $\widetilde{\text{D}}3^r$ on

$$M_{\widetilde{\text{D}}3^r} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times \{0\} \times \{l^r\} \times \{0\} \times \mathbb{R}_8 \times \{\phi^r\} \times \{\theta^r\}, \quad (4.120)$$

and M2 becomes a fundamental string $\widetilde{\text{F}}1$ on

$$M_{\widetilde{\text{F}}1} = \{t_0\} \times \{0\} \times \{0\} \times S_6^1 \times \{0\} \times \mathbb{R}_8^{\geq 0} \times \{\phi_0\} \times \{\theta_0\}. \quad (4.121)$$

4.4.2 String duality and 4d Chern-Simons theory

The N D5-branes $\widetilde{\text{D}}5_i$ support $\mathcal{N} = (1, 1)$ super Yang–Mills theory with gauge group $\text{SU}(N)$ on $\mathbb{R}_0 \times \mathbb{R}_{12}^2 \times S_6^1 \times \check{S}_9^1 \times S_{10}^1$. Crucially, this theory is deformed in the present case due to the fact that the product of S_3^1 and \mathbb{R}_{12}^2 was twisted. This is a deformation of the

type studied in [52], and in the sector in which the relevant supersymmetry is preserved, the deformed theory is actually equivalent to a four-dimensional variant of Chern–Simons theory, with Planck constant $\hbar \propto \epsilon$ [53]. Four-dimensional Chern–Simons theory, here placed on $\mathbb{R}_0 \times S_6^1 \times \check{S}_9^1 \times S_{10}^1$, depends topologically on the cylinder

$$\Sigma = \mathbb{R}_0 \times S_6^1 \quad (4.122)$$

and holomorphically on the torus

$$E = \check{S}_9^1 \times S_{10}^1. \quad (4.123)$$

The D3-branes $\widetilde{D3}^r$ create line defects extending in the longitudinal direction of Σ and located at the points

$$w^r = \phi^r + i\theta^r \quad (4.124)$$

on E . The fundamental string $\widetilde{F1}$, on the other hand, creates a Wilson line in the vector representation that winds around the circumferential direction and is located at

$$z_0 = \phi_0 + i\theta_0 \quad (4.125)$$

on E . Thus, on the cylinder Σ , we have the same situation as in figure 6(c), in which a quantum spin chain was described in terms of lines on a cylinder.

Indeed, a quantum integrable system emerges from such a configuration of line operators in four-dimensional Chern–Simons theory [54]. The Hilbert space of the integrable system is the space of states of the field theory on a time slice (where the x^0 -coordinate is constant) intersected by line operators extending in the time direction. The integrability is a consequence of the topological–holomorphic nature of the theory: by the topological invariance on Σ , one can slide line operators winding around the cylinder continuously along the longitudinal direction; and if two such line operators are located at different points on E , one can move them past each other without encountering a phase transition, thereby establishing the commutativity of transfer matrices.

It was argued in [53], based on the earlier work [50, 36], that a crossing of line defects created by a D3-brane and a fundamental string produces the elliptic L-operator (4.21) with $z = z_0$, $w = w^r$ and $y = 0$ ¹⁵ (up to shifts by constants). The parameter τ is the modulus of E :

$$\tau = i \frac{R_{10}}{\check{R}_9}. \quad (4.126)$$

Now, take the limit $\check{R}_9 \rightarrow 0$, in which \check{S}_9^1 shrinks to a point, S_9^1 decompactifies, and the five-dimensional circular quiver theory reduces to the four-dimensional one. This is the

¹⁵More generally, $D3^r$ can be split into two semi-infinite D3-branes $D3_+^r$ and $D3_-^r$, each ending on the stack of D5-branes at $x^8 = 0$. The parameter y is given by the separation of these two halves in E . In the five-dimensional circular quiver theory, the separation is proportional to the complex mass parameter m^r for the bifundamental hypermultiplet charged under the r th and $(r+1)$ th gauge groups.

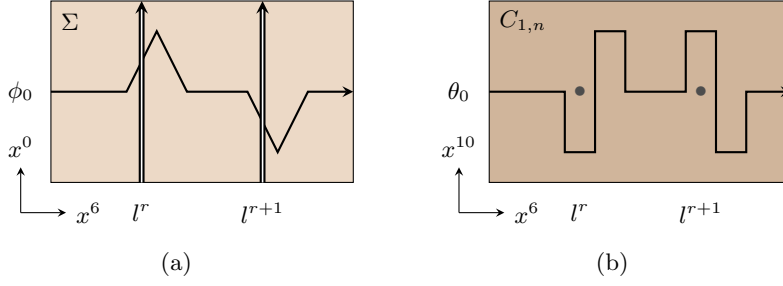


Figure 8: (a) A path in Σ bending near double lines. (b) The corresponding path in $C_{1,n}$ detours around the punctures.

trigonometric limit $\tau \rightarrow i\infty$, so we conclude that the transfer matrix constructed from the trigonometric L-operator arises from a Wilson–t Hooft line in the four-dimensional circular quiver theory.

In the previous sections we studied the transfer matrix $\mathcal{T}_{\sigma,m}$ associated with the cycle γ_σ in $C_{1,n}$ specified by an n -tuples of signs σ . The Wilson–t Hooft line considered above corresponds to a specific choice of σ . Those corresponding to the other choices can also be constructed in a similar manner, but the construction is a little more subtle. Let us explain how this construction works from the point of view of four-dimensional Chern–Simons theory.

For simplicity, let us set all $\theta^r = \theta_0$. (Since $\mathcal{T}_{\sigma,m}$ is independent of the spectral parameters z_0 and w^r , we do not lose anything by this specialization.) According to the analysis of [55], framing anomaly requires that if a Wilson line curves by an angle φ , its coordinate on S^1_{10} must be shifted by $-\epsilon N \varphi / 2\pi$. We can make use of this property to get a Wilson line supported on the cycle γ_σ : fix a small value φ_0 and let the Wilson line bends by the angle $\sigma^r \varphi_0$ right before it crosses the r th double line, as illustrated in figure 8.

The trigonometric limit $\check{R}_9 \rightarrow 0$ is equivalent to the limit $R_{10} \rightarrow \infty$. In this limit, $C_{1,n}$ is elongated by an infinite factor in the x^{10} -direction and the solid line is located at $z = l^r - \sigma^r i\infty$ when it crosses the r th double line. This is precisely the limit that appears in the definitions of the fundamental L-operators (4.38), from which $\mathcal{T}_{\sigma,m}$ is constructed.

The D5–NS5–D3 brane system (4.116)–(4.118) is another interesting duality frame. It is actually possible to introduce an additional set of NS5-branes so that the 5-brane system realizes a four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory on $\mathbb{R}^2_{12} \times_\epsilon S^1_3 \times \check{S}^1_9$. The D3-brane creates a surface defect in this theory. As expected, it acts on the partition function of the theory as an elliptic transfer matrix [50, 36].

5 Outlook

A some info.

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