

# 1 Special functions and some formulas

## 1.1 Theta functions

The theta function with characteristics is defined by

$$\theta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] (\zeta | \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n+a)^2 \tau + 2\pi i (n+a)(\zeta+b)}, \quad (1.1)$$

where  $\zeta$  is a complex variable and  $\tau$  is a complex parameter in the upper half-plane;  $\text{Im } \tau > 0$ . The Jacobi's theta functions are defined by

$$\theta_1(\zeta | \tau) = -\theta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (\zeta | \tau), \quad (1.2)$$

$$\theta_2(\zeta | \tau) = \theta_1(\zeta + 1/2 | \tau), \quad (1.3)$$

$$\theta_3(\zeta | \tau) = e^{\pi i (\zeta + \tau/4)} \theta_2(\zeta + \tau/2 | \tau), \quad (1.4)$$

$$\theta_4(\zeta | \tau) = \theta_3(\zeta + 1/2 | \tau). \quad (1.5)$$

The first of these,  $\theta_1$ , is an odd function of  $\zeta$  and satisfies

$$\theta_1(\zeta + 1 | \tau) = -\theta_1(\zeta | \tau), \quad (1.6)$$

$$\theta_1(\zeta + \tau | \tau) = -e^{\pi i (2\zeta - \tau)} \theta_1(\zeta | \tau). \quad (1.7)$$

The other three are even functions. We have

$$2\theta_1(\zeta + \zeta')\theta_1(\zeta - \zeta') = \bar{\theta}_4(\zeta)\bar{\theta}_3(\zeta') - \bar{\theta}_4(\zeta')\bar{\theta}_3(\zeta), \quad (1.8)$$

$$2\theta_2(\zeta + \zeta')\theta_2(\zeta - \zeta') = \bar{\theta}_3(\zeta)\bar{\theta}_3(\zeta') - \bar{\theta}_4(\zeta')\bar{\theta}_4(\zeta), \quad (1.9)$$

$$2\theta_3(\zeta + \zeta')\theta_3(\zeta - \zeta') = \bar{\theta}_3(\zeta)\bar{\theta}_3(\zeta') + \bar{\theta}_4(\zeta')\bar{\theta}_4(\zeta), \quad (1.10)$$

$$2\theta_4(\zeta + \zeta')\theta_4(\zeta - \zeta') = \bar{\theta}_4(\zeta)\bar{\theta}_3(\zeta') + \bar{\theta}_4(\zeta')\bar{\theta}_3(\zeta). \quad (1.11)$$

with  $\theta_a(\zeta) := \theta_a(\zeta | \tau)$  and  $\bar{\theta}_a(\zeta) := \theta_a(\zeta | \tau/2)$ .

It is useful to define another kind of theta function which we call the modified theta function:

$$\theta(z; p) = (z; p)_\infty (p/z; p)_\infty, \quad (1.12)$$

$$(z; p)_\infty := \prod_{k=0}^{\infty} (1 - p^k z), \quad |p| < 1. \quad (1.13)$$

It satisfies

$$\theta(z; p) = \theta(p/z; p). \quad (1.14)$$

Set  $z = e^{2\pi i \zeta}$ ,  $p = e^{2\pi i \tau}$  and introduce multiplicative notation

$$\theta_a(z; p) := \theta_a(\zeta | \tau). \quad (1.15)$$

Then the Jacobi's theta functions are rewritten in terms of the modified theta function as

$$\theta_1(z; p) = ip^{1/8}(p; p)_\infty z^{-1/2} \theta(z; p), \quad (1.16)$$

$$\theta_2(z; p) = p^{1/8}(p; p)_\infty z^{-1/2} \theta(-z; p), \quad (1.17)$$

$$\theta_3(z; p) = (p; p)_\infty \theta(-\sqrt{p}z; p), \quad (1.18)$$

$$\theta_4(z; p) = (p; p)_\infty \theta(\sqrt{p}z; p). \quad (1.19)$$

## 1.2 Elliptic gamma function

The elliptic gamma function is closely related to the triple gamma function and depends on two complex parameters  $p$  and  $q$ :

$$\Gamma(z; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - p^{j+1} q^{k+1} z^{-1}}{1 - p^j q^k z}; \quad |p|, |q| < 1. \quad (1.20)$$

It satisfies the identities

$$\Gamma(z; p, q) \Gamma(pq/z; p, q) = 1 \quad (1.21)$$

and

$$\Gamma(pz; p, q) = \theta(z; q) \Gamma(z; p, q), \quad (1.22)$$

$$\Gamma(qz; p, q) = \theta(z; p) \Gamma(z; p, q). \quad (1.23)$$

The function  $\Gamma(z; p, q)$  has a pole at  $z = p^{-j} q^{-k}$ , where  $j, k$  are non-negative integers. The residue at this pole is given by

$$\text{Res}_{z=p^{-j}q^{-k}} [\Gamma(z; p, q)] = \frac{(-1)^{jk+j+k} p^{(k+1)j(j+1)/2} q^{(j+1)k(k+1)/2}}{(p; p)_\infty (q; q)_\infty \theta(p, \dots, p^j; q) \theta(q, \dots, q^k; p)}, \quad (1.24)$$

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where we have introduced the notation  $\theta(z_1, \dots, z_n; q) := \theta(z_1; q) \cdots \theta(z_n; q)$ .

Let  $t_j, j = 1, \dots, 6$  be six complex parameters such that  $|t_j| < 1$  and  $\prod_{j=1}^6 t_j = pq$ . Then, we have the following identity proved in [1]:

$$\frac{(p; p)_\infty (q; q)_\infty}{2} \int_{\mathbb{T}} \frac{dz}{2\pi i z} \frac{\prod_{j=1}^6 \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} = \prod_{1 \leq j < k \leq 6} \Gamma(t_j t_k; p, q). \quad (1.25)$$

Here  $\mathbb{T}$  is the unit circle with counterclockwise orientation and

$$\Gamma(z^{\pm n}; p, q) := \Gamma(z^n; p, q) \Gamma(z^{-n}; p, q). \quad (1.26)$$

The left-hand side of the above formula is known as the elliptic beta integral.

### 1.3 The function $\Gamma_{\mathbf{b}}$ and upslon function

The function  $\Gamma_{\mathbf{b}}$  is related to the double gamma function, which is another relative of the ordinary gamma function. It can be defined by an integral representation

$$\log \Gamma_{\mathbf{b}}(x) = \int_0^\infty \frac{dt}{t} \left( \frac{e^{-xt} - e^{-Qt/2}}{(1 - e^{-bt})(1 - e^{-t/b})} - \frac{(Q - 2x)^2}{8e^t} - \frac{Q - 2x}{t} \right), \quad (1.27)$$

where  $x$  is a complex variable and  $\mathbf{b}$  is a complex parameter such that  $\text{Re } x, \text{Re } \mathbf{b} > 0$ , and  $Q = \mathbf{b} + \mathbf{b}^{-1}$ .

It is self-dual:

$$\Gamma_{\mathbf{b}}(x) = \Gamma_{1/\mathbf{b}}(x), \quad (1.28)$$

and satisfies the identity

$$\frac{\Gamma_{\mathbf{b}}(x + \mathbf{b})}{\Gamma_{\mathbf{b}}(x)} = \sqrt{2\pi} \frac{\mathbf{b}^{bx-1/2}}{\Gamma(\mathbf{b}x)}, \quad (1.29)$$

where the denominator in the right-hand side is the ordinary gamma function.

Through the function  $\Gamma_{\mathbf{b}}$ , we define upslon function as

$$\Upsilon_{\mathbf{b}}(x) = \frac{1}{\Gamma_{\mathbf{b}}(x)\Gamma_{\mathbf{b}}(Q - x)}. \quad (1.30)$$

The upslon function has an integral representation as well, which is convergent in the strip  $0 < \text{Re } x < Q$ :

$$\log \Upsilon_{\mathbf{b}}(x) = \int_0^\infty \frac{dt}{t} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2((Q/2 - x)t/2)}{\sinh(bt/2) \sinh(t/2b)} \right]. \quad (1.31)$$

The upslon function is also self-dual:

$$\Upsilon_{\mathbf{b}}(x) = \Upsilon_{1/\mathbf{b}}(x), \quad (1.32)$$

and satisfies the identity

$$\frac{\Upsilon_{\mathbf{b}}(x + \mathbf{b})}{\Upsilon_{\mathbf{b}}(x)} = \frac{\Gamma(\mathbf{b}x)}{\Gamma(1 - \mathbf{b}x)} \mathbf{b}^{1-2\mathbf{b}x}. \quad (1.33)$$

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identity

## References

- [1] V. P. Spiridonov, *On the elliptic beta function*, *Uspekhi Mat. Nauk* **56** (2001) 181–182.