Uncertainty Propagation in the Dynamics of a Nonlinear Random Bar

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Outline

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Oil Exploration: historical aspects

Modern oil exploration began in Poland in 1853 with the drilling of the first commercial oil well.

Oil demand has been increasing, since beginning of 20th century, due to a combination of several factors:

- demand for fuel (automobiles and industrial devices)
- high energy power of an oil barrel
- relative low cost of oil production (compared to coal)
- wide range of oil by-products

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Oil Exploration: economical aspects

Table: Distribution of energy supply by source for Brazil in 2010 and world in 2008.

Source	Brazil (%)	World (%)
Biomass	25.4	10.2
Coal	5.6	27.2
Hydraulic and Eletric Energy	14.7	2.3
Natural Gas	10.2	20.9
Oil and Oil by-products	38.6	32.8
Other	4.0	0.8
Uranium	1.5	5.8

Brazilian Energy Balance 2012: Year 2011.

Directional Drilling: opportunities and challenges

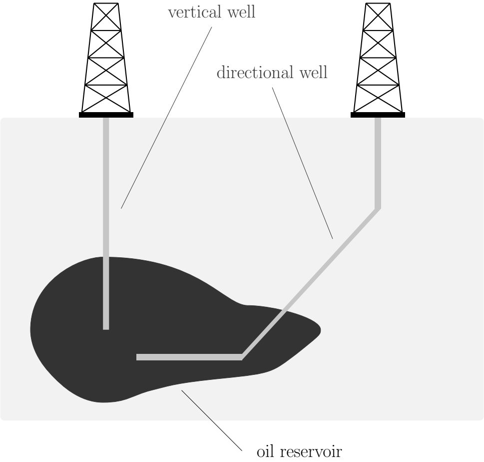


Figure: Schematic representation of two (onshore) oil wells. The left well configuration is vertical while the right one is directional.

Drillstring: an equipment for oil wells drilling

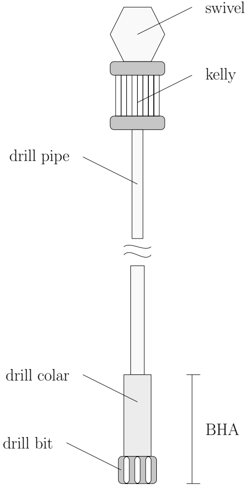


Figure: Schematic representation of a typical drillstring.

Drillstring Vibrations

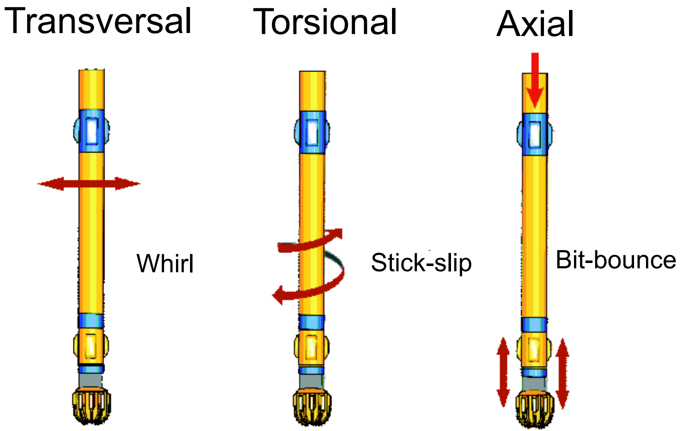


Figure: Schematic representation of the three vibration mechanisms that act in a typical drillstring.

Research Objectives

This work is a first attempt to understand the longitudinal dynamics of a horizontal drillstring.

We intend to:

- Investigate the nonlinear stochastic dynamics of an elastic bar, attached to discrete elements.

Work Plan:

- Construct a deterministic model for the system;
- Construct a stochastic model for the system;
- Compute uncertainty propagation of the random parameters;
- Investigate the influence of the lumped mass on the system dynamical behavior.

Physical System: fixed-mass-spring bar

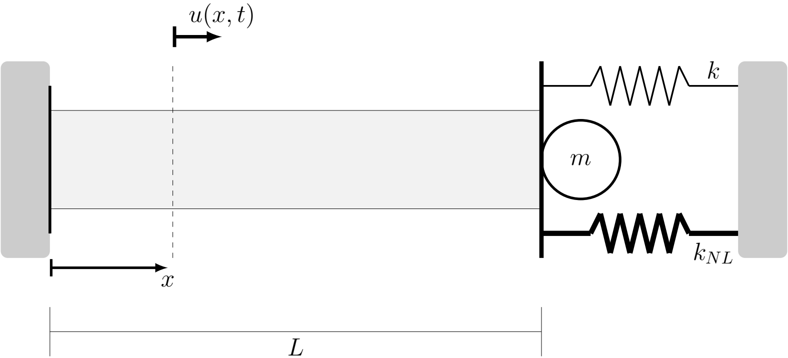


Figure: Sketch of a bar fixed at one and attached to two springs and a lumped mass on the other extreme, i.e. a fixed-mass-spring bar.

Strong Formulation

Find $u : [0, L] \times [0, T] \to \mathbb{R}$ that satisfies the equation of motion

$$\rho A \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x} \right) + f(x, t),$$

for all $(x, t) \in (0, L) \times (0, T)$, as well as the boundary conditions

$$u(0, t) = 0 \text{ and } EA \frac{\partial u}{\partial x}(L, t) = -ku(L, t) - k_{NL}\left(u(L, t) \right)^3 - m \frac{\partial^2 u}{\partial t^2}(L, t),$$

for all $t \in [0, T]$, and the initial conditions

$$u(x, 0) = u_0(x) \text{ and } \frac{\partial u}{\partial t}(x, 0) = v_0(x).$$

for all $x \in [0, L]$.

Variational Formulation: spaces of functions

Let

$$\mathcal{U}_t = \left\{ u(\cdot,t) : [0,L] \to \mathbb{R} \;\middle|\; \int_0^L \left(\frac{\partial u}{\partial x}(\cdot,t) \right)^2 dx < \infty, \quad u(0,t) = 0 \right\},$$

be the class of (time dependent) basis functions, and

$$\mathcal{W} = \left\{ w : [0,L] \to \mathbb{R} \;\middle|\; \int_0^L \left(\frac{\mathrm{d}w}{\mathrm{d}x} \right)^2 dx < \infty, \quad w(0) = 0 \right\},$$

be the class of weight functions.

Variational Formulation: weak problem

Find $u \in \mathcal{U}_t$ that satisfies the weak equation of motion

$$\mathcal{M}(\ddot{u}, w) + \mathcal{C}(\dot{u}, w) + \mathcal{K}(u, w) = \mathcal{F}(w) + \mathcal{F}_{NL}(u, w),$$

for all $w \in \mathcal{W}$, as well as the weak form of initial conditions

$$\widetilde{\mathcal{M}}(u(\cdot, 0), w) = \widetilde{\mathcal{M}}(u_0, w),$$

and

$$\widetilde{\mathcal{M}}(\dot{u}(\cdot, 0), w) = \widetilde{\mathcal{M}}(v_0, w).$$

Variational Formulation: system operators

- $\mathcal{M}$ is the mass operator

$$\mathcal{M}(\ddot{u}, w) = \int_0^L \rho A \ddot{u}(x,t) w(x) dx + m \ddot{u}(L,t) w(L)$$

- $\mathcal{C}$ is the damping operator

$$\mathcal{C}(\dot{u}, w) = \int_0^L c \dot{u}(x,t) w(x) dx$$

- $\mathcal{K}$ is the stiffness operator

$$\mathcal{K}(u, w) = \int_0^L EA u'(x,t) w'(x) dx + k u(L,t) w(L)$$

- $\widetilde{\mathcal{M}}$ is the associated mass operator

$$\widetilde{\mathcal{M}}(u, w) = \int_0^L \rho A u(x,t) w(x) dx$$

Variational Formulation: forcing operators

- $\mathcal{F}$ is the external force operator

$$\mathcal{F}(w) = \int_0^L f(x,t) w(x) dx$$

- $\mathcal{F}_{NL}$ is the nonlinear force operator

$$\mathcal{F}_{NL}(u,w) = -k_{NL} \left(u(L,t) \right)^3 w(L)$$

Linear Conservative Dynamics: harmonic solution

Consider the associated homogeneous equation to the variational problem above,

$$\mathcal{M}(\ddot{u}, w) + \mathcal{K}(u, w) = 0,$$

and supose it has a solution of the form

$$u(x,t) = e^{i\nu t}\phi(x),$$

where ν is a natural frequency, ϕ is the associated mode and $i = \sqrt{-1}$.

Linear Conservative Dynamics: an eigenvalue problem

Due to the linearity of the operators $\mathcal{M}$ and $\mathcal{K}$, one gets

$$-\nu^2 \mathcal{M}(\phi, w) + \mathcal{K}(\phi, w) = 0,$$

a generalized eigenvalue problem, with countable number of solutions.

Note that, for a fixed instant t, the $\{\phi_n\}_{n=1}^{+\infty}$ span the space of functions which contains the solution of the variational problem.

Linear Conservative Dynamics: orthogonality relations

It is possible to show that, a pair (ν_n^2, ϕ_n) and (ν_m^2, ϕ_m) with $\nu_m \neq \nu_n$, satisfy the following relations of orthogonality

$$\mathcal{M}(\phi_n, \phi_m) = 0,$$

and

$$\mathcal{K}(\phi_n, \phi_m) = 0.$$

These make then good choices for basis function when a weighted residual procedure to approximate the nonlinear weak equation solution.

Fixed-Mass-Spring Bar: orthogonal modes

The fixed-mass-spring bar has natural frequencies and orthogonal modes given by

$$\nu_n = \lambda_n \frac{\bar{c}}{L} \qquad \text{and} \qquad \phi_n(x) = \sin\left(\lambda_n \frac{x}{L}\right),$$

where $\bar{c} = \sqrt{E/\rho}$ and the λ_n are the solutions of

$$\cot(\lambda_n) + \left(\frac{kL}{AE}\right)\frac{1}{\lambda_n} - \left(\frac{m}{\rho A L}\right)\lambda_n = 0,$$

for $n = 1, \cdots, \infty$.

Fixed-Mass-Spring Bar: modes examples

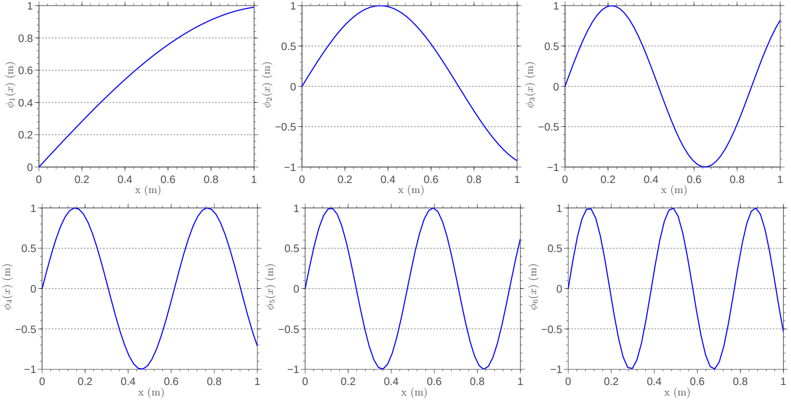


Figure: The first six orthogonal modes of a fixed-mass-spring bar.

Model Equation Discretization

The Galerkin method is used to approximate the solution of the variational problem

$$u(x,t) \approx \sum_{n=1}^{N} u_n(t)\phi_n(x),$$

where ϕ_n are orthogonal modes and u_n are time-dependent functions.

The result is a $N \times N$ set of nonlinear ordinary differential equations

$$[M]\,\ddot{\mathbf{u}}(t) + [C]\,\dot{\mathbf{u}}(t) + [K]\,\mathbf{u}(t) = \mathbf{f}(t) + \mathbf{f}_{NL}\left(\mathbf{u}(t)\right),$$

supplemented by a pair of initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{and} \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0.$$

Nonlinear ODE System Solution

An approximation for the solution of the initial value problem (IVP) above is computed by Newmark method,

$$\mathbf{v}_{n+1} = \mathbf{v}_n + (1-\gamma)\Delta t \mathbf{a}_n + \gamma \Delta t \mathbf{a}_{n+1},$$

$$\mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t \mathbf{v}_n + \left(\frac{1}{2} - \beta\right) \Delta t^2 \mathbf{a}_n + \beta \Delta t^2 \mathbf{a}_{n+1}.$$

where $\mathbf{d}_n$, $\mathbf{v}_n$ and $\mathbf{a}_n$ are approximations to $\mathbf{u}(t_n)$, $\dot{\mathbf{u}}(t_n)$ and $\ddot{\mathbf{u}}(t_n)$.

The result is a nonlinear system of algebraic equations with unknowns $\mathbf{d}_n$, $\mathbf{v}_n$ and $\mathbf{a}_n$, solved by Newton-Rapson method.

Stochastic Parameters Modeling

Consider a probability space $(\Omega, \mathbb{A}, \mathbb{P})$.

The external force is modeld as the random field

$$F : \Omega \times [0, L] \times [0, T] \to \mathbb{R},$$

such that

$$F(\omega, x, t) = \sigma \sin\left(\lambda_1 \frac{x}{L}\right) N(\omega, t),$$

where the $N(\omega, t)$ is a normalized Gaussian white noise.

The elastic modulus is modeled as a random variable

$$\mathtt{E} : \Omega \to (0, \infty).$$

Stochastic Initial–Boundary Value Problem

Therefore, the displacement of the bar is a random field

$$U : \Omega \times [0, L] \times [0, T] \to \mathbb{R},$$

such that

$$\rho A \frac{\partial^2 U}{\partial t^2} + c \frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left(\mathtt{E} A \frac{\partial U}{\partial x} \right) + F(\omega, x, t).$$

This problem has boundary and initial conditions similar to those defined in deterministic case, by changing u for U only.

Random External Force

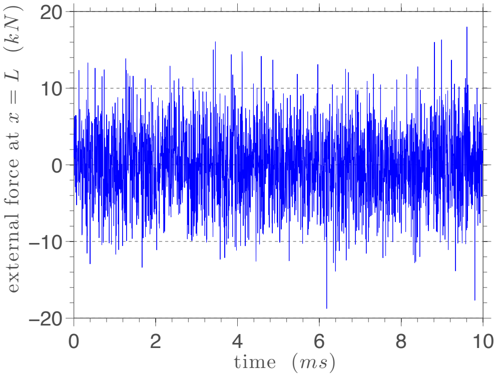


Figure: This figure illustrates a Gaussian white noise realization.

Random Variable Distribution: optimization problem

The maximum entropy principle is used to obtain the probability distribution of E.

It consists in choosing a probability density function (PDF), p_{E}, which maximizes the entropy

$$\mathbb{S}\left(p_{\mathrm{E}}\right) = -\int_{0}^{\infty} p_{\mathrm{E}}(\xi) \ln\left(p_{\mathrm{E}}(\xi)\right) d\xi,$$

subjected to the constraints:

- $\displaystyle\int_{0}^{\infty} p_{\mathrm{E}}(\xi) d\xi = 1,$
- $\mathbb{E}\left[\mathrm{E}\right] = \mu_{\mathrm{E}},$
- $\mathbb{E}\left[\mathrm{E}^2\right] = \sigma_{\mathrm{E}}^2 - \mu_{\mathrm{E}}$
- $\mathbb{E}\left[\ln\left(\mathrm{E}\right)\right] < \infty.$

Random Variable Distribution: maximum entropy PDF

The maximum entropy PDF for the optimization problem above is

$$p_{\mathrm{E}}(\xi) = \mathbb{1}_{(0,\infty)} \frac{1}{\mu_{\mathrm{E}}} \left(\frac{1}{\delta_{\mathrm{E}}^2}\right)^{\left(\frac{1}{\delta_{\mathrm{E}}^2}\right)} \frac{1}{\Gamma(1/\delta_{\mathrm{E}}^2)} \left(\frac{\xi}{\mu_{\mathrm{E}}}\right)^{\left(\frac{1}{\delta_{\mathrm{E}}^2} - 1\right)} \exp\left(-\frac{\xi}{\delta_{\mathrm{E}}^2 \mu_{\mathrm{E}}}\right),$$

which corresponds to the gamma distribution.

Random Variable Distribution: gamma PDF

An illustration for the PDF of the random variable $\mathtt{E}$, with mean $\mu_{\mathtt{E}} = 203$ *GPa* and dispersion factor $\delta_{\mathtt{E}} = 10\%$, is presented below.

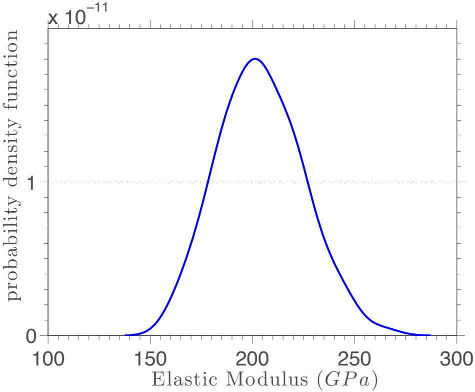


Figure: This figure illustrates the elastic modulus PDF.

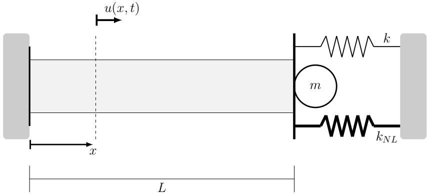
Monte Carlo Method: stochastic solver

The uncertainty propagation is computed by Monte Carlo method:

- An ensemble of 4096 realizations is used to sample the random space Ω;
- The realizations of $\mathtt{E}$ and F are generated by (Mersenne twister) pseudorandom number generator of Matlab;

Then, statistics of the random field U, such as mean value, confidence intervals and PDFs are obtained.

Simulation Parameters



Initial Conditions:

- $u_0(x) = \alpha_1 \sin\left(\lambda_3 \dfrac{x}{L}\right) + \alpha_2 x$
- $v_0(x) = 0$

Parametric Study:

- $\dfrac{m}{\rho A L} = 0.1,\ 1,\ 10,\ \textit{and}\ 50$

Deterministic Parameters:

- $\rho = 7900\ kg/m^3$
- $L = 1\ m$
- $A = 625\pi\ mm^2$
- $c = 0.1\ kNs/m$
- $k = 650\ N/m$
- $k_{NL} = 650 \times 10^{13}\ N/m^3$
- $\sigma = 5\ kN/m$
- $\alpha_1 = 0.1\ mm$
- $\alpha_2 = 0.5 \times 10^{-3}$

Study of Galerkin Approximation Convergence

Table: Residual (difference between two sucessive approximations) as function of the number of modes.

N	Residual $\|\cdot\|_{L_2}$	Residual $\|\cdot\|_{H^1}$
5	$\sim 4.1 \times 10^{-5}$	$\sim 1.1 \times 10^{-4}$
10	$\sim 6.8 \times 10^{-6}$	$\sim 8.2 \times 10^{-5}$
15	$\sim 2.2 \times 10^{-6}$	$\sim 8.2 \times 10^{-5}$
20	$\sim 2.2 \times 10^{-6}$	$\sim 1.3 \times 10^{-4}$
25	$\sim 1.8 \times 10^{-6}$	$\sim 1.4 \times 10^{-4}$
30	$\sim 1.0 \times 10^{-6}$	$\sim 9.1 \times 10^{-5}$

An approximation with 10 modes incurs an residual of $\mathcal{O}(10^{-5})$ in L_2 norm, and $\mathcal{O}(10^{-4})$ in H^1 norm,

Mean Value and Interval of Confidence for $U(\cdot, L, \cdot)$

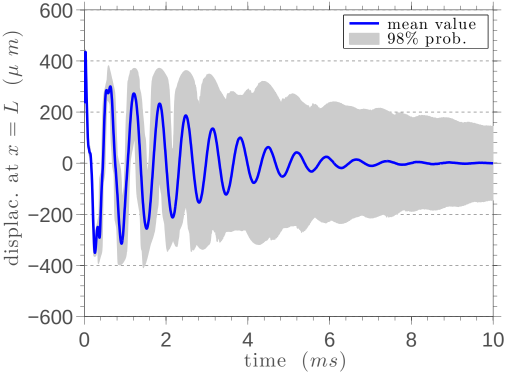


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the random process $U(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 0.1$.

Mean Value and Interval of Confidence for $U(\cdot, L, \cdot)$

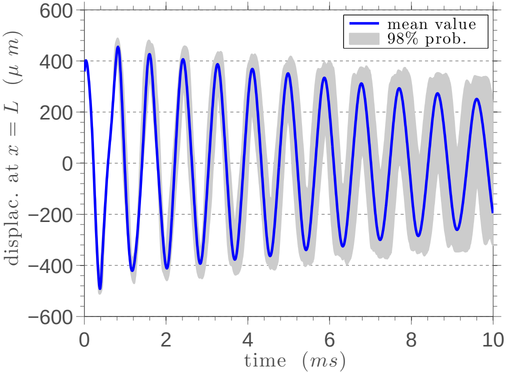


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the random process $U(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 1$.

Mean Value and Interval of Confidence for $U(\cdot, L, \cdot)$

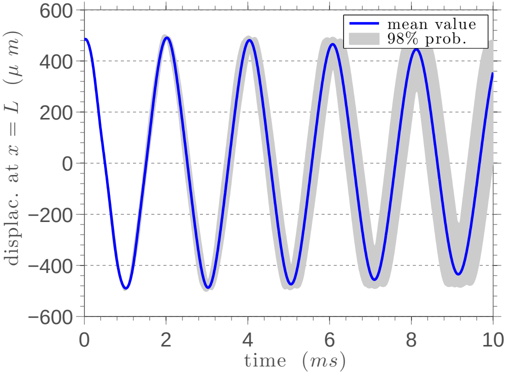


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the random process $U(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 10$.

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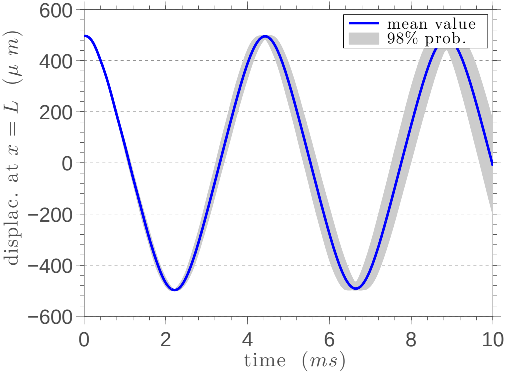

Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the random process $U(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 50$.

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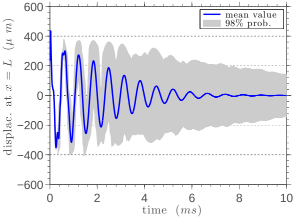

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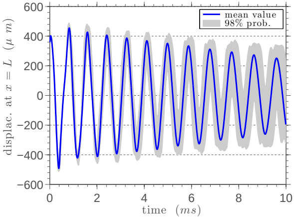
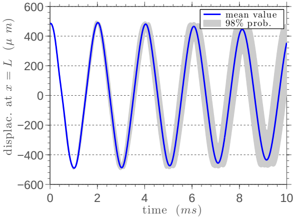
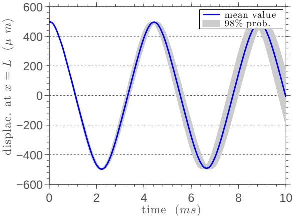


(b) $\dfrac{m}{\rho A L} = 1$

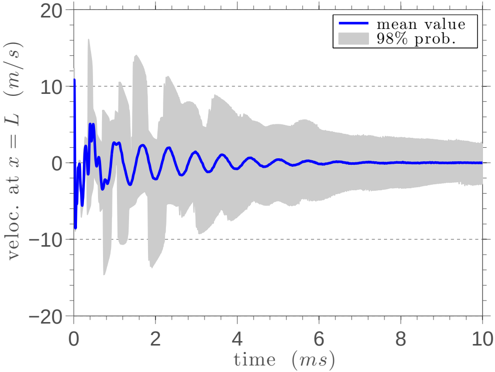


(c) $\dfrac{m}{\rho A L} = 10$



(d) $\dfrac{m}{\rho A L} = 50$

Mean Value and Interval of Confidence for $\dot{U}(\cdot, L, \cdot)$



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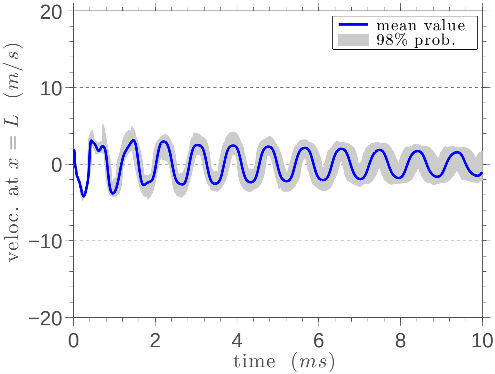

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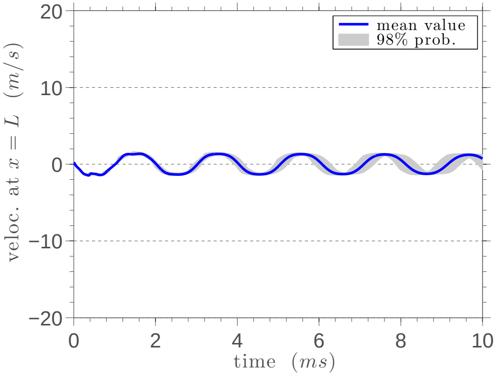

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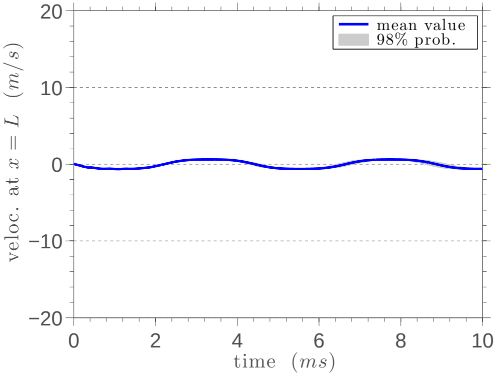

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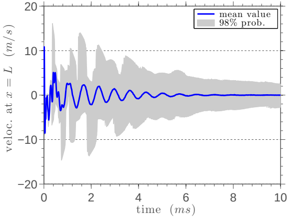

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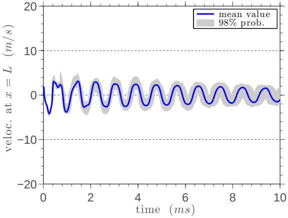

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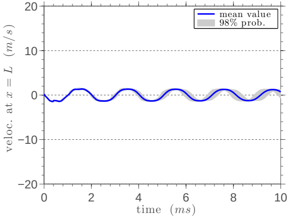

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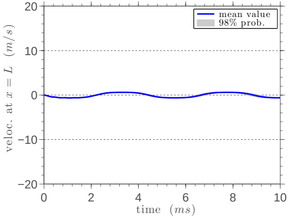

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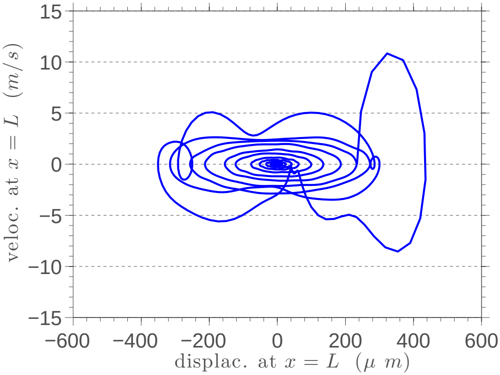
Mean Phase Space of Fixed-Mass-Spring Bar at $x = L$



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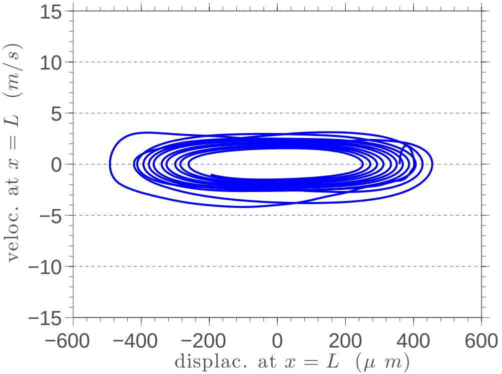

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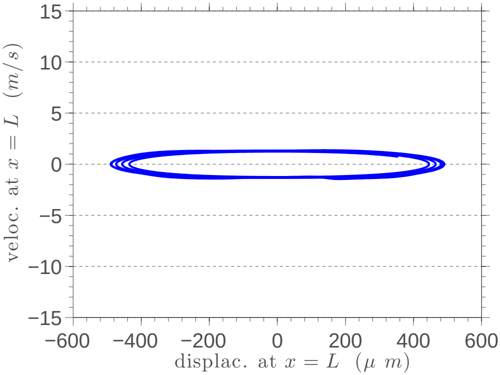

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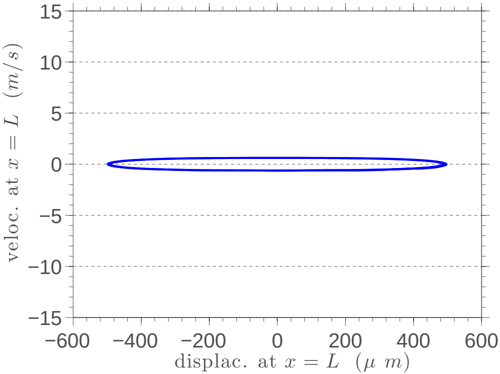

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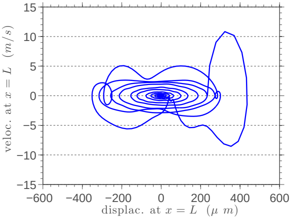

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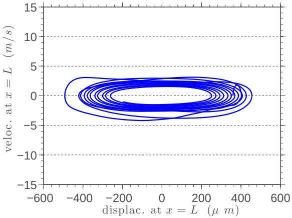
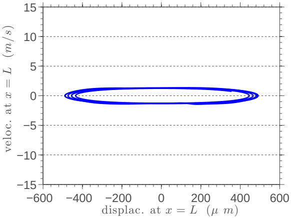
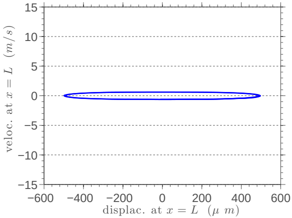


(b) $\dfrac{m}{\rho A L} = 1$



(c) $\dfrac{m}{\rho A L} = 10$



(d) $\dfrac{m}{\rho A L} = 50$

Spectral Density of $\dot{U}(\cdot, L, \cdot)$

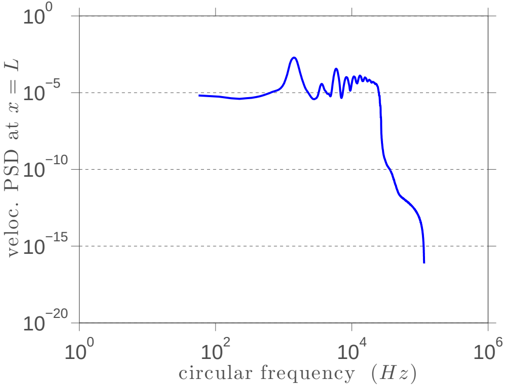


Figure: This figure illustrates the power spectral density for the random process $\dot{U}(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 0.1$.

Spectral Density of $\dot{U}(\cdot, L, \cdot)$

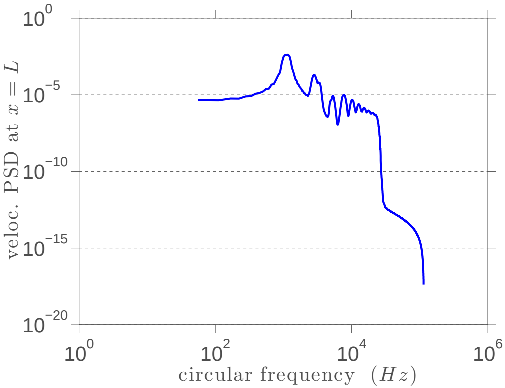


Figure: This figure illustrates the power spectral density for the random process $\dot{U}(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 1$.

Spectral Density of $\dot{U}(\cdot, L, \cdot)$

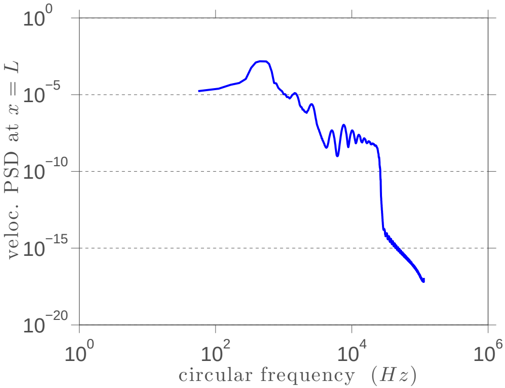


Figure: This figure illustrates the power spectral density for the random process $\dot{U}(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 10$.

Spectral Density of $\dot{U}(\cdot, L, \cdot)$

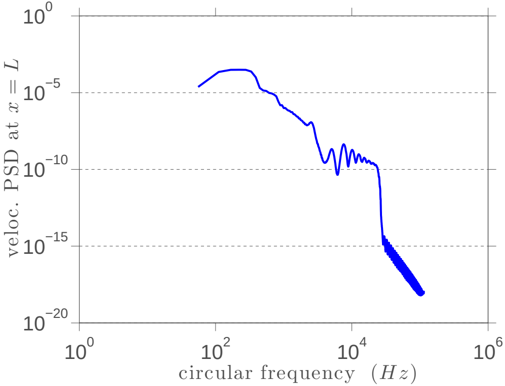
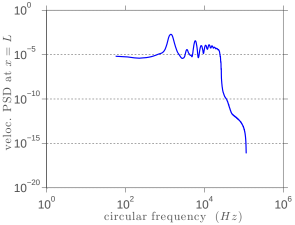
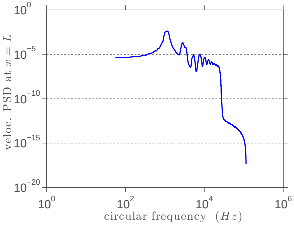


Figure: This figure illustrates the power spectral density for the random process $\dot{U}(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 50$.

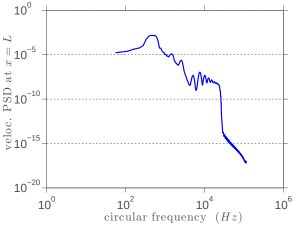
Spectral Density of $\dot{U}(\cdot, L, \cdot)$



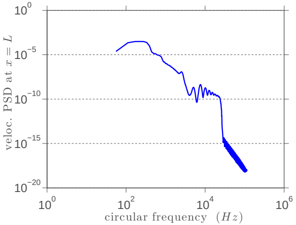
(a) $\dfrac{m}{\rho A L} = 0.1$



(b) $\dfrac{m}{\rho A L} = 1$



(c) $\dfrac{m}{\rho A L} = 10$



(d) $\dfrac{m}{\rho A L} = 50$

Probability Density Function of $U(\cdot, L, T)$

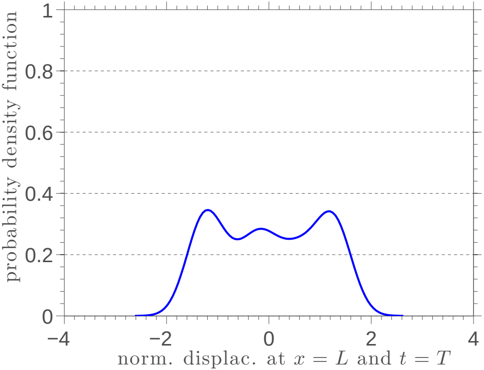


Figure: This figure illustrates an estimation to the PDF of the (normalized) random variable $U(\cdot, L, T)$, with $\dfrac{m}{\rho A L} = 0.1$.

Probability Density Function of $U(\cdot, L, T)$

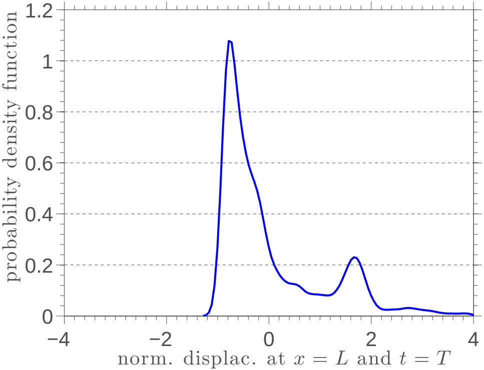


Figure: This figure illustrates an estimation to the PDF of the (normalized) random variable $U(\cdot, L, T)$, with $\dfrac{m}{\rho A L} = 1$.

Probability Density Function of $U(\cdot, L, T)$

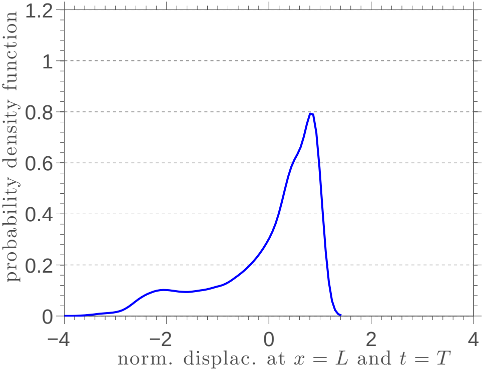


Figure: This figure illustrates an estimation to the PDF of the (normalized) random variable $U(\cdot, L, T)$, with $\dfrac{m}{\rho A L} = 10$.

Probability Density Function of $U(\cdot, L, T)$

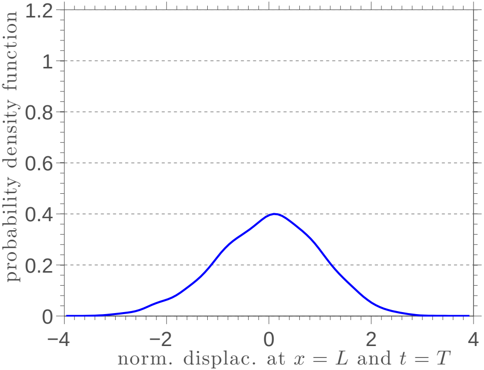
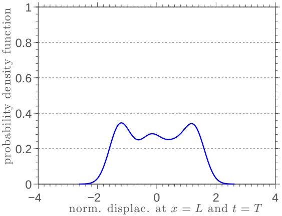
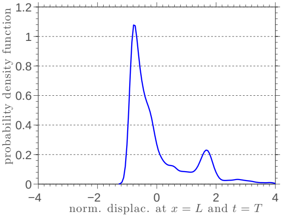


Figure: This figure illustrates an estimation to the PDF of the (normalized) random variable $U(\cdot, L, T)$, with $\dfrac{m}{\rho A L} = 50$.

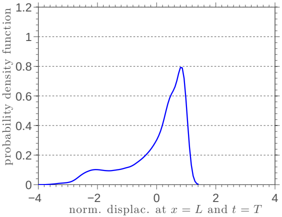
Probability Density Function of $U(\cdot, L, T)$



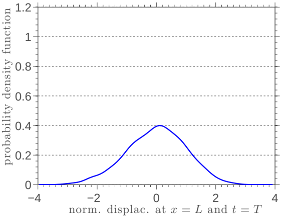
(a) $\dfrac{m}{\rho A L} = 0.1$



(b) $\dfrac{m}{\rho A L} = 1$



(c) $\dfrac{m}{\rho A L} = 10$



(d) $\dfrac{m}{\rho A L} = 50$

Probability Density Function of $\dot{U}(\cdot, L, T)$

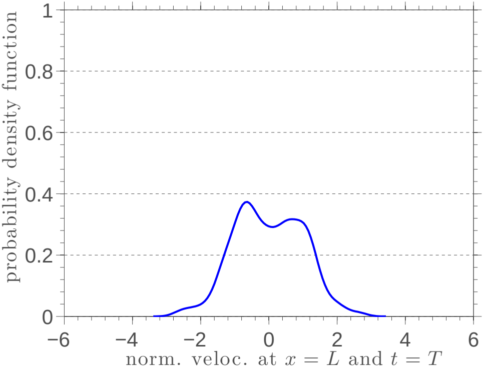


Figure: This figure illustrates an estimation to the PDF of the (normalized) random variable $\dot{U}(\cdot, L, T)$, with $\dfrac{m}{\rho A L} = 0.1$.

Probability Density Function of $\dot{U}(\cdot, L, T)$

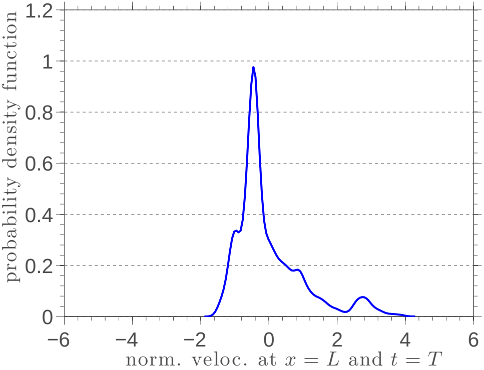


Figure: This figure illustrates an estimation to the PDF of the (normalized) random variable $\dot{U}(\cdot, L, T)$, with $\dfrac{m}{\rho A L} = 1$.

Probability Density Function of $\dot{U}(\cdot, L, T)$

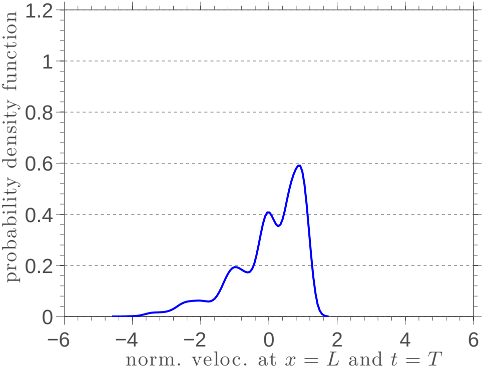


Figure: This figure illustrates an estimation to the PDF of the (normalized) random variable $\dot{U}(\cdot, L, T)$, with $\dfrac{m}{\rho A L} = 10$.

Probability Density Function of $\dot{U}(\cdot, L, T)$

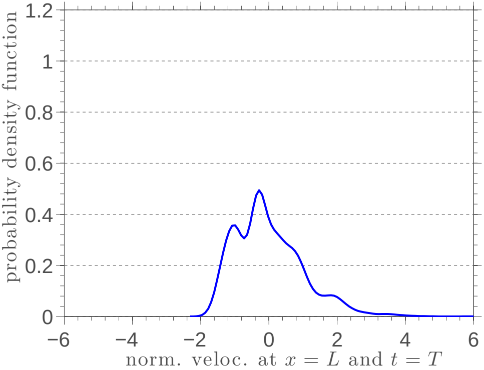
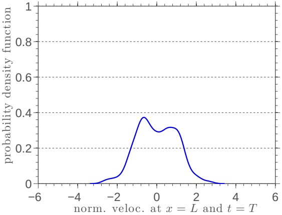
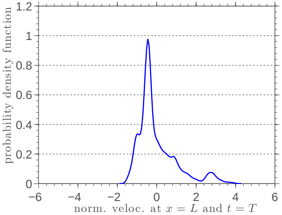


Figure: This figure illustrates an estimation to the PDF of the (normalized) random variable $\dot{U}(\cdot, L, T)$, with $\dfrac{m}{\rho A L} = 50$.

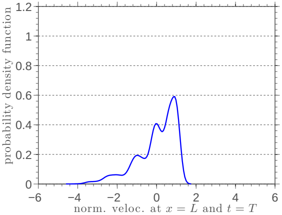
Probability Density Function of $\dot{U}(\cdot, L, T)$



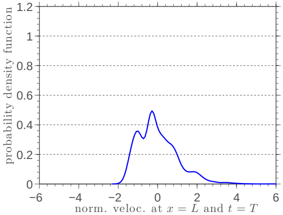
(a) $\dfrac{m}{\rho A L} = 0.1$



(b) $\dfrac{m}{\rho A L} = 1$



(c) $\dfrac{m}{\rho A L} = 10$



(d) $\dfrac{m}{\rho A L} = 50$

Final Remarks

- The system dynamics is altered when lumped mass changes;
- For large values of discrete–continuous mass ratio the system behaves like a mass-spring system;
- Irregular distribution of energy injected by random external force, due to the nonlinearity;
- Further analysis are necessary to better understand the nonlinear stochastic dynamics of this bar.

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