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EFFECTS OF A RANDOM CUBIC SPRING ON THE LONGITUDINAL DYNAMICS OF A BAR EXCITED BY A GAUSSIAN WHITE NOISE

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Abstract. *The study of the stochastic dynamics of continuous structures coupled with discrete elements is of interest in some fields of structural engineering. In this way, this papers deals with the analysis of the effects induced by a cubic spring, with gamma distributed random stiffness, on the longitudinal dynamics of an elastic bar which is also coupled with linear spring and a lumped mass. This system is subjected to a viscous dissipation mechanism and is excited by a Gaussian white noise. The model equations are discretized using the Galerkin method, and the propagation of uncertainties through the model are computed by the Monte Carlo method. Numerical simulations, indexed by the mean value of the cubic spring random stiffness, are conducted and show that this random cubic nonlinearity induces an increase in the level of uncertainty of the system response, in comparison with the linear case. Also, the random nonlinearity generates system responses which exhibit asymmetric and multimodal probability distributions.*

Keywords. *nonlinear dynamics, continuous-discrete system, uncertainty quantification, Monte Carlo method, parametric probabilistic approach*

1 INTRODUCTION

Many structures of interest in engineering applications are modeled as distributed parameter systems coupled with discrete elements. This case is extremely common when one leades with a large structure, coupled with a small structural element, and this small device influences significantly the global behavior of the structure. Just to cite some examples, drillstrings (Ritto *et al.*, 2013), carbon nanotubes (Murmu and Adhikari, 2011), naval structure (Rossit and Laura, 2001), etc. may be modeled in this way.

Due to the variability of parameters such physical constants, geometry, etc, these models are subject to a type uncertainty called *data uncertainty*. Also, wrong hypotheses about the physics of the system under analysis can be done, which results in the so called *model uncertainty*. To increase the reliability of these predictive models, it is extremely important to quantify these uncertainties (Soize, 2013).

This work intends to analyze the influence of a random cubic spring in the longitudinal dynamics of a elastic bar, excited by a Gaussian white noise. It a theoretical study, which uses a parametric probabilistic approach (Schuëller, 1997, 2007) to analyze the influence of a coupled discrete element into the stochastic dynamics of a nonlinear mechanical system.

The organization of the paper is as follows. The section 2 the deterministic and the stochastic modeling of the physical system of interest. In the section 3, some configurations of the model are analyzed in order to characterize the effect of the random cubic spring in the longitudinal dynamics of the system. Finally, in the section 4, the conclusions of the work are reemphasized.

2 MATHEMATICAL MODELING

2.1 Nominal (deterministic) model

The physical system of interest in this work, sketched in the Fig. 1, consists of an elastic bar which the left side is fixed at a rigid wall, and the right side is attached to a lumped mass and two springs (one linear and one nonlinear). This bar has a length L, cross section area A, and is made of a material with mass density ρ and elastic modulus E. It loses energy through a mechanism of viscous dissipation, with damping coefficient c. The stiffness constants of the linear and the nonlinear springs are respectively denoted by k and k_{NL}. The lumped mass on the right extreme is m.

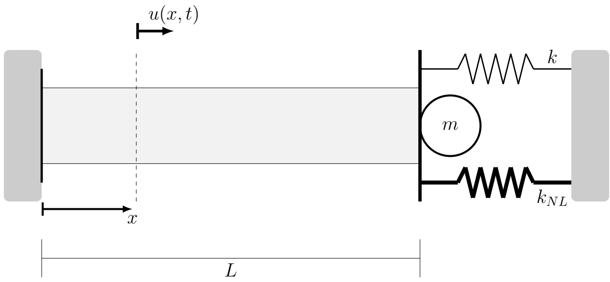


Figure 1: Sketch of a bar fixed at one and attached to two springs and a lumped mass on the other extreme.

Accordingly, the bar displacement $u(x,t)$ evolves according to

$$\begin{aligned}\rho A \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) + f(x,t) \\ &- \left(ku + k_{NL} u^3 + m \frac{\partial^2 u}{\partial t^2} \right) \delta(x-L),\end{aligned} \tag{1}$$

where the symbol $\delta(x-L)$ denotes the delta of Dirac distribution at $x = L$, and f is a distributed external force, which depends on the position x and time t.

The boundary conditions for this problem are

$$u(0,t) = 0, \quad \text{and} \quad EA \frac{\partial u}{\partial x}(L,t) = 0, \tag{2}$$

while the initial conditions read as

$$u(x,0) = u_0(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = v_0(x), \tag{3}$$

being u_0 and v_0 known functions of x.

Using the Galerkin method (Hughes, 2000) to construct an approximation to the solution of the boundary/initial value problem above, one has

$$u(x,t) \approx \sum_{n=1}^{N} u_n(t) \phi_n(x), \tag{4}$$

where the time-dependent functions u_n are the unknowns of the discretization, and the basis functions are

$$\phi_n(x) = \sin\left(\frac{\nu_n \, x}{c_L} \right), \tag{5}$$

with ν_n the $n-th$ natural frequency of the system, and $c_L = \sqrt{E/\rho}$.

This procedure results in the following initial value problem

$$[M] \, \ddot{\mathbf{u}}(t) + [C] \, \dot{\mathbf{u}}(t) + [K] \, \mathbf{u}(t) = \mathbf{f}(t) + \mathbf{f}_{NL} \left(\dot{\mathbf{u}}(t) \right), \tag{6}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{and} \qquad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \tag{7}$$

which is integrated using a Newmark scheme (Hughes, 2000). In this initial value problem, $\mathbf{u}(t)$ is the vector of unknowns, $[M]$ is the mass matrix, $[C]$ is the damping matrix, $[K]$ is the stiffness matrix. Also, $\mathbf{f}(t)$, $\mathbf{f}_{NL}\left(\mathbf{u}(t)\right)$, $\mathbf{u}_0$, and $\mathbf{v}_0$ are vectors which, respectively, represent the external force, the nonlinear force, the initial position, and the initial velocity.

2.2 Stochastic model

To introduce randomness in the physical system, the nonlinear spring stiffness k_{NL} is modeled as a gamma distributed random variable, which has probability density function (PDF) given by

$$p_{k_{NL}}(\xi) = \mathbb{1}_{(0,\infty)} \frac{1}{\mu_{k_{NL}}} \left(\frac{1}{\delta_{k_{NL}}^2}\right)^{\left(\frac{1}{\delta_{k_{NL}}^2}\right)} \frac{1}{\Gamma(1/\delta_{k_{NL}}^2)} \left(\frac{\xi}{\mu_{k_{NL}}}\right)^{\left(\frac{1}{\delta_{k_{NL}}^2}-1\right)} \exp\left(-\frac{\xi}{\delta_{k_{NL}}^2 \mu_{k_{NL}}}\right), \tag{8}$$

where $\mu_{k_{NL}}$ is the mean value, and $1 \leq \delta_{k_{NL}} \leq 1/\sqrt{2}$ a dispersion parameter.

For the sake of modeling consistency, this distribution was specified using the maximum entropy principle, as suggested by Soize (2013). This approach takes into account only the information that is assumed to know about the distribution of the random parameter. In the absence of experimental data, to the best of the authors' knowledge, this is the most conservative way to specify a probability distribution.

On the other hand, the distributed external force acting on the bar is arbitrarily assumed to be

$$F(x,t,\theta) = \sigma\phi_1(x)N(t,\theta), \tag{9}$$

where σ is the force amplitude, ϕ_1 the first elastic mode of the bar, and $N(t,\theta)$ is a Gaussian white-noise with zero mean and unit variance.

Thus, the stochastic dynamics of the structure (after the discretization in space) is described by the following stochastic initial value problem

$$[M]\,\ddot{\mathbf{U}}(t,\theta) + [C]\,\dot{\mathbf{U}}(t,\theta) + [K]\,\mathbf{U}(t,\theta) = \mathbf{F}(t,\theta) + \mathbf{f}_{NL}\left(\dot{\mathbf{U}}(t,\theta)\right), \tag{10}$$

$$\mathbf{U}(0,\theta) = \mathbf{u}_0 \qquad \text{and} \qquad \dot{\mathbf{U}}(0,\theta) = \mathbf{v}_0, \tag{11}$$

where the random processes $\mathbf{U}$ and $\mathbf{F}$ respectively represent the vector of unknowns and the external force.

To calculate the propagation of uncertainties of the random parameters through the nonlinear dynamics of the bar, the Monte Carlo (MC) method (Robert and Casella, 2010), (Cunha Jr *et al.*, 2014) is used.

3 NUMERICAL EXPERIMENTATION

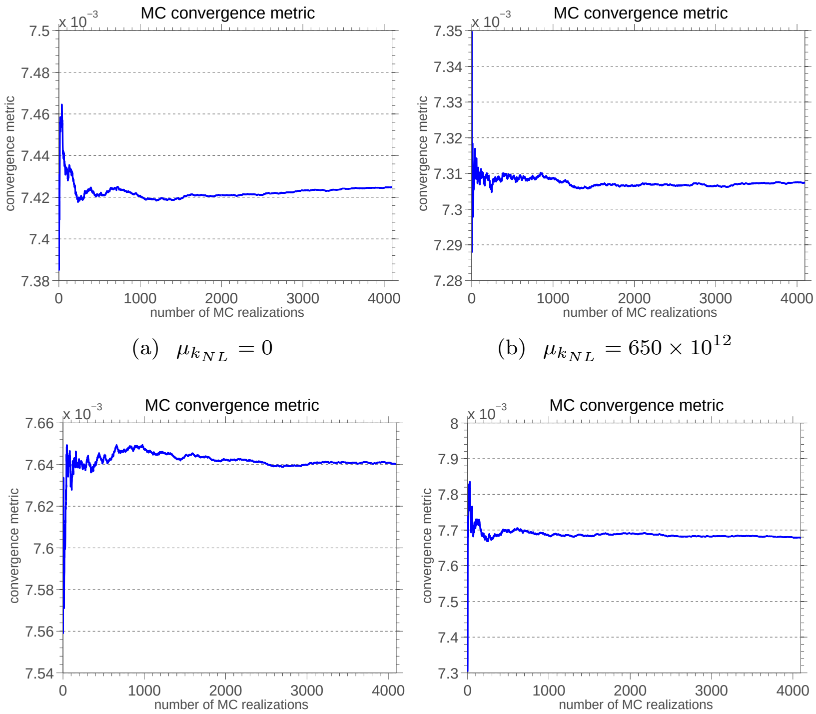
The frequency band of interest in this problem is fixed as $B = [0, 23.8]\ kHz$. In this way, the evolution of the nonlinear dynamical system is investigated for a "temporal window" defined by the interval $[t_0, t_f] = [0, 30]\ ms$, using a time step $\Delta t = 2.16 \times 10^{-5}\ s$, and the deterministic parameters presented in Table 1. The initial conditions are assumed to be zero velocity, and a non-zero displacement, which is the sum of the third order of the system with a linear displacement. This initial displacement reaches the maximum at the right end of the bar, and is used to "activate" the cubic non-linearity.

Table 1: . Nominal parameters used in the simulations.

parameter	value	unit
ρ	7900	kg/m^3
E	203	GPa
A	625π	mm^2
L	1	m
c	5	N/s
k	650	N/m
σ	5	kN

A numerical study, indexed by the mean value of k_{NL}, denoted by $\mu_{k_{NL}}$, is performed in order to investigate the effect of the nonlinearity (introduced in the system by the cubic spring) in the stochastic dynamics. To this end, four values are considered $\mu_{k_{NL}} = 650 \times \{0,\ 10^{12},\ 10^{13},\ 10^{14}\}\ N/m^3$.

Note that for the first case $\mu_{k_{NL}} = 0$ and the problem is linear. Also, in all the cases in which $\mu_{k_{NL}} \neq 0$, the dispersion factor of the distribution is $\delta_{k_{NL}} = 0.2$. It is considered as the nominal (deterministic) model, the one which has the parameters of the Table 1, $k_{NL} = \mu_{k_{NL}}$, and $f(x,t) = \sigma\phi_1(x)$.



(a) $\mu_{k_{NL}} = 0$ (b) $\mu_{k_{NL}} = 650 \times 10^{12}$

(c) $\mu_{k_{NL}} = 650 \times 10^{13}$ (d) $\mu_{k_{NL}} = 650 \times 10^{14}$

Figure 2: This figure illustrates the convergence metric of MC simulation as a function of the number of realizations, for several values of $\mu_{k_{NL}}$.

3.1 Study of convergence of the approximations

Initially it is analyzed the convergence of the approximations constructed for the deterministic and stochastic models. For the deterministic case, one takes into account the convergence of the Galerkin method, which is evaluated using as metric the L_2 norm of the difference between two successive approximations. In this case, an approximation built with 10 bases functions is sufficient to ensure a residue of $\mathcal{O}\left(10^{-6}\right)$, and is used in all the simulations reported in this work.

On the other hand, in the stochastic case, is necessary to assess the convergence of MC simulation. For this purpose, it is taken into consideration the following metric

$$\texttt{conv}(n_s) = \left(\frac{1}{n_s} \sum_{n=1}^{n_s} \int_{t=t_0}^{t_f} \left\| \mathbf{U}(t,\theta_n) \right\|^2 \, dt \right)^{1/2}, \tag{12}$$

where n_s is the number of MC realizations, and $\|\cdot\|$ denotes the standard Euclidean norm. This metric allows one to evaluate the convergence of the approximation $\mathbf{U}(t,\theta_n)$ in the mean-square sense. See Soize (2005) for further details.

The evolution of $\texttt{conv}(n_s)$ as a function of n_s, for several values of $\mu_{k_{NL}}$, can be seen in Fig. 2. Note that for $n_s = 4096$ the metric value has reached a steady value in all cases studied. So, all the stochastic simulations reported in this work use $n_s = 4096$.

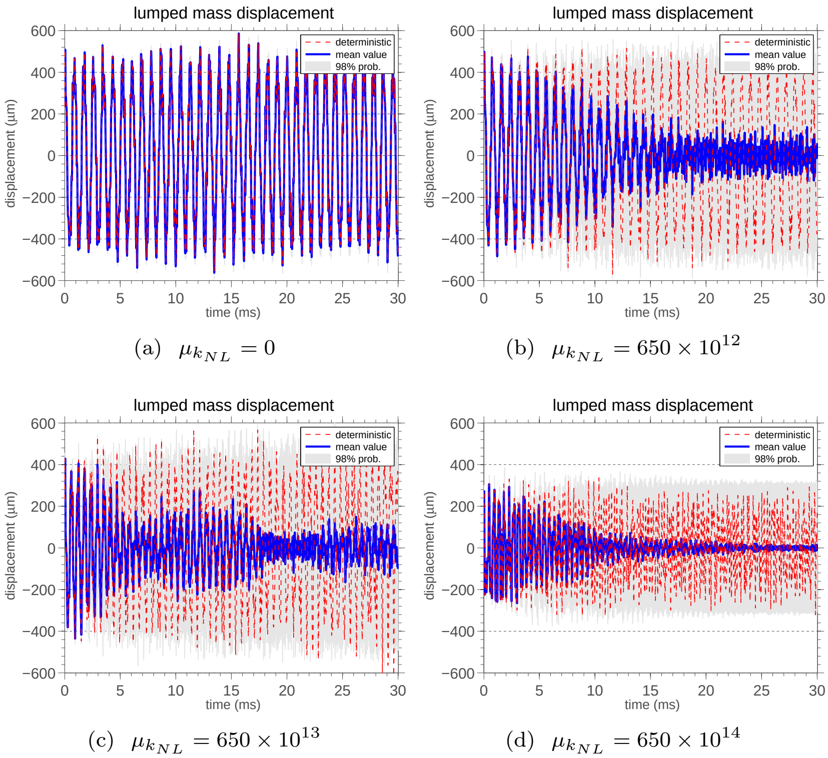
3.2 Propagation of uncertainties through the system

In this section it is analyzed, in a qualitative manner, how the uncertainties (due to randomness of k_{NL} and the external forcing) are propagated through the model.

One can observe in Fig. 3, for different values of $\mu_{k_{NL}}$, the evolution of the of the bar right extreme displacement. Are represented in the same graph, the displacement mean value (blue line), its nominal value (red line), and an envelope of reliability (grey shadow), wherein a realization of the stochastic system has 98% of probability of being contained. Figure 4 presents the same information for the bar right extreme velocity.

Note that, when $\mu_{k_{NL}} = 0$, the difference between the nominal and the mean value of the responses are virtually nonexistent. This is due to the linearity of this system, combined with the symmetry of the external forcing with respect to its mean value. This combination almost cancels the random effects.

A completely different situation can be envisioned when $\mu_{k_{NL}} \neq 0$, once now there is a nonlinearity active in the dynamical system. Note that now the confidence interval is clearly visible in the graphs, which indicates a higher level of uncertainty in the system response.

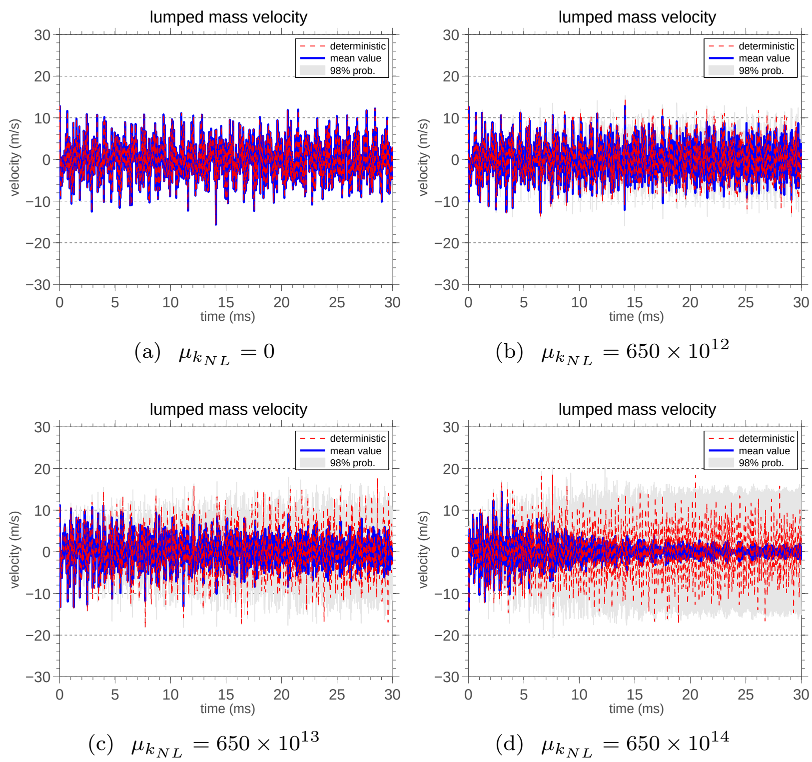


(a) $\mu_{k_{NL}} = 0$ (b) $\mu_{k_{NL}} = 650 \times 10^{12}$

(c) $\mu_{k_{NL}} = 650 \times 10^{13}$ (d) $\mu_{k_{NL}} = 650 \times 10^{14}$

Figure 3: This figure illustrates the mean value (blue line), the nominal value (red line), and a 98% of probability interval of confidence (grey shadow) for the bar right extreme displacement, as function of the time, for several values of $\mu_{k_{NL}}$.

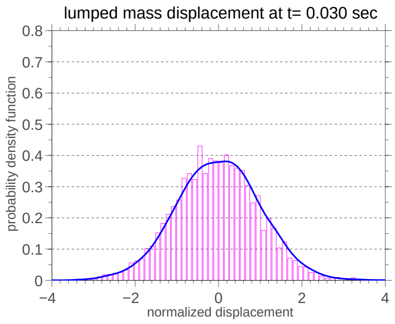
One can observe that, for all nonlinear cases studied, the amplitude of the confidence interval increases with the time, which indicates that so does the level of uncertainty. This is also shown by the difference between the nominal and the mean value of the response, which initiate close and over time become very different. This increase in the amplitude of the confidence interval, and consequently, in the level of uncertainty of the response, is a direct result of the accumulation of uncertainties over the evolution of the dynamical system.



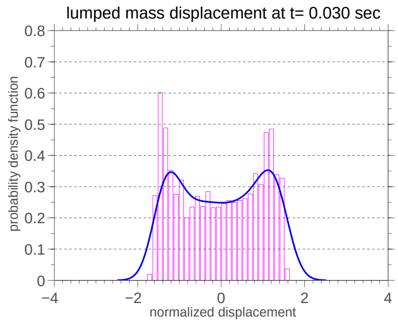
(a) $\mu_{k_{NL}} = 0$ (b) $\mu_{k_{NL}} = 650 \times 10^{12}$

(c) $\mu_{k_{NL}} = 650 \times 10^{13}$ (d) $\mu_{k_{NL}} = 650 \times 10^{14}$

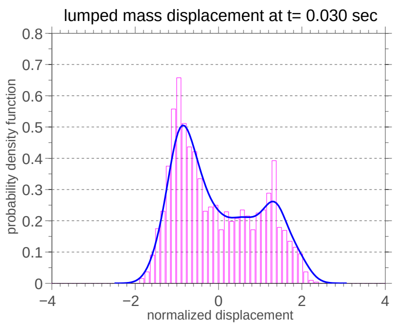
Figure 4: This figure illustrates the mean value (blue line), the nominal value (red line), and a 98% of probability interval of confidence (grey shadow) for the bar right extreme velocity, as function of the time, for several values of $\mu_{k_{NL}}$.



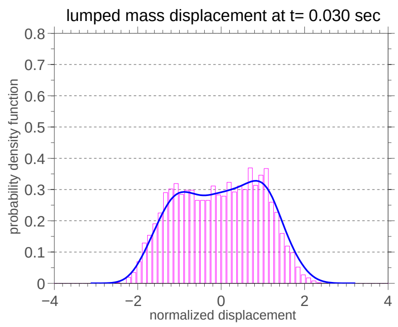
(a) $\mu_{k_{NL}} = 0$



(b) $\mu_{k_{NL}} = 650 \times 10^{12}$



(c) $\mu_{k_{NL}} = 650 \times 10^{13}$



(d) $\mu_{k_{NL}} = 650 \times 10^{14}$

Figure 5: This figure illustrates estimations to the PDFs of the (normalized) bar right extreme displacement at the instant of time $t = 30\ ms$, for several values of $\mu_{k_{NL}}$.

3.3 Probability distribution of the system response

This section presents the analysis of the probability density function (PDF) associated with the the response of the dynamical system under study. In Fig. 5, one can observe, for different values of $\mu_{k_{NL}}$, estimations for the PDFs of the (normalized[1]) displacements of the bar right extreme at the instant $t = 30\ ms$. Figure 6 presents the same information for the bar right extreme velocity at $t = 30\ ms$.

Note that for the linear system ($\mu_{k_{NL}} = 0$) the PDF estimation remind a Gaussian. This is not surprising, as the forcing acting on the system is of Gaussian nature, and this one is invariant under linear systems. However, something interesting happens in the cases where there is a nonlinearity. Asymmetries with respect to the mean value begin to be observed. When $\mu_{k_{NL}} = 650 \times 10^{12}$, the asymmetry in the displacement is small, but more pronounced for $\mu_{k_{NL}} = 650 \times \{10^{13},\ 10^{14}\}$. It is also possible to see a multimodal behavior on the displacement PDFs, which indicates a high number of realizations close to the values that correspond to the peaks. Regarding the velocity PDFs, asymmetries and multimodality are evident and, well pronounced, when $\mu_{k_{NL}} = 650 \times \{10^{12},\ 10^{13}\}$, but discrete or non-existent in the case of $\mu_{k_{NL}} = 650 \times 10^{14}$.

4 CONCLUDING REMARKS

In this work it was discussed the the effects induced by a random cubic spring, on the longitudinal damped dynamics of an elastic bar, which is also coupled with linear spring and a lumped mass, excited by a Gaussian white noise. This continuous mechanical system, coupled with discrete elements, is relevant in the study of structures that are coupled to small elements, whose dimensions are negligible when compared to the dimensions of the structure, but the coupling effects has influence on the dynamic behavior of the physical system. The analysis of the system was performed, indexed by the mean value of the cubic spring stiffness, and shows that the level of uncertainty of the system response, in comparison with the linear case, increases a lot with the time. The study developed also shows that the random cubic nonlinearity induces asymmetric and multimodal probability distributions in the system response.

5 ACKNOWLEDGMENTS

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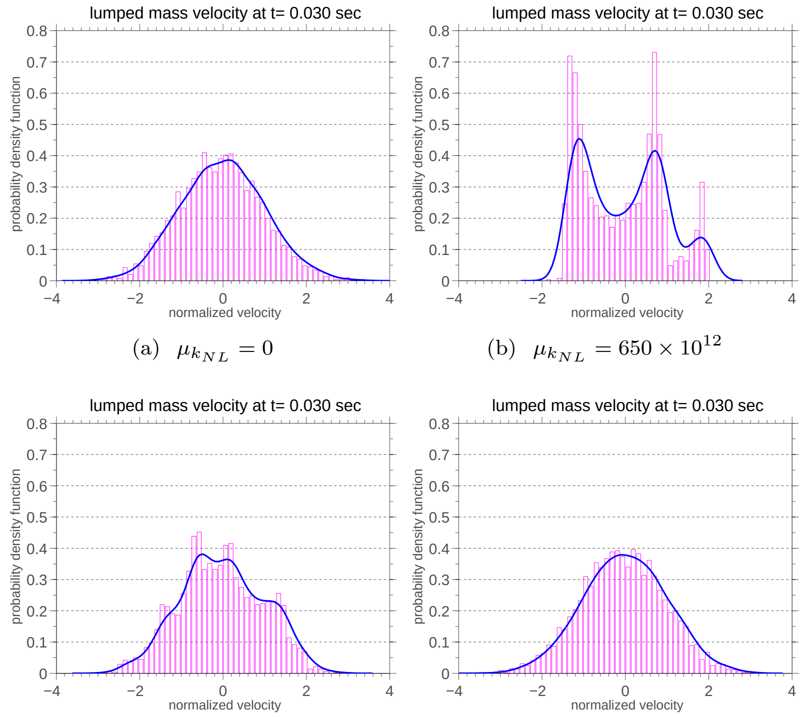
[1] In this context normalized means a random variable with zero mean and unit standard deviation.



(a) $\mu_{k_{NL}} = 0$

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Figure 6: This figure illustrates estimations to the PDFs of the (normalized) bar right extreme velocity at the instant of time $t = 30\ ms$, for several values of $\mu_{k_{NL}}$.

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