# Analysis of the Nonlinear Stochastic Dynamics of an Elastic Bar with an Attached End Mass

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# Outline



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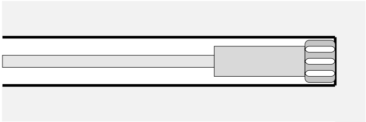
# Motivation: structures with small devices

Some engineering structures have small parts (negligible dimensions), which induce significant effects on their global behavior.

- drill-bit on a drillstring;
- stabilizers on a column;
- buckyball on a carbon nanotube;
- etc.



# Modeling and uncertainties

Typical modeling:

- continuous system coupled with discrete elements.

But these models are subjected uncertainties due to:

- variability in the system parameters;
- inaccuracies in model conception.

# Research objectives

This work intends to:

- Illustrates a consistent methodology to analyze the stochastic nonlinear dynamics of a continuous system attached to discrete elements.
- Investigate the effects of a coupled discrete elements in the dynamic response of this nonlinear stochastic system.

# Physical system: fixed-mass-spring bar

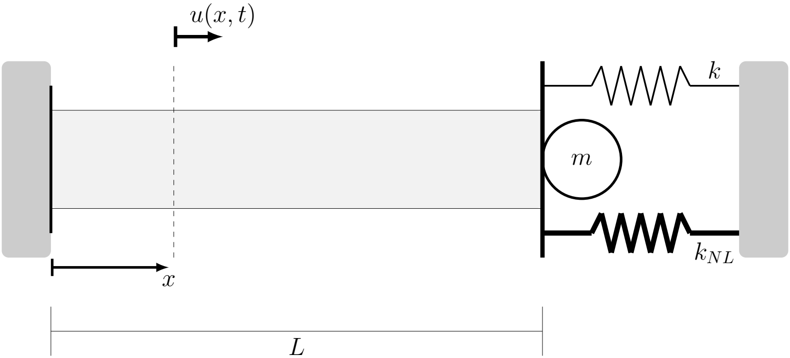



Figure: Sketch of a bar fixed at one and attached to two springs and a lumped mass on the other extreme, i.e. a fixed-mass-spring bar.

# Strong formulation

Find a *suitable* displacement field that satisfies

$$\rho A \, \frac{\partial^2 u}{\partial t^2} + c \, \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( EA \, \frac{\partial u}{\partial x} \right) + \left( ku + k_{NL} u^3 + m \, \frac{\partial^2 u}{\partial t^2} \right) \delta(x - L) = f(x, t)$$

as well as the boundary conditions

$$u(0, t) = 0, \quad \text{and} \quad EA \, \frac{\partial u}{\partial x} (L, t) = 0,$$

and the initial conditions

$$u(x, 0) = u_0(x), \quad \text{and} \quad \frac{\partial u}{\partial t} (x, 0) = v_0(x).$$

# Weak formulation

Find *suitable* $u \in \mathcal{U}_t$ that satisfies the weak equation of motion

$$\mathcal{M}(\ddot{u}, w) + \mathcal{C}(\dot{u}, w) + \mathcal{K}(u, w) = \mathcal{F}(w) + \mathcal{F}_{NL}(u, w),$$

for all weigth $w \in \mathcal{W}$, as well as the weak form of initial conditions

$$\widetilde{\mathcal{M}}(u(\cdot, 0), w) = \widetilde{\mathcal{M}}(u_0, w),$$

and

$$\widetilde{\mathcal{M}}(\dot{u}(\cdot, 0), w) = \widetilde{\mathcal{M}}(v_0, w).$$

# Model equation discretization

The model equation is discretized by the Galerkin method

$$u(x,t) \approx \sum_{n=1}^{N} u_n(t)\phi_n(x),$$

where $\phi_n$ are orthogonal modes and $u_n$ are time-dependent functions.

The result is a $N \times N$ set of nonlinear ordinary differential equations

$$[M]\,\ddot{\mathbf{u}}(t) + [C]\,\dot{\mathbf{u}}(t) + [K]\,\mathbf{u}(t) = \mathbf{f}(t) + \mathbf{f}_{NL}\left(\mathbf{u}(t)\right),$$

supplemented by a pair of initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{and} \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0.$$

# Nonlinear ODE system solution

The Newmark method is used for temporal integration

$$\mathbf{v}_{n+1} = \mathbf{v}_n + (1-\gamma)\Delta t \mathbf{a}_n + \gamma \Delta t \mathbf{a}_{n+1},$$

$$\mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t \mathbf{v}_n + \left(\frac{1}{2} - \beta\right) \Delta t^2 \mathbf{a}_n + \beta \Delta t^2 \mathbf{a}_{n+1}.$$

where $\mathbf{d}_n$, $\mathbf{v}_n$ and $\mathbf{a}_n$ are approximations to $\mathbf{u}(t_n)$, $\dot{\mathbf{u}}(t_n)$ and $\ddot{\mathbf{u}}(t_n)$.

The result is a nonlinear system of algebraic equations with unknowns $\mathbf{d}_n$, $\mathbf{v}_n$ and $\mathbf{a}_n$, solved by Newton-Rapson method.

# Stochastic initial–boundary value problem

Given a probability space $(\Omega, \mathbb{A}, \mathbb{P})$, and considering that $E$ and $f(x,t)$ are stochastic, find a *suitable* random displacement field

$$U : \Omega \times [0, L] \times [0, T] \to \mathbb{R},$$

which satisfies

$$\rho A \frac{\partial^2 U}{\partial t^2} + c \frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left( \mathbf{E} A \frac{\partial U}{\partial x} \right) + \left( kU + k_{NL} U^3 + m \frac{\partial^2 U}{\partial t^2} \right) \delta(x - L) = F(\omega, x, t)$$

as well as *appropriate* boundary and initial conditions.

# Random external force

The external force is modeled as the random field such that

$$F(\omega, x, t) = \sigma \sin\left(\lambda_1 \frac{x}{L}\right) N(\omega, t),$$

where the $N(\omega, t)$ is a normalized Gaussian white noise.

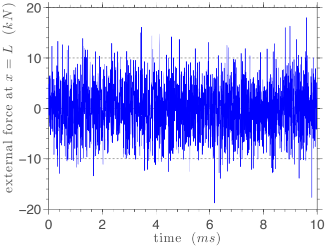



Figure: This figure illustrates a Gaussian white noise realization.

# Random elastic modulus

The elastic modulus is modeled as a random variable.

According to the maximum entropy principle, the probability density function of **E** maximizes the entropy

$$\mathbb{S}\left(p_{\mathbf{E}}\right) = - \int_0^\infty p_{\mathbf{E}}(\xi) \ln\left(p_{\mathbf{E}}(\xi)\right) d\xi,$$

subjected to the constraints:

- $\displaystyle\int_0^\infty p_{\mathbf{E}}(\xi) d\xi = 1,$
- $\mathbb{E}\left[\mathbf{E}\right] = \mu_{\mathbf{E}},$
- $\mathbb{E}\left[\ln\left(\mathbf{E}\right)\right] < \infty \quad \Longrightarrow \quad \mathbb{E}\left[\mathbf{E}^2\right] < \infty.$

# Random elastic modulus

The maximum entropy PDF corresponds to a gamma distribution

$$p_{\mathbf{E}}(\xi) = \mathbb{1}_{(0,\infty)} \frac{1}{\mu_{\mathbf{E}}} \left( \frac{1}{\delta_{\mathbf{E}}^2} \right)^{\left( \frac{1}{\delta_{\mathbf{E}}^2} \right)} \frac{1}{\Gamma(1/\delta_{\mathbf{E}}^2)} \left( \frac{\xi}{\mu_{\mathbf{E}}} \right)^{\left( \frac{1}{\delta_{\mathbf{E}}^2} - 1 \right)} \exp\left( - \frac{\xi}{\delta_{\mathbf{E}}^2 \mu_{\mathbf{E}}} \right).$$

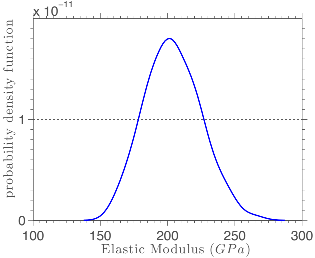


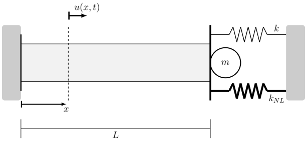

Figure: This figure illustrates the elastic modulus PDF.

# Stochastic simulation

Monte Carlo method, with 4096 realizations of $\mathbf{E}$ and $F$, is used to compute the uncertainty propagation.

deterministic parameters:

- $\rho = 7900 \; kg/m^3$
- $L = 1 \; m$
- $A = 625\pi \; mm^2$
- $c = 5 \; kNs/m$
- $k = 650 \; N/m$
- $k_{NL} = 650 \times 10^{13} \; N/m^3$
- $\sigma = 5 \; kN/m$
- $\alpha_1 = 0.1 \; mm$
- $\alpha_2 = 0.5 \times 10^{-3}$




initial conditions:

- $u_0(x) = \alpha_1 \sin\left(\lambda_3 \dfrac{x}{L}\right) + \alpha_2 x$
- $v_0(x) = 0$

parametric study:

- $\dfrac{m}{\rho A L} = 0.1, \; 1, \; 10, \; and \; 50$

# Mean value and interval of confidence for $U(\cdot, L, \cdot)$

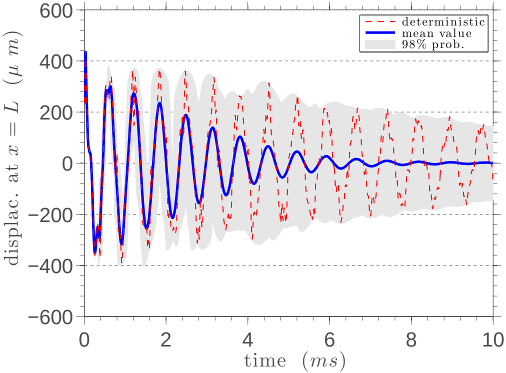
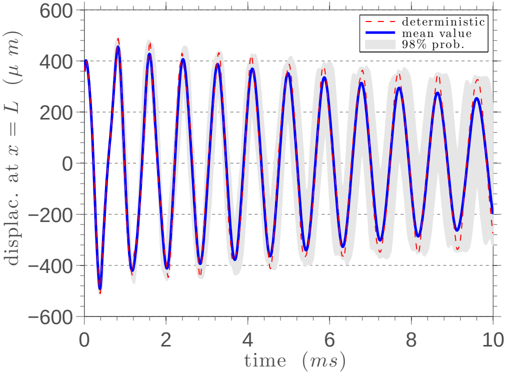



Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the random process $U(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 0.1$.

# Mean value and interval of confidence for $U(\cdot, L, \cdot)$

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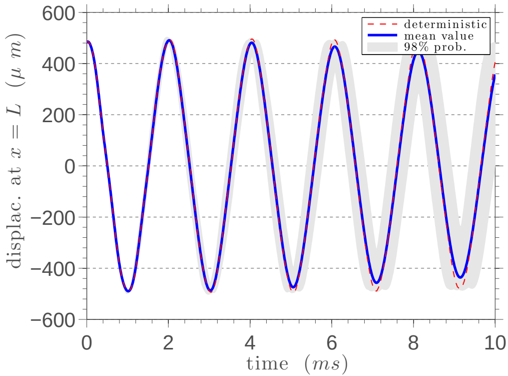


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the random process $U(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 10$.

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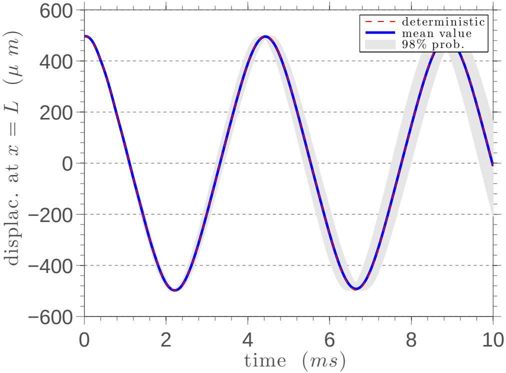


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the random process $U(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 50$.

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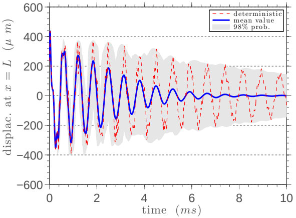
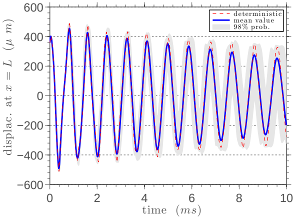
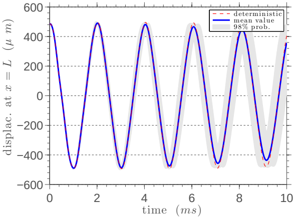


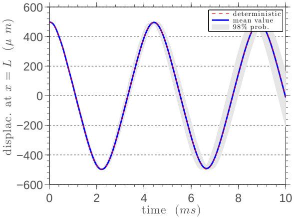

(a) $\dfrac{m}{\rho A L} = 0.1$




(b) $\dfrac{m}{\rho A L} = 1$




(c) $\dfrac{m}{\rho A L} = 10$




(d) $\dfrac{m}{\rho A L} = 50$

# Mean phase space of fixed-mass-spring bar at $x = L$

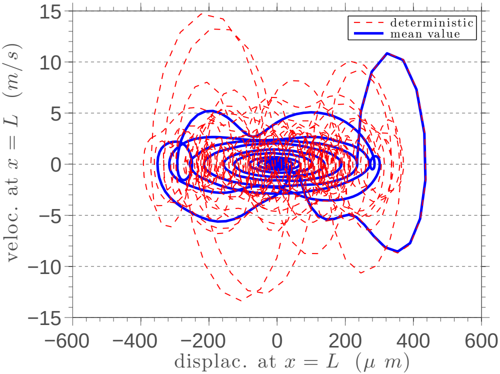



Figure: This figure illustrates the mean phase space of fixed-mass-spring bar at $x = L$, with $\dfrac{m}{\rho A L} = 0.1$.

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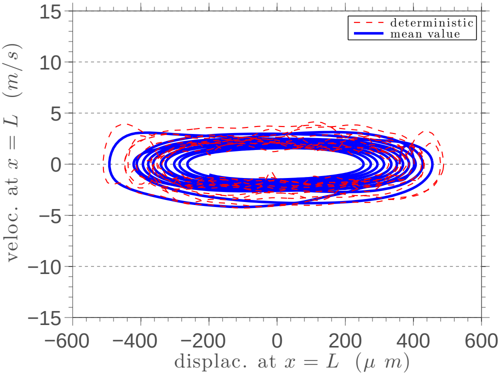


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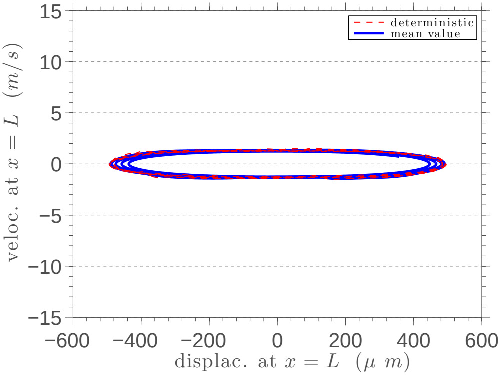


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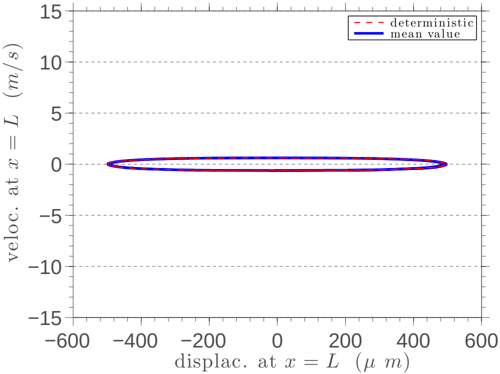


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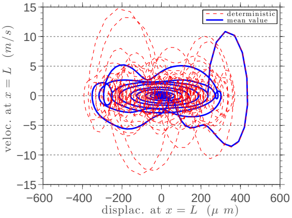


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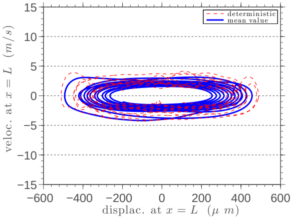


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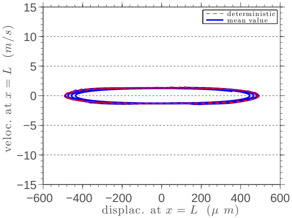
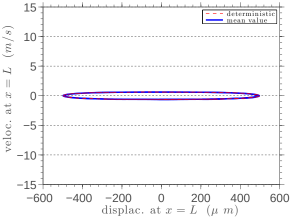



(c) $\dfrac{m}{\rho A L} = 10$




(d) $\dfrac{m}{\rho A L} = 50$

# Power spectral density of $\dot{U}(\cdot, L, \cdot)$

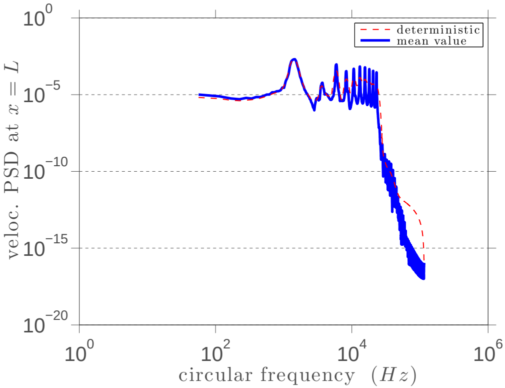



Figure: This figure illustrates the power spectral density for the random process $\dot{U}(\cdot, L, \cdot)$, with $\dfrac{m}{\rho A L} = 0.1$.

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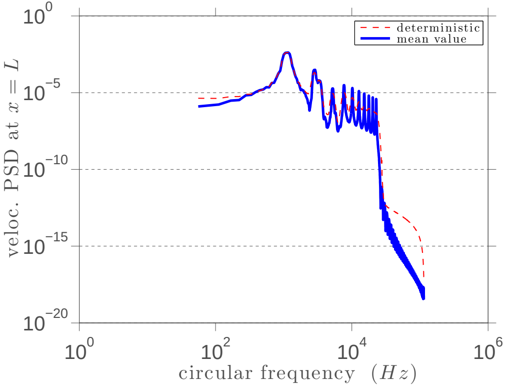


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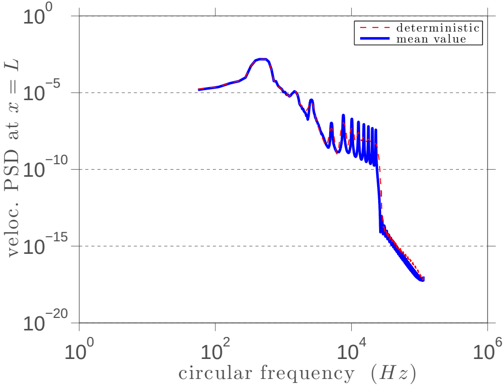


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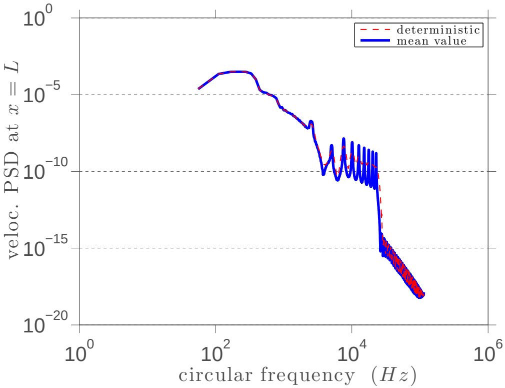


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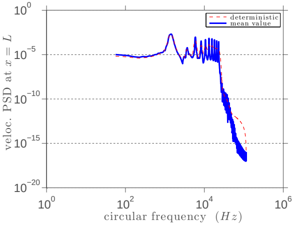


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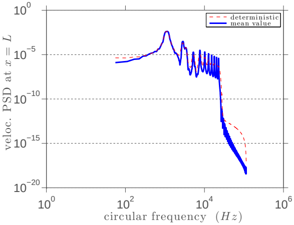


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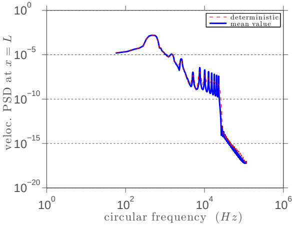
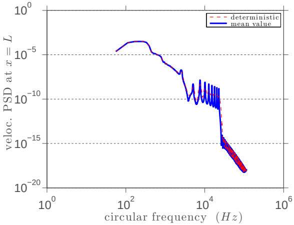



(c) $\dfrac{m}{\rho A L} = 10$

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# Probability density function of $U(\cdot, L, T)$

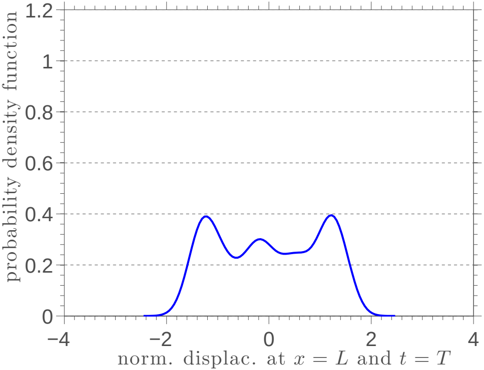


Figure: This figure illustrates an estimation to the PDF of the (normalized) random variable $U(\cdot, L, T)$, with $\dfrac{m}{\rho A L} = 0.1$.

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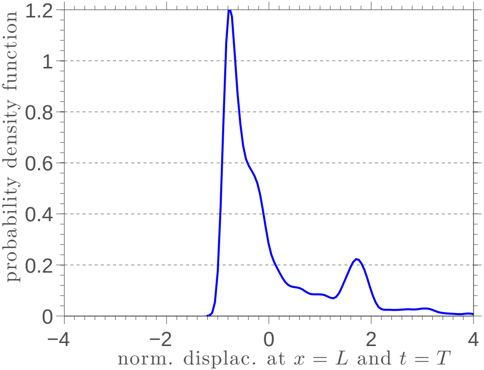


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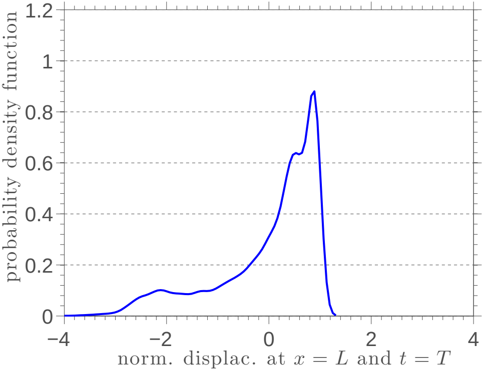


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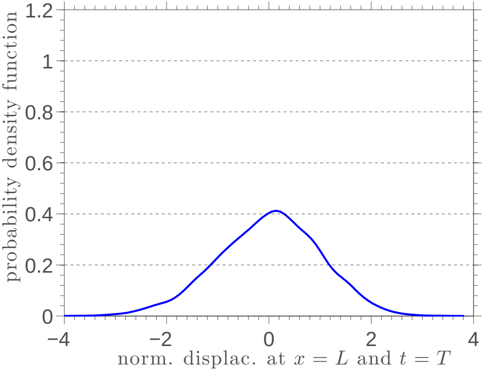


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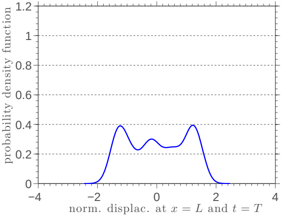


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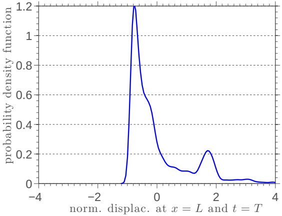


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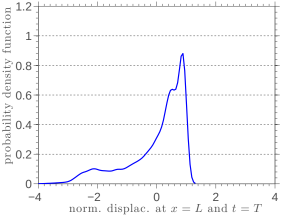
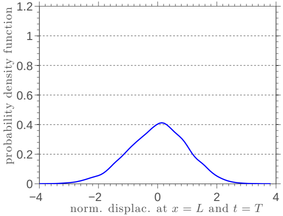



(c) $\dfrac{m}{\rho A L} = 10$




(d) $\dfrac{m}{\rho A L} = 50$

# Concluding remarks

- This work illustrates a consistent methodology to analyze the stochastic dynamics of a continuos/discrete system;
- Large values of the lumped mass makes the behavior of the continuous system like a mass-spring oscillator;
- A reduction of uncertainty in the system response is observed when the lumped mass increases;
- An irregular energy distribution through the spectrum occurs maybe due to the spring cubic nonlinearity;
- The system response presents multimodal distributions.

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- FAPERJ

# References


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