Effects of a random cubic spring on the longitudinal dynamics of a bar excited by a Gaussian white noise

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Outline

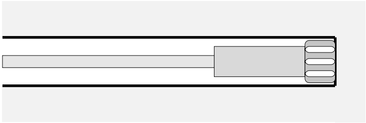
Section 1

Introduction

Motivation: structures with small devices

Some engineering structures have small parts (negligible dimensions), which induce significant effects on their global behavior.

- drill-bit on a drillstring;
- stabilizers on a column;
- buckyball on a carbon nanotube;
- etc.



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Modeling and uncertainties

Typical model:

- continuous system coupled with discrete elements.

But these models are subjected uncertainties due to:

- variability in the system parameters;
- inaccuracies in model conception.

Research objectives

This work intends to:

- Illustrates a consistent methodology to analyze the stochastic nonlinear dynamics of a continuous system attached to discrete elements.
- Investigate the effects of a coupled discrete elements in the dynamic response of this nonlinear stochastic system.

Section 2

Deterministic Model

Mechanical system of interest

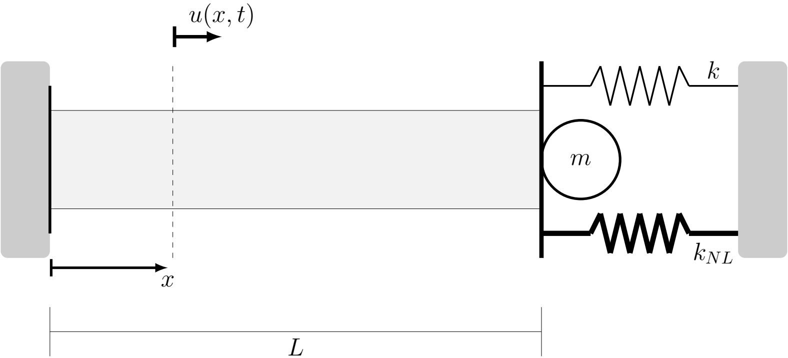


Figure: Sketch of a bar fixed at one and attached to two springs and a lumped mass on the other extreme, i.e. a fixed-mass-spring bar.

Mathematical model for the mechanical system

Equation of motion

$$\rho A \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x} \right) = f(x,t)$$

Boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad EA \frac{\partial u}{\partial x}(L,t) = -ku - k_{NL} u^3 - m \frac{\partial^2 u}{\partial t^2}$$

Initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = v_0(x)$$

Weak formulation of the problem

Find a “suitable” displacement field such that

$$\mathcal{M}(\ddot{u}, w) + \mathcal{C}(\dot{u}, w) + \mathcal{K}(u, w) = \mathcal{F}(w) + \mathcal{F}_{NL}(u, w),$$

$$\widetilde{\mathcal{M}}(u(\cdot, 0), w) = \widetilde{\mathcal{M}}(u_0, w),$$

and

$$\widetilde{\mathcal{M}}(\dot{u}(\cdot, 0), w) = \widetilde{\mathcal{M}}(v_0, w),$$

for any weight function.

Model equation discretization

For discretization, the Galerkin method is employed, which results in the nonlinear initial value problem

$$[M]\,\ddot{\mathbf{u}}(t) + [C]\,\dot{\mathbf{u}}(t) + [K]\,\mathbf{u}(t) = \mathbf{f}(t) + \mathbf{f}_{NL}\left(\mathbf{u}(t)\right),$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{and} \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0,$$

which is integrated using Newmark method.

Section 3

Stochastic Model

Uncertainties in the model

In each case of study, two sources of uncertainty are considered:

- Case 1
  - external force (arbitrarily specified)
    Gaussian white noise
  - nonlinear spring stiffness (maximum entropy principle)
    Gamma distribution
- Case 2
  - external force (arbitrarily specified)
    Gaussian white noise
  - elastic modulus (maximum entropy principle)
    Gamma distribution

Stochastic initial value problem

The stochastic nonlinear dynamics is described by

$$[M]\,\ddot{\mathbf{U}}(t,\theta) + [C]\,\dot{\mathbf{U}}(t,\theta) + [K]\,\mathbf{U}(t,\theta) = \mathbf{F}(t,\theta) + \mathbf{f}_{NL}\left(\dot{\mathbf{U}}(t,\theta)\right),$$

$$\mathbf{U}(0,\theta) = \mathbf{u}_0 \qquad \text{and} \qquad \dot{\mathbf{U}}(0,\theta) = \mathbf{v}_0,$$

which the uncertainty propagation is computed through Monte Carlo method.

Section 4

Numerical Experiments

Stochastic simulation - Case 1

Case 1:

- Random external force
- Random nonlinear spring stiffness

A parametric study is performed, indexed by

$$\mu_{k_{NL}} = 650 \times \{0,\ 10^{12},\ 10^{13},\ 10^{14}\}\ N/m^3$$

Evolution of the lumped mass displacement

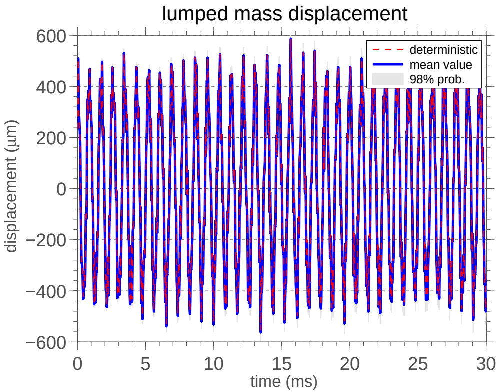


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass displacement, with $\mu_{k_{NL}} = 0$.

Evolution of the lumped mass displacement

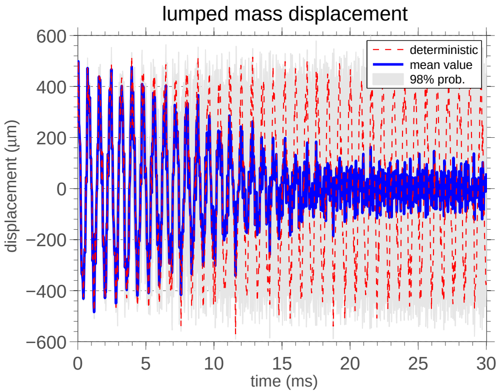
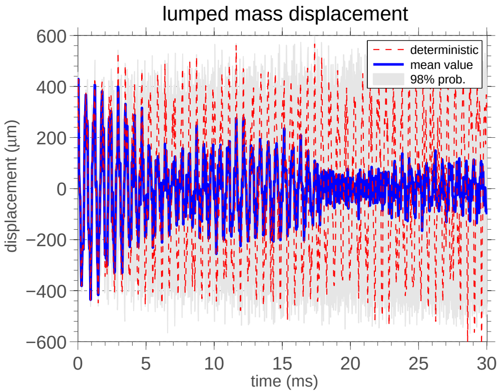


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass displacement, with $\mu_{k_{NL}} = 10^{12}$.

Evolution of the lumped mass displacement



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Evolution of the lumped mass displacement

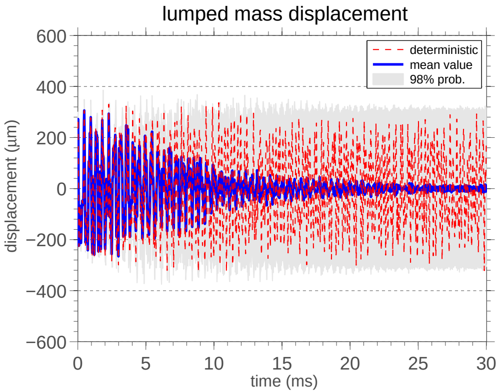
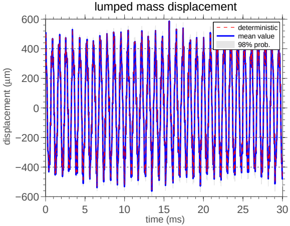
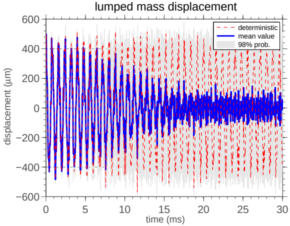


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass displacement, with $\mu_{k_{NL}} = 10^{14}$.

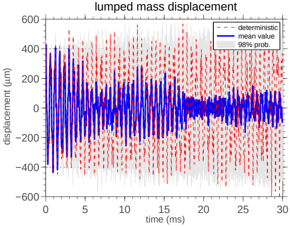
Evolution of the lumped mass displacement



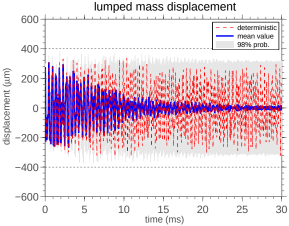
(a) $\mu_{k_{NL}} = 0$



(b) $\mu_{k_{NL}} = 10^{12}$



(c) $\mu_{k_{NL}} = 10^{13}$



(d) $\mu_{k_{NL}} = 10^{14}$

Evolution of the lumped mass velocity

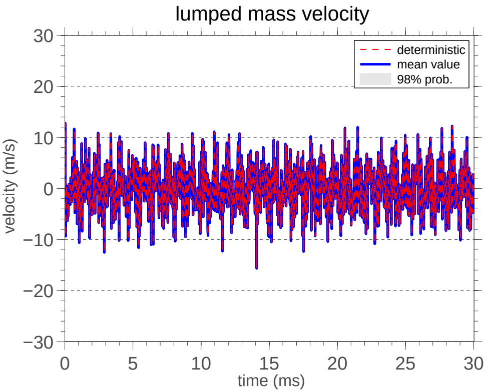
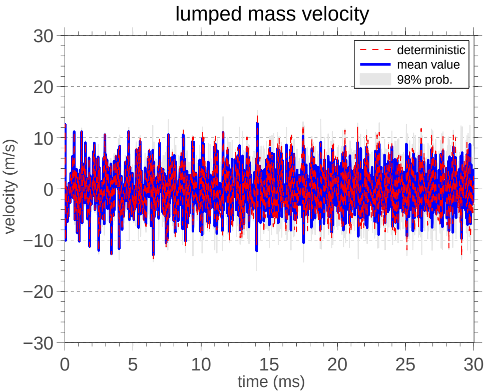


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass velocity, with $\mu_{k_{NL}} = 0$.

Evolution of the lumped mass velocity



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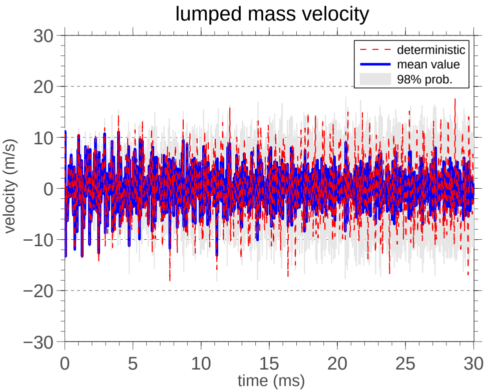

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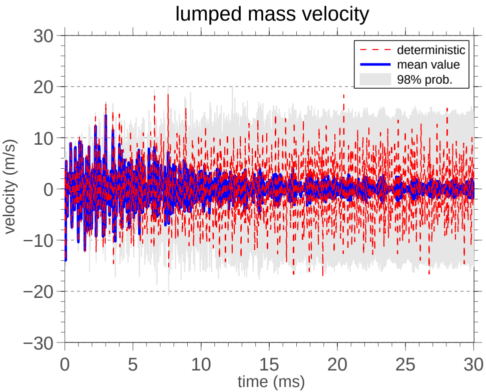

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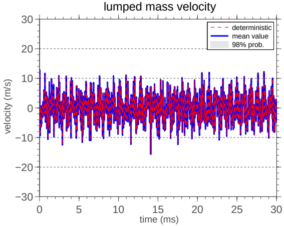

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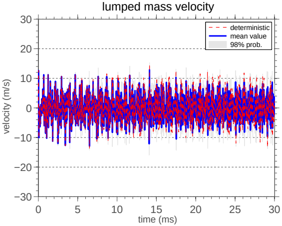
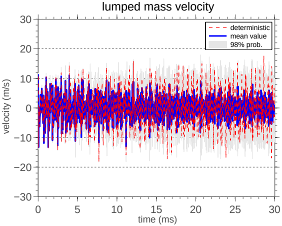
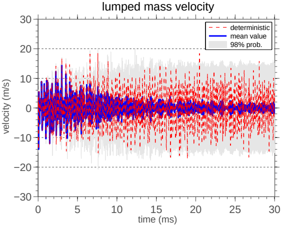


(b) $\mu_{k_{NL}} = 10^{12}$



(c) $\mu_{k_{NL}} = 10^{13}$



(d) $\mu_{k_{NL}} = 10^{14}$

Probability distribution of the lumped mass displacement

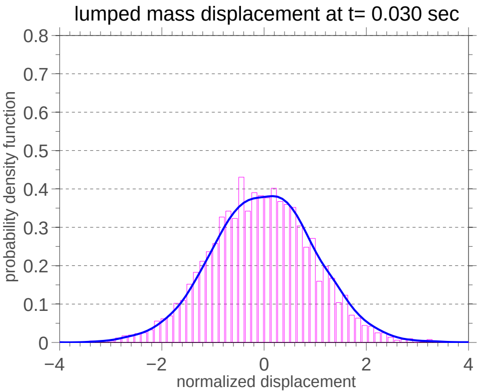
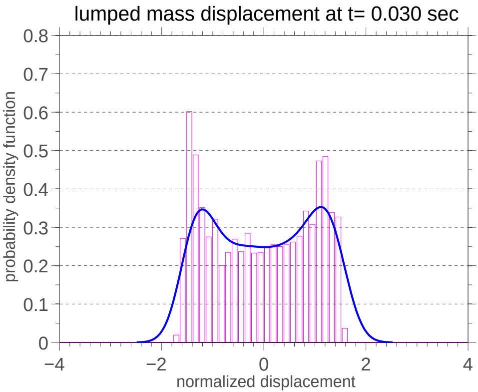


Figure: This figure illustrates the PDF of the (normalized) lumped mass displacement, with $\mu_{k_{NL}} = 0$.

Probability distribution of the lumped mass displacement



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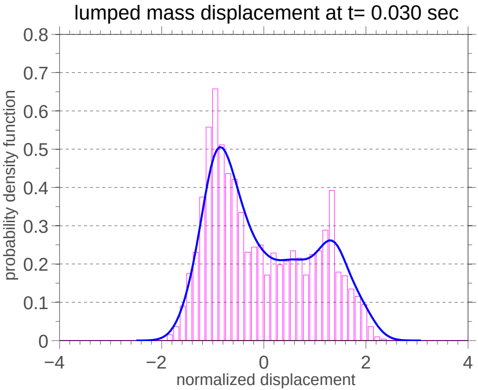

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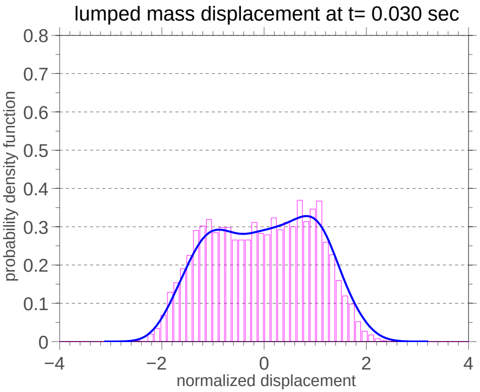

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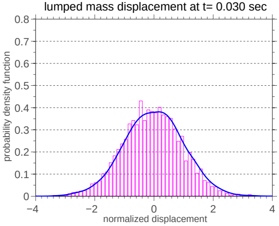
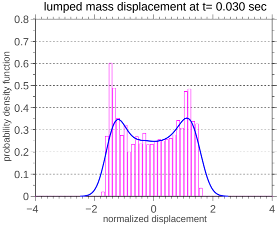
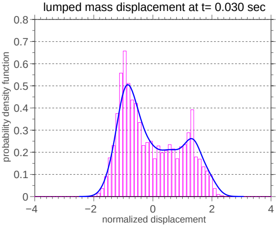
Probability distribution of the lumped mass displacement



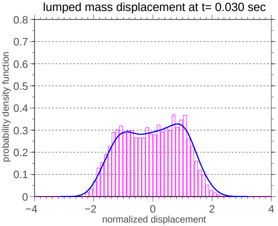
(a) $\mu_{k_{NL}} = 0$



(b) $\mu_{k_{NL}} = 10^{12}$



(c) $\mu_{k_{NL}} = 10^{13}$



(d) $\mu_{k_{NL}} = 10^{14}$

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Stochastic simulation - Case 2

Case 2:

- Random external force
- Random elastic modulus

A parametric study is performed, indexed by

$$m^* = \frac{\text{lumped mass}}{\text{continuous mass}} = \{0.1,\ 1,\ 10,\ 50\}$$

Evolution of the lumped mass displacement

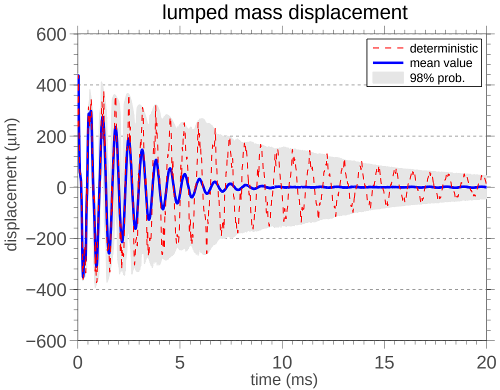


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass displacement, with $m^* = 0.1$.

Evolution of the lumped mass displacement

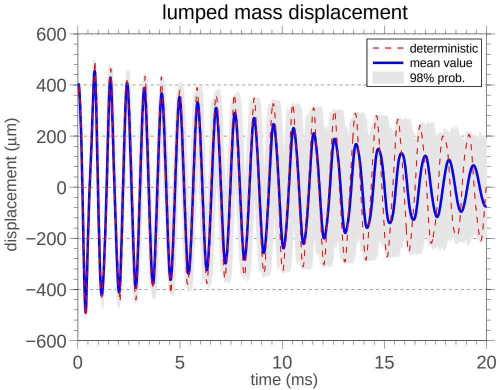


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass displacement, with $m^* = 1$.

Evolution of the lumped mass displacement

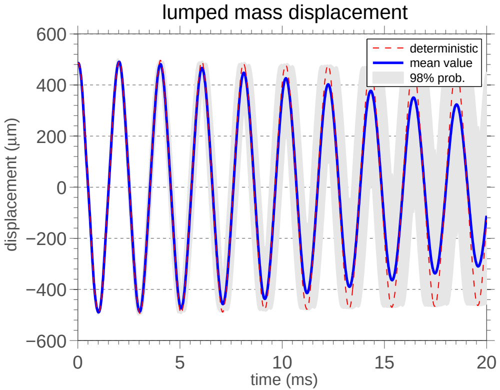


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass displacement, with $m^* = 10$.

Evolution of the lumped mass displacement

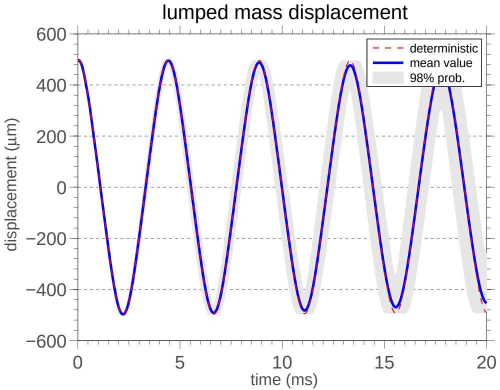
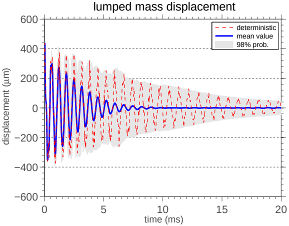
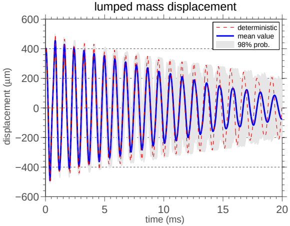


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass displacement, with $m^* = 50$.

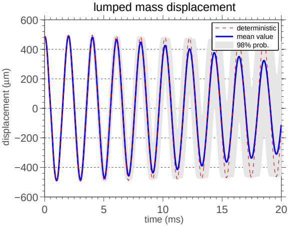
Evolution of the lumped mass displacement



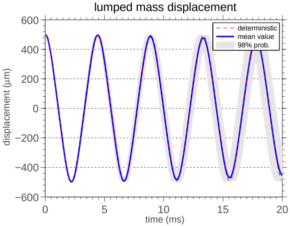
(a) $m^* = 0.1$



(b) $m^* = 1$



(c) $m^* = 10$



(d) $m^* = 50$

Evolution of the lumped mass velocity

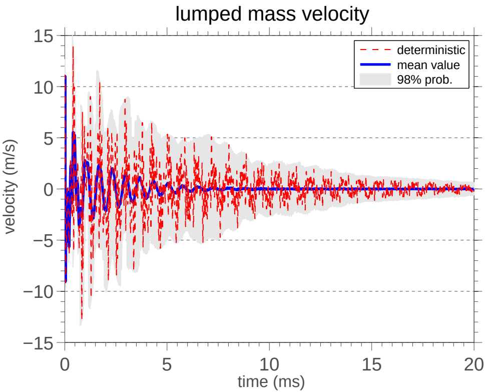


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass velocity, with $m^* = 0.1$.

Evolution of the lumped mass velocity

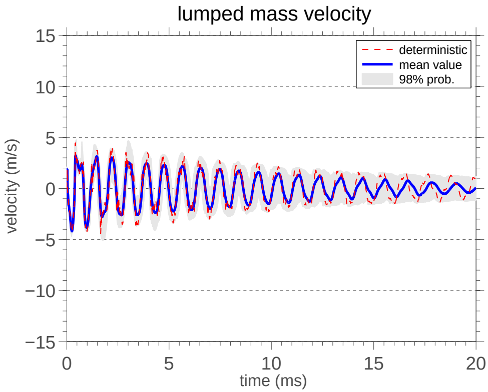


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass velocity, with $m^* = 1$.

Evolution of the lumped mass velocity

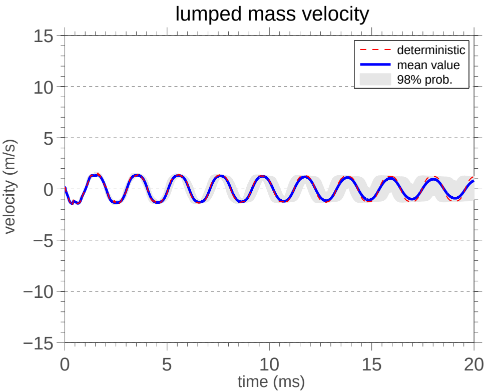


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass velocity, with $m^* = 10$.

Evolution of the lumped mass velocity

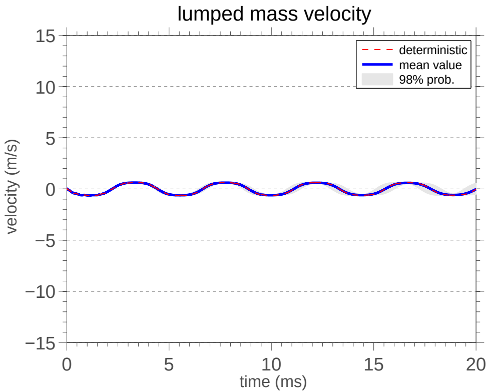
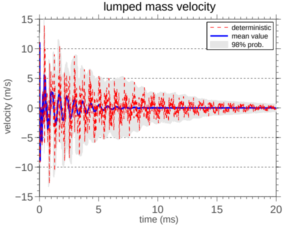
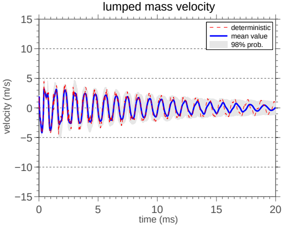


Figure: This figure illustrates the mean value (blue line) and an interval of confidence (grey shadow) for the lumped mass velocity, with $m^* = 50$.

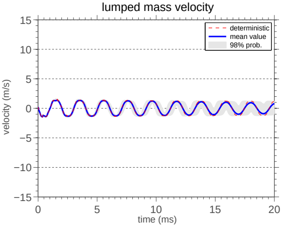
Evolution of the lumped mass velocity



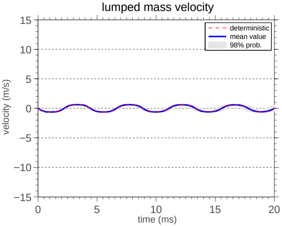
(a) $m^* = 0.1$



(b) $m^* = 1$



(c) $m^* = 10$



(d) $m^* = 50$

Orbit of the lumped mass displacement

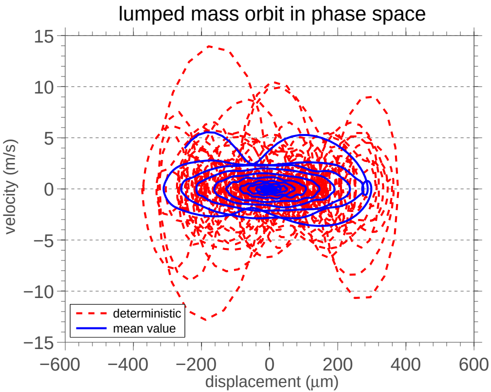


Figure: This figure illustrates the mean orbit, in the phase space, of the lumped mass with $m^* = 0.1$.

Orbit of the lumped mass displacement

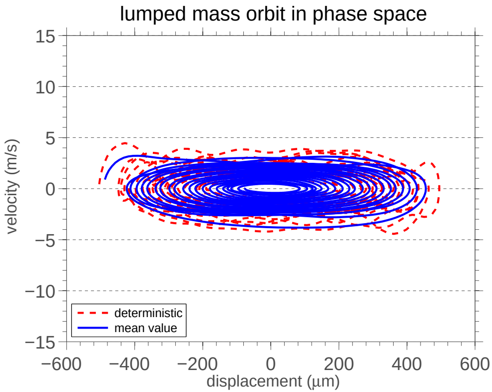


Figure: This figure illustrates the mean orbit, in the phase space, of the lumped mass with $m^* = 1$.

Orbit of the lumped mass displacement

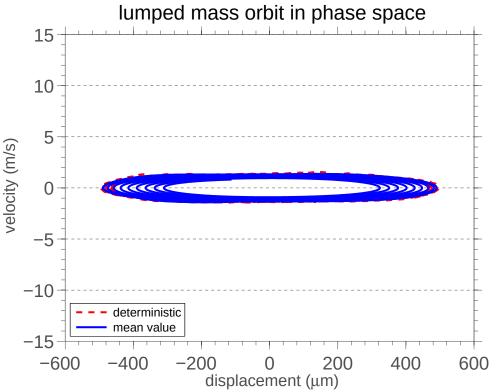


Figure: This figure illustrates the mean orbit, in the phase space, of the lumped mass with $m^* = 10$.

Orbit of the lumped mass displacement

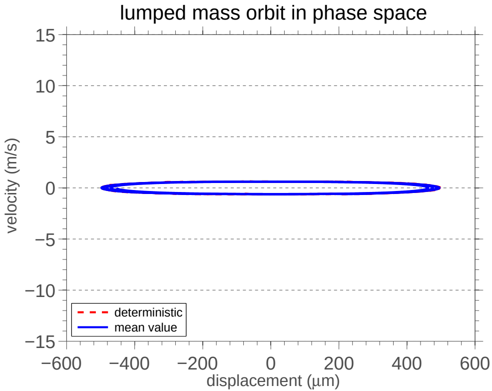
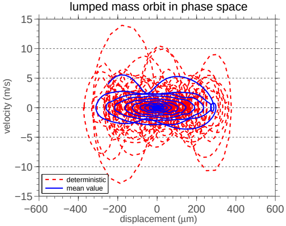
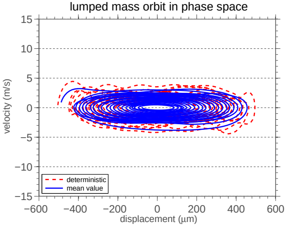


Figure: This figure illustrates the mean orbit, in the phase space, of the lumped mass with $m^* = 50$.

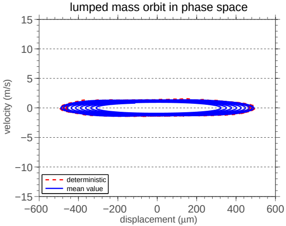
Orbit of the lumped mass displacement



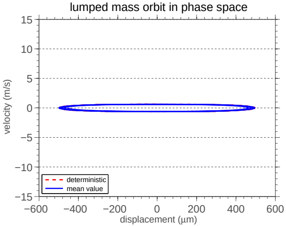
(a) $m^* = 0.1$



(b) $m^* = 1$



(c) $m^* = 10$



(d) $m^* = 50$

Probability distribution of the lumped mass displacement

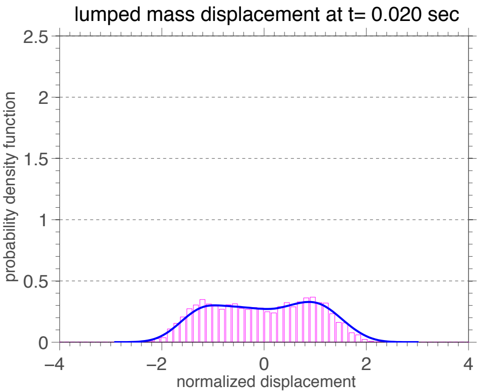


Figure: This figure illustrates the PDF of the (normalized) lumped mass displacement, with $m^* = 0.1$.

Probability distribution of the lumped mass displacement

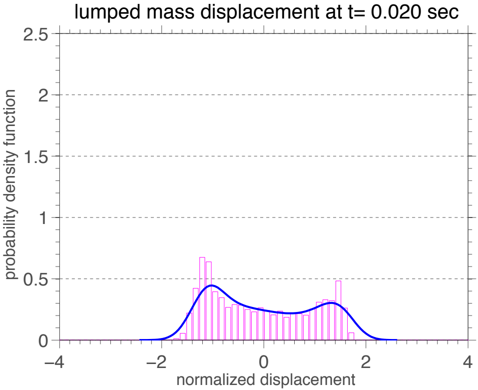


Figure: This figure illustrates the PDF of the (normalized) lumped mass displacement, with $m^* = 1$.

Probability distribution of the lumped mass displacement

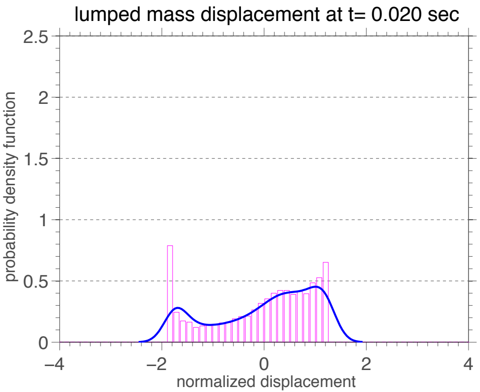


Figure: This figure illustrates the PDF of the (normalized) lumped mass displacement, with $m^* = 10$.

Probability distribution of the lumped mass displacement

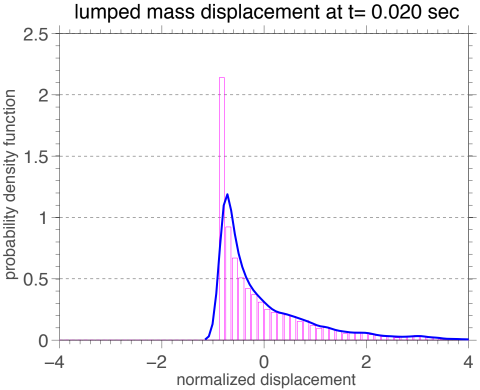
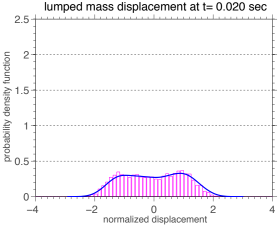
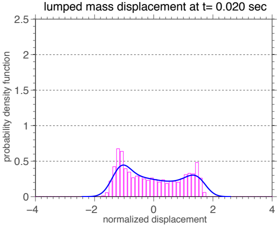


Figure: This figure illustrates the PDF of the (normalized) lumped mass displacement, with $m^* = 50$.

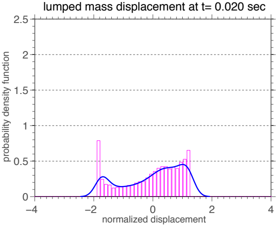
Probability distribution of the lumped mass displacement



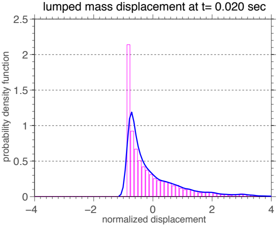
(a) $m^* = 0.1$



(b) $m^* = 1$



(c) $m^* = 10$



(d) $m^* = 50$

Section 5

Conclusion

Concluding remarks

- Uncertainties accumulation is observed with time increase;
- Large values of the lumped mass makes the behavior of the continuous system like a mass-spring oscillator;
- A reduction of uncertainty in the system response is observed when the lumped mass increases;
- Cubic nonlinearity induces asymmetric and multimodal distribution of probability.

Paper with the results

The results shown here can be seen in:

A. Cunha Jr and R. Sampaio
On the nonlinear stochastic dynamics of a continuous system with discrete attached elements.
Applied Mathematical Modelling (under review)

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