Question 1

Part A

In this part we want to determine the probability that after 1 month we have at least one functioning bulb
We can note the following; $1\ month\ \approx 730\ hours$, to keep the units consistent

We can begin by defining the following random variables…

$X :=\ 60W\ bulb\ burns\ out\ after\ X\ hours$

$Y :=\ 120W\ bulb\ burns\ out\ after\ Y\ hours$

To help us define the joint PDF of the above 2 random variables, we can use $E[x]$ to obtain λ…

$E[x]\ = \frac{1}{\lambda} \Rightarrow \lambda_X = \frac{1}{2000},\ \lambda_Y = \frac{1}{1200}$

We can then use the above to define the following joint PDF for the random variables X and Y…

$f_{X,Y}(x, y)\ = \left[\lambda_X e^{-(\lambda_X)x}\right]\left[\lambda_Y e^{-(\lambda_Y)y}\right] = \lambda_1\lambda_2 e^{-\left(\lambda_X x + \lambda_Y y\right)};\ 0\ \leq\ x, y\ \leq\ \infty$

$f_{X,Y}(x, y)\ =\ 0 \qquad\qquad\qquad ;\ e.w.$

We can simplify the problem by taking the complement of the desired probability…

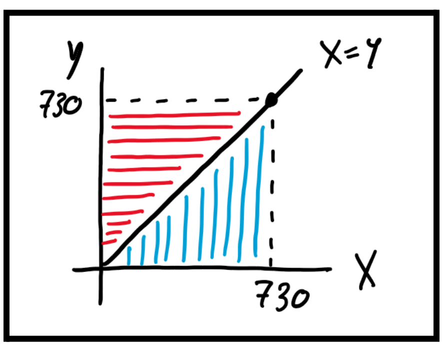
$P(at\ least\ 1\ does\ not\ fail)\ =\ 1\ -\ P(both\ fail)\ =\ 1\ -\ P(x, y\ \leq\ 730)$

Computing the above, we obtain…

$P(at\ least\ 1\ does\ not\ fail)\ =\ 1\ -\left\{\int_0^{730}\left(\int_0^{730}\left(\lambda_1\lambda_2 e^{-\left(\lambda_X x + \lambda_Y y\right)}\right)dx\right)dy\right\};\lambda_X = \frac{1}{2000},\ \lambda_Y = \frac{1}{1200}$

$P(at\ least\ 1\ does\ not\ fail)\ =\ 0.86$

Thus, we can say that the probability that after 1 month we have at least one functioning lightbulb is 0.86



Part B

Begin by noting that fact that $1\ year\ =\ 8760\ hours$, to keep our units consistent

The problem asks us to determine the probability that we will not need to change a lightbulb for at least 1 year

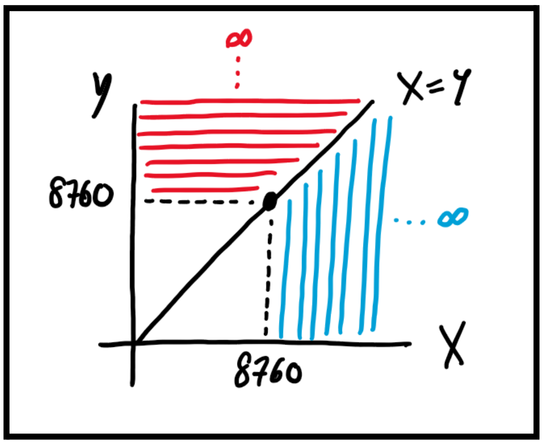
$$P(no\ change\ after\ 1\ year)\ =\ P(x, y\ \geq\ 8760)$$

Computing the above, we obtain…

$$P(no\ change\ after\ 1\ year)\ =\ \int_{8760}^{\infty}\left(\int_{8760}^{\infty}\left(\lambda_1\lambda_2 e^{-\left(\lambda_X x + \lambda_Y y\right)}\right)dx\right)dy;\ \lambda_X = \frac{1}{2000},\ \lambda_Y = \frac{1}{1200}$$

$$P(no\ change\ after\ 1\ year)\ =\ 8.461\ \times\ 10^{-6}$$

Thus, we can say that the probability that we will not have to change bulbs after one year is 0.000008461



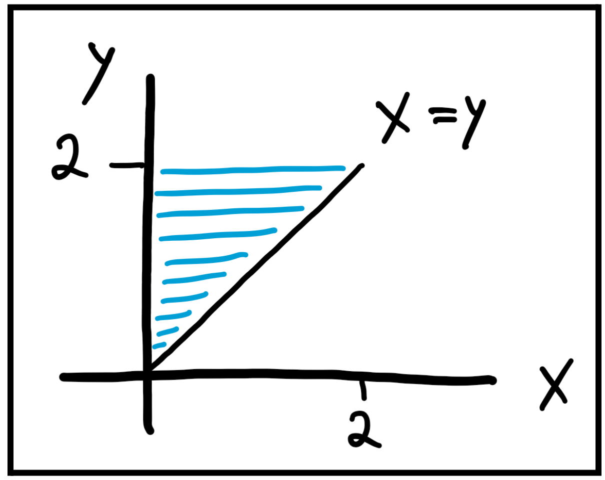
Question 2

Part A

We know the following…

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy\,dx = 1$$

We can use the above to solve for the unknown constant, c



$$\int_{0}^{2}\int_{x}^{2} cx^2 y\,dy\,dx = 1 \Leftrightarrow \int_{0}^{2}\left\{cx^2\int_{x}^{2} ydy\right\}dx = 1$$

$$\int_{0}^{2}\left(cx^2\right)\left(\frac{2^2}{2} - \frac{x^2}{2}\right)dx = 1$$

$$\int_{0}^{2}\left(\frac{4cx^2}{2} - \frac{cx^4}{2}\right)dx = 1$$

$$1 = \frac{32}{15}c$$

$$c = \frac{15}{32}$$

Part B

We know the following…

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

We can use the above to solve for the following marginal distributions…

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{x}^{2}\frac{15}{32}x^2 y\,dy = \frac{-15}{64}\left(x^2\right)\left(x^2 - 4\right)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_{0}^{y}\frac{15}{32}x^2 y\,dx = \frac{5}{32}y^4$$

Part C

We know the following…

$$f_{X|Y}(x|y) = \frac{f_{x,y}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f_{x,y}(x,y)}{f_X(x)}$$

We can use the above to solve for the following conditional distributions…

$$f_{Y|X}(x|y) = \frac{f_{x,y}(x,y)}{f_X(x)} = \frac{\frac{15}{32}x^2y}{\frac{-15}{64}(x^2)(x^2-4)} = \frac{-2y}{x^2-4}$$

$$f_{X|Y}(y|x) = \frac{f_{x,y}(x,y)}{f_Y(y)} = \frac{\frac{15}{32}x^2y}{\frac{5}{32}y^4} = 3\frac{x^2}{y^3}$$

Part D

We can determine independence using the following condition…

$$f_X(x)f_Y(y) = f_{X,Y}(x,y) \Rightarrow independance$$

We can use the above to determine if our derived marginals satisfy…

$$f_X(x)f_Y(y) = \left[\frac{-15}{64}\left(x^2\right)\left(x^2 - 4\right)\right]\left[\frac{5}{32}y^4\right] = \frac{-75}{2048}\left(x^2\right)\left(x^2 - 4\right)\left(y^4\right) \neq f_{X,Y}(x,y)$$

Thus, we can conclude that X and Y are NOT independent

Question 3

We know the following information…

$John \sim U(7{:}55,\ 8{:}10)$, taking $J\ =\ 0\ @\ 8{:}00 \Rightarrow John \sim U(-\ 5,\ 10)$

$Susan \sim EXP(0.5),\ taking\ S\ =\ 0\ @\ 8{:}00$

We can begin by defining the following random variables…

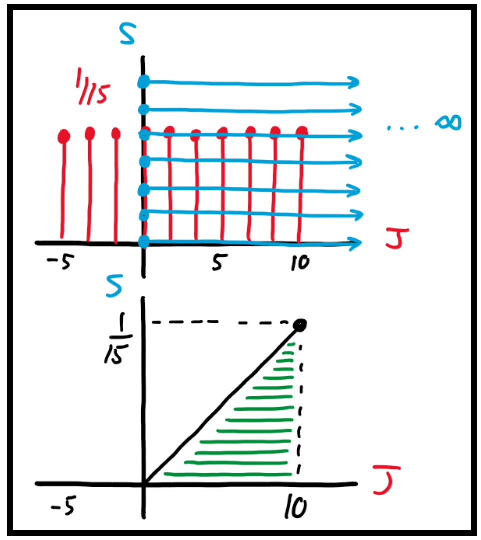
$J := \ John\ arrives\ J\ minutes\ after\ 7{:}55pm$

$S := \ Susan\ arrives\ S\ minutes\ after\ 8{:}00pm$

Using the above, we can define the PDF's for Kevin and Susans arrival times…

$f_J(j)\ = \frac{1}{15};\ -5\ \le\ j\ \le\ 10$

$f_S(s)\ =\ 0.5e^{-0.5s};\ 0\ \le\ s\ \le\ \infty$



We can further form the joint PDF of Kevin and Susan's arrival times…

$f_{S,J}(s,j)\ =\ \left(\frac{1}{15}\right)\left(0.5e^{-0.5s}\right);\ -5\ \le\ j\ \le\ 10,\ 0\ \le\ s\ \le\ \infty$

$f_{S,J}(s,j)\ =\ 0 \qquad\qquad ;\ e.w.$

The probability that Susan arrives before John can be stated as…

$$P(s\ <\ j) = \int_0^{10}\int_0^{j} f_{S,J}(s,j)\ ds\ dj$$

$$P(s\ <\ j)\ =\ \int_0^{10}\left(\int_0^{j}\left(\frac{1}{15}\right)\left(0.5e^{-0.5s}\right)ds\right)dj$$

Evaluating the above, we obtain…

$P(s\ <\ j)\ \approx\ 0.53$

Thus, we can conclude that the probability that Susan arrives before John is about 0.53