Question 1.
Consider two independent Gaussian random variables with expected values of 0 and variances 1. Let random variable U = 2X + 3Y and random variable V=X+2Y. Find the bivariate distribution of U,V.

Solution 1.
We can begin by calculating the related expected value and variance of our given distributions, U and V...

$$\mathbf{E(u)} = 2\mathbf{E(x)} + 3\mathbf{E(y)} = 0 \qquad \mathbf{Var(u)} = 2^2\mathbf{Var(x)} + 3^2\mathbf{Var(y)} = 13$$
$$\mathbf{E(v)} = \mathbf{E(x)} + 2\mathbf{E(y)} = 0 \qquad \mathbf{Var(v)} = 1^2\mathbf{Var(x)} + 2^2\mathbf{Var(y)} = 5$$

We can then compute the Covariance of U and V as follows...

$$\mathbf{Cov}(u,v) = \mathbf{Cov}(2x+3y, x+2y)$$
$$\mathbf{Cov}(u,v) = \mathbf{Cov}(2x,x) + \mathbf{Cov}(2x,2y) + \mathbf{Cov}(3y,x) + \mathbf{Cov}(3y,2y)$$
$$\mathbf{Cov}(u,v) = 2\mathbf{Var}(x) + 0 + 0 + 6\mathbf{Var}(y)$$
$$\mathbf{Cov}(u,v) = 8$$

Knowing the above, we can compute the correlation coefficient of U and V as follows...

$$\rho_{u,v} = \frac{\mathbf{Cov}(u,v)}{\sqrt{\mathbf{E}(x)\mathbf{E}(y)}} = \frac{8}{\sqrt{65}}$$

Recall the general form Bivariate Gaussian Equation...

$$f_{X,Y}(x,y) = \frac{exp(\frac{(\frac{-(x-\mu_X)}{\sigma_X})^2 - \frac{2\rho_{X,Y}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_X} + (\frac{(y-\mu_Y)}{\sigma_Y})^2}{2(1-\rho_{X,Y}^2)})}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$

Taking the relevant variables and plugging them into the Bivariate Gaussian Equation above yields...

$$\boxed{f_{U,V}(u,v) = \frac{\frac{exp(\frac{-u^2}{13} - \frac{16uv}{65} + \frac{v^2}{5})}{2/65}}{2\pi}}$$

Question 2.
Consider two Gaussian random variables U and V such that:
a) the expected value of U is 2 and variance of U is 9 and
b) the expected value of V is 1 and variance is 16.
In addition the covariance Cov(U,V) is 3. Find coefficients a,b,c and d so that X=aU+bV and Y=cU+dV are jointly Gaussian and independent. What are the corresponding distributions of X and Y.

Solution 2.
We can begin by determining an algebraic expression for the covariance of X and Y. To ensure independence, we can set the covariance to 0...

$$\begin{aligned}
&\mathbf{Cov}(X,Y) = \mathbf{Cov}(aU+bV, cU+dV) = 0\\
&\mathbf{Cov}(X,Y) = ac\mathbf{Var}(U) + ad\mathbf{Cov}(U,V) + bc\mathbf{Cov}(U,V) + bd\mathbf{Var}(V) = 0\\
&\mathbf{Cov}(X,Y) = ac\mathbf{Var}(U) + 3ad + 3bc + bd\mathbf{Var}(V) = 0\\
&\mathbf{Cov}(X,Y) = 9ac + 3ad + 3bc + 16bd = 0
\end{aligned}$$

We can choose coefficients for a, b, and c...

$$\begin{aligned}
&a = 1\\
&b = -1\\
&c = 13
\end{aligned}$$

Plugging the above into our simplified Covarience equation, we can solve for d...

$$\begin{aligned}
&(9)(1)(13) + (3)(1)(d) + (3)(1)(13) + (16)(1)(d) = 0\\
&\implies d = 6
\end{aligned}$$

We can use the above coefficients to form the completed equations for X and Y...

$$\begin{aligned}
&X = U - V\\
&Y = 13U + 6V
\end{aligned}$$

We can determine the Expected Value and Variance of X and Y as follows...

$$\begin{aligned}
&\mathbf{E(X)} = \mathbf{E(U)} - \mathbf{E(V)} = 1 && \mathbf{Var(X)} = \mathbf{Var(U)} + \mathbf{Var(V)} = 25\\
&\mathbf{E(Y)} = 13\mathbf{E(U)} + 6\mathbf{E(V)} = 25 && \mathbf{Var(V)} = 13^2\mathbf{Var(U)} + 6^2\mathbf{Var(V)} = 2097
\end{aligned}$$

We can further conclude that X and Y are jointly Gaussian because they are made up of a linear combination of Gaussian Random Variables.

Finally, we can form the distributions of X and Y as follows...

$$\boxed{X \sim \mathcal{N}(1, 25) \Longrightarrow f_X(x) = \frac{1}{\sqrt{2\pi} \cdot 5} e^{-\frac{1}{2}\left(\frac{x-1}{5}\right)^2}}$$

$$\boxed{Y \sim \mathcal{N}(25, 2097) \Longrightarrow f_Y(y) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{2097}} e^{-\frac{1}{2}\left(\frac{y-25}{\sqrt{2097}}\right)^2}}$$