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Digital Communications

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Abbreviations

ADM	Adaptive Delta Modulation
ASK	Amplitude Shift Keying
AWGN	Additive White Gaussian Noise
BER	Bit Error Rate
BPF	Bandpass Filter
BSC	Binary Symmetric Channel
BW	Bandwidth
BPSK	Binary Phase Shift Keying
BFSK	Binary Frequency Shift Keying
CW	Continuous Wave
CDM	Code Division Multiplexing
CDMA	Code Division Multiple Access
CDF	Cumulative Distribution Function
dB	Decibel
DM	Delta Modulation
DPCM	Differential Pulse Code Modulation
DPSK	Differential Phase Shift Keying
DSB	Double Sided Modulation
erf	Error function
erfc	Complementary error function
exp	Exponential (e)
esd	Energy spectral density
FDM	Frequency Division Multiplexing

FDMA	Frequency Division Multiple Access
FSK	Frequency Shift Keying
ISI	Inter Symbol Interference
ISDN	Integrated Services Digital Network
MSK	Minimum Shift Keying
NRZ	Nonreturn to Zero
OOK	On-off Keying
PAM	Pulse Amplitude Modulation
PDM	Pulse Duration Modulation
PDF	Probability Density Function
psd	Power spectral density
PSK	Phase Shift Keying
PLL	Phase Locked Loop
PPM	Pulse Position Modulation
PWM	Pulse Width Modulation
QAM	Quadrature Amplitude Modulation
QPSK	Quadrature Phase Shift Keying
RZ	Return to Zero
SNR	Signal to Noise Ratio
$\frac{S}{N}$	Signal to Noise Ratio
TDM	Time Division Multiplexing
TDMA	Time Division Multiple Access

Symbols

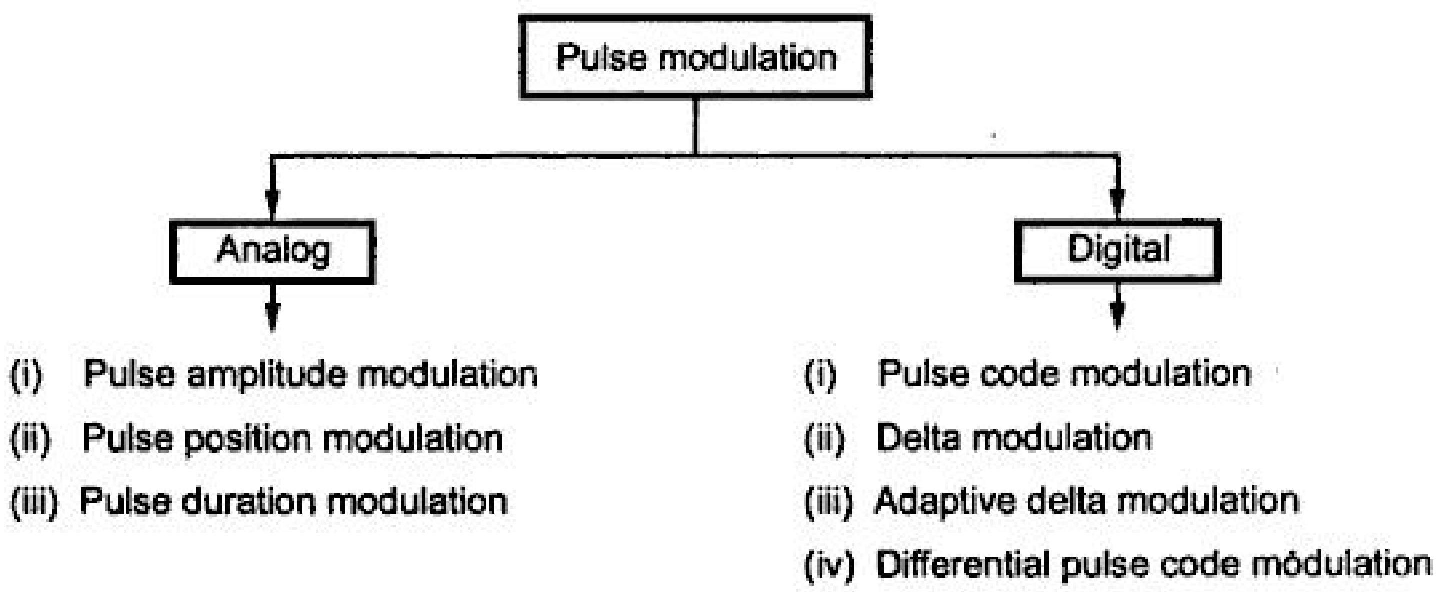
$\| \|$	Magnitude of the complex quantity contained within
$\leftrightarrow$	Fourier transform pair
$< x(t) >$	Time average of $x(t)$
X^*	* denotes complex conjugate function. Here X^* is complex conjugate of X
$x(t) * y(t)$	Convolution of x (t) and y (t)
arg ( )	Phase angle of the complex quantity contained within
Re [ ]	Real part of quantity contained within
Im [ ]	Imaginary part of the quantity contained within
F [x (t)]	Fourier transform of x (t)
F^{-1} [X (f)]	Inverse fourier transform of X (f)
IFT [X(f)]	Inverse fourier transform of X (f)
$\oplus$	Modulo - 2 Addition i.e. Exclusive-OR operation
$\hat{x}(t)$	Symbol ^ represents that $\hat{x}(t)$ is the reconstructed value of x (t) in receiver
$\{a_k\}$	Represents sequence whose k^{th} value is a_k
E [ ]	Expected value or mean value of the random variable contained within
P_e or P(e)	Probability of error of symbol or bit
$f_X(x)$	Probability density function of a continuous random variable at X = x.
$f_{XY}(x, y)$	Probability density function of random variables X & Y at X = x & Y = y
$P(X \leq x)$	Probability of a random variable X for $X \leq x$.
$F_X(x)$	Cumulative Distribution function of a random variable X at $X \leq x$

m_x	Mean or average value of a random variable X
σ_X	Standard deviation of a random variable X
σ_X^2	Variance of a random variable X
$\psi(f)$	Energy spectral density of a signal x(t)
S (f)	Power spectral density of a signal x (t)
h (t)	Impulse response of the linear system
H (f)	Transfer function of the linear system
T_b	Duration of one bit
B or B_T or B_o	Transmission channel bandwidth in digital transmission
r	Signaling rate in digital transmission. It indicates bit rate or code rate
v	Number of binary bits used for encoding a sample value
q	Number of digital levels used to encode a sample value
δ	Step size of the quantizer used in Digital Modulation Methods i.e. PCM, DPCM, DM, ADM etc
P_s	Symbol power
J	Average interference power
ε	Quantization error in digital modulation methods
$\phi(t)$	Coherent / noncoherent carrier & reference signal used in digital passband transmission
G (p)	Generator polynomial in cyclic codes
X (p)	Code vector polynomial
M (p)	Message bits polynomial
C (p)	Check bits polynomial

1 Pulse Digital Modulation

Introduction

- There are three types of modulation
  (i) Amplitude modulation
  (ii) Angle modulation
  (iii) Pulse modulation
- Pulse modulation can be further classified as,
  (i) Pulse analog modulation
  (ii) Pulse digital modulation
- The above two techniques can be further classified as,



- In the above techniques following points are studied :
  (i) Principle of operation
  (ii) Transmitter and receiver block diagram
  (iii) Error analysis
  (iv) Signal to quantization noise ratio.

1.1 Advantages of Digital Communication System

Presently most of the communication is digital. For example cellular (mobile phone) communication, satellite communication, radar and sonar signals, Facsimile, data transmission over internet etc all use digital communication. Paractically, after 20 years, analog communication will be totally replaced by digital communication.

Why digital communication is so popular ?

There are few reasons due to which people are prefering digital communication over analog communication.

1. Due to advancements in VLSI technology, it is possible to manufacture very high speed embedded circuits. Such circuits are used in digital communications.
2. High speed computers and powerful software design tools are available. They make the development of digital communication systems feasible.
3. Internet is spread almost in every city and towns. The compatibility of digital communication systems with internet has opened new area of applications.

Advantages and Disadvantages of Digital Communication

Advantages :

1. Because of the advances in digital IC technologies and high speed computers, digital communication systems are simpler and cheaper compared to analog systems.
2. Using data encryption, only permitted receivers can be allowed to detect the transmitted data. This is very useful in military applications.
3. Wide dynamic range is possible since the data is converted to the digital form.
4. Using multiplexing, the speech, video and other data can be merged and transmitted over common channel.
5. Since the transmission is digital and channel encoding is used, the noise does not accumulate from repeater to repeater in long distance communication.
6. Since the transmitted signal is digital, a large amount of noise interference can be tolerated.
7. Since channel coding is used, the errors can be detected and corrected in the receivers.
8. Digital communication is adaptive to other advanced branches of data processing such as digital signal processing, image processing, data compression etc.

Disadvantages :

Eventhough digital communication offer many advantages as given above, it has some drawbacks also. But the advantages of digital communication outweigh disadvantages. They are as follows -

1. Because of analog to digital conversion, the data rate becomes high. Hence more transmission bandwidth is required for digital communication.
2. Digital communication needs synchronization in case of synchronous modulation.

1.2 Elements of Digital Communication System

Fig. 1.2.1 shows the basic operations in digital communication system. The source and the destination are the two physically separate points. When the signal travels in the communication channel, noise interferes with it. Because of this interference, the smeared or disturbed version of the input signal is received at the receiver. Therefore the signal received may not be correct. That is errors are introduced in the received signal. Thus the effects of noise due to the communication channel limit the rate at which signal can be transmitted. The probability of error in the received signal and transmission rate are normally used as performance measures of the digital communication system.

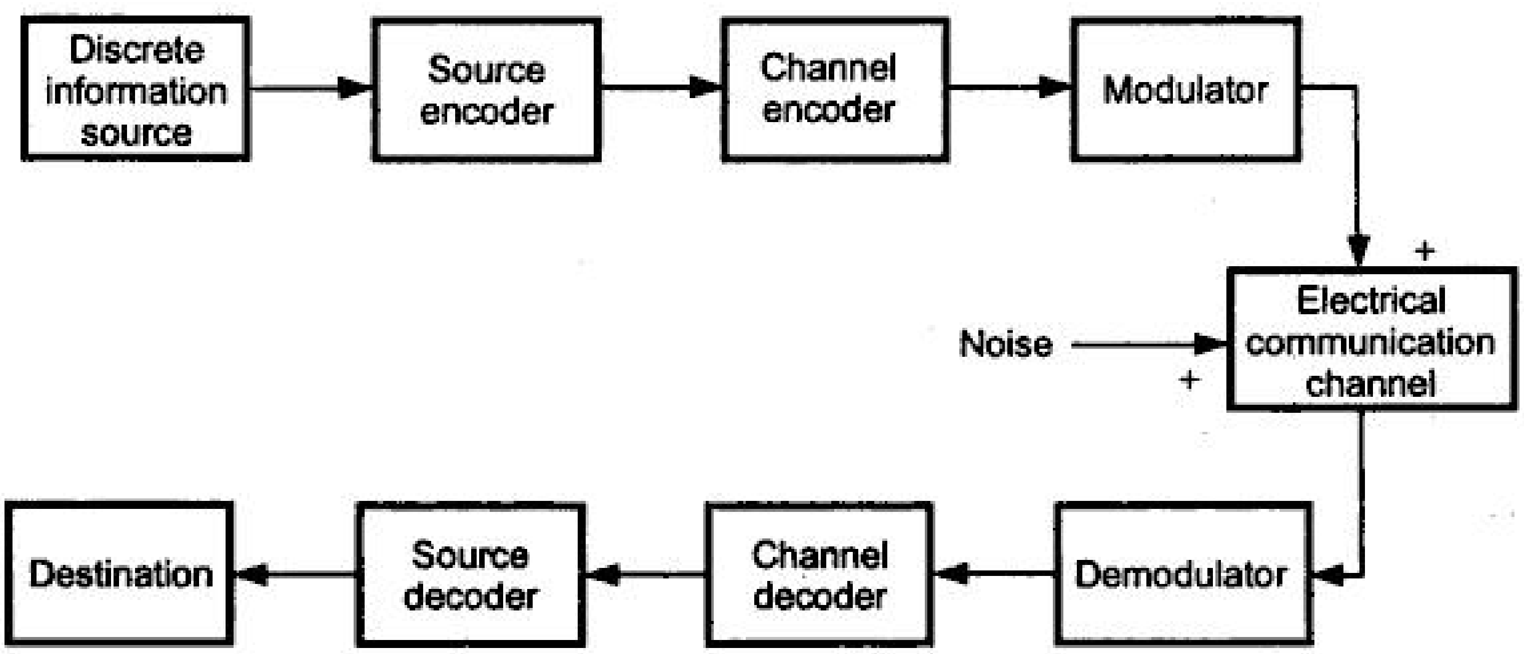


Fig. 1.2.1 Basic digital communication system

1.2.1 Information Source

The information source generates the message signal to be transmitted. In case of analog communication, the information source is analog. In case of digital communication, the information source produces a message signal which is not continuously varying with time. Rather the message signal is intermittent with respect to time. The examples of discrete information sources are data from computers, teletype etc. Even the message containing text is also discrete. The analog signal can be converted to discrete signal by sampling and quantization. In sampling, the analog signal is chopped off at regular time intervals. Those chopped samples form a discrete signal. The discrete information sources have following important parameters :

a) **Source alphabet :** These are the letters, digits or special characters available from the information source.

b) **Symbol rate :** It is the rate at which the information source generates source alphabets. It is normally represented in symbols/sec unit.

c) **Source alphabet probabilities :** Each source alphabet from the source has independent occurrence rate in the sequence. For example, letters A, E, I etc. occur frequently in the sequence. Thus probability of the occurrence of each source alphabet can become one of the important property which is useful in digital communication.

d) **Probabilistic dependence of symbols in a sequence :** The information carrying capacity of each source alphabet is different in a particular sequence. This parameter defines average information content of the symbols. The entropy of a source refers to the average information content per symbol in long messages. Entropy is defined in terms of bits per symbol. Bit is the abbreviation for binary digit. The source information rate is thus the product of symbol rate and source entropy i.e..

$$\underset{\text{(Bits/sec)}}{\text{Information rate}} = \underset{\text{(Symbols/sec)}}{\text{Symbol rate}} \times \underset{\text{(Bits/Symbol)}}{\text{Source entropy}}$$

The information rate represents minimum average data rate required to transmit information from source to the destination.

1.2.2 Source Encoder and Decoder

The symbols produced by the information source are given to the source encoder. These symbols cannot be transmitted directly. They are first converted into digital form (i.e. Binary sequence of 1's and 0's) by the source encoder. Every binary '1' and '0' is called a bit. The group of bits is called a codeword. The source encoder assigns codewords to the symbols. For every distinct symbol there is a unique codeword. The codeword can be of 4, 8, 16 or 32 bits length. As the number of bits are increased in each codeword, the symbols that can be represented are increased.

For example, 8 bits will have $2^8 = 256$ distinct codewords. Therefore 8 bits can be used to represent 256 symbols, 16 bits can represent $2^{16} = 65536$ symbols and so on. In both of the above examples the number of bits in every codeword is same throughout. That is 8 in first case and 16 in next case respectively. This is called fixed length coding. Fixed length coding is efficient only if all the symbols occur with equal probabilities in a statistically independent sequence. In the practical situations, the symbols in the sequence are statistically dependent and they have unequal probabilities of occurrence. For example, let us assume that the symbol sequence represents the percentage marks of the students. The 02%, 08%, 20%, 98%, 99% etc. symbols will have minimum probability of occurrence. But 60%, 55%, 70%, 75% will have more probability. For such symbols normally variable length codewords are assigned. More bits (More length) are assigned to rarely occurring symbols and less bits are assigned to frequently occurring symbols. Typical source encoders are pulse code modulators, delta modulators, vector quantizers etc. We will come across these codewords in detail in the subsequent chapters. Source encoders have following important parameters.

a) **Block size :** This gives the maximum number of distinct codewords that can be represented by the source encoder. It depends upon maximum number of bits in the codeword. For example, the block size of 8 bits source encoder will have $2^8 = 256$ codewords.

b) **Codeword length :** This is the number of bits used to represent each codeword. For example, if 8 bits are assigned to every codeword, then codeword length is 8 bits.

c) **Average data rate :** It is the output bits per second from the source encoder. The source encoder assigns multiple number of bits to every input symbol. Therefore the data rate is normally higher than the symbol rate. For example let us consider that the symbols are given to the source encoder at the rate of 10 symbols/sec and the length of codeword is 8 bits. Then the output data rate from the source encoder will be,

$$\text{Date rate} = \text{Symbol rate} \times \text{Codeword length}$$

$$= 10 \times 8 = 80 \text{ bits/sec}$$

Information rate is the minimum number of bits per second needed to convey information from source to destination as stated earlier. Therefore optimum data rate is equal to information rate. But because of practical limitations, designing such source encoder is difficult. Hence average data rate is higher than information rate and hence symbol rate also.

d) **Efficiency of the encoder :** This is the ratio of minimum source information rate to the actual output data rate of the source encoder.

At the receiver, some decoder is used to perform the reverse operation to that of source encoder. It converts the binary output of the channel decoder into a symbol sequence. Both variable length and fixed length decoders are possible. Some decoders use memory to store codewords. The decoders and encoders can be synchronous or asynchronous.

1.2.3 Channel Encoder and Decoder

At this stage we know that the message or information signal is converted in the form of binary sequence (i.e. 1's and 0's). The communication channel adds noise and interference to the signal being transmitted.

Therefore errors are introduced in the binary sequence received at the receiver. Hence errors are also introduced in the symbols generated from these binary codewords. To avoid these errors, channel coding is done. The channel encoder adds some redundant binary bits to the input sequence. These redundant bits are added with some properly defined logic. For example consider that the codeword from the source encoder is three bits long and one redundant bit is added to make it 4-bit long. This 4^{th} bit is added (either 1 or 0) such that number of 1's in the encoded word remain even (also called even parity). Following table gives output of source encoder, the 4^{th} bit depending upon the parity, and output of channel encoder.

Output of source encoder	Bit to be added by channel encoder for even parity	Output of channel encoder
b_3 b_2 b_1	b_0	b_3 b_2 b_1 b_0
1 1 0	0	1 1 0 0
0 1 0	1	0 1 0 1
0 0 0	0	0 0 0 0
1 1 1	1	1 1 1 1
:	:	:

Table 1.2.1 Even parity coding

Observe in the above table that every codeword at the output of channel encoder contains "even" number of 1's. At the receiver, if odd number of 1's are detected, then receiver comes to know that there is an error in the received signal. The channel decoder at the receiver is thus able to detect error in the bit sequence, and reduce the effects of channel noise and distortion. The channel encoder and decoder thus serve to increase the reliability of the received signal. The extra bits which are added by the channel encoders carry no information, rather, they are used by the channel decoder to detect and correct errors if any. These error correcting bits may be added recurrently after the block of few symbols or added in every symbol as shown in Table 1.2.1. The example of parity coding given above is just illustrative. There are many advanced and efficient coding techniques available. We will discuss them in the book.

The coding and decoding operation at encoder and decoder needs the memory (storage) and processing of binary data. Because of microcontrollers and computers, the complexity of encoders and decoders is nowadays very much reduced. The important parameters for channel encoder are -

a) The method of coding used.

b) Coding rate, which depends upon the redundant bits added by the channel encoder.

c) Coding efficiency, which is the ratio of data rate at the input to the data rate at the output of encoder.

d) Error control capabilities, i.e. detecting and correcting errors

e) Feasibility or complexity of the encoder and decoder.

The time delay involved in the decoding is also an important parameter for channel decoder.

1.2.4 Digital Modulators and Demodulators

Whenever the modulating signal is discrete (i.e. binary codewords), then digital modulation techniques are used. The carrier signal used by digital modulators is always continuous sinusoidal wave of high frequency. The digital modulators maps the input binary sequence of 1's and 0's to analog signal waveforms. If one bit at a time is to be transmitted, then digital modulator signal is $s_1(t)$ to transmit binary '0' and $s_2(t)$ to transmit binary '1'. For example consider the output of digital modulator shown in Fig. 1.2.2.

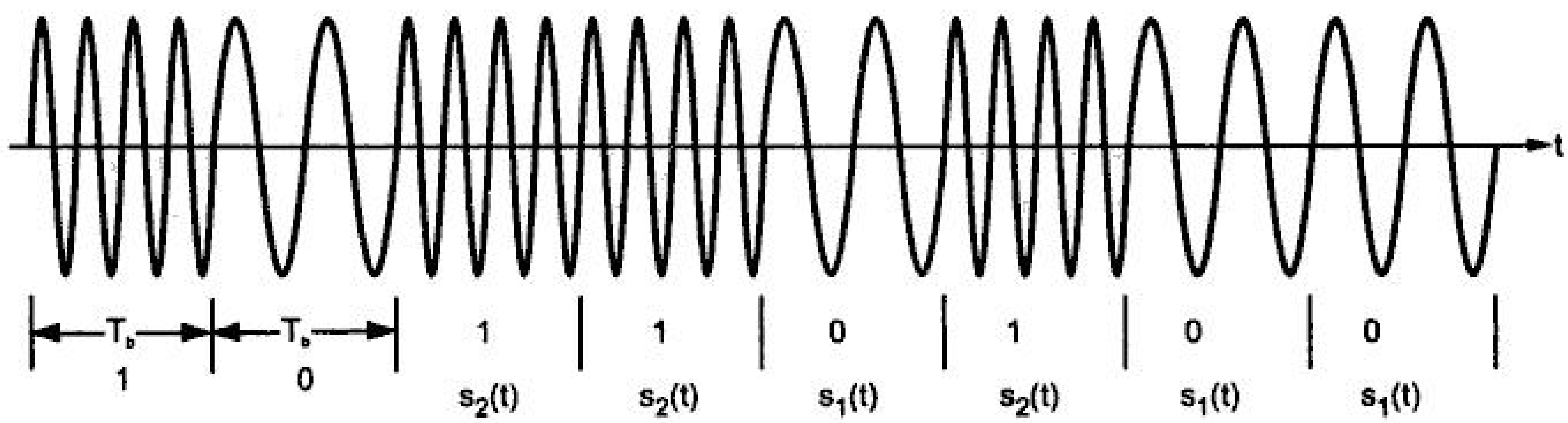


Fig. 1.2.2 Frequency modulated output of a digital modulator

The signal $s_1(t)$ has low frequency compared to signal $s_2(t)$. It is frequency modulation (FM) in two steps corresponding to binary symbols '0' and '1'. Thus even though the modulated signal appears to be continuous, the modulation is discrete (or in steps). Single carrier is converted into two waveforms $s_1(t)$ and $s_2(t)$ because of digital modulation.

If the codeword contains two bits and they are to be transmitted at a time, then there will be $M = 2^2 = 4$ distinct symbols (or codewords). These four codewords will require four distinct waveforms for transmission. Such modulators are called M-ary modulators. Frequency Shift Keying (FSK), Phase Shift Keying (PSK), Amplitude Shift Keying (ASK), Differential Phase Shift Keying (DPSK), Minimum Shift Keying (MSK) are the examples of various digital modulators. Since these modulators use continuous carrier wave, they are also called digital CW modulators.

In the receiver, the digital demodulator converts the input modulated signal to the sequence of binary bits. The most important parameter for the demodulator is the method of demodulation. The other parameters for the selection of digital modulation method are,

a) Probability of symbol or bit error.

b) Bandwidth needed to transmit the signal.

c) Synchronous or asynchronous method of detection and

d) Complexity of implementation.

1.2.5 Communication Channel

As we have seen in the preceding sections, the connection between transmitter and receiver is established through communication channel. We have seen that the communication can take place through wirelines, wireless or fiber optic channels. The other media such as optical disks, magnetic tapes and disks etc. can also be called as communication channel, because they can also carry data through them. Every communication channel has got some problems. Following are the common problems associated with the channels :

a) **Additive noise interference :** This noise is generated due to internal solid state devices and resistors etc. used to implement the communication system.

b) **Signal attenuation :** It occurs due to internal resistance of the channel and fading of the signal.

c) **Amplitude and phase distortion :** The signal is distorted in amplitude and phase because of non-linear characteristics of the channel.

d) **Multipath distortion :** This distortion occurs mostly in wireless communication channels. Signals coming from different paths tend to interfere with each other.

There are two main resources available with the communication channels. These two resources are -

a) **Channel bandwidth :** This is the maximum possible range of frequencies that can be used for transmission. For example, the bandwidth offered by wireline channels is less compared to fiber optic channels.

b) **Power in the transmitted signal :** This is the power that can be put in the signal being transmitted. The effect of noise can be minimized by increasing the power. But this cannot be increased to very high value because of the equipment and other constraints. For example, the power in the wireline channel is limited because of the cables.

The power and bandwidth limit the data rate of the communication channel. As we know, the fiber optic channel transports light signals from one place to another just like a metallic wire carriers an electric signal. There is no current or metallic conductor in optical fiber. The optical fiber has following advantages :

a) Very large bandwidths are possible.

b) Transmission losses are very small.

c) Electromagnetic interference is absent.

d) They have small size and weight.

e) They offer ruggedness and flexibility.

f) Optical fibers are low cost and cheap.

Satellites essentially perform wireless communication. Mainly satellites are repeaters. Broad area coverage is the main advantage of satellites. The power requirement is also less, since solar energy is used by satellites. Global communication is very easily possible through satellite channel. The interference on satellite channels is present but it is minimum.

Theory Question

1. *Explain with neat block diagram the essential and non essential features of a digital communication system.*

1.3 Sampling Process

1.3.1 Representation of CT Signals by its Samples

Why CT signals are represented by samples ?

- A CT signal cannot be processed in the digital processor or computer.
- To enable digital transmission of CT signals.

Fig. 1.3.1 shows the CT signal and its sampled DT signal. In this figure observe that the CT signal is sampled at t = 0, T_s, $2T_s$, $3T_s$, ... and so on.

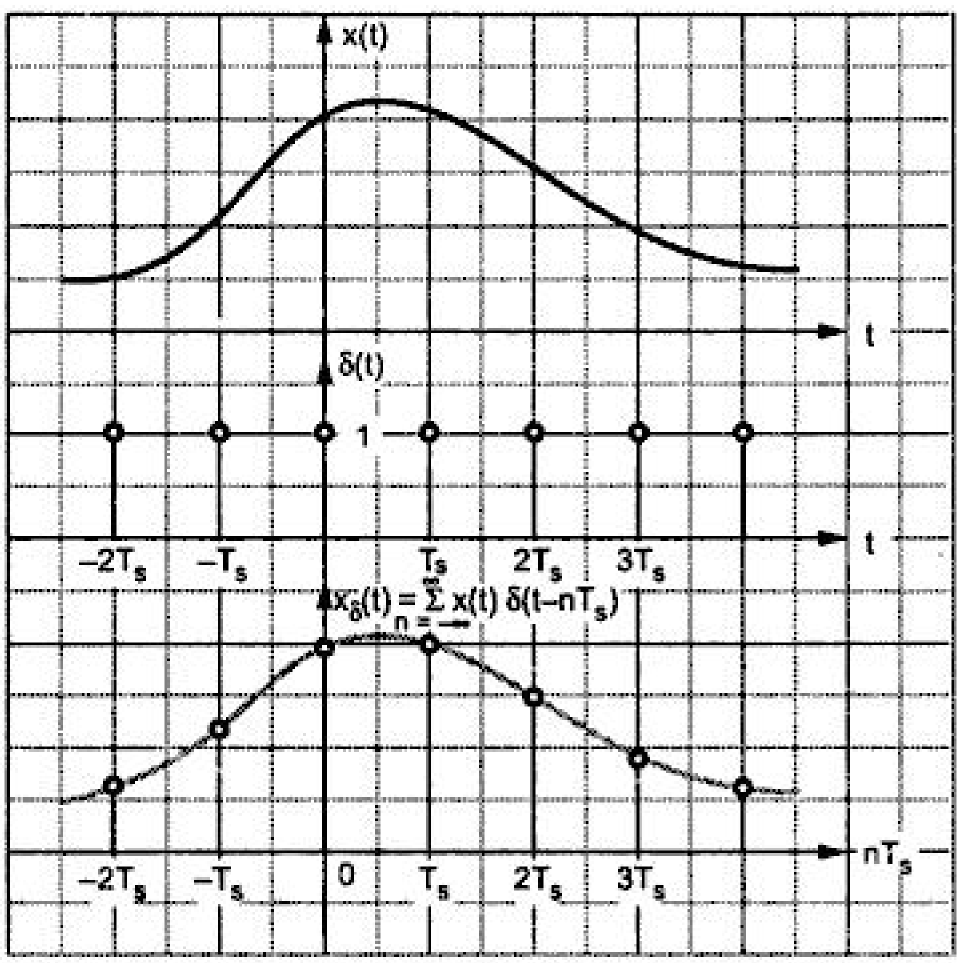


Fig. 1.3.1 CT and its DT signal

- Here sampling theorem gives the criteria for spacing 'T_s' between two successive samples.
- The samples $x_\delta(t)$ must represent all the information contained in x(t).

The sampled signal $x_\delta(t)$ is called discrete time (DT) signal. It is analyzed with the help of DTFT and z-transform.

1.3.2 Sampling Theorem for Lowpass (LP) Signals

A lowpass or LP signal contains frequencies from 1 Hz to some higher value.

Statement of sampling theorem

1) ***A band limited signal of finite energy, which has no frequency components higher than W Hertz, is completely described by specifying the values of the signal at instants of time separated by $\frac{1}{2W}$ seconds and***
2) ***A band limited signal of finite energy, which has no frequency components higher than W Hertz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second.***

The first part of above statement tells about sampling of the signal and second part tells about reconstruction of the signal. Above statement can be combined and stated alternately as follows :

A continuous time signal can be completely represented in its samples and recovered back if the sampling frequency is twice of the highest frequency content of the signal. i.e.,

$$f_s \geq 2W$$

Here f_s is the sampling frequency and

W is the higher frequency content

Proof of sampling theorem

There are two parts : (I) Representation of $x(t)$ in terms of its samples

(II) Reconstruction of $x(t)$ from its samples.

Part I : Representation of $x(t)$ in its samples $x(nT_s)$

Step 1 : Define $x_\delta(t)$

Step 2 : Fourier transform of $x_\delta(t)$ i.e. $X_\delta(f)$

Step 3 : Relation between $X(f)$ and $X_\delta(f)$

Step 4 : Relation between $x(t)$ and $x(nT_s)$

Step 1 : Define $x_\delta(t)$

Refer Fig. 1.3.1. The sampled signal $x_\delta(t)$ is given as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s) \quad \text{... (1.3.1)}$$

Here observe that $x_\delta(t)$ is the product of x_δ and impulse train $\delta(t)$ as shown in Fig. 1.3.1. In the above equation $\delta(t-nT_s)$ indicates the samples placed at $\pm T_s$, $\pm 2T_s$, $\pm 3T_s$ ... and so on.

Step 2 : FT of $x_\delta(t)$ i.e. $X_\delta(f)$

Taking FT of equation (1.3.1).

$$X_\delta(f) = \text{FT}\left\{\sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)\right\}$$

$$= \text{FT \{Product of } x(t) \text{ and impulse train\}}$$

We know that FT of product in time domain becomes convolution in frequency domain. i.e.,

$$X_\delta(f) = \text{FT}\{x(t)\} * FT\{\delta(t-nT_s)\} \quad \text{...(1.3.2)}$$

By definitions, $x(t) \xleftrightarrow{FT} X(f)$ and

$$\delta(t-nT_s) \xleftrightarrow{FT} f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Hence equation (1.3.2) becomes,

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Since convolution is linear,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f-nf_s)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f-nf_s) \qquad \textit{By shifting property of impulse function}$$

$$= \cdots f_s\, X(f-2f_s) + f_s\, X(f-f_s) + f_s\, X(f) + f_s\, X(f-f_s) + f_s\, X(f-2f_s) + \cdots$$

Comments

(i) The RHS of above equation shows that X(f) is placed at $\pm f_s, \pm 2f_s, \pm 3f_s, \cdots$

(ii) This means X(f) is periodic in f_s.

(iii) If sampling frequency is $f_s = 2W$, then the spectrums $X(f)$ just touch each other.

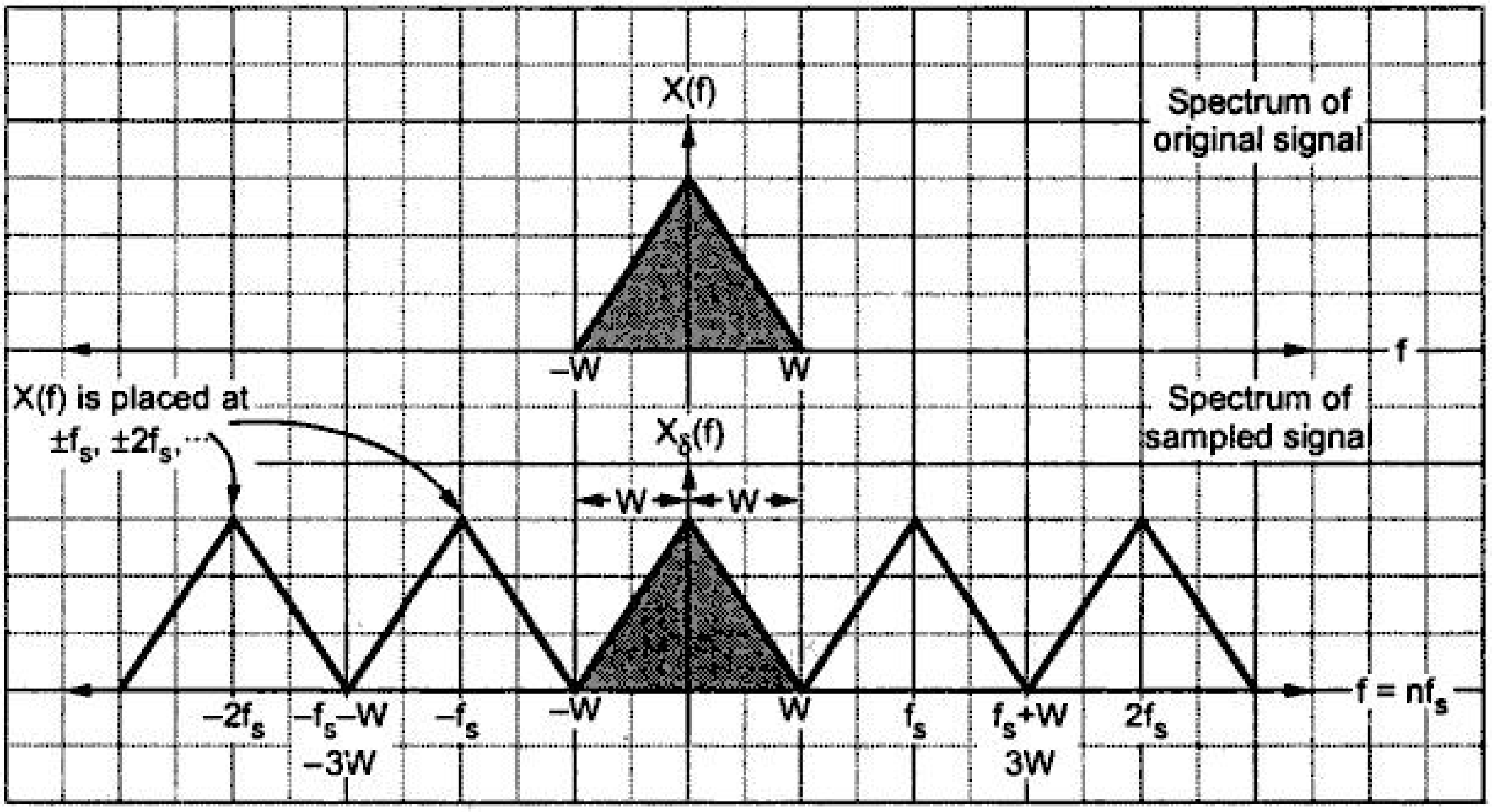


Fig. 1.3.2 Spectrum of original signal and sampled signal

Step 3 : Relation between $X(f)$ and $X_\delta(f)$

Important assumption : Let us assume that $f_s = 2W$, then as per above diagram.

$$X_\delta(f) = f_s X(f) \qquad \text{for } -W \le f \le W \text{ and } f_s = 2W$$

or
$$X(f) = \frac{1}{f_s} X_\delta(f) \qquad \ldots (1.3.3)$$

Step 4 : Relation between $x(t)$ and $x(nT_s)$

DTFT is,
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$\therefore \quad X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \qquad \ldots (1.3.4)$$

In above equation 'f' is the frequency of DT signal. If we replace $X(f)$ by $X_\delta(f)$, then 'f' becomes frequency of CT signal. i.e.,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \frac{f}{f_s} n}$$

In above equation 'f' is frequency of CT signal. And $\frac{f}{f_s}$ = Frequency of DT signal in equation (1.3.4). Since $x(n) = x(nT_s)$, i.e. samples of $x(t)$, then we have,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \quad \text{since } \frac{1}{f_s} = T_s$$

Putting above expression in equation (1.3.3),

$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}$$

Inverse Fourier Transform (IFT) of above equation gives $x(t)$ i.e.,

$$x(t) = IFT\left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}\right\} \qquad \dots (1.3.5)$$

Comments :

i) Here $x(t)$ is represented completely in terms of $x(nT_s)$.

ii) Above equation holds for $f_s = 2W$. This means if the samples are taken at the rate of 2W or higher, $x(t)$ is completely represented by its samples.

iii) First part of the sampling theorem is proved by above two comments.

Part II : Reconstruction of $x(t)$ from its samples

Step 1 : Take inverse Fourier transform of $X(f)$ which is in terms of $X_\delta(f)$.

Step 2 : Show that $x(t)$ is obtained back with the help of interpolation function.

Step 1 : The IFT of equation (1.3.5) becomes,

$$x(t) = \int_{-\infty}^{\infty} \left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}\right\} e^{j2\pi f t}\, df$$

Here the integration can be taken from $-W \le f \le W$. Since $X(f) = \frac{1}{f_s} X_\delta(f)$ for $-W \le f \le W$. (See Fig. 1.3.2).

$$\therefore \qquad x(t) = \int_{-W}^{W} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f nT_s} \cdot e^{j2\pi ft}\, df$$

Interchanging the order of summation and integration,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{f_s} \int_{-W}^{W} e^{j2\pi f(t-nT_s)}\, df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \cdot \left[\frac{e^{j2\pi f(t-nT_s)}}{j2\pi(t-nT_s)} \right]_{-W}^{W}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \left\{ \frac{e^{j2\pi W(t-nT_s)} - e^{-j2\pi W(t-nT_s)}}{j2\pi(t-nT_s)} \right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \cdot \frac{\sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi(2Wt-2WnT_s)}{\pi(f_s t - f_s nT_s)}$$

Here $f_s = 2W$, hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$. Simplifying above equation,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}(2Wt-n) \qquad \text{since } \frac{\sin \pi\theta}{\pi\theta} = \operatorname{sinc}\theta \qquad \ldots(1.3.6)$$

Step 2 : Let us interpret the above equation. Expanding we get,

$$x(t) = \cdots + x(-2T_s)\operatorname{sinc}(2Wt+2) + x(-T_s)\operatorname{sinc}(2Wt+1) + x(0)\operatorname{sinc}(2Wt) + x(T_s)\operatorname{sinc}(2Wt-1) + \cdots$$

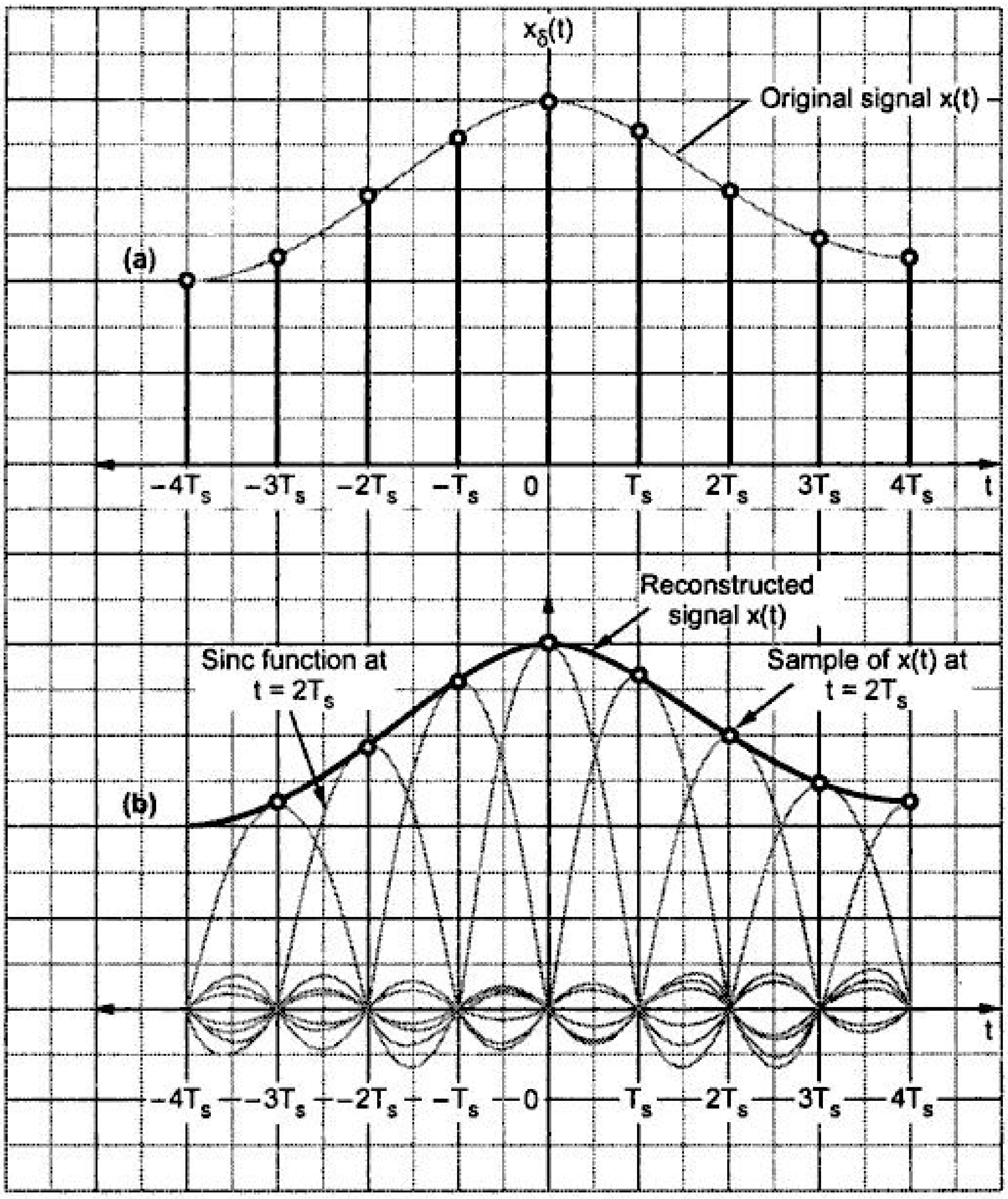


Fig. 1.3.3 (a) Sampled version of signal $x(t)$
(b) Reconstruction of $x(t)$ from its samples

Comments :

i) The samples $x(nT_s)$ are weighted by sinc functions.

ii) The sinc function is the interpolating function. Fig. 1.3.3 shows, how $x(t)$ is interpolated.

Step 3 : Reconstruction of $x(t)$ by lowpass filter

When the interpolated signal of equation (1.3.6) is passed through the lowpass filter of bandwidth $-W \leq f \leq W$, then the reconstructed waveform shown in above Fig. 1.3.3 (b) is obtained. The individual sinc functions are interpolated to get smooth $x(t)$.

1.3.3 Effects of Undersampling (Aliasing)

While proving sampling theorem we considered that $f_s = 2W$. Consider the case of $f_s < 2W$. Then the spectrum of $X_\delta(f)$ shown in Fig. 1.3.4 will be modified as follows :

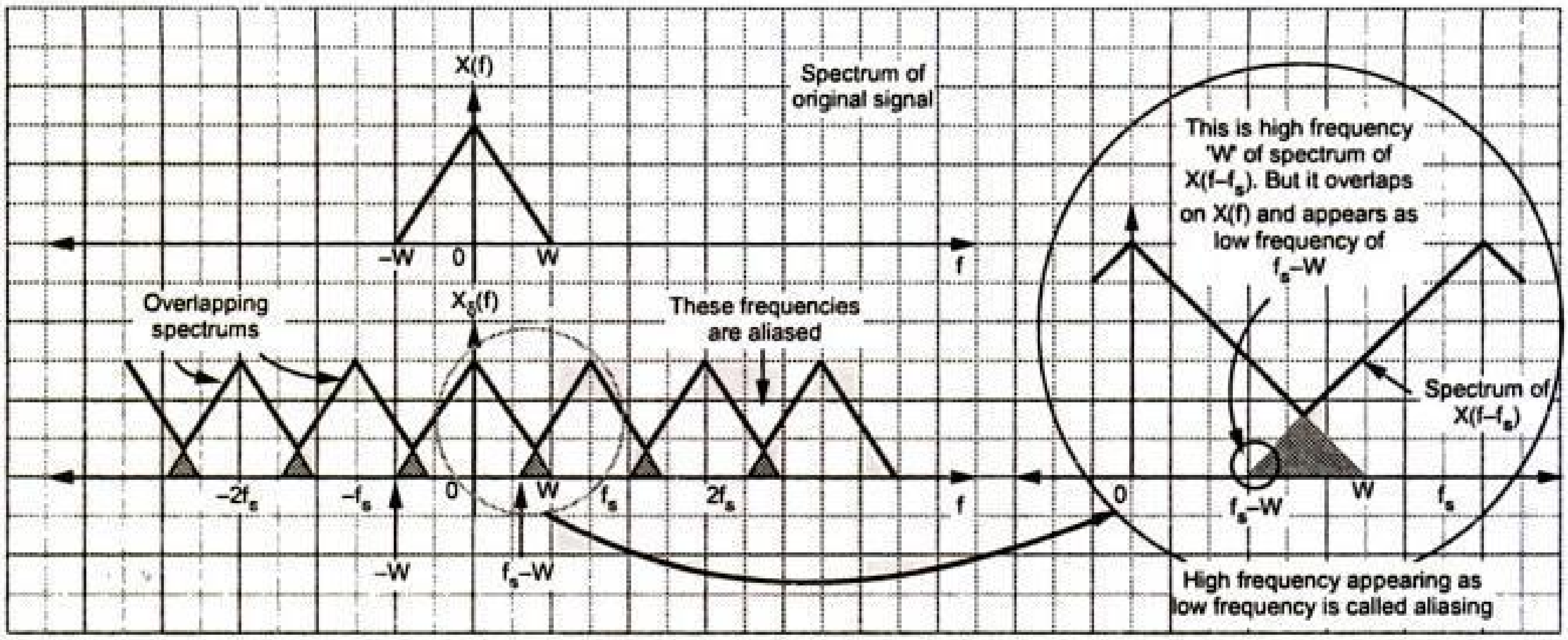


Fig. 1.3.4 Effects of undersampling or aliasing

Comments :

i) The spectrums located at $X(f), X(f-f_s), X(f-2f_s), \ldots$ overlap on each other.

ii) Consider the spectrums of $X(f)$ and $X(f-f_s)$ shown as magnified in above figure. The frequencies from $(f_s - W)$ to W are overlapping in these spectrums.

iii) The high frequencies near 'ω' in $X(f-f_s)$ overlap with low frequencies $(f_s - W)$ in $X(f)$.

Definition of aliasing : When the high frequency interferes with low frequency and appears as low frequency, then the phenomenon is called aliasing.

Effects of aliasing : i) Since high and low frequencies interfere with each other, distortion is generated.

ii) The data is lost and it cannot be recovered.

Different ways to avoid aliasing

Aliasing can be avoided by two methods :

i) Sampling rate $f_s \geq 2W$.

ii) Strictly bandlimit the signal to 'W'.

i) Sampling rate $f_s \geq 2W$

When the sampling rate is made higher than 2W, then the spectrums will not overlap and there will be sufficient gap between the individual spectrums. This is shown in Fig. 1.3.5.

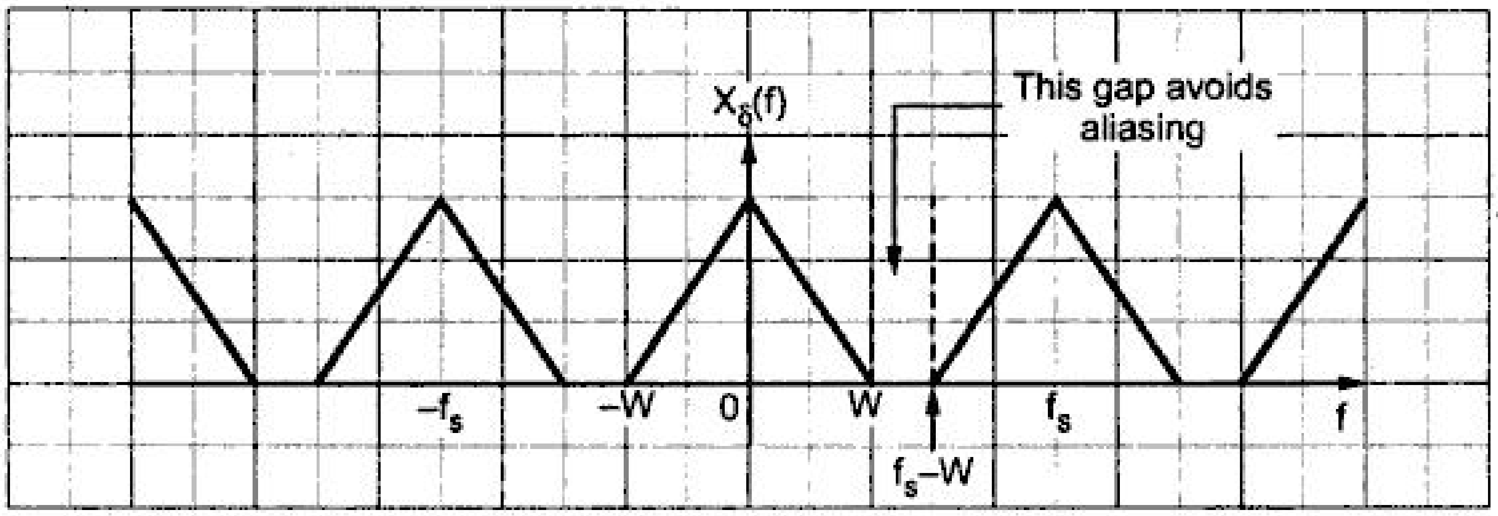


Fig. 1.3.5 $f_s \geq 2W$ avoids aliasing by creating a bandgap

ii) Bandlimiting the signal

The sampling rate is, $f_s = 2W$. Ideally speaking there should be no aliasing. But there can be few components higher than 2W. These components create aliasing. Hence a lowpass filter is used before sampling the signals as shown in Fig. 1.3.6. Thus the output of lowpass filter is strictly bandlimited and there are no frequency components higher than 'W'. Then there will be no aliasing.

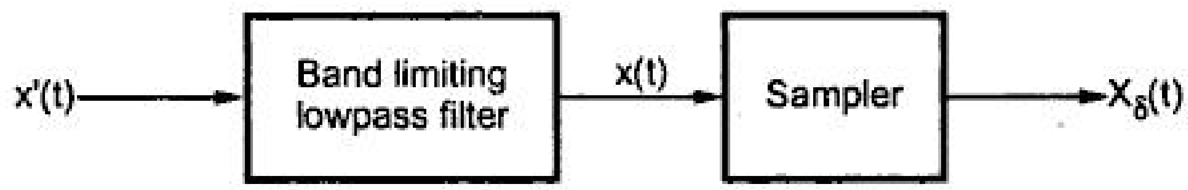


Fig. 1.3.6 Bandlimiting the signal. The bandlimiting LPF is called prealins filter

1.3.4 Nyquist Rate and Nyquist Interval

Nyquist rate : When the sampling rate becomes exactly equal to '2W' samples/sec, for a given bandwidth of W Hertz, then it is called Nyquist rate.

Nyquist interval : It is the time interval between any two adjacent samples when sampling rate is Nyquist rate.

$$\boxed{\begin{aligned} \text{Nyquist rate} &= 2W \text{ Hz} \\ \text{Nyquist interval} &= \frac{1}{2W} \text{ seconds} \end{aligned}}$$

... (1.3.7)

... (1.3.8)

1.3.5 Reconstruction Filter (Interpolation Filter)

Definition

In section 1.3.2 we have shown that the reconstructed signal is the succession of sinc pulses weighted by $x(nT_s)$. These pulses are interpolated with the help of a lowpass filter. It is also called *reconstruction* filter or *interpolation* filter.

Ideal filter

Fig. 1.3.7 shows the spectrum of sampled signal and frequency response of required filter. When the sampling frequency is exactly 2W, then the spectrums just touch each other as shown in Fig. 1.3.7. The spectrum of original signal, $X(f)$ can be filtered by an ideal filter having passband from $-W \leq f \leq W$.

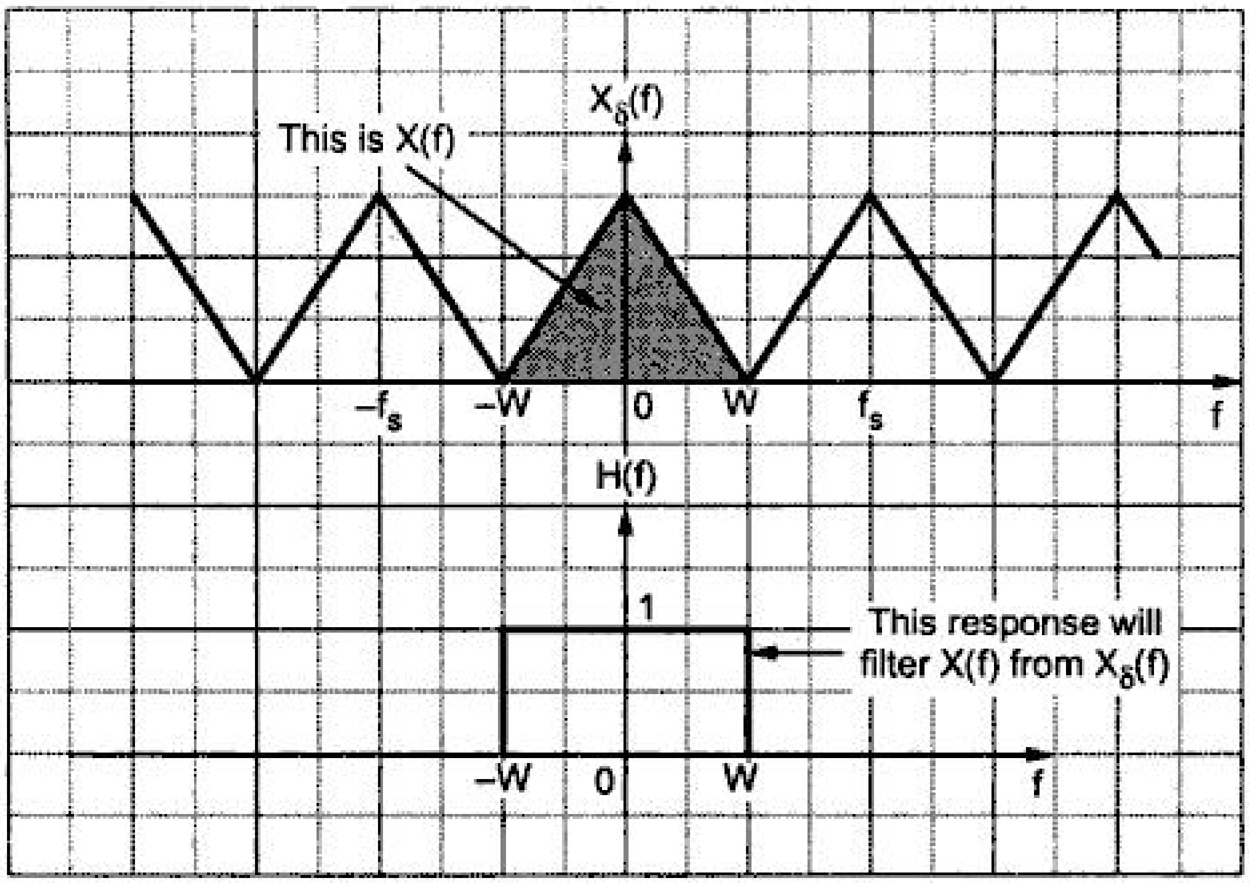


Fig. 1.3.7 Ideal reconstruction filter

Non-ideal filter

As discussed above, an ideal filter of bandwidth 'W' filters out an original signal. But practically ideal filter is not realizable. It requires some transition band. Hence f_s must be greater than 2W. It creates the gap between adjacent spectrums of $X_\delta(f)$. This gap can be used for the transition band of the reconstruction filter. The spectrum $X(f)$ is then properly filtered out from $X_\delta(f)$. Hence the sampling frequency must be greater than '2W' to ensure sufficient gap for transition band.

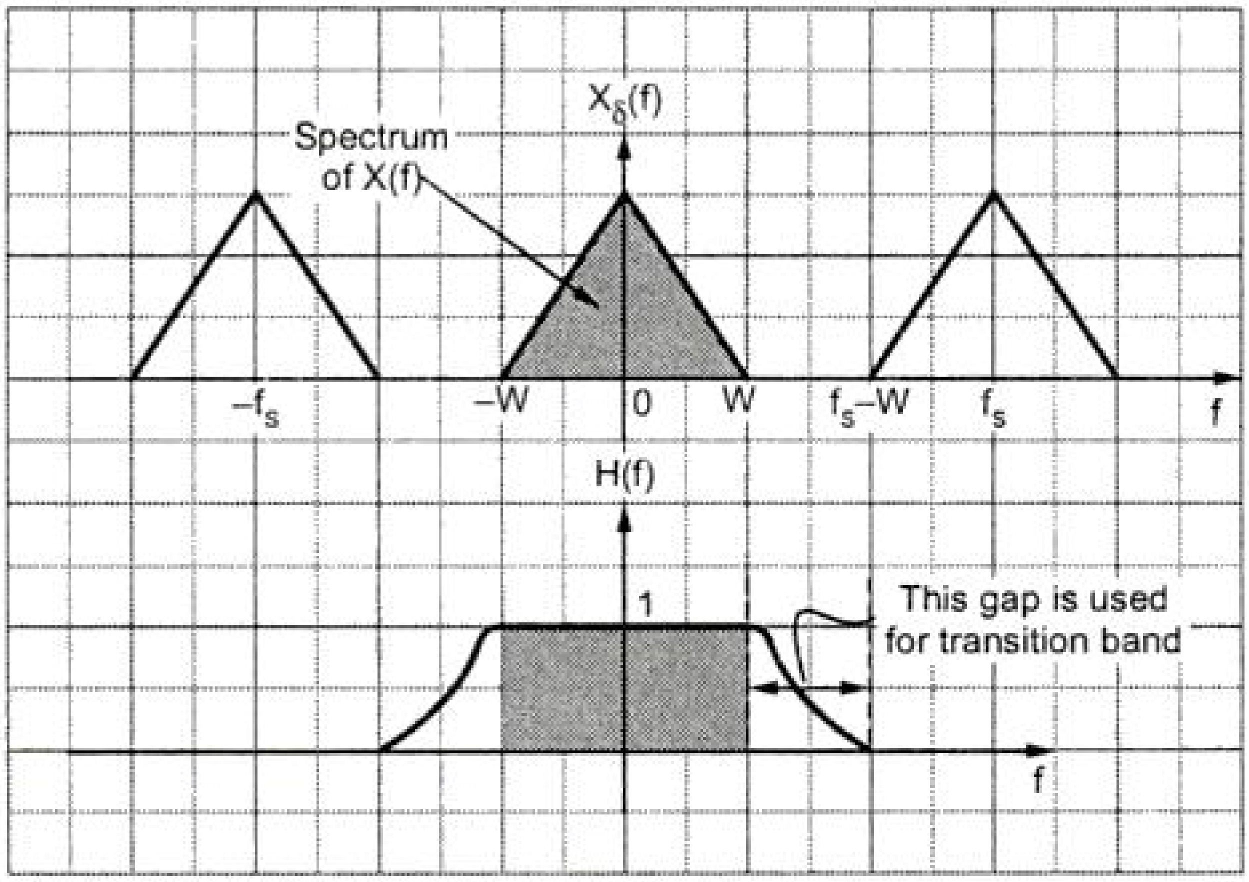


Fig. 1.3.8 Practical reconstruction filter

1.3.6 Zero-Order Hold for Practical Reconstruction

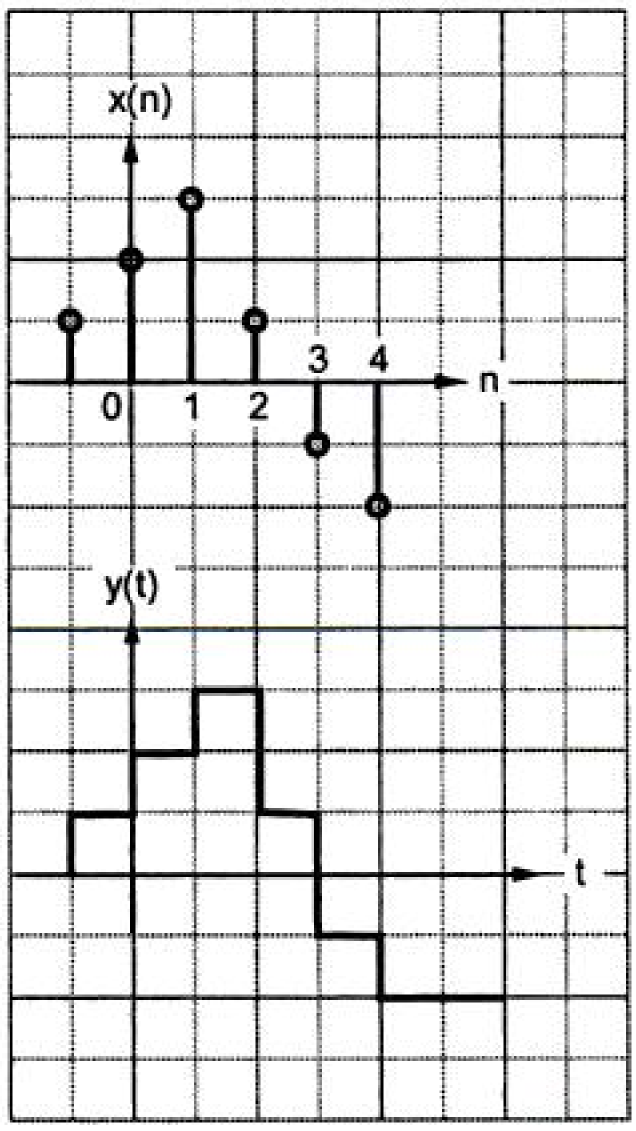


Fig. 1.3.9 Input and output of zero order hold

- The zero-order hold circuit is used for practical reconstruction. It simply hold the value $x(n)$ for 'T' seconds. Here 'T' is the sampling period .
- The output of the zero-order hold is staircase signal that approximates $x(n)$. This is shown in Fig. 1.3.9.
- Let the impulse response of zero-order hold be represented as,

$$h(t) = \begin{cases} 1, & 0 \le t \le T \\ 0 & Otherwise \end{cases}$$

Then the output $y(t)$ of the zero-order hold will be convolution of $h(t)$ and sampled input $x_\delta(t)$. i.e,

$$y(t) = h(t) * \sum_{h=-\infty}^{\infty} x(n)\, \delta(t-nT)$$

$$= h(t) * x_\delta(t)$$

$$\therefore \quad Y(\omega) = H(\omega) \cdot X(\omega)$$

Here
$$H(\omega) = 2e^{-j\omega T/2} \frac{\sin\frac{\omega T}{2}}{\omega}$$

$$\therefore \quad Y(\omega) = 2e^{-j\frac{\omega T}{2}} \frac{\sin\frac{\omega T}{2}}{\omega} \cdot X(\omega)$$

The above equation shows that spectrum $X(\omega)$ is changed due to convolution or passing through zero order hold. These changes are,

(i) There is linear phase shift corresponding to time delay of $\frac{T}{2}$ sec.

(ii) The main lobe of $\frac{\sin\frac{\omega T}{2}}{\omega}$ modifies the shape of $X(\omega)$.

- The above modifications can be reduced by increasing the sampling frequency ω_s or reducing the time 'T'.
- Sometimes anti-imaging filter is used for compensating the modifications. It's spectrum is given as,

$$H_c(\omega) = \begin{cases} \dfrac{\omega T}{2\sin\frac{\omega T}{2}}, & |\omega| \le \omega_m \\ 0 & |\omega| > \omega_s - \omega_m \end{cases}$$

This filter provides reverse action to that of zero-order hold. Fig. 1.3.10 shows the block diagram with anit-imaging filter.

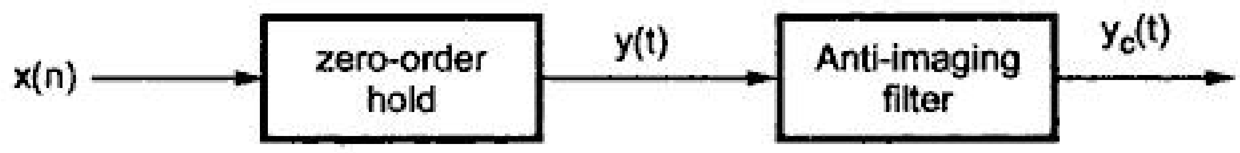


Fig. 1.3.10 Block diagram of practical reconstruction

1.3.7 Sampling Theorem in Frequency Domain

Statement

We have seen that if the bandlimited signal is sampled at the rate of ($f_s > 2W$) in time domain, then it can be fully recovered from its samples. This is sampling theorem in time domain. A dual of this also exists and it is called sampling theorem in frequency domain. It states that,

> *"A timelimited signal which is zero for $|t| > T$ is uniquely determined by the samples of its frequency spectrum at intervals less than $\frac{1}{2T}$ Hertz apart".*

- **Explanation :** Thus the spectrum is sampled at $f_s < \frac{1}{2T}$ in the frequency domain. T is the maximum time limit above which signal $x(t)$ goes to zero. 'f_s' represents the sampling frequency interval in the frequency spectrum of the signal. Note that here f_s does not represent number of samples taken per second. But it represents the frequency interval at which the samples are separated in frequency domain.
- Fig. 1.3.11 illustrates the sampling theorem in frequency domain. We can see from 1.3.11 (a) that a rectangular pulse is time limited to $\pm\frac{T}{2}$ seconds i.e., $x(t) = A$ for $-\frac{T}{2} \le t < \frac{T}{2}$. The spectrum of rectangular pulse is shown in Fig. 1.3.11 (b). This spectrum $X(f)$ of Fig. 1.3.11 (b) is sampled at the uniform intervals less than $\frac{1}{2T}$ Hz. The sampled version of this spectrum is shown in Fig. 1.3.11 (c) and called $X_\delta(f)$. Thus each frequency sample of $x_\delta(f)$ is separated by 'f_s' Hz with respect to the neighbouring frequency samples.

As shown in Fig. 1.3.11 (c); since $X(f)$ is sampled in frequency domain, this is called sampling theorem in frequency domain. Therefore now we can state sampling theorem in frequency domain as,

> *"A timelimited spectrum $X(f)$ can be represented and recovered fully from its samples if the samples are taken at the intervals less than $\frac{1}{2T}$ Hz apart".*

We have represented $x(t)$ in terms of its samples $x(nT_s)$ and an interpolation function (sinc function) in equation 1.3.6 for time domain sampling theorem. Similarly a continuous frequency spectrum $X(f)$ can be represented in terms of its samples $X(kf_s)$ and an interpolation function (sinc function) as follows :

$$\boxed{X(f) = \sum_{k=-\infty}^{\infty} X(kf_s)\, sinc[T(f - kf_s)]} \qquad \dots (1.3.9)$$

Aliasing can also occur in frequency domain sampling. The replicas are in the time domain and are located at multiple of

$$T_s = \frac{1}{f_s}$$

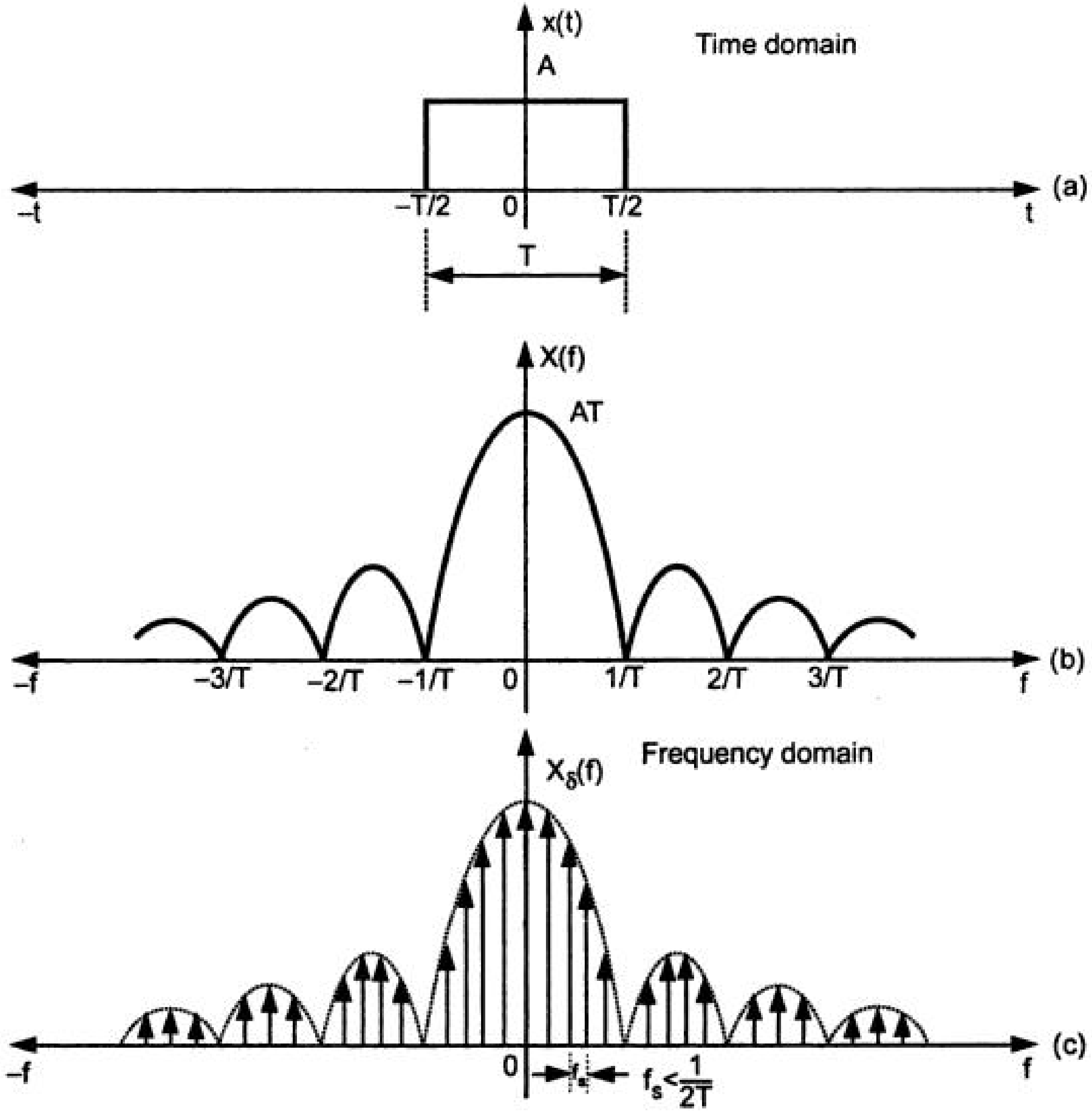


Fig. 1.3.11 (a) Signal x(t) time limited to $\pm\frac{T}{2}$

(b) Continuous spectrum of $x(t)$

(c) Sampled spectrum X_δ (f)

1.3.8 Sampling of Bandpass Signals

Statement :

In the last sections we discussed sampling theorem for lowpass signals. However when the signal is bandpass in nature, then different criteria should be used to sample the signal. The sampling theorem for bandpass signals can be written as follows

The bandpass signal $x(t)$ whose maximum bandwidth is 2W can be completely represented into and recovered from its samples if it is sampled at the minimum rate of twice the bandwidth.

- **Spectrum of bandpass signal :** Thus if bandwidth is 2W, then minimum sampling rate for bandpass signal should be 4W samples per second. Fig. 1.3.12 shows the spectrum of bandpass signal.

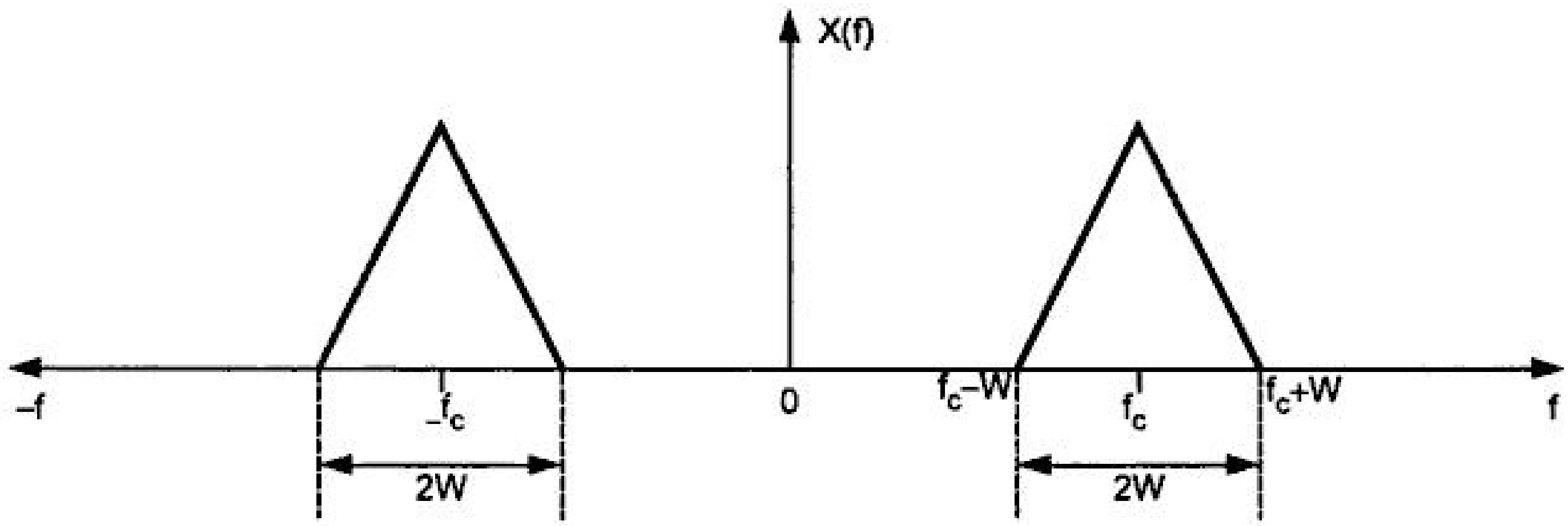


Fig. 1.3.12 Spectrum of bandpass signal

The spectrum is centered around frequency f_c. The bandwidth is 2W. Thus the frequencies in the bandpass signal are from $f_c - W$ to $f_c + W$. That is the highest frequency present in the bandpass signal is $f_c + W$. Normally the centre frequency $f_c > W$.

- **Inphase and quadrature components :** This bandpass signal is first represented in terms of its inphase and quadrature components.

Let $x_I(t)$ = Inphase component of $x(t)$

and $x_Q(t)$ = Quadrature component of $x(t)$

Then we can write $x(t)$ in terms of inphase and quadrature components as,

$$x(t) = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t) \quad \dots (1.3.10)$$

The inphase and quadrature components are obtained by multiplying $x(t)$ by $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ and then suppressing the sum frequencies by means of low-pass filters. Thus inphase $x_I(t)$ and quadrature $x_Q(t)$ components contain only low frequency components. The spectrum of these components is limited between – W to +W. This is shown in Fig. 1.3.13.

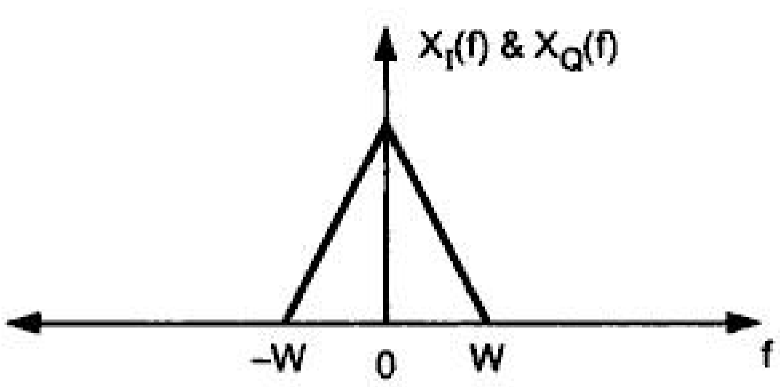


Fig. 1.3.13 Spectrum of inphase and quadrature components of bandpass signal x(t)

- **Representation in terms of samples** : After some mathematical manipulations, on equation 1.3.10. We obtain the reconstruction formula as,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{4W}\right) \operatorname{sinc}\left(2Wt - \frac{n}{2}\right) \cos\left[2\pi f_c\left(t - \frac{n}{4W}\right)\right] \quad \ldots (1.3.11)$$

Compare this reconstruction formula with that of lowpass signals given by equation (1.3.6). It is clear that $x(t)$ is represented by $x\left(\frac{n}{4W}\right)$ completely. Here,

$$x\left(\frac{n}{4W}\right) = x(nT_s) = \text{Sampled version of bandpass signal}$$

and $$T_s = \frac{1}{4W}$$

- Thus if 4W samples per second are taken, then the bandpass signal of bandwidth 2W can be completely recovered from its samples.

Thus, for bandpass signals of bandwidth 2W,

$$\text{Minimum sampling rate} = \text{Twice of bandwidth}$$
$$= \text{4W samples per second}$$

Example 1.3.1 : *Show that a bandlimited signal of finite energy which has no frequency components higher than W Hz is completely described by specifying values of the signals at instants of time separated by 1 / 2W seconds and also show that if the instantaneous values of the signal are separated by intervals larger than $\frac{1}{2W}$ seconds, they fail to describe the signal. A bandpass signal has spectral range that extends from 20 to 82 kHz. Find the acceptable range of sampling frequency f_s.*

Solution :

Step 1 : Define $x_\delta(t)$.

Let $x(t)$ be the bandlimited signal which has no frequency components higher than W Hz. Let it be sampled by a sampling function

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The sampling function is the train of impulses with T_s as distance between successive impulses. Let $x(nT_s)$ be the instantaneous amplitude of signal $x(t)$ at instant $t = T_s$. The sampled version of $x(t)$ can be represented as multiplication of $x(nT_s)$ and $\delta(t)$ i.e.

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s) \qquad \text{... (1.3.12)}$$

Step 2 : Fourier transform of $x_\delta(t)$ i.e. $X_\delta(f)$

Fourier transform of this sampled signal can be obtained as,

$$X_\delta(f) = FT\{x_\delta(t)\}$$

$$\therefore \quad X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s) \qquad \text{... (1.3.13)}$$

Here f_s is the sampling rate which is given as $f_s = \frac{1}{T_s}$

And, $X(f-nf_s) = X(f)$ at $nf_s = 0, \ \pm f_s, \pm 2f_s, \pm 3f_s \ldots$

Thus the same spectrum $X(f)$ appears at $f=0, f=\pm f_s, f=\pm 2f_s$ etc. This means that a periodic spectrum with period equal to f_s is generated in frequency domain because of sampling $x(t)$ in time domain. Therefore equation 1.3.13 can be written as,

$$X_\delta(f) = f_s\,X(f) + f_s\,X(f\pm f_s) + f_s\,X(f\pm 2f_s) + f_s\,X(f\pm 3f_s) + f_s\,X(f\pm 4f_s) + \ldots.. \qquad \text{... (1.3.14)}$$

or

$$X_\delta(f) = f_s\,X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} f_s\,X(f-nf_s) \qquad \text{... (1.3.15)}$$

Step 3 : Relation between $X(f)$ and $X_\delta(f)$.

By definition of Fourier transform, $X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt$

For sampled version of $x(t)$, we have $t = nT_s$. Then above equation becomes,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\,e^{-j2\pi fnT_s} \qquad \text{... (1.3.16)}$$

It is given that the signal is band limited to W Hz and,

$$T_s = \frac{1}{2W} \text{ seconds}, \quad \therefore \quad f_S = \frac{1}{T_s} = 2W \qquad \text{... (1.3.17)}$$

From equation 1.3.14 we know that $X_\delta(f)$ is periodic in f_s. The spectrum $X(f)$ and $X_\delta(f)$ are shown in Fig. 1.3.14.

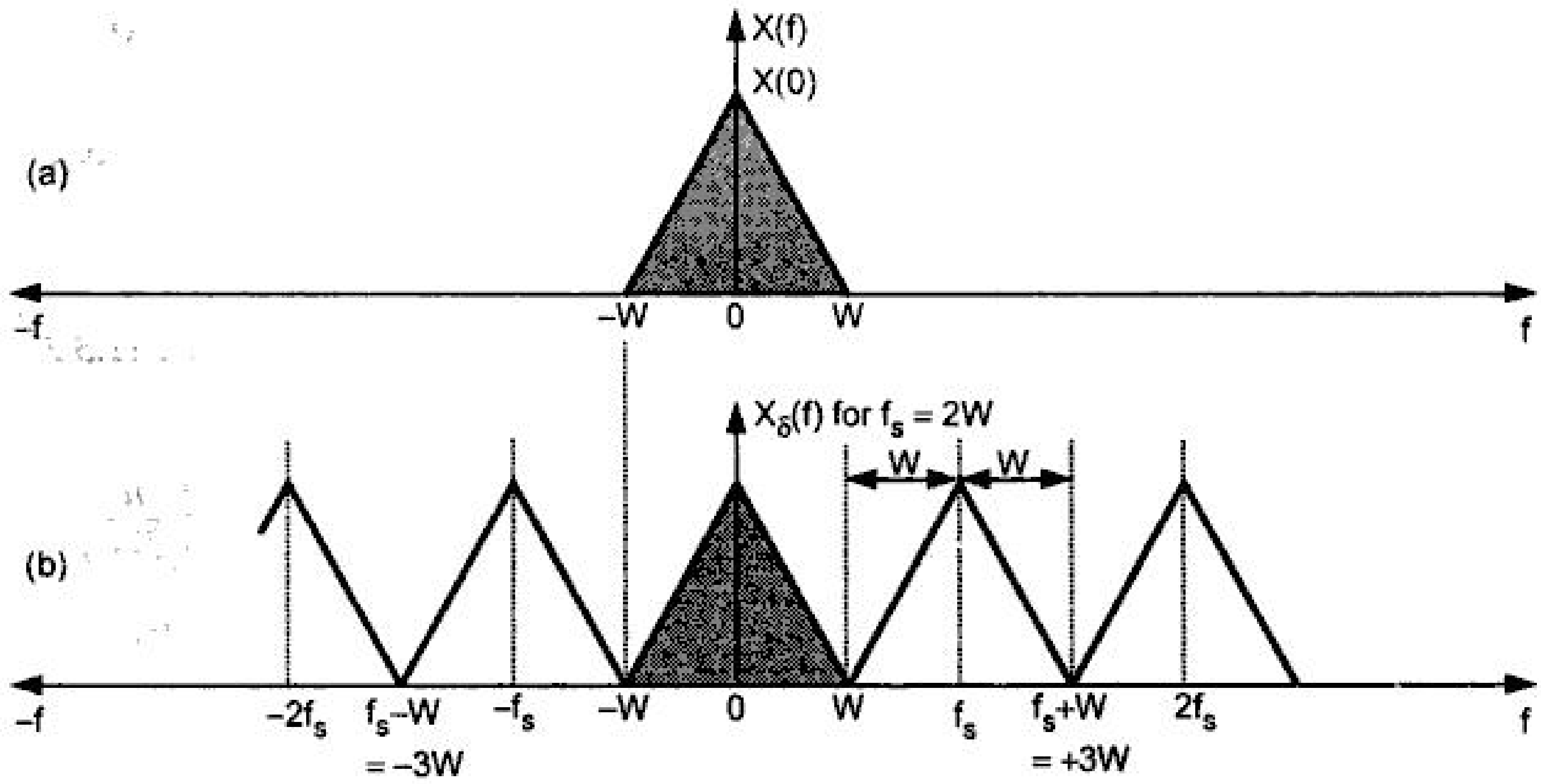


Fig. 1.3.14 (a) Spectrum of $x(t)$
(b) Spectrum of $X_\delta(f)$ with $f_s = 2W$

Since $f_s = 2W$; $f_s - W = W$ and $f_s + W = 3W$

Thus the periodic spectrums $X(f)$ just touch $\pm W, \pm 3W, \pm 5W$ .... etc.

Thus there is no aliasing. From equation 1.3.15. we can write,

$$X(f) = \frac{1}{f_s} X_\delta(f) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} X(f - nf_s) \qquad \text{... (1.3.18)}$$

With $f_s = 2W$ in above equation,

$$X(f) = \frac{1}{2W} X_\delta(f) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} X(f - nf_s)$$

i.e.

$$X(f) = \frac{1}{2W} X_\delta(f) \qquad \text{For } -W \le f \le W \qquad \text{... (1.3.19)}$$

Step 4 : Relation between $x(t)$ and $x(nT_s)$ or $x\left(\frac{n}{2W}\right)$.

Putting the value of $X_\delta(f)$ from equation (1.3.16) in the above equation,

$$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi fn \cdot T_s}$$

Since

$$T_s = \frac{1}{2W}$$

i.e. $$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi fn/W} \qquad \text{...(1.3.20)}$$

$x(t)$ can be recovered from $X(f)$ by taking Inverse Fourier Transform of above equation.

i.e. $$x(t) = IFT\left\{\frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi fn/W}\right\} \qquad \text{... (1.3.21)}$$

This equation shows that $x(t)$ is represented completely by its samples $x\left(\frac{n}{2W}\right)$ for $-\infty < n < \infty$. That is the sequence $x\left(\frac{n}{2W}\right)$ has all the information contained in $x(t)$.

Reconstruction of signal from samples :

Consider equation 1.3.21,

$$x(t) = IFT\left\{\frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi fn/W}\right\}$$

- By definition of Inverse Fourier Transform (IFT) the above equation becomes,

$$x(t) = \int_{-W}^{W} \frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi fn/W} e^{j2\pi ft}\, dt$$

Interchanging the order of summation of integration,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{1}{2W} \int_{-W}^{W} e^{j2\pi f\left(t - \frac{n}{2W}\right)}\, dt$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{\sin(2\pi Wt - n\pi)}{(2\pi Wt - n\pi)}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{\sin \pi (2Wt - n)}{\pi (2Wt - n)}$$

Since $\text{sinc}\, \theta = \frac{\sin \pi\theta}{\pi\theta}$, above equation becomes,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \text{sinc}(2Wt - n) \qquad -\infty < n < \infty$$

This is interpolation formula to reconstruct $x(t)$ from its samples $x(nT_s)$.

- Thus the above discussion shows that the signal can be completely represented into and recovered from its samples if the spacing between the successive samples is $\frac{1}{2W}$ seconds. i.e. $f_s = 2W$ samples per second.

Sampling frequency for bandpass signal :

The spectral range of the bandpass signal is 20 to 82 kHz.

$$\text{Bandwidth} = 2W = 82 \text{ kHz} - 20 \text{ kHz} = 62 \text{ kHz}$$

$$\begin{aligned} \text{Minimum sampling rate} &= 2 \times \text{ Bandwidth} \\ &= 2 \times 62 \text{ kHz} \\ &= 124 \text{ kHz} \end{aligned}$$

Normally the range of minimum sampling frequencies is specified for bandpass signals. It lies between 4 W to 8 W samples per second.

$\therefore$ Range of minimum sampling frequencies

$$\begin{aligned} &= (2 \times \text{Bandwidth}) \text{ to } (4 \times \text{Bandwidth}) \\ &= 2 \times 62 \text{ kHz to } 4 \times 62 \text{ kHz} \\ &= 124 \text{ kHz to } 248 \text{ kHz} \end{aligned}$$

Example 1.3.2 : *Find the Nyquist rate and Nyquist interval for following signals.*

i) $m(t) = \dfrac{1}{2\pi} \cos(4000\pi t)\cos(1000\pi t)$

ii) $m(t) = \dfrac{\sin 500\pi t}{\pi t}$

Solution : i)

$$\begin{aligned} m_1(t) &= \frac{1}{2\pi} \cos(4000\pi t)\cos(1000\pi t) \\ &= \frac{1}{2\pi}\left\{\frac{1}{2}\left[\cos(4000\pi t - 1000\pi t) + \cos(4000\pi t + 1000\pi t)\right]\right\} \\ &= \frac{1}{4\pi}\left[\cos 3000\pi t + \cos 5000\pi t\right] \\ &= \frac{1}{4\pi}\left[\cos 2\pi f_1 t + \cos 2\pi f_2 t\right] \end{aligned}$$

Comparing, we get, $f_1 = 1500$ Hz and $f_2 = 2500$ Hz

Here highest frequency $W = f_2 = 2500$ Hz

$$\therefore \quad \text{Nyquist rate} = 2W = 2\times 2500 = 5000 \text{ Hz}$$

$$\text{Nyquist interval} = \frac{1}{2W} = \frac{1}{2\times 2500} = 0.2 \text{ msec}$$

ii)
$$m_2(t) = \frac{\sin 500\pi t}{\pi t}$$

$$= \frac{\sin 2\pi f t}{\pi t}$$

Comparing, we get, f = 250 Hz or W = 250 Hz

$$\therefore \quad \text{Nyquist rate} = 2W = 2\times 250 = 500 \text{ Hz}$$

$$\text{Nyquist interval} = \frac{1}{2W} = \frac{1}{2\times 250} = 2 \text{ msec}$$

Example 1.3.3 : *Fig. 1.3.15 shows the spectrum of a message signal. The signal is sampled at the rate of $f_s = 1.5 f_{max}$, where $f_{max} = 1$ Hz, is maximum signal frequency. Sketch the spectrum of the sampled version of the signal. If the sampled signal is passed through an ideal low-pass filter of bandwidth f_{max}, sketch the spectrum of the output signal from this filter.*

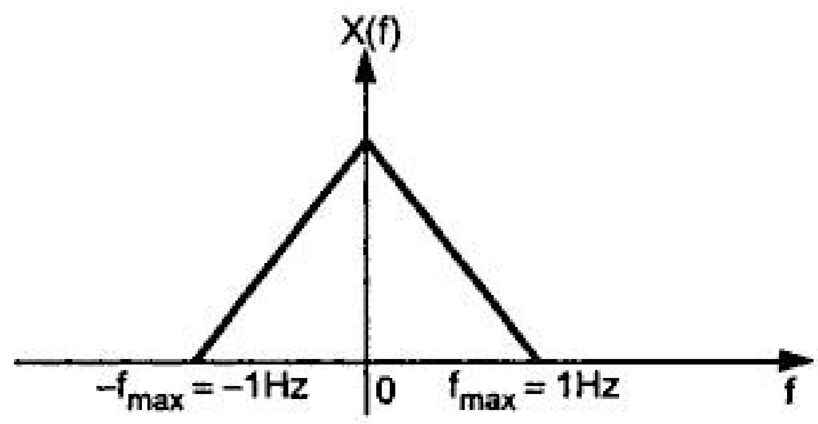


Fig. 1.3.15 Spectrum of message signal *x*(*f*)

Solution : When $x(t)$ is sampled instantaneously its spectrum is given as,

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} f_s X(f - n f_s)$$

Here $f_s = 1.5 f_{max}$. Then above equation will be,

$$X_\delta(f) = 1.5 f_{max} \cdot X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} 1.5 f_{max} X(f - 1.5\, n f_{max})$$

with $f_{max} = 1$ Hz, above equation will be,

$$X_\delta(f) = 1.5 X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} 1.5 X(f - 1.5\, n)$$

Fig. 1.3.16 (a) shows the plot of above equation.

Because of under sampling ($f_s = 1.5 f_{max}$), there is aliasing effect and it is visible in Fig. 1.3.16 (a). When the sampled signal is passed through an ideal lowpass filter of bandwidth f_{max} [Fig. 1.3.16 (b)], the spectrum of the corresponding output signal is shown in Fig. 1.3.16 (c). It shows clearly that the spectrum of Fig. 1.3.15 and Fig. 1.3.16(c) are not same because of aliasing.

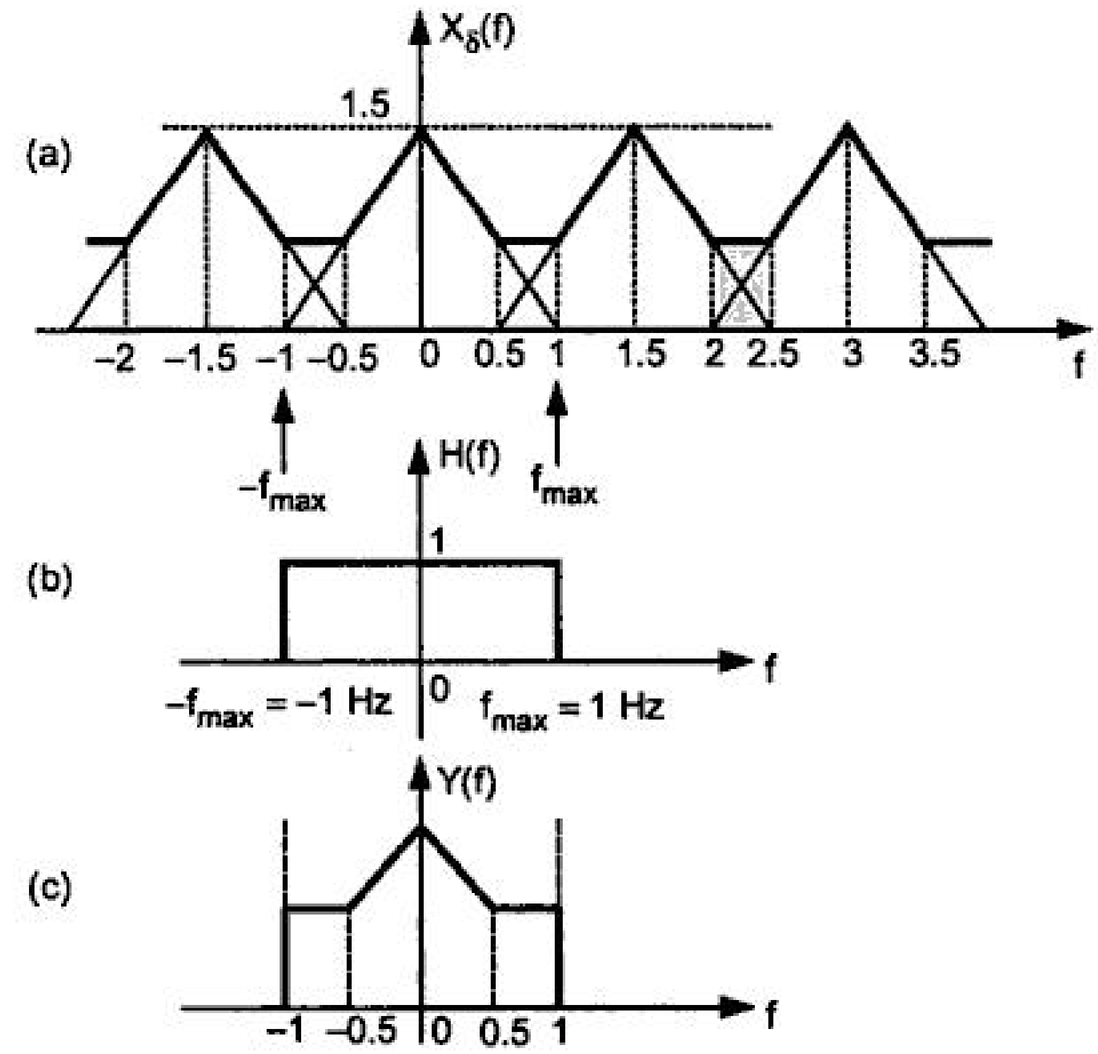


Fig. 1.3.16 (a) Spectrum of the sampled signal at $f_s = 1.5 f_{max}$
(b) Response of ideal low-pass filter with $|H(f)| = 1$ for $-f_{max} < f < f_{max}$
(c) Spectrum of signal at the output of the lowpass filter

Theory Questions

1. *State and prove the sampling theorem in time domain. What is Nyquist rate ?*
2. *Explain the function of low pass filter in sampling.*
3. *State and prove the sampling theorem in frequency domain. Show that the effect of sampling is to produce double sided spectra around each harmonic of sampling frequency.*
4. *A bandlimited signal $f(t)$ is sampled by train of rectangular pulses of width τ and period T.*

   (a) Find an expression for the sampled signal.

   (b) Determine the spectrum of the sampled signal and sketch it.
5. *What is aliasing and how it is reduced ?*

Unsolved Examples

1. *Specify the Nyquist rate and the Nyquist interval for each of the following signals.*

   (a) $x(t) = sinc\,(200\,t)$

   (b) $x(t) = sinc^2\,(200\,t)$ *(c)* $x(t) = sinc\,(200\,t) + sinc^2\,(200\,t)$

   [Ans : (a) rate : 200, interval : $\frac{1}{200}$, (b) rate : 400, interval : $\frac{1}{400}$

   (c) rate : 400, interval : $\frac{1}{400}$]
2. *Consider the analog signal* $x(t) = 3\cos 50\,\pi t + 10\sin 300\,\pi t - \cos 100\,\pi t$. *What is the Nyquist rate for this signal ?* **(Ans. : Nyquist rate = 300 Hz)**
3. *Find the Nyquist rate and Nyquist interval for the following.*

   i) $m(t) = \frac{1}{2\pi}\cos\,(400\,\pi\,t)\cdot\cos\,(200\,\pi\,t)$ **(Ans. : rate = 600 Hz, Interval = 1.6667 ms]**

   ii) $m(t) = \frac{\sin \pi t}{\pi t}$ **[Ans. : rate 1 Hz, Interval = 1 sec]**
4. *The signal* $x\,(t) = \cos 200\pi t + 0.25\cos 700\pi t$ *is sampled at the rate of 400 samples per second. Sampled waveform is then passed through an ideal lowpass filter with 200 Hz bandwidth. Write an expression for filter output. Sketch the frequency spectrum of sampled waveform.*

   [Ans. : $y(t) = c\ \ s(2\pi \times 100t) + 0.2\ \ c\ \ s(2\pi \times\ \ 0f)$
5. *A waveform* $[20 + 20\sin(500t + 30°)]$ *is to be sampled periodically and reproduced from these sample values. Find maximum allowable time interval between sample values. How many sample values are needed to be stored in order to reproduce 1 sec of this waveform ?*

   (Ans. : $\dot{T}_s$ = 6.283 ms, samples = 160)

1.4 Pulse Amplitude Modulation (PAM)

- **Definition** : The amplitude of the pulse change according to amplitude of modulation signal at the sampling instant.
- **Types of PAM** : Depending upon the shape of the pulse of PAM, there are three types of PAM :

(i) Ideally or instantaneously sampled PAM.

(ii) Naturally sampled PAM.

(iii) Flat top sampled PAM.

1.4.1 Ideal Sampling or Instantaneous Sampling or Impulse Sampling

Basic Principle

Ideal sampling is same as instantaneous sampling. Fig. 1.4.1 (a) shows the switching sampler. If closing time 't' of the switch approaches zero the output $x_\delta(t)$ gives only instantaneous value. The waveforms are shown in Fig. 1.4.1 (b). Since the width of the pulse approaches zero, the instantaneous sampling gives train of impulses in $x_\delta(t)$. The area of each impulse in the sampled version is equal to instantaneous value of input signal $x(t)$.

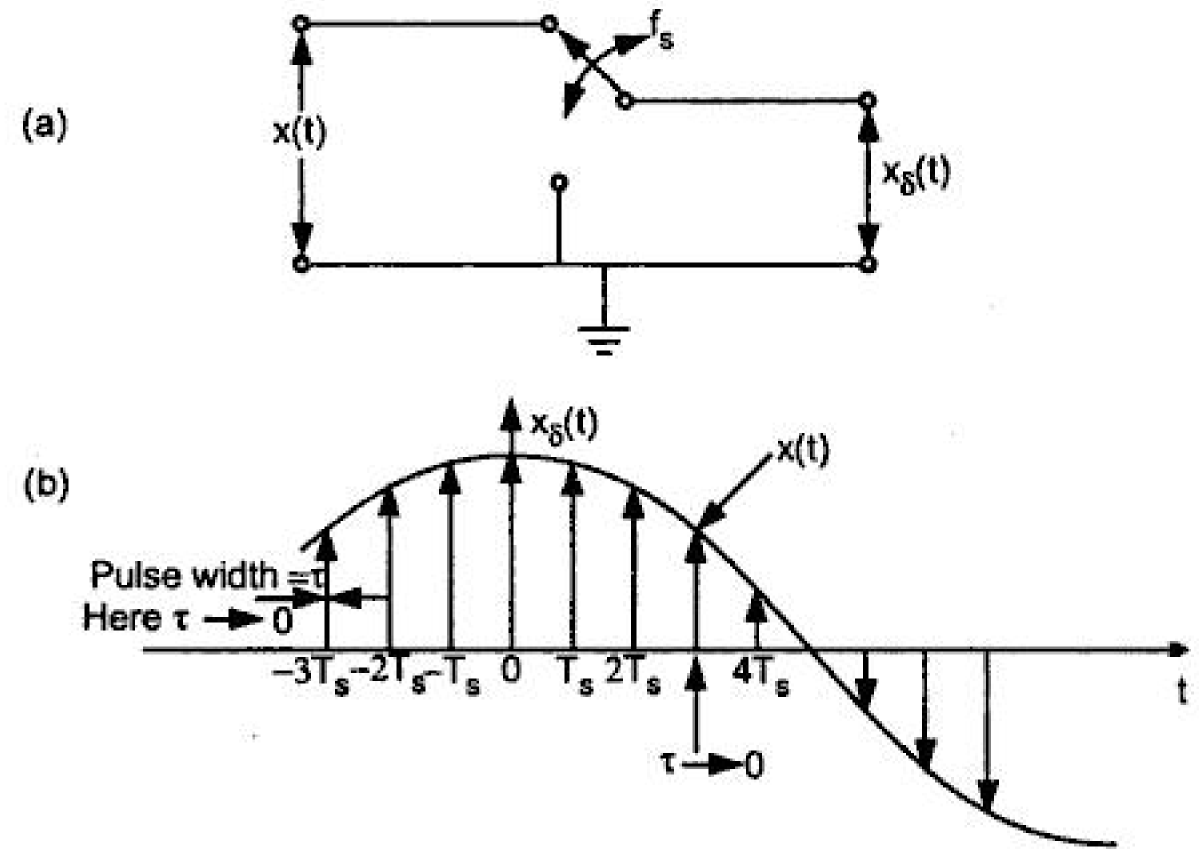


Fig. 1.4.1 (a) Functional diagram of a switching sampler (b) Waveforms of $x(t)$ and its sampled version $x_\delta(t)$ giving instantaneous sampling

Explanation

- We know that the train of impulses can be represented mathematically as,

$$s_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \text{... (1.4.1)}$$

- This is called sampling function. The sampled signal $x_\delta(t)$ is given by multiplication of $x(t)$ and $s_\delta(t)$.

Therefore, $x_\delta(t) = x(t)\, s_\delta(t)$

$$= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s) \quad \text{... (1.4.2)}$$

The above equation we have directly written previously as equation 1.3.1.

The Fourier transform of the ideally sampled signal given by above equation can be written as,

$$\boxed{\textbf{Spectrum of Ideally Sampled Signal : } X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)} \quad \text{... (1.4.3)}$$

Comments

i) $X(f)$ is periodic in f_s and weighed by f_s.

ii) Instantaneous sampling is possible only in theory because it is not possible to have a pulse whose width approaches zero

1.4.2 Natural Sampling or Chopper Sampling

- **Basic Principle**

In natural sampling the pulse has a finite width τ. Natural sampling is some times called chopper sampling because the waveform of the sampled signal appears to be chopped off from the original signal waveform.

- **Explanation**

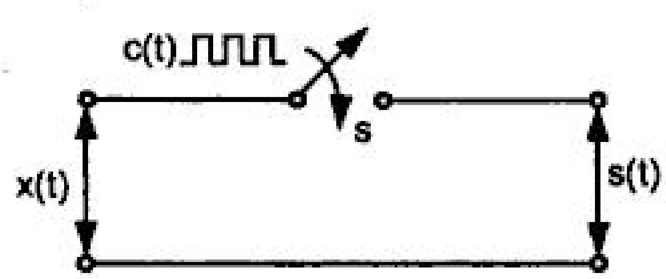


Fig. 1.4.2 Natural sampler

Let us consider an analog continuous time signal $x(t)$ to be sampled at the rate of f_s Hz and f_s is higher than Nyquist rate such that sampling theorem is satisfied. A sampled signal

$s(t)$ is obtained by multiplication of a sampling function and signal $x(t)$. Sampling function $c(t)$ is a train of periodic pulses of width τ and frequency equal to f_s Hz. Fig. 1.4.2 shows a functional diagram of natural sampler. When $c(t)$ goes high, a switch 's' is closed. Therefore,

$$s(t) = x(t) \quad \text{when} \quad c(t) = A$$

$$s(t) = 0 \quad \text{when} \quad c(t) = 0$$

Here A is amplitude of $c(t)$.

- The waveforms of $x(t), c(t)$ and $s(t)$ are shown in Fig. 1.4.3 (a), 1.4.3 (b) and 1.4.3 (c) respectively. Signal $s(t)$ can also be defined mathematically as,

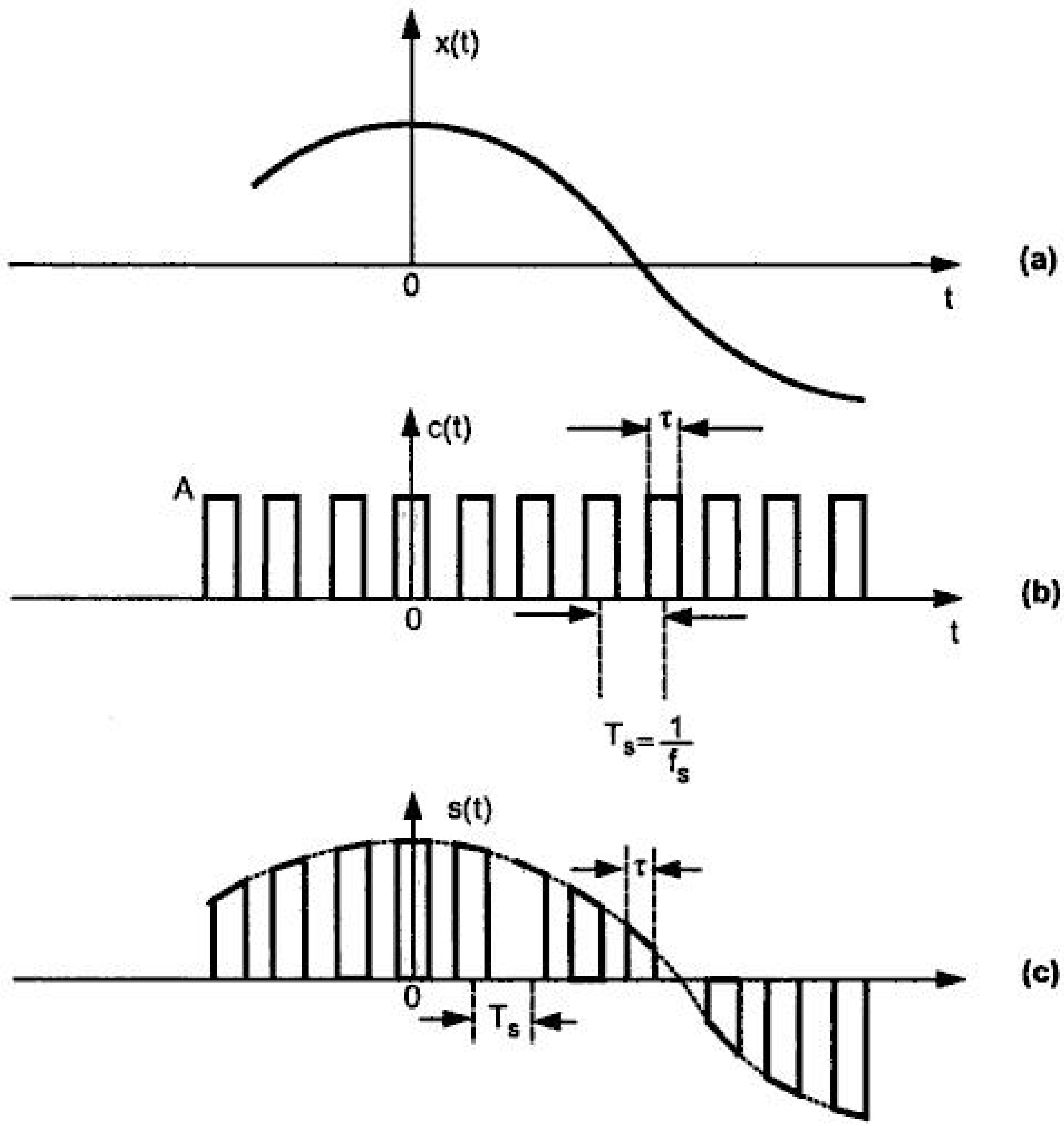


Fig. 1.4.3 (a) Continuous time signal $x(t)$
(b) Sampling function waveform i.e. periodic pulse train
(c) Naturally sampled signal waveform $s(t)$

$$s(t) = c(t) \cdot x(t) \quad \text{... (1.4.4)}$$

Here, $c(t)$ is the periodic train of pulses of width τ and frequency f_s.

Spectrum of Naturally Sampled Signal

- Exponential Fourier Series for a periodic waveform is given as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{j2\pi nt/T_0}$$

For the periodic pulse train of $c(t)$ we have,

$$T_0 = T_s = \frac{1}{f_s} = \text{Period of } c(t).$$

$\therefore$ or $$f_0 = f_s = \frac{1}{T_0} = \frac{1}{T_s} = \text{Frequency of } c(t).$$

$\therefore$ Above equation will be, [with $x(t) = c(t)$],

$$c(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{j2\pi f_s nt} \qquad \text{Putting } \frac{1}{T_0} = f_s \quad \text{... (1.4.5)}$$

$c(t)$ is a rectangular pulse train. C_n for this waveform is given as :

$$C_n = \frac{TA}{T_0} \, sinc\,(f_n \, T)$$

Here T = Pulse width = τ

and f_n = Harmonic frequency. Here $f_n = n\, f_s$ or $f_n = \frac{n}{T_0} = nf_0$

$\therefore$ $$C_n = \frac{\tau A}{T_s} \, sinc\,(f_n \tau) \quad \text{... (1.4.6)}$$

$\therefore$ Fourier series for periodic pulse train will be written from equation 1.4.5 and equation 1.4.6 as,

$$c(t) = \sum_{n=-\infty}^{\infty} \frac{\tau A}{T_s} \, sinc\,(f_n \, \tau) \, e^{j2\pi f_s nt} \quad \text{... (1.4.7)}$$

On putting the value of $c(t)$ in equation 1.4.4 we get,

$$s(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} sinc\,(f_n \tau) \, e^{j2\pi f_s nt} \cdot x(t) \quad \text{... (1.4.7 (a))}$$

This equation represents naturally sampled signals.

Now Fourier transform of $s(t)$ is obtained by definition of FT as,

$$S(f) = FT\{s(t)\}$$

$$= \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} sinc(f_n \tau)\, FT\left\{e^{j2\pi f_s nt} \cdot x(t)\right\} \qquad \text{... (1.4.8)}$$

We know from frequency shifting property of Fourier transform that,

$$e^{j2\pi f_s nt}\, x(t) \leftrightarrow X(f - f_s n) \qquad \text{... (1.4.9)}$$

$$S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} sinc(f_n \tau)\, X(f - f_s n) \qquad \text{... (1.4.10)}$$

We know that $f_n = nf_s$ i.e. harmonic frequency

∴ Above equation becomes,

Spectrum of Naturally Sampled Signal : $S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} sinc(nf_s \tau)\, X(f - nf_s)$

... (1.4.11)

Comments :

(i) $X(f)$ are periodic in f_s and are weighed by the sinc function. Fig. 1.4.4 (a) shows some arbitrary spectra for $x(t)$ and corresponding spectrum $S(f)$ is shown in Fig. 1.4.4 (b).

(ii) Thus unlike the spectrum of instantaneously sampled signal given by Fig.1.3.2 (b), the spectrum of naturally sampled signal is weighted by sinc function. But spectrum of instantaneously sampled signal given by Fig. 1.3.2 (b) remains constant throughout the frequency range.

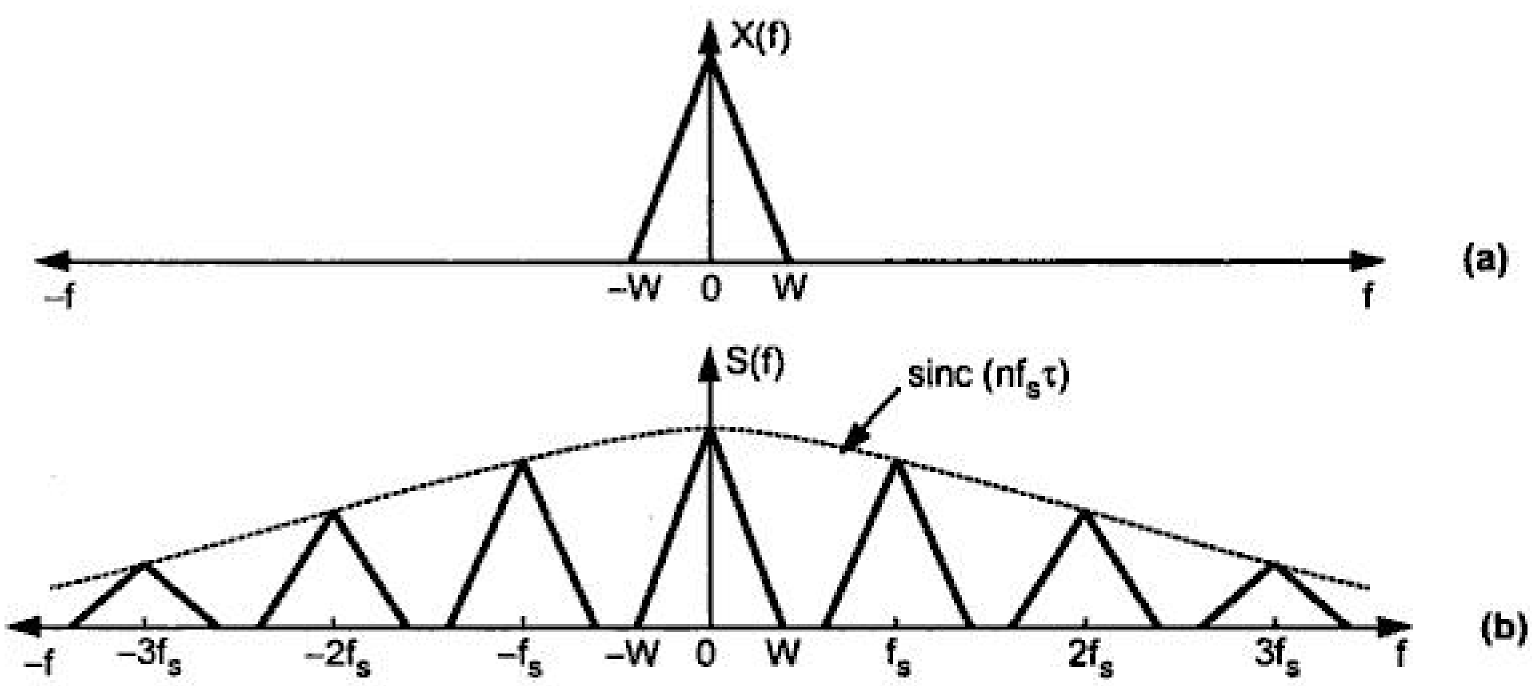


Fig. 1.4.4 (a) Spectrum of continuous time signal $x(t)$
(b) Spectrum of naturally sampled signal

1.4.3 Flat Top Sampling or Rectangular Pulse Sampling

Basic Principle

This is also a practically possible sampling method. Natural sampling is little complex, but it is very easy to get flat top samples. The top of the samples remains constant and equal to instantaneous value of baseband signal $x(t)$ at the start of sampling. The duration of each sample is τ and sampling rate is equal to $f_s = \frac{1}{T_s}$.

Generation of flat top samples

Fig.1.4.5 (a) shows the functional diagram of sample and hold circuit generating flat top samples and Fig. 1.4.5 (b) shows waveforms.

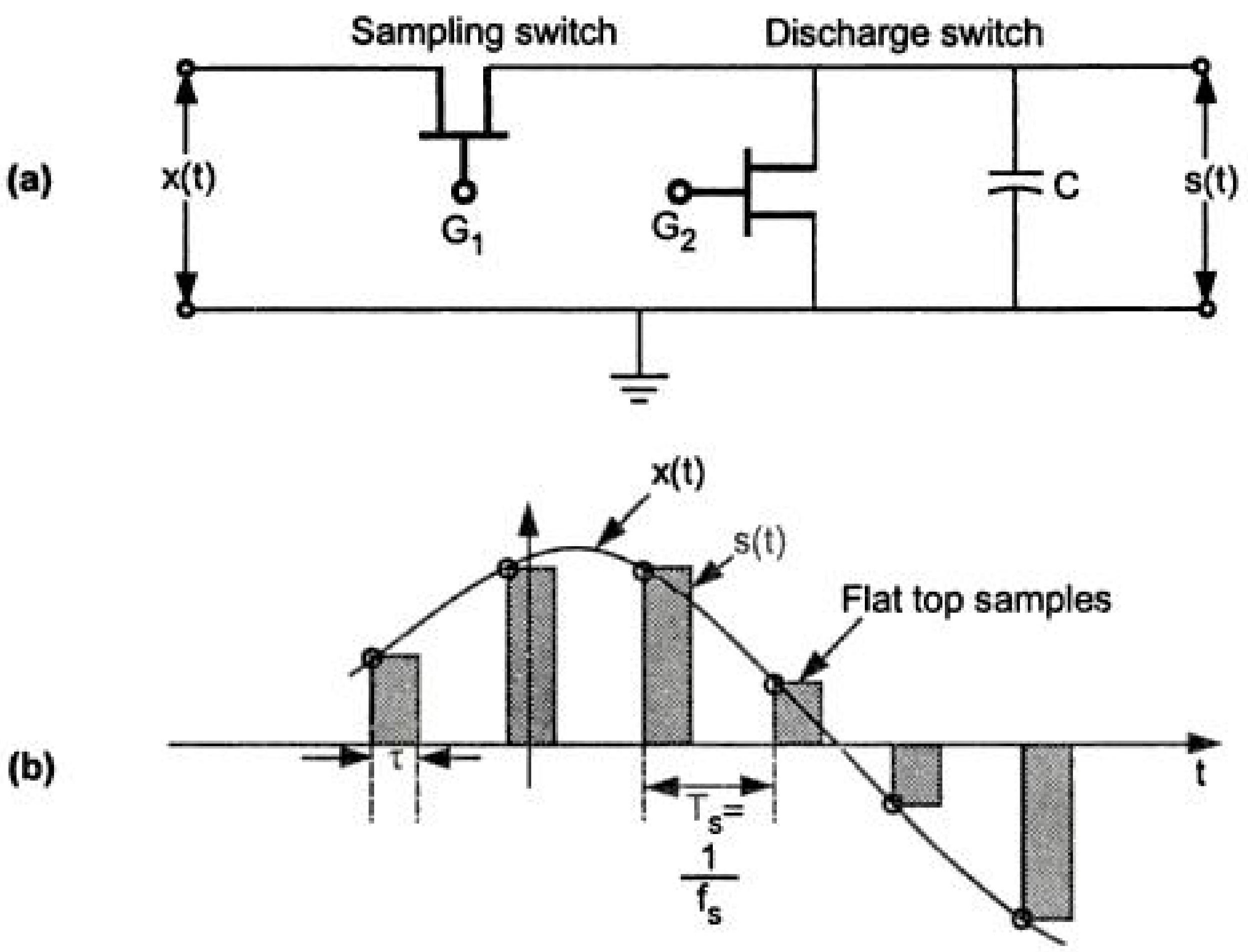


Fig. 1.4.5 (a) Sample and hold circuit generating flat top sampling
(b) Waveforms of flat top sampling

Normally the width of the pulse in flat top sampling and natural sampling is increased as far as possible to reduce the transmission bandwidth.

Explanation of Flat top Sampled PAM

Here we can see from Fig. 1.4.5 (b) that only starting edge of the pulse represents instantaneous value of the baseband signal $x(t)$. The flat top pulse of $s(t)$ is mathematically equivalent to the convolution of instantaneous sample and pulse $h(t)$ as shown in Fig. 1.4.6.

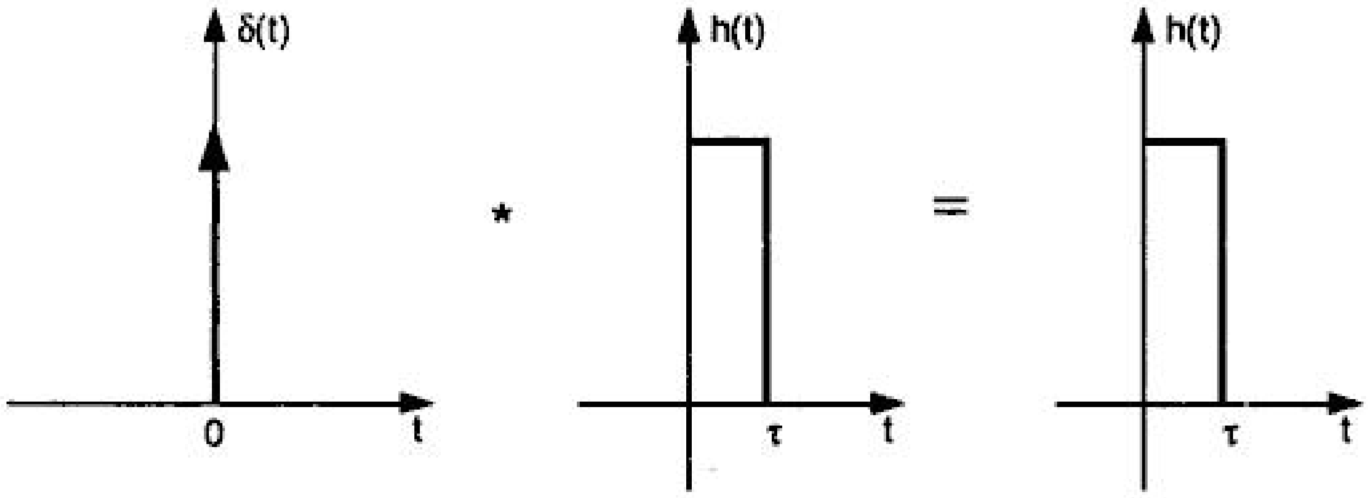


Fig. 1.4.6 Convolution of any function with delta function is equal to that function

- That is width of the pulse in $s(t)$ is determined by width of $h(t)$, and sampling instant is determined by delta function. In the waveforms shown in Fig. 1.4.5 (b), the starting edge of pulse represents the point where baseband signal is sampled and width is determined by function $h(t)$. Therefore $s(t)$ will be given as,

$$s(t) = x_\delta(t) * h(t) \qquad \text{... (1.4.12)}$$

The meaning of this equation is further explained by Fig. 1.4.7.

By the replication property of delta function we know that

$$x(t) * \delta(t) = x(t) \qquad \text{... (1.4.13)}$$

This is explained in Fig. 1.4.6 also. The same property is used to obtain flat top samples.

- The delta function in equation 1.4.13 is instantaneously sampled signal $x_\delta(t)$, and function $h(t)$ is convolved with $x_\delta(t)$. Clearly observe that we are not directly applying equation 1.4.13 here, but we are using it similarly. In equation 1.4.13, $\delta(t)$ is constant amplitude delta function. But in Fig. 1.4.7 (b), $x_\delta(t)$ is varying amplitude train of impulses. Therefore on convolution of $x_\delta(t)$ and $h(t)$ we get a pulse whose duration is equal to $h(t)$ only but amplitude is defined by $x_\delta(t)$.

From equation 1.3.1 $x_\delta(t)$ is given as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) \qquad \text{... (1.4.14)}$$

∴ From equation 1.4.12 we can write the convolution as,

$$s(t) = x_\delta(t) * h(t)$$

i.e.,
$$= \int_{-\infty}^{\infty} x_\delta(u)\, h(t-u)\, du$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(u-nT_s)\, h(t-u)\, du \quad \text{From equation (1.4.14)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \int_{-\infty}^{\infty} \delta(u-nT_s)\, h(t-u)\, du \quad \text{... (1.4.15)}$$

From the sifting property of delta function we know that,

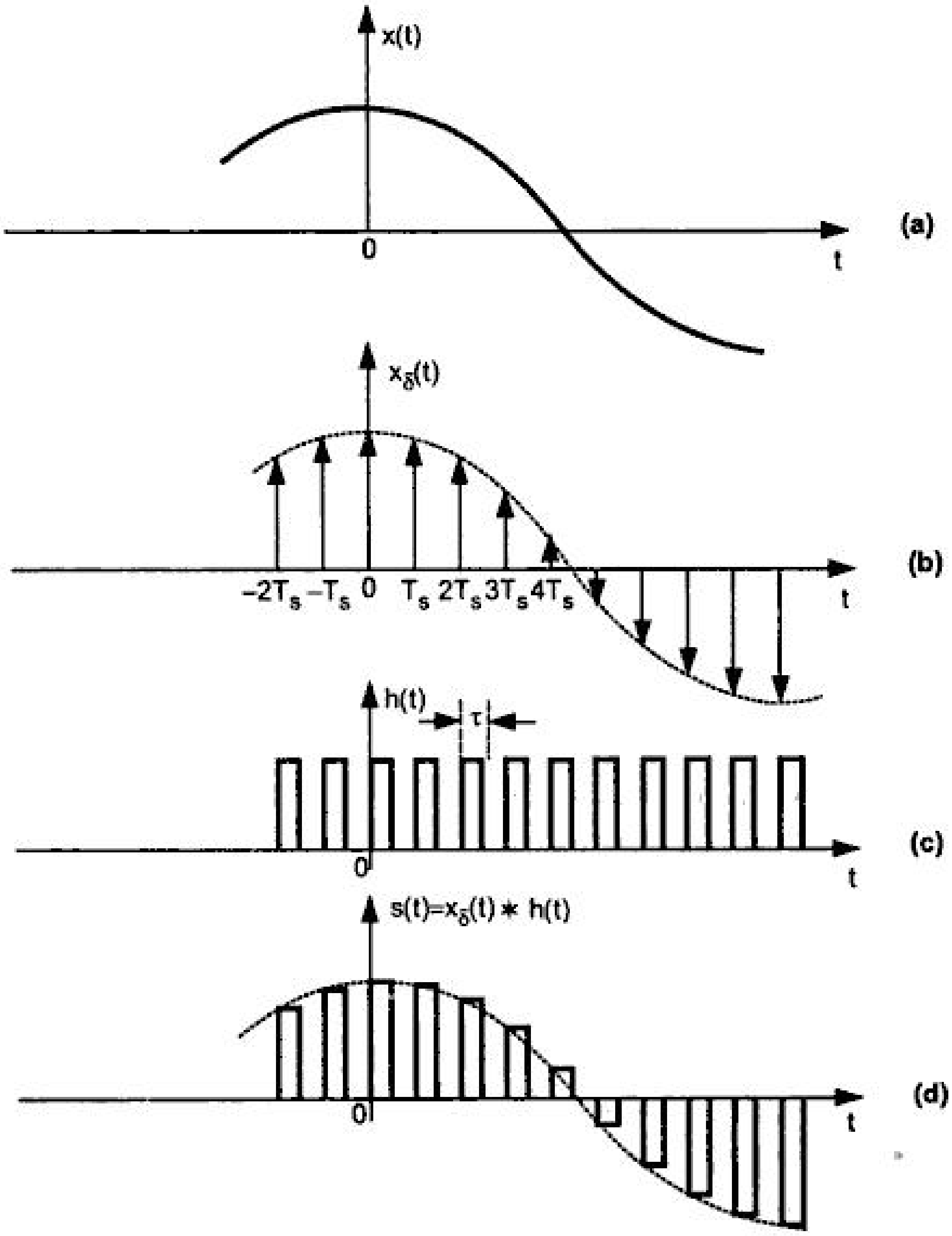


Fig. 1.4.7 **(a) Baseband signal $x(t)$**
(b) Instantaneously sampled signal $x_\delta(t)$
(c) Constant pulse width function $h(t)$
(d) Flat top sampled signal $s(t)$ obtained through convolution of $h(t)$ and $x_\delta(t)$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0) = f(t_0) \qquad ...(1.4.16)$$

Using this equation we can write equation 1.4.15 as,

$$s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,h(t-nT_s) \qquad ...(1.4.17)$$

- This equation represents value of $s(t)$ in terms of sampled value $x(nT_s)$ and function $h(t-nT_s)$ for flat top sampled signal.

we also know from equation 1.4.12 that,

$$s(t) = x_\delta(t) * h(t)$$

By taking Fourier transform of both sides of above equation,

$$S(f) = X_\delta(f)\,H(f) \qquad ...(1.4.18)$$

Convolution in time domain is converted to multiplication in frequency domain.

$X_\delta(f)$ is given as,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s) \qquad ...(1.4.19)$$

$\therefore$ Equation 1.4.18 becomes,

$$\boxed{\textbf{Spectrum of Flat Top Sampled Signal : } S(f) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s)\,H(f)} \qquad ...(1.4.20)$$

This equation represents the spectrum of flat top sampled signal.

1.4.3.1 Aperture Effect

Definition

The spectrum of flat top sampled signal is given by equation 1.4.20 above. This equation shows that the signal $s(t)$ is obtained by passing through a filter having transfer function $H(f)$. The corresponding impulse response $h(t)$, in time domain is shown in Fig. 1.4.8 (a). This pulse is one pulse of rectangular pulse train shown in Fig. 1.4.7 (c). Every sample of $x(t)$ is convolved with this pulse. Equation 1.4.20 represents that spectrum of this rectangular pulse is multiplied with that of $x_\delta(t)$. Fig. 1.4.8 (b) shows the spectrum of one rectangular pulse of $h(t)$.

The spectrum of a rectangular pulse is given as,

$$H(f) = \tau\, sinc(f\tau)\, e^{-j\pi f\tau} \qquad \because A = 1 \qquad ...(1.4.21)$$

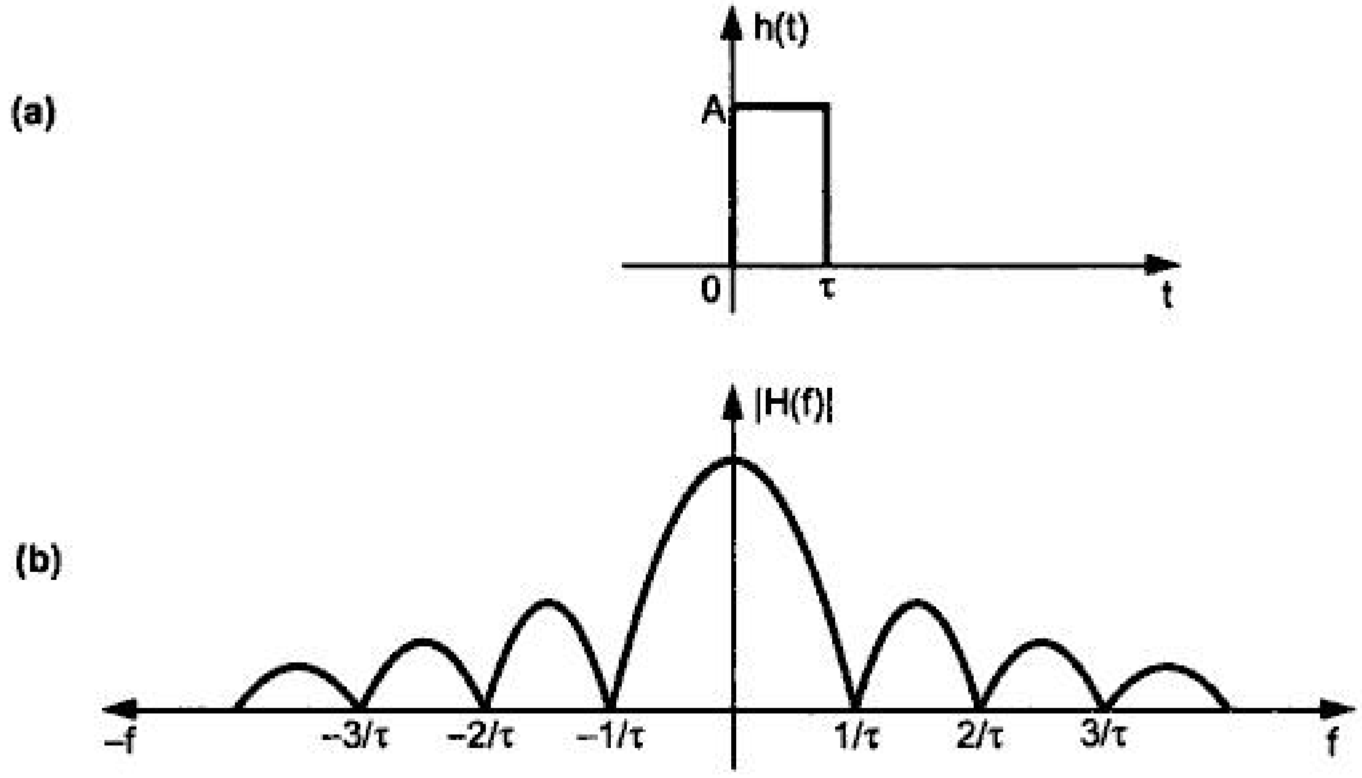


Fig. 1.4.8 (a) One pulse of rectangular pulse train
(b) Spectrum of the pulse of Fig. (a)

Thus we can see from Fig. 1.4.8 (b) that by using flat top samples an amplitude distortion is introduced in reconstructed signal $x(t)$ from $s(t)$. The high frequency rolloff of $H(f)$ acts like a lowpass filter and attenuates upper portion of message spectrum. These high frequencies of $x(t)$ are affected. This effect is called *aperture effect*.

Compensation for Aperture Effect

As the duration 'τ' of the pulse increases, aperture effect is more prominent. Therefore during reconstruction an equalizer is required to compensate for this effect. As shown in Fig. 1.4.9, the receiver consists of lowpass reconstruction filter with cutoff frequency slightly higher than the maximum frequency in message signal. The equalizer compensates for the aperture effect. It also compensates for the attenuation by a low-pass reconstruction filter.

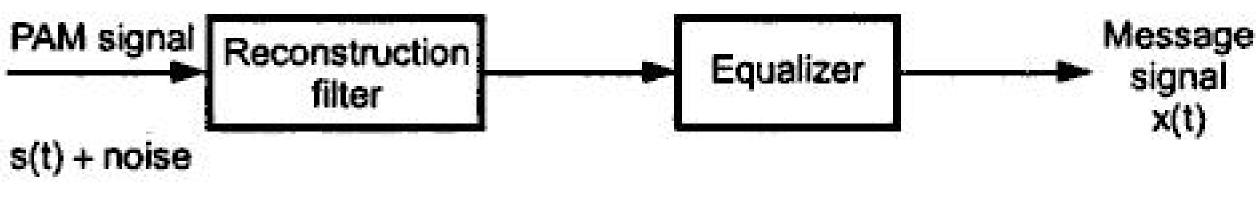


Fig. 1.4.9 Recovering $x(t)$

From equation 1.4.21 we know that the sample function $h(t)$ acts like a lowpass filter where Fourier transform is given as,

$$H(f) = \tau \, sinc(f\tau)\, e^{-j\pi f\tau} \text{ from equation 1.4.21} \qquad \text{... (1.4.22)}$$

This spectrum is plotted in Fig. 1.4.8. Equalizer used in cascade with the reconstruction filter has the effect of decreasing the inband loss of the reconstruction filter as the frequency increases in such a manner as to compensate for the aperture effect. The transfer function of the equalizer is given by,

$$H_{eq}(f) = \frac{K e^{-j2\pi f t_d}}{H(f)} \quad \text{... (1.4.23)}$$

Here 't_d' is the delay introduced by lowpass filter which is equal to $\tau/2$

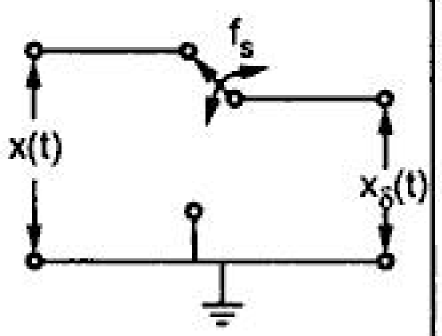
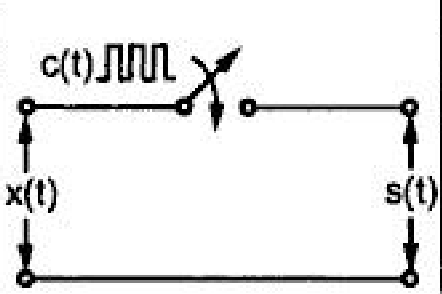
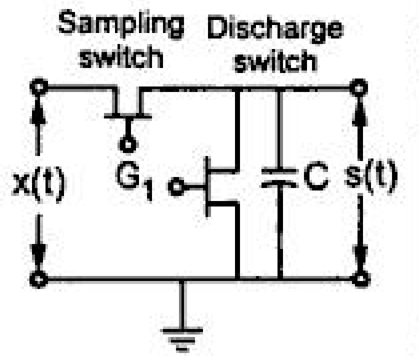
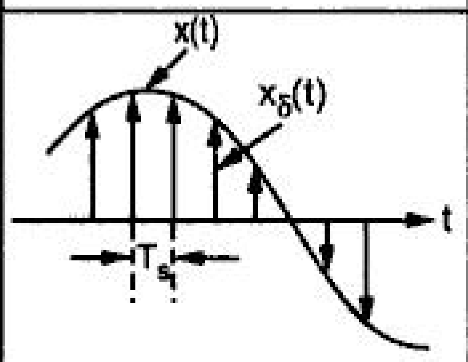
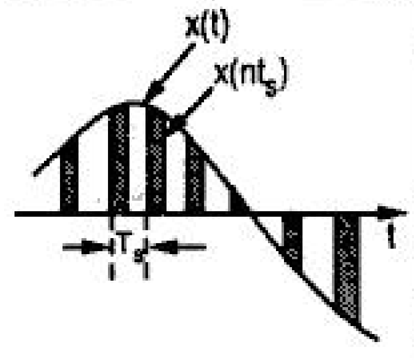
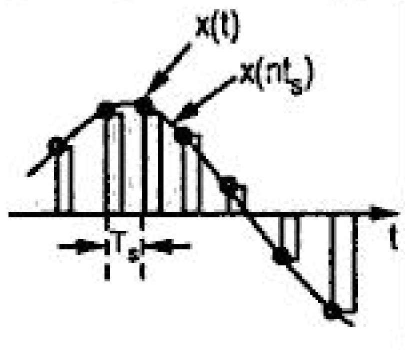
$$\therefore \quad H_{eq}(f) = \frac{K e^{-j\pi f\tau}}{\tau \sin c(f\tau)\, e^{-j\pi f\tau}}$$

$$= \frac{K}{\tau \, sinc(f\tau)} \quad \text{... (1.4.24)}$$

This is the transfer function of an equalizer.

1.4.4 Comparison of Various Sampling Techniques

Various sampling techniques can be compared on the basis of their method, noise interference, spectral properties etc. The following table lists some of the important points of comparison.

Sr. No.	Parameter of comparison	Ideal or instantaneous sampling	Natural sampling	Flat top sampling
1	Principle of sampling	It uses multiplication by an impulse function	It uses chopping principle	It uses sample and hold circuit
2	Circuit of sampler			
3	Waveforms			
4	Realizability	This is not practically possible method	This method is used practically	This method is used practically

5	Sampling rate	Sampling rate tends to infinity	Sampling rate satisfies Nyquist criteria	Sampling rate satisfies Nyquist criteria
6	Noise interference	Noise interference is maximum	Noise interference is minimum	Noise interference is maximum
7	Time domain representation	$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)$	$s(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} x(t)\,\text{sinc}(n f_s \tau)\, e^{j 2\pi n f_s t}$	$s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,h(t-nT_s)$
8	Frequency domain representation	$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s)$	$S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \text{sinc}(n f_s \tau)\, X(f-nf_s)$	$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s)\,H(f)$

Table 1.4.1 Comparison of sampling techniques

Example 1.4.1 : *The spectrum of signal x(t) is shown below. This signal is sampled at the Nyquist rate with a periodic train of rectangular pulses of duration 50/3 milliseconds. Find the spectrum of the sampled signal for frequencies upto 50 Hz giving relevant expression.*

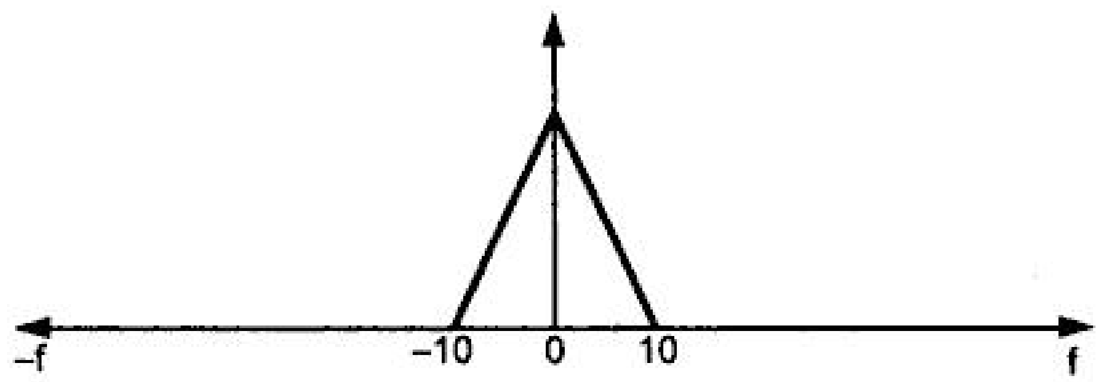


Fig. 1.4.10

Solution : It is clear from Fig. 1.4.10 that the signal is bandlimited to 10 Hz.

$$\therefore \quad W = 10 \text{ Hz}$$

$$\therefore \quad \text{Nyquist rate} = 2 \times W = 2 \times 10 = 20\, Hz$$

Since the signal is sampled at Nyquist rate, the sampling frequency will be,

$$f_s = 20 \text{ Hz}$$

Rectangular pulses are used for sampling. That is flat top sampling is used. The spectrum of flat top sampled signal is given by equation 1.4.20 as,

$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - n f_s)\, H(f) \qquad \text{... (1.4.25)}$$

Value of $H(f)$ is given by equation 1.4.21 as,

$$H(f) = \tau \, sinc(f\tau) \, e^{-j\pi f \tau} \qquad \text{... (1.4.26)}$$

Here τ is the width of the rectangular pulse used for sampling. The given value of rectangular sampling pulse is 50/3 milliseconds. i.e.,

$$\tau = \frac{50}{3} \times 10^{-3}$$

or $$\tau = \frac{0.05}{3} \text{ seconds}$$

Putting the value of τ in equation 1.4.26 we get,

$$H(f) = \frac{0.05}{3} sinc\left(\frac{0.05f}{3}\right) e^{-j\,0.05\pi f/3}$$

Put this value of $H(f)$ and f_s in equation 1.4.25

$$S(f) = 20 \sum_{n=-\infty}^{\infty} X(f-20n) \times \frac{0.05}{3} sinc\left(\frac{0.05f}{3}\right) e^{-j\,0.05\pi f/3}$$

(since $f_s = 20$)

$$S(f) = \frac{1}{3} \sum_{n=-3}^{3} X(f-20n) \times sinc\left(\frac{0.05f}{3}\right) e^{-j\,0.05\pi f/3}$$

This expression gives the spectrum up to 60 Hz

(since $n = \pm 3$) for the sampled signal.

Example 1.4.2 : *A flat top sampling system samples a signal of maximum 1 Hz with 2.5 Hz sampling frequency. The duration of the pulse is 0.2 seconds. Calculate the amplitude distortion due to aperture effect at highest signal frequency. Also find out the equalization characteristic.*

Solution : It is given that

Sampling frequency $f_s = 2.5$ Hz

Maximum signal frequency $f_{max} = 1$ Hz

Pulse width $\tau = 0.2$ sec.

By equation 1.4.22, the aperture effect is given by a transfer function $H(f)$ as,

$$H(f) = \tau \, sinc(f\tau) \, e^{-j\pi f \tau}$$

The magnitude of the above equation is given as,

$$|H(f)| = \tau\, sinc(f\tau) \quad \text{... (1.4.27)}$$

$$|H(f)| = 0.2\, sinc(f \times 0.2)$$

Aperture effect at highest frequency will be obtained by putting $f = f_{\max} = 1\ Hz$ in above equation i.e.,

$$|H(1)| = 0.2 \text{ sinc } (0.2) = 0.18709$$

or $$|H(1)| = 18.70\% \quad \text{... (Ans)}$$

From equation 1.4.24 the equalizer characteristic is given as,

$$H_{eq}(f) = \frac{k}{\tau\, sinc(f\tau)}$$

Putting $\tau = 0.2$ second and assuming $k = 1$, the above equation will be,

$$H_{eq}(f) = \frac{1}{0.2\, sinc(0.2\, f)} \quad \text{... (1.4.28)}$$

This equation is the plot of $H_{eq}(f)\ Vs\ f$ and it represents the equalization characteristic to overcome aperture effect.

1.4.5 Transmission Bandwidth of PAM Signal

The pulse duration 'τ' is supposed to be very very small compared to time period T_s between the two samples. If the maximum frequency in the signal $x(t)$ is 'W' then by sampling theorem, f_s should be higher than Nyquist rate or,

$$f_s \geq 2W \text{ or}$$

$$T_s \leq \frac{1}{2W} \quad \text{since } f_s = \frac{1}{T_s}$$

and $$t << T_s \leq \frac{1}{2W} \quad \text{... (1.4.29)}$$

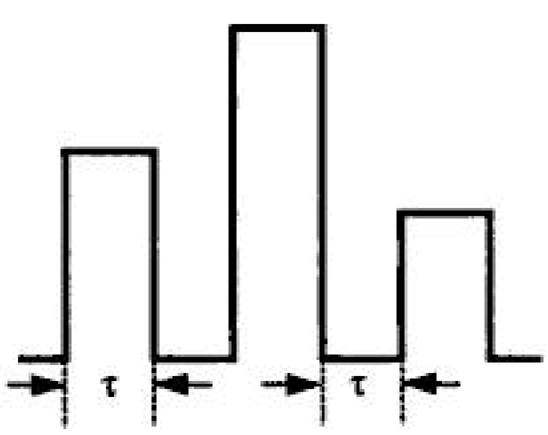


Fig. 1.4.11 Maximum frequency of PAM signal

If ON and OFF time of the pulse is same, then frequency of the PAM pulse becomes,

$$f = \frac{1}{\tau + \tau} = \frac{1}{2\tau} \quad \text{...(1.4.30)}$$

Thus Fig. 1.4.11 shows that if ON and OFF times of PAM signal are same, then maximum frequency of

PAM signal is given by equation 1.4.30 i.e.,

$$f_{max} = \frac{1}{2\tau} \quad \text{... (1.4.31)}$$

$\therefore$ Bandwidth required for transmission of PAM signal will be equal to maximum frequency f_{max} given by above equation. This bandwidth gives adequate pulse resolution i.e.,

$$B_T \geq f_{max}$$

$$\therefore \quad B_T \geq \frac{1}{2\tau} \quad \text{... (1.4.32)}$$

Since $\tau << \frac{1}{2W}$ $B_T \geq \frac{1}{2\tau} >> W$ i.e.,

$$\boxed{\textbf{Transmission bandwidth of PAM signal : } B_T >> W} \quad \text{... (1.4.33)}$$

Thus the transmission bandwidth B_T of PAM signal is very very large compared to highest frequency in the signal $x(t)$.

1.4.6 Disadvantages of PAM

1. As we have seen just now, the bandwidth needed for transmission of PAM signal is very very large compared to its maximum frequency content.
2. The amplitude of PAM pulses varies according to modulating signal. Therefore interference of noise is maximum for the PAM signal and this noise cannot be removed very easily.
3. Since amplitude of PAM signal varies, this also varies the peak power required by the transmitter with modulating signal.

Theory Questions

1. *Distinguish between instantaneous sampling, natural sampling and flat top sampling. With functional block diagram explain the working of a circuit that provides flat top sampling.*
2. *Show that a bandlimited signal of finite energy, which has no frequency components higher than W Hz may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second. How the recovered signal will differ in amplitude if samples are taken by*

   (a) Natural samplmg (b) Flat top sampling ?
3. *What is aperture effect ? How it can be reduced ?*

1.5 Other Forms of Pulse Modulation

There are two more types of pulse modulation other than PAM :

(i) Pulse Duration Modulation (PDM)

In this technique the width of the pulse changes according to amplitude of the modulating signal at sampling instant. Fig. 1.5.1 (c) shows such signal.

(ii) Pulse Position Modulation (PPM)

In this technique the position of the pulse changes according to amplitude of the modulating signal of sampling instant. Fig. 1.5.1(d) shows such signal.

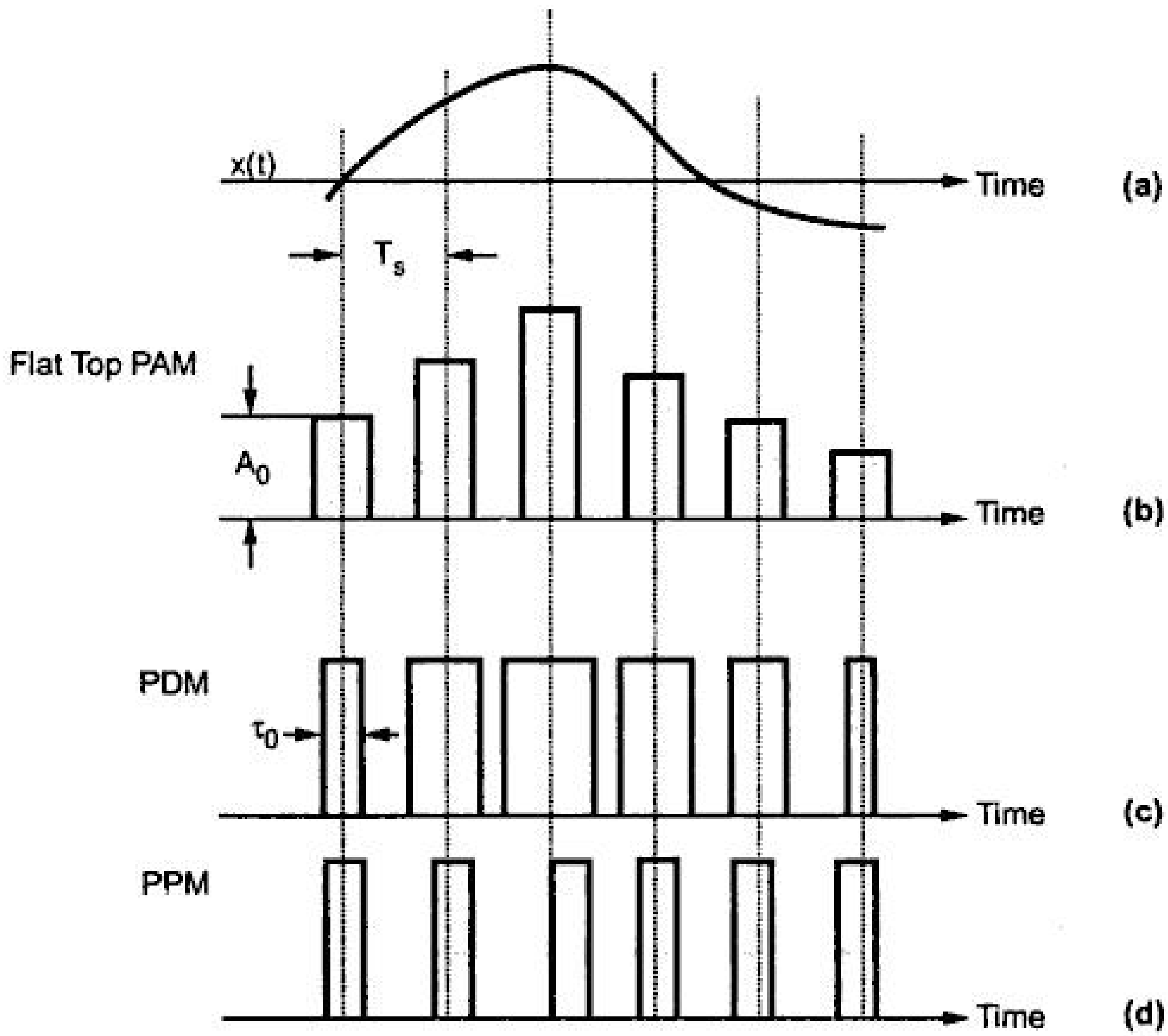


Fig. 1.5.1 Various pulse modulation methods

- Pulse position modulation (PPM) and pulse duration modulation (PDM or PWM) both modulate the time parameter of the pulses. PPM has fixed width pulses where as width of PDM pulses varies. Both the methods are of constant amplitude.

1.5.1 Generation of PPM and PDM

The block diagram of Fig. 1.5.2 (a) shows the scheme to generate PDM and PPM. The corresponding waveforms are shown in Fig. 1.5.2 (b). The scheme of Fig.1.5.2(a) combines both sampling and modulation operation. The sawtooth generator generates the sawtooth signal of frequency f_s (i.e. period T_s). The sawtooth signal, also called sampling signal is applied to the inverting input of comparator.

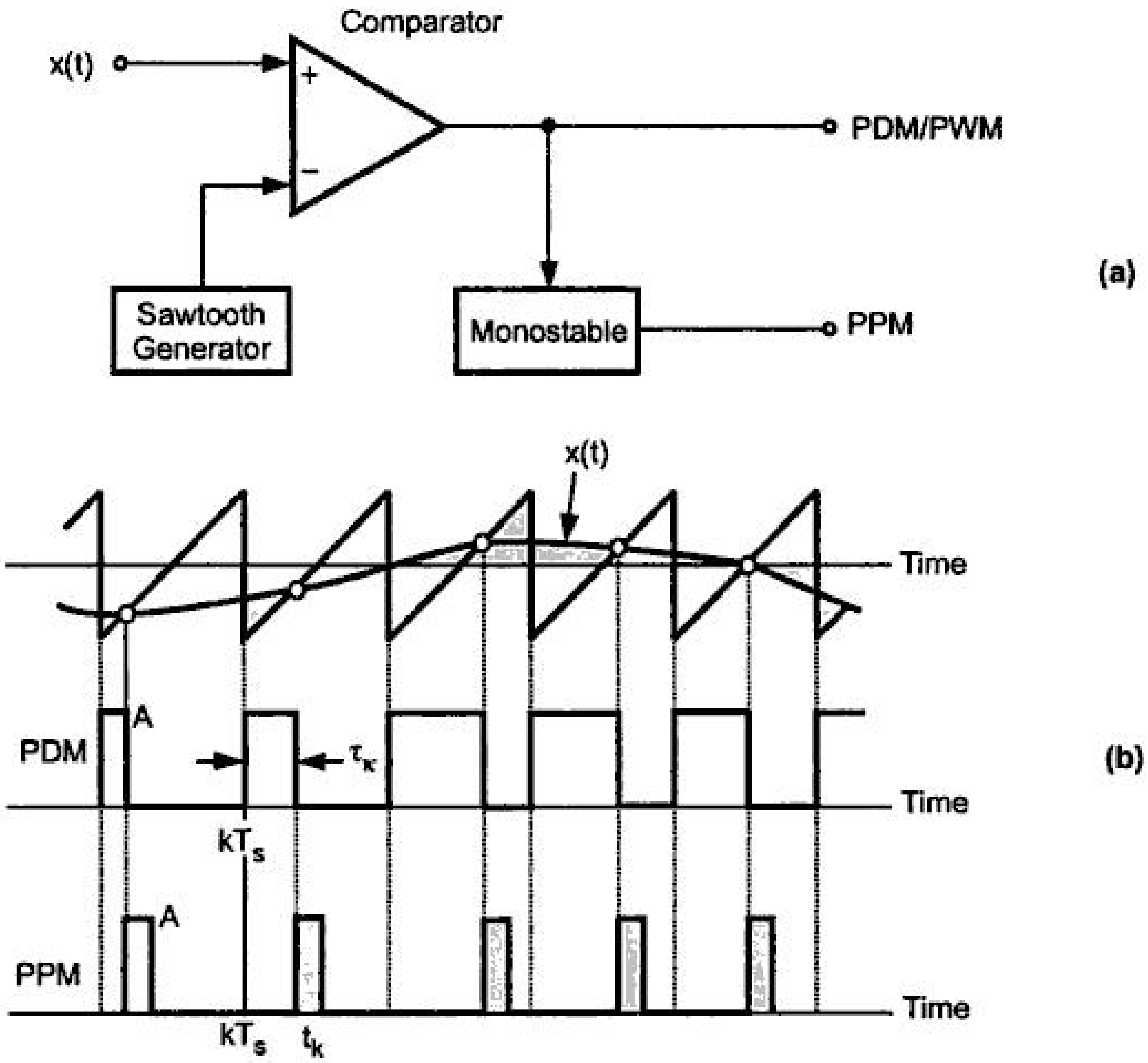


Fig. 1.5.2 Generator of PPM and PDM (a) Block diagram (b) Waveforms

The modulating signal $x(t)$ is applied to the noninverting input of the comparator. The output of the comparator is high only when instantaneous value of $x(t)$ is higher than that of sawtooth waveform. Thus the leading edge of PDM signal occurs at the fixed time period i.e. kT_s the trailing edge of output of comparator depends on the amplitude of signal $x(t)$. When sawtooth waveform voltage is greater than voltage of $x(t)$ at that instant, the output of comparator remains zero. The trailing edge of the output of comparator (PDM) is modulated by the signal $x(t)$. If the sawtooth waveform is reversed, then trailing edge will be fixed and leading edge will be modulated. If sawtooth waveform is replaced by triangular waveform, then both leading and trailing edges will be modulated.

The pulse duration modulation (PDM) or PWM signal is nothing but output of the comparator. The amplitude of this PDM or PWM signal will be positive saturation of the comparator, which is shown as 'A' in the waveforms. The amplitude is same for all pulses.

To generate pulse position modulation (PPM), PDM signal is used as the trigger input to one monostable multivibrator. The monostable output remains zero untill it is triggered. The monostable is triggered on the falling (trailing) edge of PDM. The output of monostable then switches to positive saturation level 'A'. This voltage remains high for the fixed period then goes low. The width of the pulse can be determined by monostable. The pulse is this delayed from sampling time kT_s depending on the amplitude of signal $x(t)$ at kT_s.

1.5.2 Transmission Bandwidth of PPM and PDM

As can be seen from the waveform, both PPM and PDM possess DC value. The amplitude of all the pulses is same. Therefore nonlinear amplitude distortion as well as noise interference does not affect the detection at the receiver. However both PPM and PDM needs a sharp rise time and fall time for pulses in order to preserve the message information. Rise time should be very very less than T_s i.e.,

$$t_r \ll T_s$$

And transmission bandwidth should be,

$$B_T \geq \frac{1}{2t_r}$$

Thus the transmission bandwidth of PPM and PDM is higher than PAM. The power requirement of PPM is less than that of PDM because of short duration pulses. It can be further reduced by transmitting only edges rather than pulses.

$$\text{Transmission bandwidth of PDM and PPM : } B_T \geq \frac{1}{2t_r} \quad \text{... (1.5.1)}$$

1.5.3 Comparison between Various Pulse Modulation Methods

Following table shows the comparison among various pulse modulation techniques.

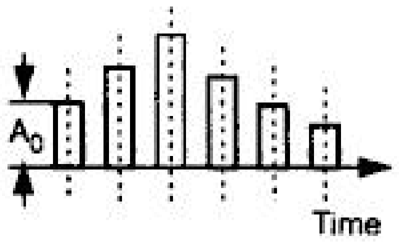
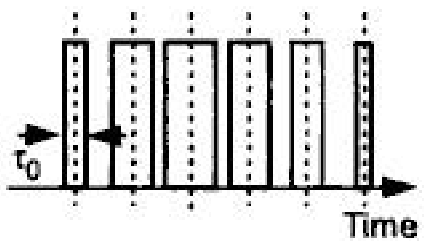
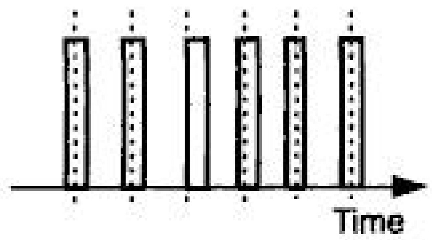
Sr. No.	Pulse Amplitude Modulation	Pulse Width/Duration Modulation	Pulse Position Modulation
1	Waveform 	Waveform 	Waveform 
2	Amplitude of the pulse is proportional to amplitude of modulating signal.	Width of the pulse is proportional to amplitude of modulating signal.	The relative position of the pulse is proprotional to the amplitude of modulating signal.
3	The bandwidth of the transmission channel depends on width of the pulse.	Bandwidth of transmission channel depends on rise time of the pulse.	Bandwidth of transmission channel depends on rising time of the pulse.
4	The instantaneous power of the transmitter varies.	The instantaneous power of the transmitter varies.	The instantaneous power of the transmitter remains constant.
5	Noise interference is high.	Noise interference is minimum.	Noise interference is minimum.
6	System is complex.	Simple to implement.	Simple to implement.
7	Similar to amplitude modulation.	Similar to frequency modulation.	Similar to phase modulation.

Table 1.5.1 Comparison of PAM, PPM and PDM

Example 1.5.1 : *For a PAM transmission of voice signal with W = 3 kHz. Calculate B_T if $f_s = 8\,kHz$ and $\tau = 0.1\,T_s$.*

Solution :

T_s is given as,
$$T_s = \frac{1}{f_s} = \frac{1}{8 \times 10^3}\ sec$$

$$\therefore \quad \tau = 0.1\ T_s = \frac{0.1}{8 \times 10^3}\ sec$$

From equation 1.5.1, the transmission bandwidth B_T is given as,

$$B_T \geq \frac{1}{2\tau} \geq \frac{1}{2 \times \frac{0.1}{8 \times 10^3}} = 40\ \text{kHz}$$

Example 1.5.2 : *For the signal given in example 1.5.1, if the rise time is 1% of the width of the pulse, find out the minimum transmission bandwidth needed for PDM and PPM.*

Solution : In example 1.5.1 we obtained the pulse width $\tau = \frac{0.1}{8 \times 10^3}$ *sec.* The rise time is given as 1% of width of pulse i.e.,

$$t_r = \tau \times 0.01 = \frac{0.1}{8 \times 10^3} \times 0.01 = 1.25 \times 10^{-7} \text{ } sec$$

We know that transmission bandwidth is given as,

$$B_T \geq \frac{1}{2t_r} \geq \frac{1}{2 \times 1.25 \times 10^{-7}} \geq 4 \text{ MHz}$$

Theory Questions

1. *Compare PAM, PPM and PDM.*
2. *Explain the scheme to generate PDM and PPM.*
3. *Explain how to generate PAM signal for various types of sampling techniques.*

1.6 Bandwidth Noise Trade-off

The noise analysis of PPM and FM have similar results as follows :

i) For both systems, the figure of merit is proportional to square of the ratio $\left(\frac{B_T}{W}\right)$.

ii) As the signal to noise ratio is reduced, both the systems exhibit threshold effect.

- With digital pulse modulation, the better noise performance than square law can be obtained.
- The digital pulse modulation such as pulse code modulation gives negligible noise effect by increasing the average power in binary PCM signal.
- With PCM, the bandwidth noise trade-off can be related by exponential law.

1.7 Time Division Multiplexing (PAM/TDM System)

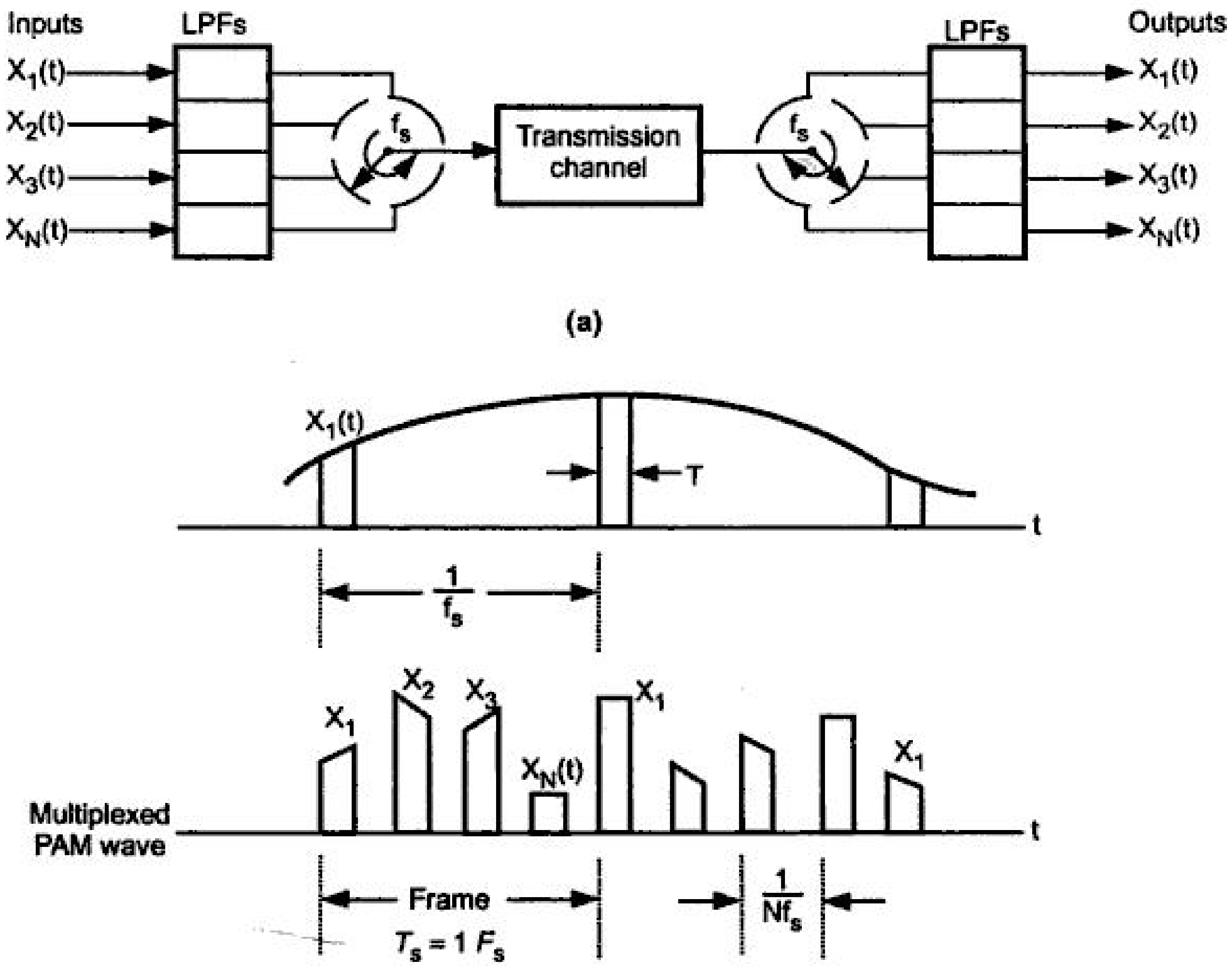
In PAM, PPM and PDM the pulse is present for short duration and form most of the time between the two pulses, no signal is present. This free space between the pulses can be occupied by pulses from other channels. This is called Time Division Multiplexing (TDM). It makes maximum utilization of the transmission channel.

1.7.1 Block Diagram of PAM / TDM

Fig.1.7.1 (a) shows the block diagram of a simple TDM system and Fig. 1.7.1 (b) shows the waveforms of the system.

The system shows the time division multiplexing of 'N' PAM channels. Each channel to be transmitted is passed through the lowpass filter. The outputs of the lowpass filters are connected to the rotating sampling switch or commutator. It takes the sample from each channel per revolution and rotates at the rate of f_s.

Thus the sampling frequency becomes f_s. The single signal composed due to multiplexing of input channels is given to the transmission channel. At the receiver the decommutator separates (decodes) the time multiplexed input channels. These channel signals are then passed through lowpass reconstruction filters.



(b)

Fig. 1.7.1 TDM system (PAM/TDM system)
(a) Block diagram (b) Waveforms

If the highest signal frequency present in all the channels is 'W', then by sampling theorem the sampling frequency f_s should be,

$$f_s \geq 2W \quad \text{... (1.7.1)}$$

Therefore the time space between successive samples from any one input will be

$$T_s = \frac{1}{f_s} \quad \text{... (1.7.2)}$$

$$\therefore \quad T_s \leq \frac{1}{2W} \quad \text{... (1.7.3)}$$

Thus the time interval T_s contains one sample from each input. This time interval is called frame. Let there be 'N' input channels. Then in each frame there will be one sample from each of the 'N' channels. That is one frame of T_s seconds contain total 'N' samples. Therefore pulse to pulse spacing between two samples in the frame will be equal to $\frac{T_s}{N}$.

$$\therefore \text{ Spacing between two samples} = \frac{T_s}{N} \quad \text{... (1.7.4)}$$

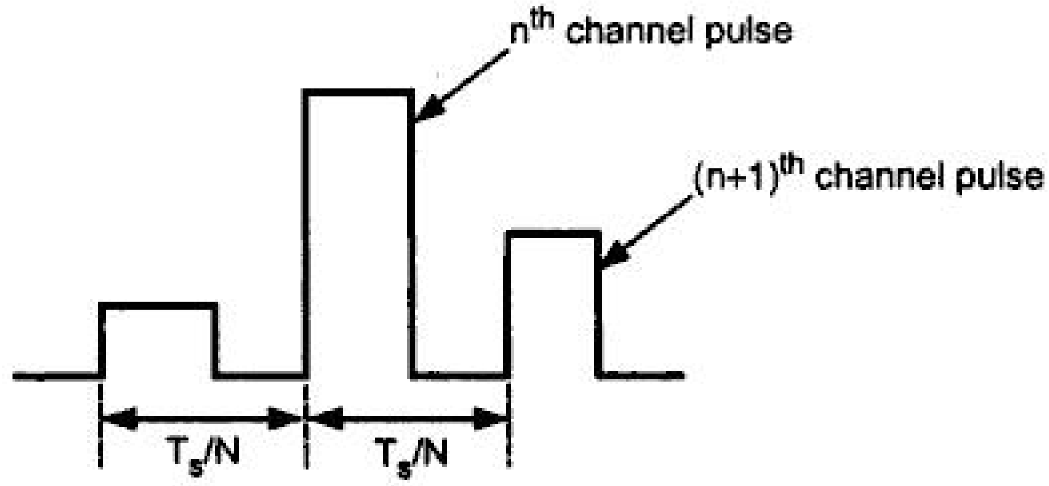


Fig. 1.7.2 Calculation of number of pulses per second in TDM

From the above figure we can very easily calculate the number of pulses per second or pulse frequency as,

$$\text{Number of pulses per second} = \frac{1}{\text{Spacing between two pulses}}$$

$$= \frac{1}{T_s / N}$$

$$= \frac{N}{T_s}$$

We know that $T_s = \frac{1}{f_s}$

$\therefore$ Number of pulses per second $= \frac{N}{1/f_s} = N f_s$ ... (1.7.5)

These number of pulses per second is also called signalling rate of TDM signal and is denoted by 'r' i.e.,

$$\text{Signalling rate} = r = N f_s \quad \text{... (1.7.6)}$$

Since $f_s \geq 2W$, then signaling rate becomes,

Signalling rate in PAM/TDM system : $r \geq 2NW$ ... (1.7.7)

The RF transmission of TDM needs modulation. That is TDM signal should modulate some carrier. Before modulation, the pulsed signal in TDM is converted to baseband signal. That is pulsed TDM signal is converted to smooth modulating waveform $x_b(t)$; the baseband signal that modulates the carrier. The baseband signal $x_b(t)$ passes through all the individual sample values baseband signal is obtained by passing pulsed TDM signal through lowpass filter. The bandwidth of this lowpass filter is given by half of the signalling rate. i.e.,

$$B_b = \frac{1}{2} r = \frac{1}{2} N f_s \quad \text{... (1.7.8)}$$

$\therefore$ Transmission bandwidth of TDM channel will be equal to bandwidth of the lowpass filter,

$$\therefore \quad B_T = \frac{1}{2} N f_s \quad \text{from above equation}$$

If sampling rate becomes equal to Nyquist rate i.e.,

$$f_s\,(\text{min}) = \text{Nyquist rate} = 2W, \text{ then}$$

$$B_T = \frac{1}{2} N \times 2W$$

Minimum transmission bandwidth of TDM channel : $B_T = NW$ ... (1.7.9)

This equation shows that if there are total 'N' channels in TDM which are bandlimited to 'W' Hz, then minimum bandwidth of the transmission channel will be equal to NW.

➡ **Example 1.7.1 :** *'N' number of independent baseband signal samples are transmitted over a channel of bandwidth = f_c Hz. If each sample is bandlimited to f_m Hz, show that the channel need not have a bandwidth larger than Nf_m in order to avoid crosstalk.*

Solution : Here we have to show that, the bandwidth of the transmission channel in PAM/TDM system should be minimum of Nf_m in order to avoid crosstalk between successive channel samples. From Fig. 1.7.1 we know that samples from various channels are interlaced one after another. The figure is reproduced here for convenience.

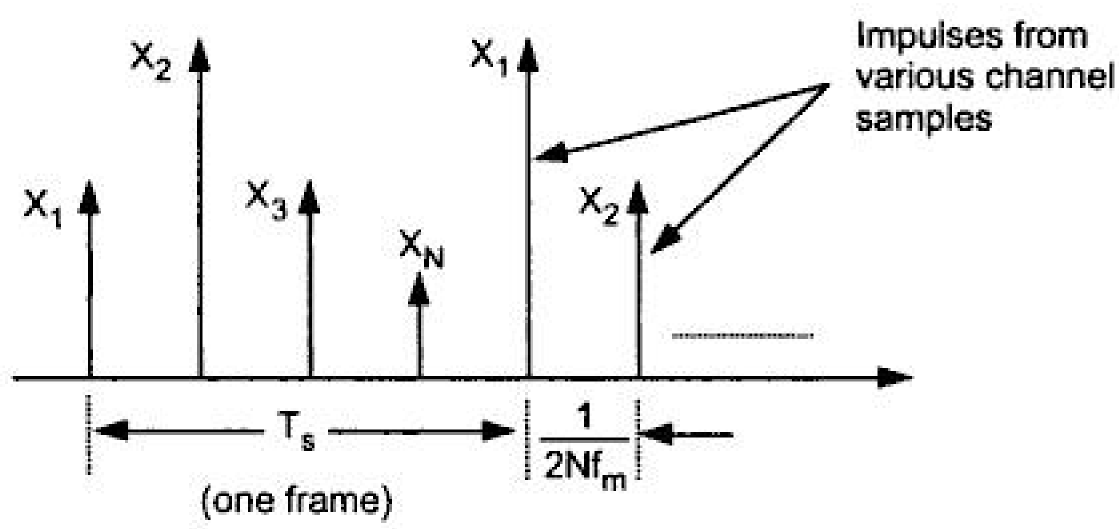


Fig. 1.7.3 PAM/TDM samples with instantaneous sampling

Here we will assume that the samples from various channels are instantaneously sampled. Thus the samples are impulses of various height.

One frame is of 'T_s' duration. In this frame there are impulses from 'N' channels. Therefore the time space between any two consecutive samples will be,

$$\text{Spacing between two consecutive samples} = \frac{T_s}{N} \quad \text{... (1.7.10)}$$

Since maximum signal frequency is f_m, the minimum sampling frequency will be $f_s = 2f_m$ (i.e. minimum sampling rate or Nyquist rate).

$$T_s = \frac{1}{f_s} = \frac{1}{2f_m}$$

Therefore equation 1.7.10 will be,

$$\text{Spacing between two consecutive samples} = \frac{1}{2N f_m} \quad \text{... (1.7.11)}$$

The impulse train of Fig. 1.7.3 is given to PAM/TDM transmission channel. This channel is lowpass type of channel as shown in Fig. 1.7.4.

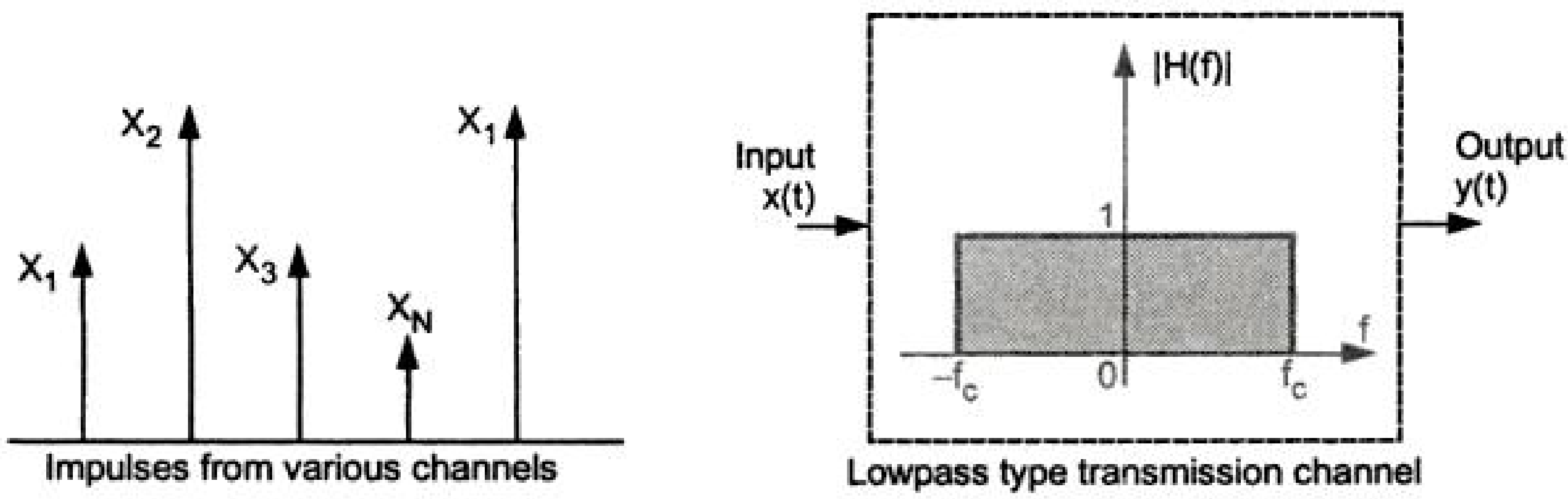


Fig. 1.7.4 PAM/TDM transmission channel

As shown in the above figure, the transmission channel is lowpass type and it has bandwidth of 'f_c' Hz. Therefore it is approximated by an ideal lowpass filter response. The response of the channel is $|H(f)| = 1$ over $-f_c \leq f \leq f_c$.

The input $x(t)$ to the transmission channel are impulses from various channels. Those impulses are passed through the transmission channel. Hence output $y(t)$ will be impulse response of the transmission channel. We know that the transfer function $H(f)$ is the Fourier transform of impulse response $h(t)$. Therefore,

Impulse response of the transmission channel = $h(t) = IFT\,[H(f)]$

Since output $y(t)$ is nothing but impulse response of transmission channel (since input $x(t)$ is train of impulses),

$$y(t) = h(t) = IFT\,[H(f)]$$

$$= \int_{-\infty}^{\infty} H(f)\, e^{j2\pi ft}\, df \qquad \text{By definition of IFT.}$$

$$= \int_{-f_c}^{f_c} 1 \cdot e^{j2\pi ft}\, df \qquad \text{Since } H(f) = 1 \text{ for } -f_c \leq f \leq f_c$$

$$= \left[\frac{e^{j2\pi ft}}{j2\pi t}\right]_{-f_c}^{f_c} = \frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{j2\pi t}$$

$$= \frac{1}{\pi t}\left[\frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{2j}\right]$$

$$= \frac{1}{\pi t}\, sin\,(2\pi f_c t) \qquad \text{[By Euler's theorem]} \qquad \ldots (1.7.12)$$

$$= 2f_c \cdot \frac{sin\,(2\pi f_c t)}{2\pi f_c t} \qquad \text{By rearranging the equation}$$

$$= 2f_c\, sinc\,(2f_c t) \qquad \ldots (1.7.13)$$

Thus the output is a sinc function and we know that it has zero values when

$$2f_c t = \pm 1, \pm 2, \pm 3, \pm 4, \ldots\ldots\ldots\ldots$$

i.e.
$$t = \pm\frac{1}{2f_c}, \pm\frac{2}{2f_c}, \pm\frac{3}{2f_c}, \pm\frac{4}{2f_c}, \ldots\ldots\ldots\ldots$$

This can also be verified from equation 1.7.12. At above given values of t, $sin(2\pi f_c t)$ has zero values, Fig. 1.7.5 shows the plot of sinc function.

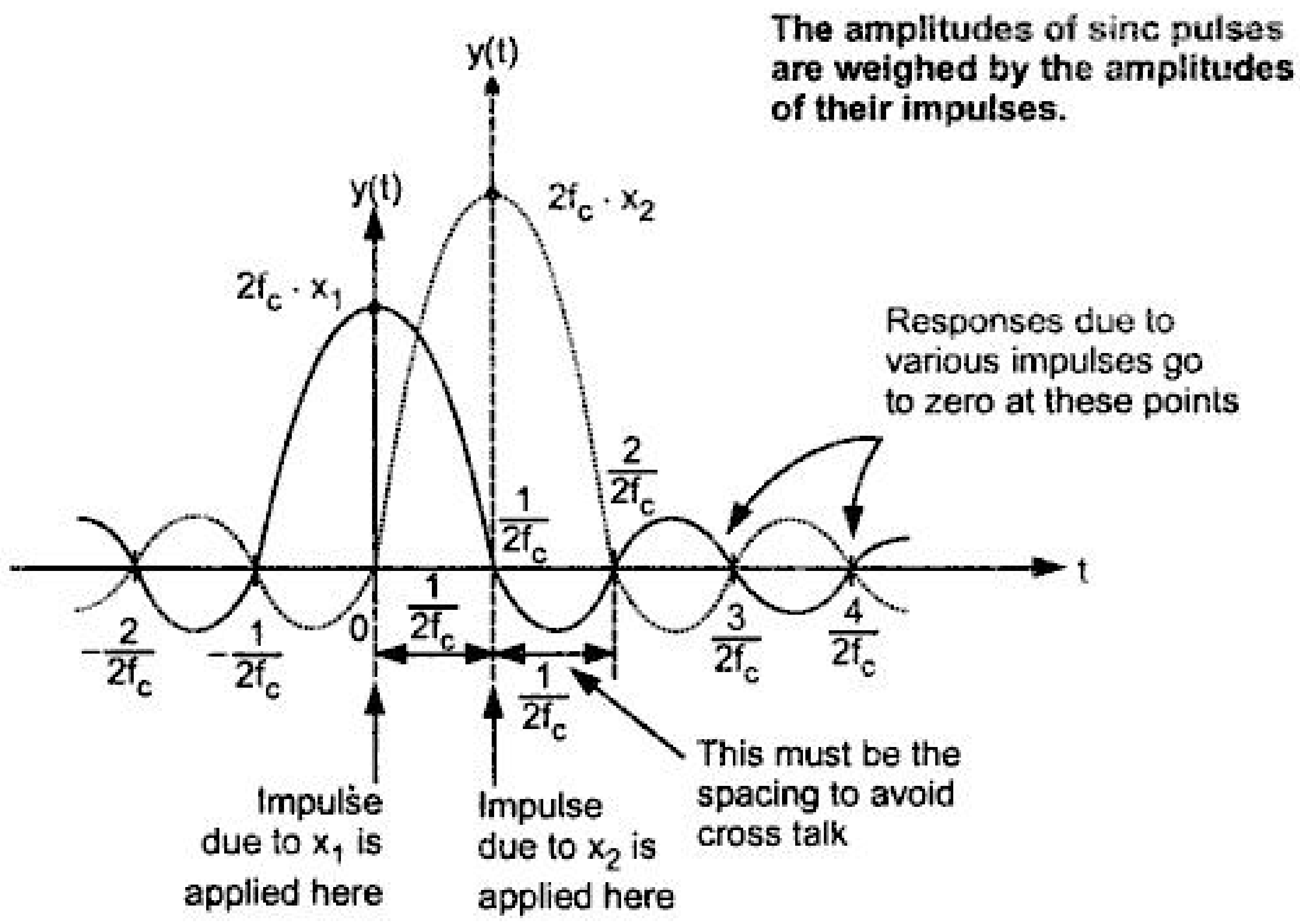


Fig. 1.7.5 Signal at the output of transmission channel which has a bandwidth of f_c Hz

Thus if impulse from channel X, is applied at $t = 0$, then its corresponding output (i.e. its impulse response given by equation 1.7.13) is shown by solid line in above figure. It shows that the response due to one impulse at $t = 0$ persists over a long time. Consider that second impulse due to second channel is applied at $t = \frac{1}{2f_c}$. The response due to this impulse also persists over long period. This means at any time the responses due to other impulses are present. Therefore there is possibility of crosstalk. But a careful observation of Fig. 1.7.5 shows that responses due to all the other impulses are at $t = \pm\frac{1}{2f_c}, \pm\frac{2}{2f_c}, \pm\frac{3}{2f_c}, \pm\frac{4}{2f_c}, \ldots\ldots\ldots$ except that of impulse sent at that time. For example at $t = 0$, responses due to all other impulses are zero except impulse response due to x_1, it has peak value of $t=0$. Similarly at $t=\frac{1}{2f_c}$, impulse response due to x_2 is at peak whereas all other responses are zero. This shows that if impulses are transmitted at $t=0, \pm\frac{1}{2f_c}, \pm\frac{2}{2f_c}$ ...., the crosstalk will be zero. In other words we can say that the spacing between two consecutive samples should be $\frac{1}{2f_c}$ in order to avoid crosstalk, i.e.,

$$\text{spacing between two consecutive samples in order to avoid crosstalk} = \frac{1}{2f_c} \quad \text{... (1.7.14)}$$

Comparing the above equation with equation 1.7.11 (which also gives spacing between two consecutive samples),

$$\frac{1}{2f_c} = \frac{1}{2N f_m}$$

$$\therefore \quad f_c = N f_m$$

Thus,

$$\boxed{\textbf{Minimum channel bandwidth to avoid crosstalk} : f_c = N f_m} \quad \text{... (1.7.15)}$$

Observe that this equation is similar to the relation we obtained earlier given by equation 1.7.9.

1.7.2 Synchronization in TDM System

From the discussion of TDM system it is clear that the receiver should operate in perfect synchronization with the transmitter. Normally markers are inserted to indicate the separation between the frames. Fig. 1.7.6 shows the TDM signals with markers.

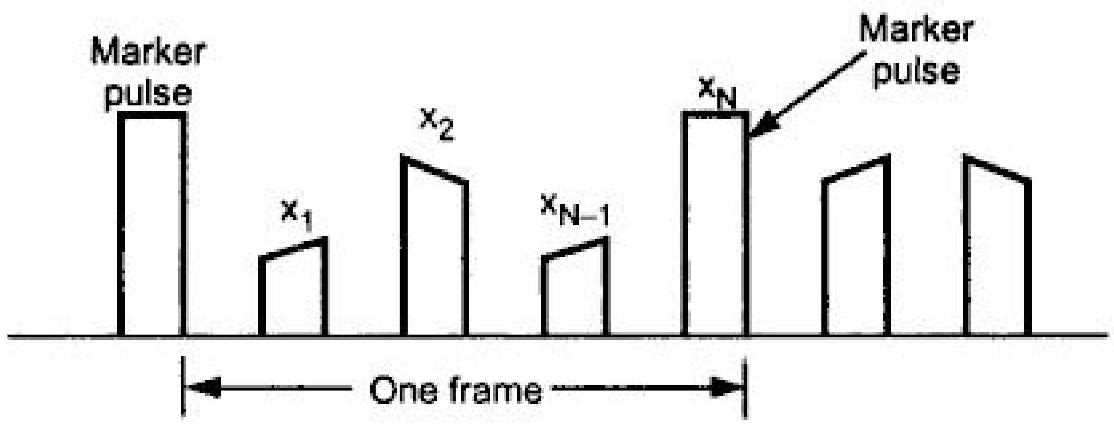


Fig. 1.7.6 Marker pulses for synchronization in TDM

The above figure shows that a marker pulse is inserted at the end of the frame. Because of the marker pulse, synchronization is obtained but number of channels to be multiplexed is reduced by one (i.e. N–1 channels can be multiplexed).

1.7.3 Crosstalk and Guard Times

We have seen that RF transmission of TDM needs modulation. Hence the TDM signal is converted to a smooth modulating waveform (i.e. baseband signal) by passing through a baseband filter. Fig. 1.7.7 shows the TDM transmission with baseband filtering and the baseband waveform.

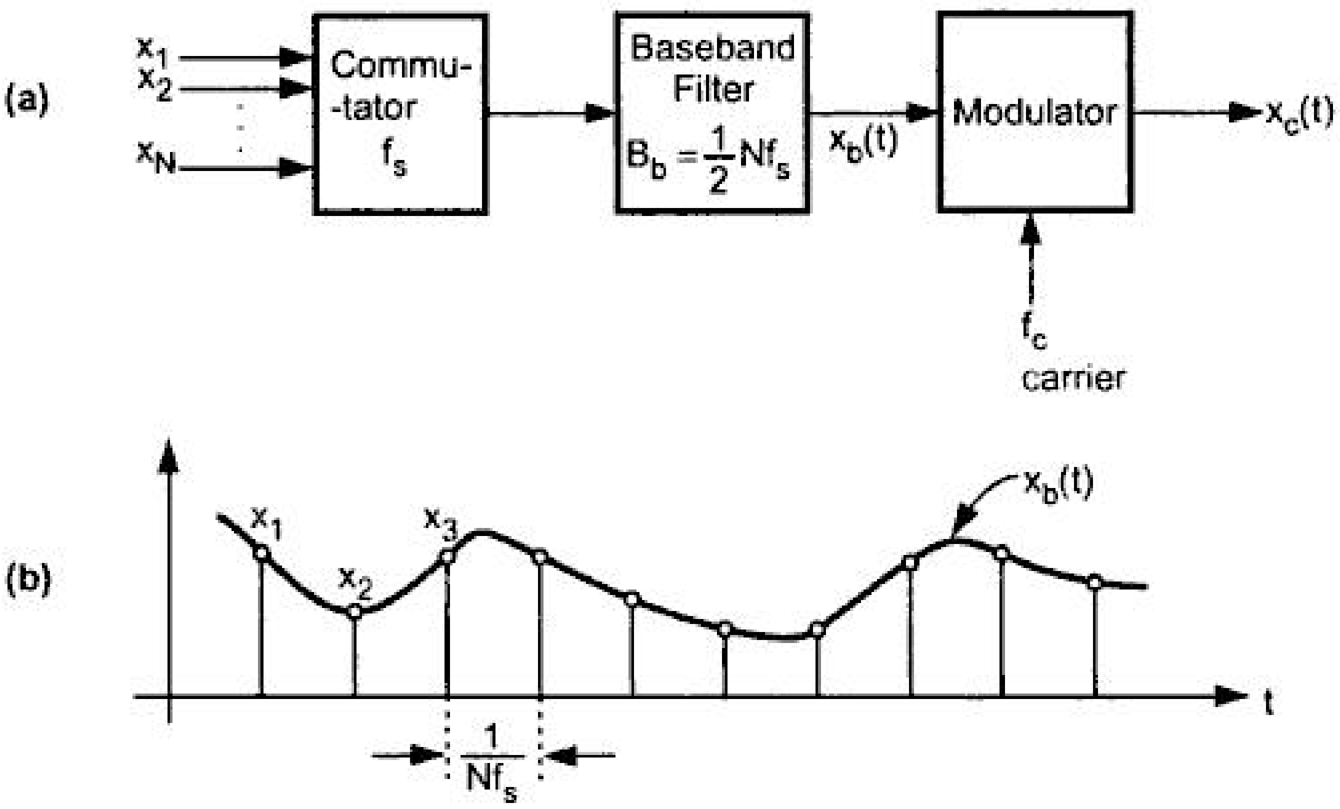


Fig. 1.7.7 (a) TDM transmission with baseband filtering
(b) Baseband waveform

Thus the baseband waveform passes through the values of all the individual samples. The baseband filtering gives rise to interchannel crosstalk from one sample value to the next. In other words *crosstalk* means the individual signal sample amplitudes interfere with each other. This interference can be reduced by increasing the distance between individual signal samples. The minimum distance between the individual signal samples to avoid crosstalk is called *guard time.*

Now let us derive an expression for guard time in TDM. Let us assume that the transmission channel acts like a first order lowpass filter with 3-dB bandwidth 'B'. And assume that every pulse transmitted in TDM is a rectangular pulse. When this pulse is applied to the channel, its response is shown in Fig. 1.7.8 (b).

In the Fig. 1.7.8 observe that even after the pulse is removed, the response of the channel decays from its value of 'A'. The response then decays for long period. The guard time T_g represents the minimum pulse spacing. At the end of guard time, the value of pulse tail is less than A_{ct}, where it is given as,

$$A_{ct} = A e^{-2\pi B T_g} \qquad \text{... (1.7.16)}$$

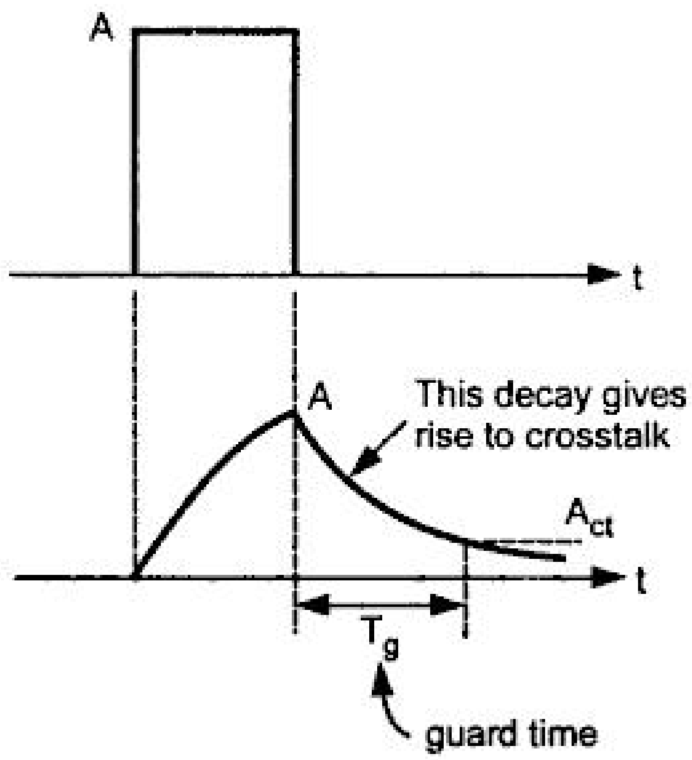


Fig. 1.7.8 (a) A rectangular pulse applied to the lowpass channel
(b) Response of the lowpass channel to the rectangular pulse

And the *cross talk reduction factor* is defined as,

$$K_{ct} = 10\,Log\left(\frac{A_{ct}}{A}\right)^2$$

$$\approx -54.5\,B\,T_g\ dB \qquad \text{... (1.7.17)}$$

This equation shows that to keep cross talk below $-30\,dB$, T_g should be greater than $\frac{1}{2B}$. The guard times are very much important particularly in pulse duration or pulse position modulation techniques.

Example 1.7.2 : *Twelve different message signals, each of bandwidth 10 kHz are to be multiplexed and transmitted. Determine the minimum bandwidth required for PAM/TDM system.*

Solution : Here the number of channels N = 12.

Bandwidth of each channel f_m = 10 kHz

Minimum channels bandwidth to avoid crosstalk in PAM/TDM system is,

$$f_c = N f_m \qquad \text{(By equation 1.7.15)}$$

$$= 12 \times 10\,kHz$$

$$= 120 \text{ kHz}$$

Example 1.7.3 : *Twenty four voice signals are sampled uniformly and then time division multiplexed. The highest frequency component for each voice signal is 3.4 kHz.*

i) If the signals are pulse amplitude modulated using Nyquist rate sampling, what is the minimum channel bandwidth required?

ii) If the signals are pulse code modulated with an 8 bit encoder, what is the sampling rate ? The bit rate of system is 1.5×10^6 bits/sec.

Solution : i) We know that if N channels are time division multiplexed, then minimum transmission bandwidth is given as,

$$B_T = NW$$

Here W is the maximum frequency in the signals.

$$\therefore \quad B_T = 24 \times 3.4 \text{ kHz} = 81.6 \text{ kHz} \quad \text{... (Ans)}$$

ii) The signalling rate of the system is given as,

$$r = 1.5 \times 10^6 \text{ bits/sec}$$

Since there are 24 channels, the bit rate of an individual channel is,

$$\text{r (one channel)} = \frac{1.5 \times 10^6}{24}$$

$$= 62500 \text{ bits/sec}$$

Since each sample is encoded using 8 bits, the samples per second will be,

$$\text{Sample/sec} = \frac{\text{r (one channel) bits / sec}}{\text{bits / sample}}$$

Samples per seconds is nothing but sampling frequency.

$$\text{i.e.} \quad f_s = \frac{62500 \text{ bits / sec}}{8 \text{ bits / sample}}$$

$$= 7812.5 \text{ Hz or samples per second} \quad \text{... (Ans)}$$

Example 1.7.4 : *Twenty four voice signals are sampled uniformly and then time division multiplexed. The sampling operation uses flat samples with 1 μsec duration. The multiplexing operation provides for synchronization by adding an extra pulse of* 1μ sec *duration. Assuming sampling rate of 8 kHz, calculate spacing between successive pulses of multiplexed signal and setup a scheme for accomplishing a multiplexing requirement.*

Solution : There are 24 voice signal pulses plus one synchronization pulse. Hence there are total 25 pulses. Sampling rate is 8 kHz. Hence duration of one frame will be,

$$T_s = \frac{1}{f_s} = \frac{1}{8000}$$

$$= 125 \ \mu sec$$

Thus in 125 μsec time there are 25 pulses at uniform distances. This is illustrated in Fig. 1.7.9.

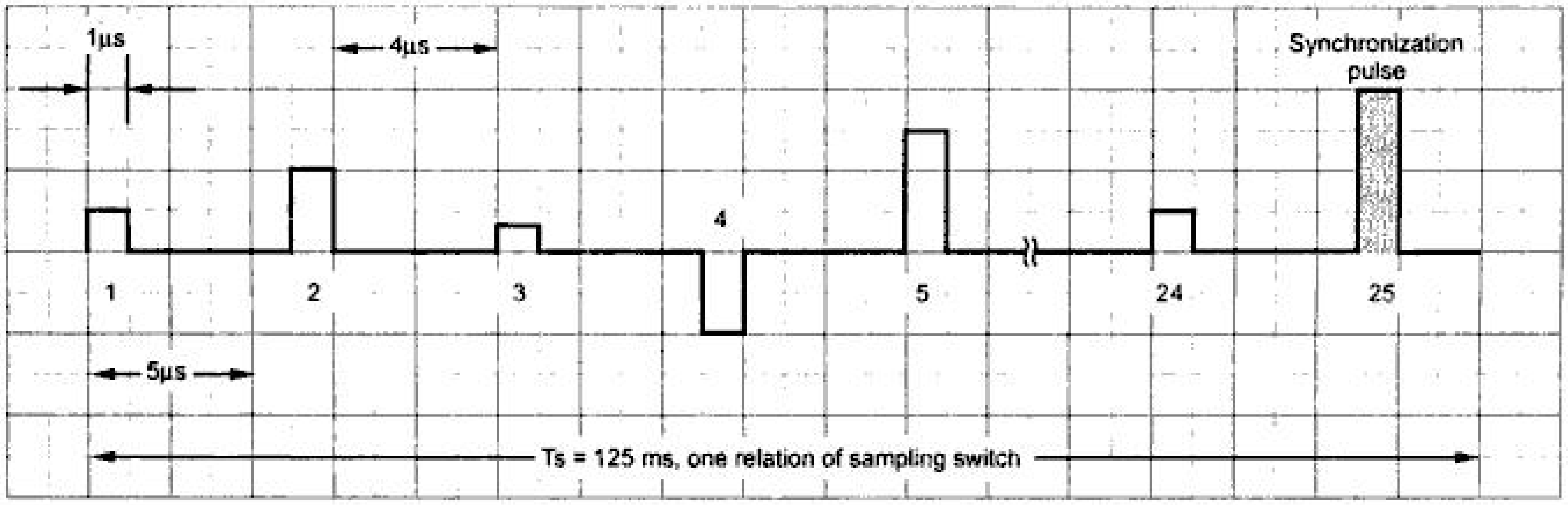


Fig. 1.7.9 Multiplexing of 24 voice signals

As shown in above figure, the pulses are separated by $\frac{125 \mu s}{25} = 5\ \mu s$. Width of the pulse is 1 μs. Hence,

Spacing between pulses = 5 – 1 = 4 μsec.

Fig. 1.7.10 shows the multiplexing scheme.

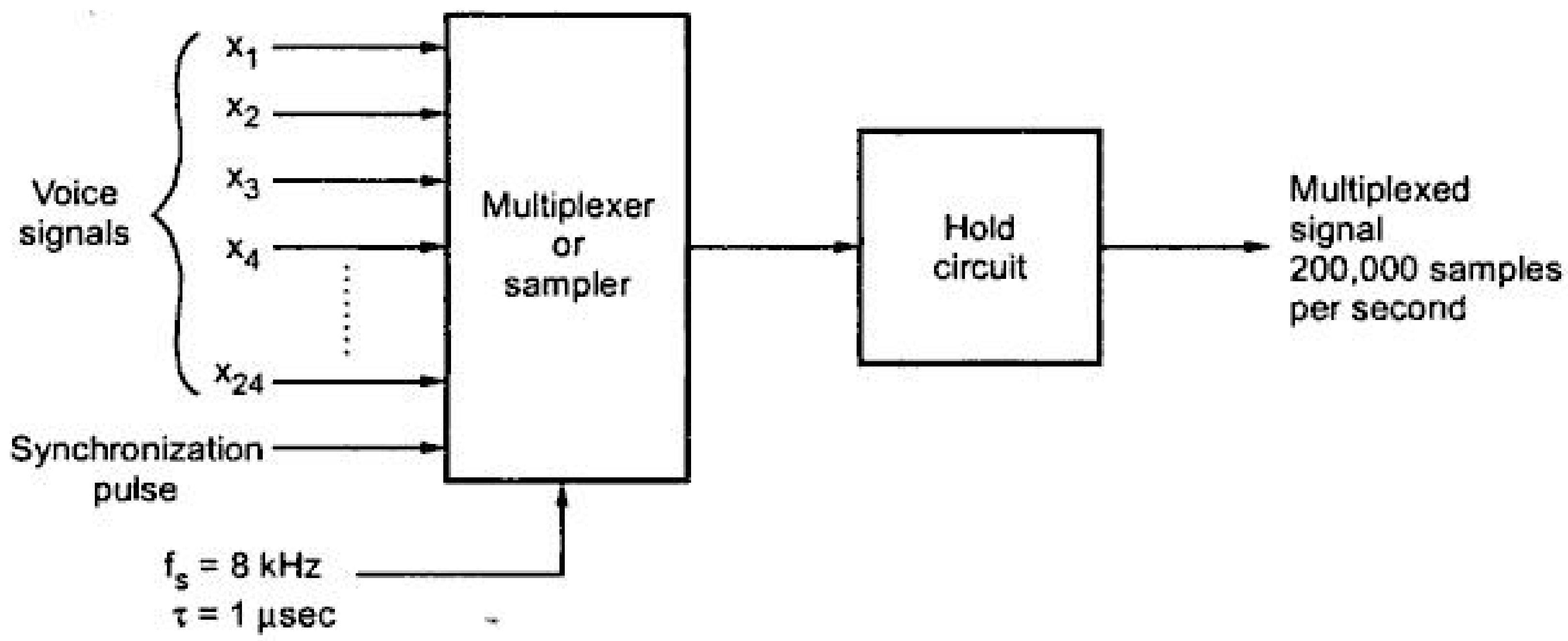


Fig. 1.7.10 PAM-TDM system

Theory Questions

1. *Explain PAM/TDM system for 'N' number of channels.*
2. *Derive the relation for minimum bandwidth to transmit 'N' channels in PAM/TDM system such that crosstalk is avoided.*
3. *Explain the importance of synchronization in TDM systems.*

Unsolved Examples

1. *Twenty four voice signals are sampled uniformly and then time division multiplexed, the sampling operation uses flat top samples with 1 μsec duration. The synchronization is provided by adding an extra pulse of 1 μsec duration. The highest frequency component of each voice signal is 3.4 kHz.*

   (a) For sampling rate of 8 kHz, calculate spacing between successive pulses of multiplexed signal.

   (b) For Nyquist rate repeat part (a).

1.8 Pulse Code Modulation

1.8.1 PCM Generator

The pulse code modulator technique samples the input signal $x(t)$ at frequency $f_s \geq 2W$. This sampled 'Variable amplitude' pulse is then digitized by the analog to digital converter. The parallel bits obtained are converted to a serial bit stream. Fig.1.8.1 shows the PCM generator.

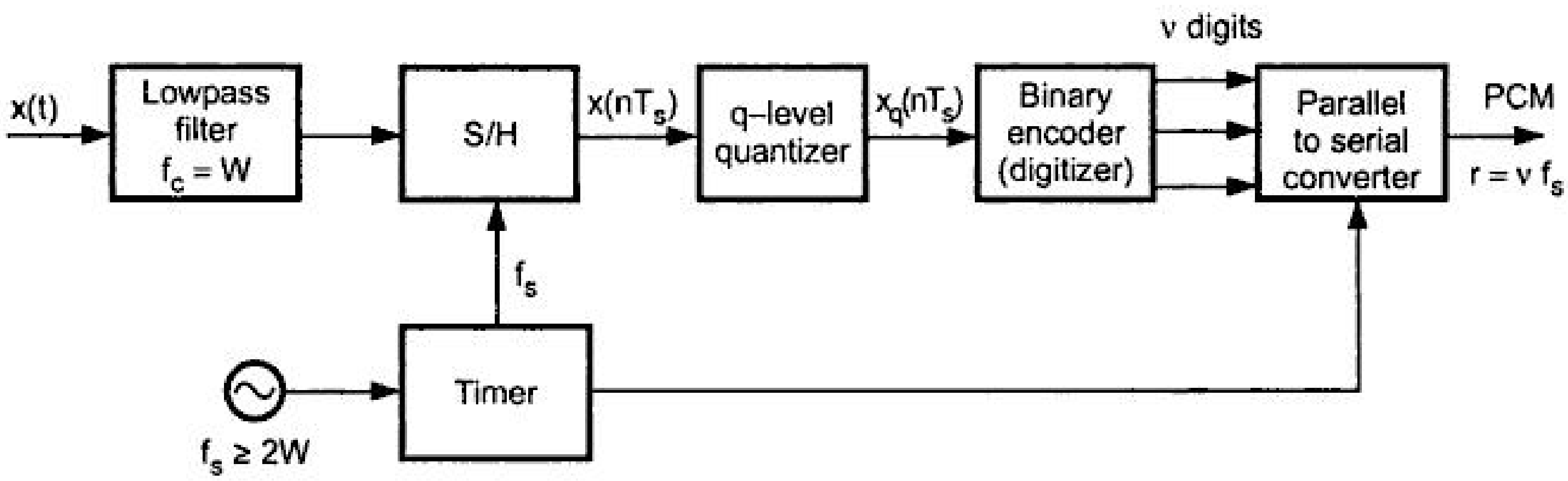


Fig. 1.8.1 PCM generator

In the PCM generator of above figure, the signal $x(t)$ is first passed through the lowpass filter of cutoff frequency 'W' Hz. This lowpass filter blocks all the frequency components above 'W' Hz. Thus $x(t)$ is bandlimited to 'W' Hz. The sample and hold circuit then samples this signal at the rate of f_s. Sampling frequency f_s is selected sufficiently above Nyquist rate to avoid aliasing i.e.,

$$f_s \geq 2W$$

In Fig. 1.8.1 output of sample and hold is called $x(nT_s)$. This $x(nT_s)$ is discrete in time and continuous in amplitude. A q-level quantizer compares input $x(nT_s)$ with its fixed digital levels. It assigns any one of the digital level to $x(nT_s)$ with its fixed digital levels. It then assigns any one of the digital level to $x(nT_s)$ which results in minimum distortion or error. This error is called *quantization error*. Thus output of quantizer is a digital level called $x_q(nT_s)$.

Now coming back to our discussion of PCM generation, the quantized signal level $x_q(nT_s)$ is given to binary encoder. This encoder converts input signal to 'v' digits binary word. Thus $x_q(nT_s)$ is converted to 'V' binary bits. The encoder is also called digitizer.

It is not possible to transmit each bit of the binary word separately on transmission line. Therefore 'v' binary digits are converted to serial bit stream to generate single baseband signal. In a parallel to serial converter, normally a shift register does this job. The output of PCM generator is thus a single baseband signal of binary bits.

An oscillator generates the clocks for sample and hold an parallel to serial converter. In the pulse code modulation generator discussed above ; sample and hold, quantizer and encoder combinely form an analog to digital converter.

1.8.2 Transmission Bandwidth in PCM

Let the quantizer use 'v' number of binary digits to represent each level. Then the number of levels that can be represented by 'v' digits will be,

$$q = 2^v \qquad \text{... (1.8.1)}$$

Here 'q' represents total number of digital levels of q-level quantizer.

For example if $\nu = 3$ bits, then total number of levels will be,

$$q = 2^3 = 8 \text{ levels}$$

Each sample is converted to 'v' binary bits. i.e. Number of bits per sample = v

We know that, Number of samples per second = f_s

$\therefore$ Number of bits per second is given by,

$$\begin{aligned}(\text{Number of bits per second}) &= (\text{Number of bits per samples}) \\ &\quad \times (\text{Number of samples per second}) \\ &= v \text{ bits per sample} \times f_s \text{ samples per second} \qquad \text{... (1.8.2)}\end{aligned}$$

The number of bits per second is also called signaling rate of PCM and is denoted by 'r' i.e.,

$$\boxed{\textbf{Signaling rate in PCM : } r = v f_s} \qquad \text{... (1.8.3)}$$

Here $f_s \geq 2W$.

Bandwidth needed for PCM transmission will be given by half of the signaling rate i.e.,

Transmission Bandwidth of PCM :
$$\begin{cases} B_T \geq \frac{1}{2} r & \text{... (1.8.4)} \\ B_T \geq \frac{1}{2} v f_s \quad \text{Since } f_s \geq 2W & \text{... (1.8.5)} \\ B_T \geq v W & \text{... (1.8.6)} \end{cases}$$

1.8.3 PCM Receiver

Fig. 1.8.2 (a) shows the block diagram of PCM receiver and Fig. 1.8.2 (b) shows the reconstructed signal. The regenerator at the start of PCM receiver reshapes the pulses and removes the noise. This signal is then converted to parallel digital words for each sample.

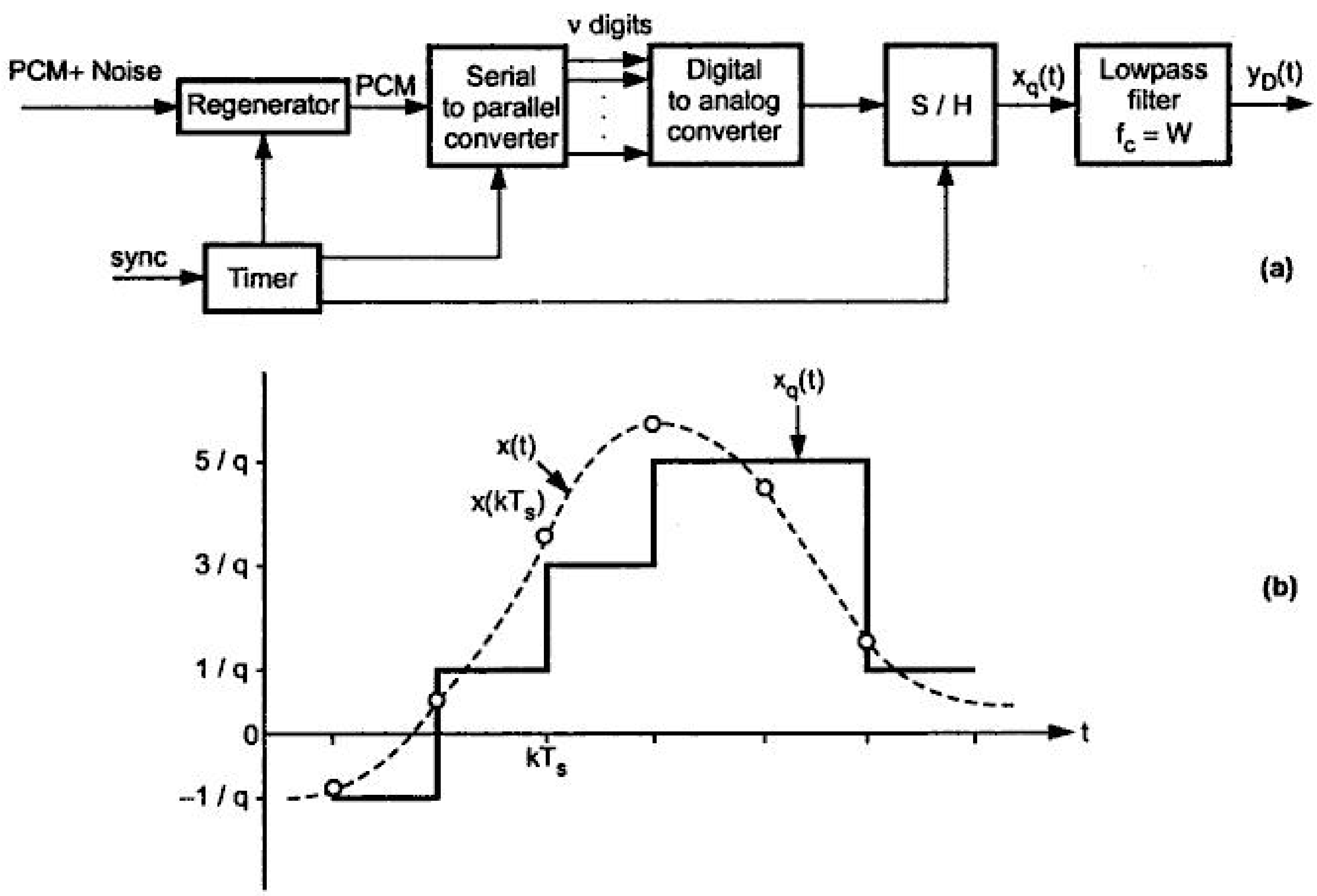


Fig. 1.8.2 (a) PCM receiver

(b) Reconstructed waveform

The digital word is converted to its analog value $x_q(t)$ along with sample and hold. This signal, at the output of S/H is passed through lowpass reconstruction filter to get $y_D(t)$. As shown in reconstructed signal of Fig. 1.8.2 (b), it is impossible to reconstruct exact original signal $x(t)$ because of permanent quantization error introduced during quantization at the transmitter. This quantization error can be reduced by increasing the binary levels. This is equivalent to increasing binary digits (bits) per sample. But increasing bits 'v' increases the signaling rate as well as transmission bandwidth as we have seen in equation 1.8.3 and equation 1.8.6. Therefore the choice of these parameters is made, such that noise due to quantization error (called as quantization noise) is in tolerable limits.

1.8.4 Uniform Quantization (Linear Quantization)

We know that input sample value is quantized to nearest digital level. This quantization can be uniform or nonuniform. In uniform quantization, the quantization step or difference between two quantization levels remains constant over the complete amplitude range. Depending upon the transfer characteristic there are three types of uniform or linear quantizers as discussed next.

1.8.4.1 Midtread Quantizer

The transfer characteristic of the midtread quantizer is shown in Fig. 1.8.3.

As shown in this figure, when an input is between $-\delta/2$ and $+\delta/2$ then the quantizer output is zero. i.e.,

For $\qquad -\delta/2 \le x(nT_s) < \delta/2 \; ; \quad x_q(nT_s) = 0$

Here 'δ' is the step size of the quantizer.

for $\qquad \delta/2 \le x(nT_s) < 3\,\delta/2 \; ; \quad x_q(nT_s) = \delta$

Similarly other levels are assigned. It is called midtread because quantizer output is zero when $x(nT_s)$ is zero. Fig.1.8.3 (b) shows the quantization error of midtread quantizer. Quantization error is given as,

$$\varepsilon = x_q(nT_s) - x(nT_s) \qquad \text{... (1.8.7)}$$

In Fig. 1.8.3 (b) observe that when $x(nT_s) = 0$, $x_q(nT_s) = 0$. Hence quantization error is zero at origin. When $x(nT_s) = \delta/2$, quantizer output is zero just before this level. Hence error is $\delta/2$ near this level. From Fig. 1.8.3 (b) it is clear that,

$$-\delta/2 \le \varepsilon \le \delta/2 \qquad \text{... (1.8.8)}$$

Thus quantization error lies between $-\delta/2$ and $+\delta/2$. And maximum quantization error is, maximum quantization error, $\varepsilon_{max} = \left|\frac{\delta}{2}\right|$ ... (1.8.9)

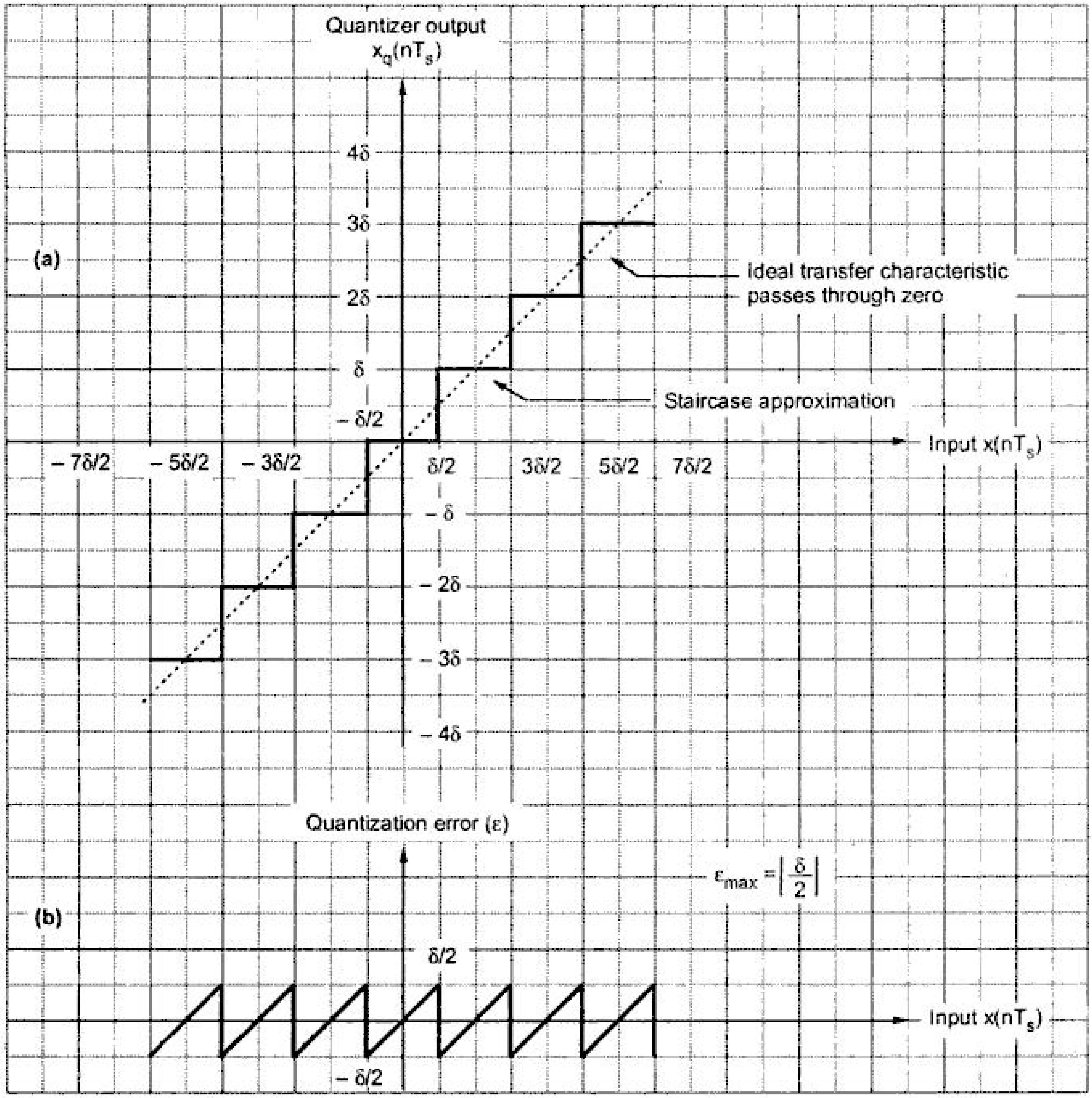


Fig. 1.8.3 (a) Quantization characteristic of midtread quantizer (b) Quantization error

1.8.4.2 Midriser Quantizer

The transfer characteristic of the midriser quantizer is shown in Fig. 1.8.4.

In Fig. 1.8.4 observe that, when an input is between 0 and δ, the output is $\delta/2$. Similarly when an input is between 0 and $-\delta$, the output is $-\delta/2$. i.e.,

$$\text{For} \quad 0 \leq x(nT_s) < \delta\,; \quad x_q(nT_s) = \delta/2$$

$$-\delta \leq x(nT_s) < 0\,; \quad x_q(nT_s) = -\delta/2$$

Similarly when an input is between $3\,\delta$ and $4\,\delta$, the output is $7\,\delta/2$. This is called midriser quantizer because its output is either $+\delta/2$ or $-\delta/2$ when input is zero.

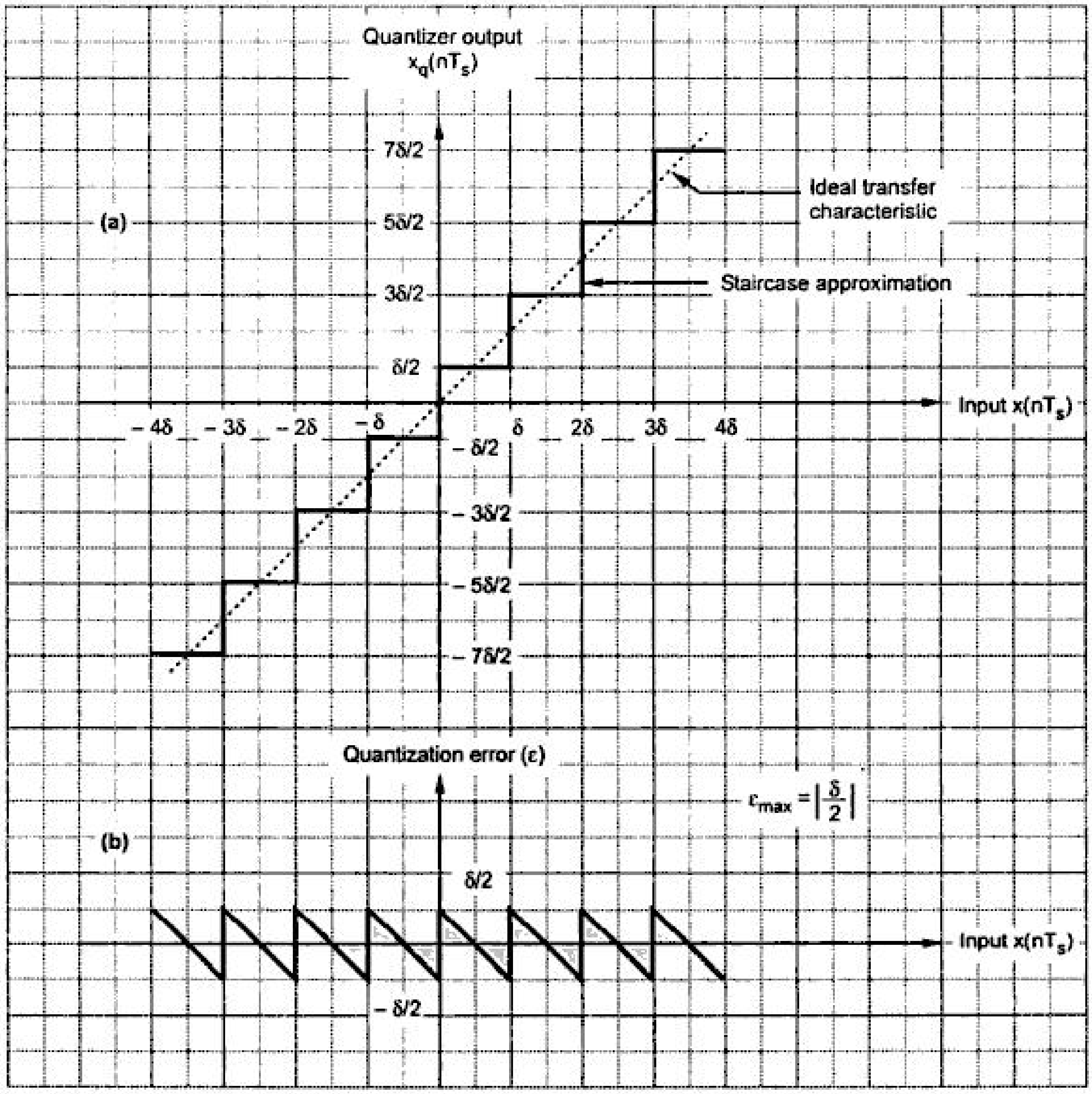


Fig. 1.8.4 (a) Transfer characteristic of midriser quantizer (b) Quantization error

Fig. 1.8.4 (b) shows the quantization error in midriser quantization. When input $x(nT_s) = 0$, the quantizer will assign the level of $\delta/2$. Hence quantization error will be,

$$\begin{aligned} \varepsilon &= x_q(nT_s) - x(nT_s) \\ &= \delta/2 - 0 = \delta/2 \end{aligned}$$

Thus the quantization error lies between $-\delta/2$ and $+\delta/2$. i.e.,

$$-\delta/2 \leq \varepsilon \leq \delta/2 \qquad \text{... (1.8.10)}$$

And the maximum quantization error is,

$$\varepsilon_{max} = \left|\frac{\delta}{2}\right| \qquad \text{... (1.8.11)}$$

In both the midriser and midtread quantizers, the dotted line of unity slope pass through origin. It represents ideal nonquantized input output characteristic. The staircase characteristic is an approximation of this line. The difference between the staircase and unity slope line represents the quantization error.

1.8.4.3 Biased Quantizer

Fig. 1.8.5 shows the transfer characteristic of biased uniform quantizer.

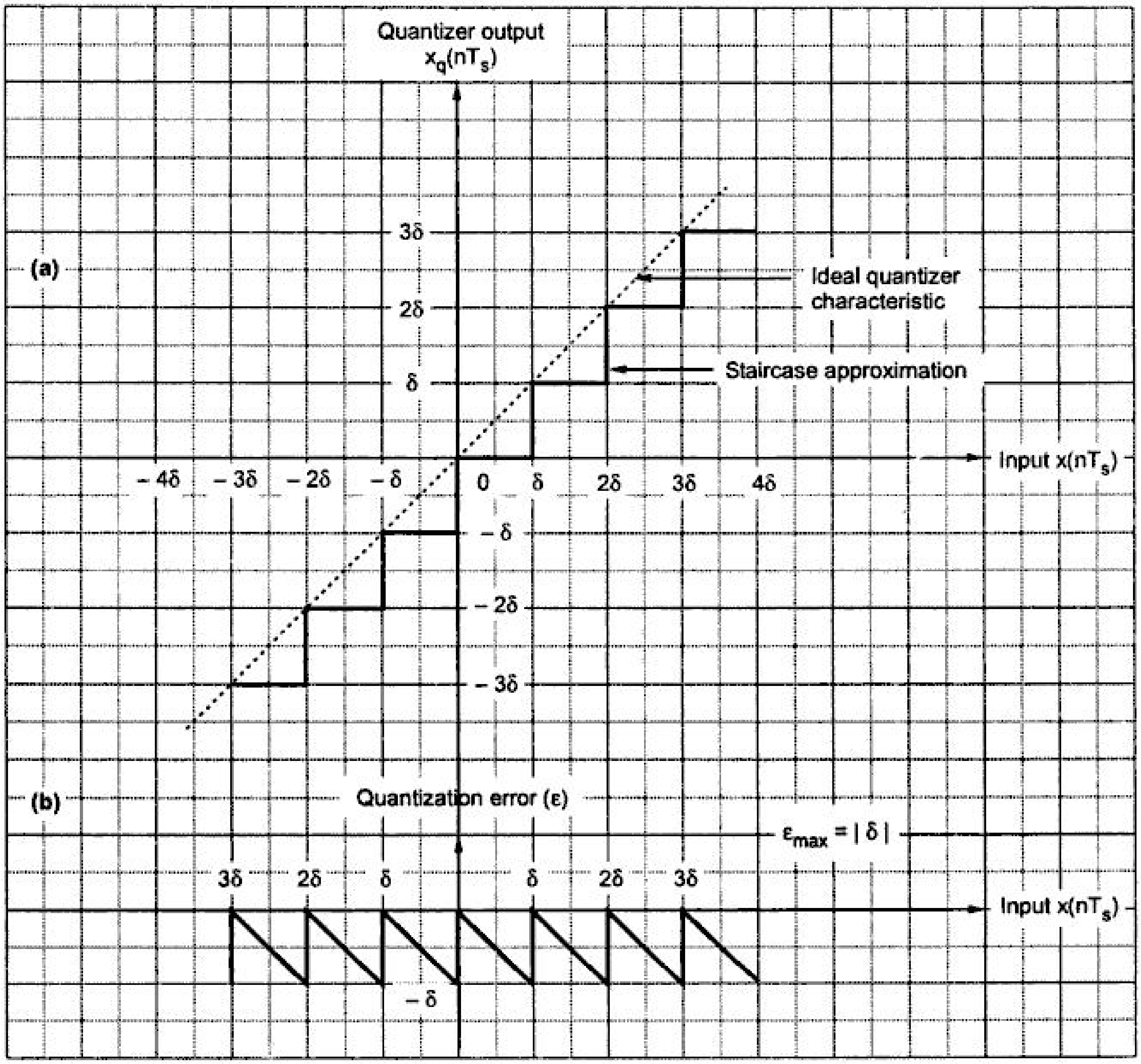


Fig. 1.8.5 (a) Biased quantizer transfer characteristic
(b) Quantization error

The midriser and midtread quantizers are rounding quantizers. But biased quantizer is truncation quantizer. This is clear from above diagram. When input is between 0 and δ, the output is zero. i.e.,

for $\qquad 0 \leq x(nT_s) < \delta \,; \quad x_q(nT_s) = 0$

Similarly, for $-\delta \leq x(nT_s) < 0$; $x_q(nT_s) = -\delta$

Fig. 1.8.5 shows quantization error. When input is δ, output is zero. Hence quantization error is,

$$\varepsilon = x_q(nT_s) - x(nT_s)$$
$$= 0 - \delta = -\delta$$

Thus the quantization error lies between 0 and – δ. i.e.,

$$-\delta \leq \varepsilon \leq 0 \quad \text{... (1.8.12)}$$

And the maximum quantization error is,

$$\varepsilon_{max} = |\delta| \quad \text{... (1.8.13)}$$

Thus the quantization error is more in biased quantizer compared to midriser and midtread quantizers. The unity slope dotted line passes through origin. It represents ideal nonquantized transfer characteristic. The difference between staircase and dotted line gives quantization error.

1.8.5 Quantization Noise and Signal to Noise Ratio in PCM

1.8.5.1 Derivation of Quantization Error/Noise or Noise Power for Uniform (Linear) Quantization

Step 1 : Quantization Error

Because of quantization, inherent errors are introduced in the signal. This error is called *quantization error*. We have defined quantization error as,

$$\varepsilon = x_q(nT_s) - x(nT_s) \quad \text{... (1.8.14)}$$

Step 2 : Step size

Let an input $x(nT_s)$ be of continuous amplitude in the range $-x_{max}$ to $+x_{max}$.

From Fig. 1.8.4 (a) we know that the total excursion of input $x(nT_s)$ is mapped into 'q' levels on vertical axis. That is when input is 4 δ, output is $\frac{7}{2}\delta$ and when input is -4δ, output is $-\frac{7}{2}\delta$. That is $+x_{max}$ represents $\frac{7}{2}\delta$ and $-x_{max}$ represents $-\frac{7}{2}\delta$. Therefore the total amplitude range becomes,

$$\text{Total amplitude range} = x_{max} - (-x_{max})$$
$$= 2x_{max} \quad \text{... (1.8.15)}$$

If this amplitude range is divided into 'q' levels of quantizer, then the step size 'δ' is given as,

$$\delta = \frac{x_{max} - (-x_{max})}{q}$$
$$= \frac{2x_{max}}{q} \quad \text{... (1.8.16)}$$

If signal $x(t)$ is normalized to minimum and maximum values equal to 1, then

$$x_{max} = 1$$
$$-x_{max} = -1 \qquad \text{... (1.8.17)}$$

Therefore step size will be,

$$\delta = \frac{2}{q} \quad \text{(for normalized signal)} \qquad \text{... (1.8.18)}$$

Step 3 : Pdf of Quantization error

If step size 'δ' is sufficiently small, then it is reasonable to assume that the quantization error 'ε' will be uniformly distributed random variable. The maximum quantization error is given by equation 1.8.11 as,

$$\varepsilon_{max} = \left|\frac{\delta}{2}\right| \qquad \text{... (1.8.19)}$$

i.e.

$$-\frac{\delta}{2} \geq \varepsilon_{max} \geq \frac{\delta}{2} \qquad \text{... (1.8.20)}$$

Thus over the interval $\left(-\frac{\delta}{2}, \frac{\delta}{2}\right)$ quantization error is uniformly distributed random variable.

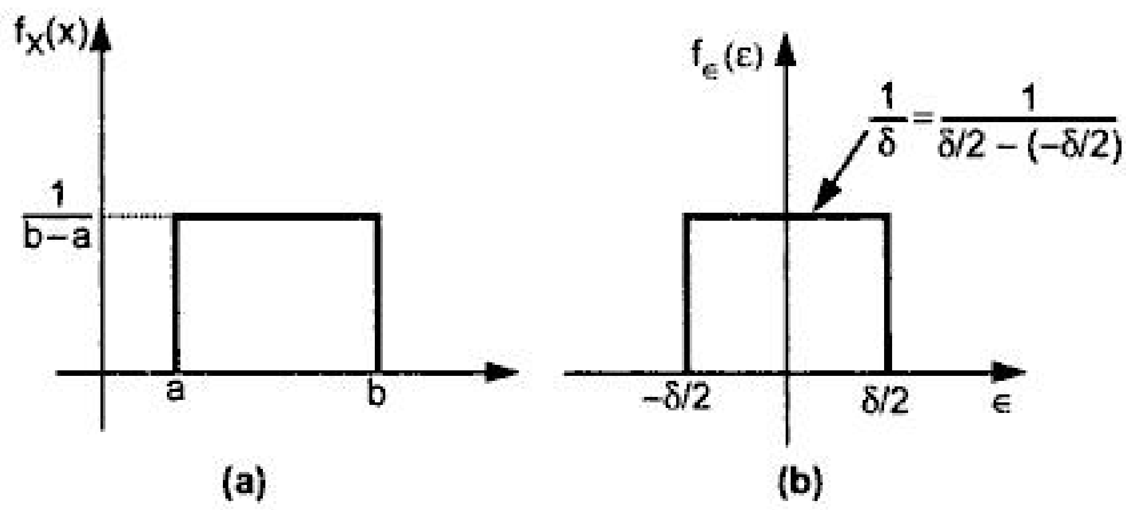


Fig. 1.8.6 (a) Uniform distribution
(b) Uniform distribution for quantization error

In above figure, a random variable is said to be uniformly distributed over an interval (a, b). Then PDF of 'X' is given by, (from equation of Uniform PDF).

$$f_X(x) = \begin{cases} 0 & for \quad x \leq a \\ \dfrac{1}{b-a} & for \quad a < x \leq b \\ 0 & for \quad x > b \end{cases} \qquad \text{... (1.8.21)}$$

Thus with the help of above equation we can define the probability density function for quantization error 'ε' as,

$$f_\varepsilon(\varepsilon) = \begin{cases} 0 & for \quad \varepsilon \leq \dfrac{\delta}{2} \\ \dfrac{1}{\delta} & for \quad -\dfrac{\delta}{2} < \varepsilon \leq \dfrac{\delta}{2} \\ 0 & for \quad \varepsilon > \dfrac{\delta}{2} \end{cases} \qquad \text{... (1.8.22)}$$

Step 4 : Noise Power

From Fig. 1.8.4 (b) we can see that quantization error 'ε' has zero average value. That is mean 'm_ε' of the quantization error is zero.

The signal to quantization noise ratio of the quantizer is defined as,

$$\frac{S}{N} = \frac{\text{Signal power (normalized)}}{\text{Noise power (normalized)}} \qquad \text{... (1.8.23)}$$

If type of signal at input i.e., $x(t)$ is known, then it is possible to calculate signal power.

The noise power is given as,

$$\text{Noise power} = \frac{V_{noise}^2}{R} \qquad \text{... (1.8.24)}$$

Here V_{noise}^2 is the mean square value of noise voltage. Since noise is defined by random variable 'ε' and PDF $f_\varepsilon(\varepsilon)$, its mean square value is given as,

$$\text{mean square value} = E[\varepsilon^2] = \bar{\varepsilon}^2 \qquad \text{... (1.8.25)}$$

The mean square value of a random variable 'X' is given as,

$$\bar{X}^2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \quad \text{By definition} \qquad \text{... (1.8.26)}$$

Here $$E[\varepsilon^2] = \int_{-\infty}^{\infty} \varepsilon^2 f_\varepsilon(\varepsilon)\, d\varepsilon \qquad \text{... (1.8.27)}$$

From equation 1.8.22 we can write above equation as,

$$E[\varepsilon^2] = \int_{-\delta/2}^{\delta/2} \varepsilon^2 \times \frac{1}{\delta} d\varepsilon$$

$$= \frac{1}{\delta}\left[\frac{\varepsilon^3}{3}\right]_{-\delta/2}^{\delta/2} = \frac{1}{\delta}\left[\frac{(\delta/2)^3}{3} + \frac{(\delta/2)^3}{3}\right]$$

$$= \frac{1}{3\delta}\left[\frac{\delta^3}{8} + \frac{\delta^3}{8}\right] = \frac{\delta^2}{12} \quad \text{... (1.8.28)}$$

∴ From equation 1.8.25, the mean square value of noise voltage is,

$$V_{noise}^2 = \text{mean square value} = \frac{\delta^2}{12}$$

When load resistance, $R = 1$ ohm, then the noise power is normalized i.e.,

$$\text{Noise power (normalized)} = \frac{V_{noise}^2}{1} \quad \text{[with } R = 1 \text{ in equation 1.8.24]}$$

$$= \frac{\delta^2/12}{1} = \frac{\delta^2}{12}$$

Thus we have,

Normalized noise power

or Quantization noise power $= \frac{\delta^2}{12}$ **; For linear quantization.**

or Quantization error (in terms of power) ... (1.8.29)

1.8.5.2 Derivation of Maximum Signal to Quantization Noise Ratio for Linear Quantization

From equation 1.8.23 signal to quantization noise ratio is given as,

$$\frac{S}{N} = \frac{\text{Normalized signal power}}{\text{Normalized noise power}}$$

$$= \frac{\text{Normalized signal power}}{(\delta^2/12)} \quad \text{... (1.8.30)}$$

The number of bits 'v' and quantization levels 'q' are related as,

$$q = 2^v \quad \text{... (1.8.31)}$$

Putting this value in equation 1.8.16 we have,

$$\delta = \frac{2\,x_{max}}{2^{v}} \qquad \text{... (1.8.32)}$$

Putting this value in equation 1.8.30 we get,

$$\frac{S}{N} = \frac{\text{Normalized signal power}}{\left(\frac{2\,x_{max}}{2^{v}}\right)^{2} + 12}$$

Let normalized signal power be denoted as 'P'.

$$\frac{S}{N} = \frac{P}{\frac{4\,x_{max}^{2}}{2^{2v}} \times \frac{1}{12}} = \frac{3P}{x_{max}^{2}} \cdot 2^{2v}$$

This is the required relation for maximum signal to quantization noise ratio. Thus,

$$\boxed{\textbf{Maximum signal to quantization noise ratio : } \frac{S}{N} = \frac{3P}{x_{max}^{2}} \cdot 2^{2v}} \qquad \text{... (1.8.33)}$$

This equation shows that signal to noise power ratio of quantizer increases exponentially with increasing bits per sample.

If we assume that input $x(t)$ is normalized, i.e.,

$$x_{max} = 1 \qquad \text{... (1.8.34)}$$

Then signal to quantization noise ratio will be,

$$\frac{S}{N} = 3 \times 2^{2v} \times P \qquad \text{... (1.8.35)}$$

If the destination signal power 'P' is normalized, i.e.,

$$P \leq 1 \qquad \text{... (1.8.36)}$$

Then the signal to noise ratio is given as,

$$\frac{S}{N} \leq 3 \times 2^{2v} \qquad \text{... (1.8.37)}$$

Since $x_{max} = 1$ and $P \leq 1$, the signal to noise ratio given by above equation is normalized.

Expressing the signal to noise ratio in decibels,

$$\left(\frac{S}{N}\right) dB = 10 \log_{10}\left(\frac{S}{N}\right) dB \quad \text{since power ratio.}$$

$$\leq 10\,log_{10}\,[3 \times 2^{2v}]$$

$$\leq (4.8 + 6\,v)\,dB$$

Thus,

Signal to Quantization noise ratio
for normalized values of power : $\left(\frac{S}{N}\right) dB \le (4.8 + 6\, v)\, dB$
'P' and amplitude of input $x(t)$

... (1.8.38)

Example 1.8.1 : *Derive the expression for signal to quantization noise ratio for PCM system that employs linear quantization technique. Assume that input to the PCM system is a sinusoidal signal.*

OR

A PCM system uses a uniform quantizer followed by a v bit encoder. Show that rms signal to quantization noise ratio is approximately given by (1.8 + 6v) dB.

Solution : Assume that the modulating signal be a sinusoidal voltage, having peak amplitude A_m. Let this signal cover the complete excursion of representation levels.

The power of this signal will be,

$$P = \frac{V^2}{R} \qquad \text{Here V = rms value}$$

$$= \left[A_m / \sqrt{2}\right]^2 \qquad ... (1.8.39)$$

When $R = 1$, the power P is normalized, i.e.,

Normalized power : $$P = \frac{A_m^2}{2} \qquad \text{with } R = 1 \text{ in above equation.}$$

$\therefore$ Signal to quantization noise ratio is given by equation 1.8.33 as,

$$\frac{S}{N} = \frac{3P}{x_{\max}^2} \times 2^{2v}$$

Here $$P = \frac{A_m^2}{2} \text{ and } x_{\max} = A_m$$

Putting these values in the above equation,

$$\frac{S}{N} = \frac{3 \times \frac{A_m^2}{2}}{A_m^2} \times 2^{2v} = \frac{3}{2} \times 2^{2v} = 1.5 \times 2^{2v}$$

Expressing signal to noise power ratio in dB,

$$\left(\frac{S}{N}\right)dB = 10\log_{10}\left(\frac{S}{N}\right) = 10\log_{10}\left(1.5\times 2^{2v}\right)$$

$$= 10\log_{10}(1.5) + 10\log_{10} 2^{2v}$$

$$= 1.76 + 2v\times 10\times 0.3$$

Thus,

$$\boxed{\left(\frac{S}{N}\right)dB \text{ in PCM : } \left(\frac{S}{N}\right)dB = 1.8 + 6v \text{ ; for sinusoidal signal}} \quad \text{... (1.8.40)}$$

Example 1.8.2 : *A Television signal with a bandwidth of 4.2 MHz is transmitted using binary PCM. The number of quantization levels is 512.*
Calculate,
i) Code word length ii) Transmission bandwidth
iii) Final bit rate iv) Output signal to quantization noise ratio.

[March-2003, 10 Marks]

Solution : The bandwidth is 4.2 MHz, means highest frequency component will have frequency of 4.2 MHz i.e.,

$$W = 4.2 \text{ MHz}$$

Quantization levels $q = 512$

i) Number of bits and quantization levels are related in binary PCM as,

$$q = 2^v$$

i.e.
$$512 = 2^v$$

$$\log 512 = v\log 2$$

or
$$v = \frac{\log 512}{\log 2}$$

$$= 9 \text{ bits} \quad \text{... (Ans)}$$

Thus the code word length is 9 bits.

ii) From equation 1.8.6 the transmission channel bandwidth is given as,

$$B_T \geq vW$$

$$\geq 9\times 4.2\times 10^6 \; Hz$$

$$B_T \geq 37.8 \text{ MHz} \quad \text{... (Ans)}$$

iii) The final bit rate will equal to signaling rate. From equation 1.8.3 signaling rate is given as,

$$r = v f_s$$

Sampling frequency $f_s \geq 2W$ by sampling theorem.

$$\therefore \quad f_s \geq 2 \times 4.2\ MHz \quad \text{since } W = 4.2 \text{ MHz}$$

$$\therefore \quad f_s \geq 8.4 \text{ MHz}$$

Putting this value of 'f_s' in equation for signaling rate,

$$r = 9 \times 8.4 \times 10^6$$

$$= 75.6 \times 10^6 \text{ bits/sec} \qquad \text{... (Ans)}$$

From equation 1.8.4 transmission bandwidth is also obtained as,

$$B_T \geq \frac{1}{2} r$$

$$\geq \frac{1}{2} \times 75.6 \times 10^6 \text{ bits/sec}$$

or $\quad B_T \geq 37.8$ MHz which is same as the value obtained earlier.

iv) The signal to noise ratio

$$\left(\frac{S}{N}\right) dB \leq 4.8 + 6v\ dB$$

$$\leq 4.8 + 6 \times 9$$

$$\leq 58.8 \text{ dB} \qquad \text{... (Ans)}$$

Example 1.8.3 : *The bandwidth of signal input to the PCM is restricted to 4 kHz. The input varies from –3.8 V to + 3.8 V and has the average power of 30 mW. The required signal to noise ratio is 20 dB. The modulator produces binary output. Assume uniform quantization.*

i) Calculate the number of bits required per sample.

ii) Outputs of 30 such PCM coders are time multiplexed. What is the minimum required transmission bandwidth for the multiplexed signal ?

Solution : The given value of signal to noise ratio is 20 dB.

i.e. $$\left(\frac{S}{N}\right) dB = 10 \log_{10}\left(\frac{S}{N}\right) = 20\ dB$$

$$\therefore \quad \frac{S}{N} = 100$$

i) The signal to quantization noise ratio is given as,

$$\frac{S}{N} = \frac{3P \cdot 2^{2v}}{x_{max}^2} \quad \text{By equation 1.8.33}$$

Here $\quad x_{max} = 3.8\,V,\, P = 30\,mW \text{ and } \frac{S}{N} = 100$

$$\therefore \quad 100 = \frac{3 \times 30 \times 10^{-3}\, 2^{2v}}{(3.8)^2}$$

$$\therefore \quad v = 6.98 \text{ bits}$$

$$= 7 \text{ bits}$$ ... **(Ans)**

ii) The maximum frequency is,

$$W = 4 \text{ kHz}$$

The transmission bandwidth is given by equation 1.8.6 as,

$$B_T \geq vW$$

Since there are 30 PCM coders which are time multiplexed, the transmission bandwidth will be,

$$B_T \geq 30 \times v \cdot W$$

$$\geq 30 \times 7 \times 4\, kHz$$

$$\geq 840 \text{ kHz}$$ ... **(Ans)**

Signaling rate is two times the transmission bandwidth as given by equation 1.8.4 i.e.,

$$\text{Signaling rate } r = 840 \times 2 \text{ bits/sec} = 1680 \text{ bits/sec.}$$

Example 1.8.4 : *The information in an analog signal voltage waveform is to be transmitted over a PCM system with an accuracy of ± 0.1% (full scale). The analog voltage waveform has a bandwidth of 100 Hz and an amplitude range of –10 to +10 volts.*

a) Determine the maximum sampling rate required.

b) Determine the number of bits in each PCM word.

c) Determine minimum bit rate required in the PCM signal.

d) Determine the minimum absolute channel bandwidth required for the transmission of the PCM signal.

Solution : Here an accuracy is given as $\pm 0.1\%$. That is quantization error should be $\pm 0.1\%$.

or the maximum quantization error should be $\pm 0.1\%$

or $$\varepsilon_{max} = \pm 0.1\% = \pm 0.001$$

The maximum quantization error for an uniform quantizer is given as,

$$\varepsilon_{max} = \left|\frac{\delta}{2}\right|$$

or $$\left|\frac{\delta}{2}\right| = 0.001$$

That is

$$\text{Step size } \delta = 2 \times 0.001 = 0.002$$

The step size, number of levels and maximum value of the signal are related as (By equation 1.8.16)

$$\delta = \frac{2\,x_{max}}{q} \quad \text{Here} \quad |x_{max}| = 10 \text{ volts}$$

$\therefore$ Putting values of δ and x_{max},

$$0.002 = \frac{2 \times 10}{q}$$

or $$q = \frac{20}{0.002}$$

$$= 10{,}000$$

That is the number of levels are 10,000.

a) The maximum frequency in the signal is 100 Hz i.e.,

$$W = 100 \text{ Hz}$$

By sampling theorem,minimum sampling frequency should be,

$$f_s \geq 2W$$

$$\geq 2 \times 100 \geq 200 \; Hz$$ **... (Ans)**

b) We know that minimum 10,000 levels should be used to quantize the signal. If binary PCM is used, then number of bits for each samples can be calculated as,

$$q = 2^v$$

Here, q = number of levels

$$v = \text{bits in PCM,}$$

$$\therefore \quad 10{,}000 = 2^v$$

$$\log_{10} 10{,}000 = v \log_{10} 2$$

or
$$v = \frac{\log 10{,}000}{\log 2} = 13.288$$

or
$$v = 14 \text{ bits} \qquad \text{... (Ans)}$$

c) From equation 1.8.3 the bit rate or signaling rate is given as,

$$r = v f_s$$

$$= 14 \times 200$$

$$\geq 2800 \text{ bits per second.}$$

d) The transmission channel for PCM is given by equation 1.8.4 as,

$$B_T \geq \frac{1}{2} r$$

$$\geq \frac{1}{2} \times 2800$$

$$\geq 1400 \text{ Hz} \qquad \text{... (Ans)}$$

Example 1.8.5 : *A PCM system uses a uniform quantizer followed by a 7-bit binary encoder. The bit rate of the system is equal to* 50×10^6 *bits/sec.*

a) What is the maximum message bandwidth for which the system operates satisfactorily ?

b) Determine the output signal to quantization noise ratio when a full load sinusoidal modulating wave of frequency 1 MHz is applied to the input.

Solution : a) Let us assume that the message bandwidth be W Hz. Therefore sampling frequency should be,

$$f_s \geq 2W$$

The number of bits $v = 7$ bits

From equation 1.8.3 the signaling rate is given as,

$$r = v \cdot f_s$$

$$\therefore \quad r \geq 7 \times 2W$$

$$\therefore \quad 50 \times 10^6 \geq 14\,W \qquad \text{(putting value of r)}$$

$$\therefore \quad W \leq 3.57 \text{ MHz} \qquad \text{... (Ans)}$$

Solution : The signal is uniformly distributed in the range $\pm x_{max}$. Therefore we can write its PDF (using the Standard Uniform Distribution) as,

$$f_X(x) = 0 \quad \text{for } x < -x_{max}$$

$$= \frac{1}{2x_{max}} \quad \text{for } -x_{max} < x < x_{max}$$

$$= 0 \quad \text{for } x > x_{max}$$

Fig. 1.8.7 shows this PDF,

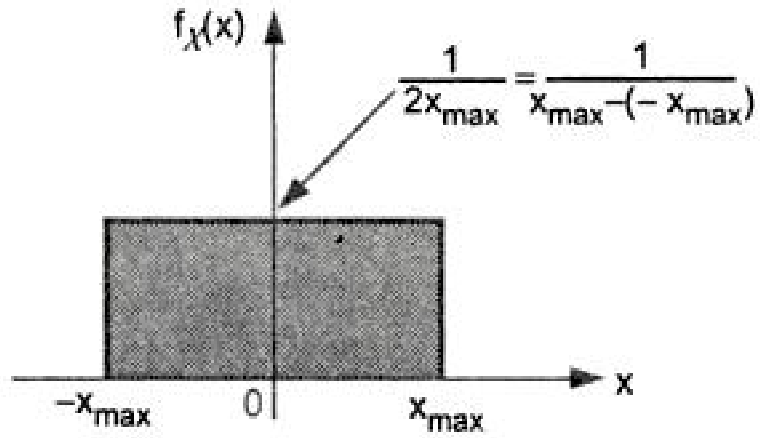


Fig. 1.8.7 PDF of a uniformly distributed random variable

The mean square value of a random variable X is given as,

$$\overline{X^2} = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx$$

Therefore mean square value of $x(t)$ will be,

$$\overline{x^2(t)} = \int_{-x_{max}}^{x_{max}} x^2(t) \cdot \frac{1}{2x_{max}}\, dx$$

$$= \frac{1}{2x_{max}} \cdot \left[\frac{x^3(t)}{3}\right]_{-x_{max}}^{x_{max}}$$

$$= \frac{x_{max}^2}{3}$$

The signal power $P = \dfrac{\overline{x^2(t)}}{R}$

Normalized signal power $P = \dfrac{\overline{x^2(t)}}{1}$ [since $R = 1$]

$$= \frac{x_{max}^2}{3}$$

Step size $\delta = \dfrac{2\,x_{max}}{q}$ By equation 1.8.16

$$\therefore \qquad x_{max} = \frac{\delta\, q}{2}$$

$$\therefore \text{Normalized signal power, } P = \frac{\left(\frac{\delta\, q}{2}\right)^2}{3} = \frac{\delta^2 q^2}{12}$$

Normalized noise power $= \dfrac{\delta^2}{12}$ By equation 1.8.29

$$\therefore \text{Signal to noise power ratio } \frac{S}{N} = \frac{\text{Normalized signal power}}{\text{Normalized noise power}}$$

$$= \frac{\delta^2 q^2 / 12}{\delta^2 / 12} = q^2$$

Since $q = 2^v$, above equation will be,

$$\frac{S}{N} = 2^{2v}$$

or

$$\left(\frac{S}{N}\right) dB = 10 \log_{10}(2^{2v})\, dB$$

$$\approx 6v$$

This is the required expression for maximum value of signal to noise ratio.

Example 1.8.9 : *Consider an audio signal comprised of the sinusoidal term* $s(t) = 3\cos(500\pi t)$

i) Find the signal to quantization noise ratio when this is quantized using 10 bit PCM.

ii) How many bits of quantization are needed to achieve a signal to quantization noise ratio of atleast 40 dB ?

Solution : Here $s(t) = 3\cos(500\,\pi t)$

That is sinusoidal signal applied to the quantizer.

i) Let us assume that peak value of cosine wave defined by $s(t)$ covers the complete range of quantizer.

i.e. $A_m = 3V$ covers complete range.

We know that signal to noise ratio for sinusoidal signal is given by

$$\left(\frac{S}{N}\right)dB = 1.8+6v$$

Here 10 bit PCM is used i.e.,

$$v = 10$$

$$\left(\frac{S}{N}\right)dB = 1.8+6\times 10 = 61.8 \text{ dB}$$

ii) For sinusoidal signal again we will use the same relation. i.e.

i.e. $$\left(\frac{S}{N}\right)dB = 1.8+6v\ dB$$

To get signal to noise ratio of at least 40 dB we can write above equation as,

$$1.8+6v \geq 40 \text{ dB}$$

$$\therefore \quad v \geq 6.36 \text{ bits} \approx 7 \text{ bits}$$

Thus at least 7 bits are required to get signal to noise ratio of 40 dB.

Example 1.8.10 : *A 7 bit PCM system employing uniform quantization has an overall signaling rate of 56 k bits per second. Calculate the signal to quantization noise ratio that would result when its input is a sine wave with peak to peak amplitude equal to 5. Calculate the dynamic range for the sine wave inputs in order that the signal to quantization noise ratio may be less than 30 dBs. What is the theoretical maximum frequency that this system can handle ?*

Solution : The number of bits in the PCM system are

$$v = 7 \text{ bits}$$

Assume that 5 V peak to peak voltage utilizes complete range of quantizer. Then we can find the signal to quantization noise ratio as,

$$\left(\frac{S}{N}\right)dB = 1.8+6v\ dB = 1.8+6\times 7$$

$$= 43.8 \text{ dB}$$

By equation 1.8.3 signaling rate is given as,

$$r = v f_s$$

Putting $r = 56 \times 10^3$ bits/second and $v = 7$ bits in above equation we get,

$$56 \times 10^3 = 7 \cdot f_s$$

$\therefore$ Sampling frequency, $f_s = 8 \times 10^3 \ Hz$

By sampling theorem, $f_s \geq 2W$

$\therefore$ Maximum frequency that can be handled is given as,

$$W \leq \frac{f_s}{2} \leq \frac{8000}{2}$$

$\therefore$ $W \leq 4000$ Hz **(Ans)**

Example 1.8.11 : *The bandwidth of TV video plus audio signal is 4.5 MHz. If the signal is converted to PCM bit stream with 1024 quantization levels, determine the number of bits/sec generated by the PCM system. Assume that the signal is sampled at the rate of 20% above nyquist rate. If above linear PCM system is converted to companded PCM, will the output bit rate change? Justify.*

Solution : The given data is,

$$W = 4.5 \text{ MHz}$$

$$q = 1024 \text{ levels}$$

The Nyquist rate is,

$$\text{Nyquist rate} = 2W = 2 \times 4.5 = 9 \text{ MHz}$$

The sampling rate is 20% above the nyquist rate. i.e.

$$\text{Sampling rate, } f_s = 1.2 \times 9 = 10.8 \text{ MHz}$$

We know that quantization levels q and number of bits v are related as,

$$q = 2^v$$

$$\therefore \quad 1024 = 2^v$$

$$\therefore \quad v = 10 \text{ bits}$$

The number of bits/sec generated by PCM system is called bit rate or signaling rate. i.e.,

$$\text{Signaling rate, r} = v f_s$$

$$= 10 \times 10.8 \times 10^6 \text{ bits/sec.}$$

$$= 108 \times 10^6 \text{ bits / sec.}$$

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shown in Fig. 1.8.9. That is nonlinear transfer characteristic means compression and expansion curves.

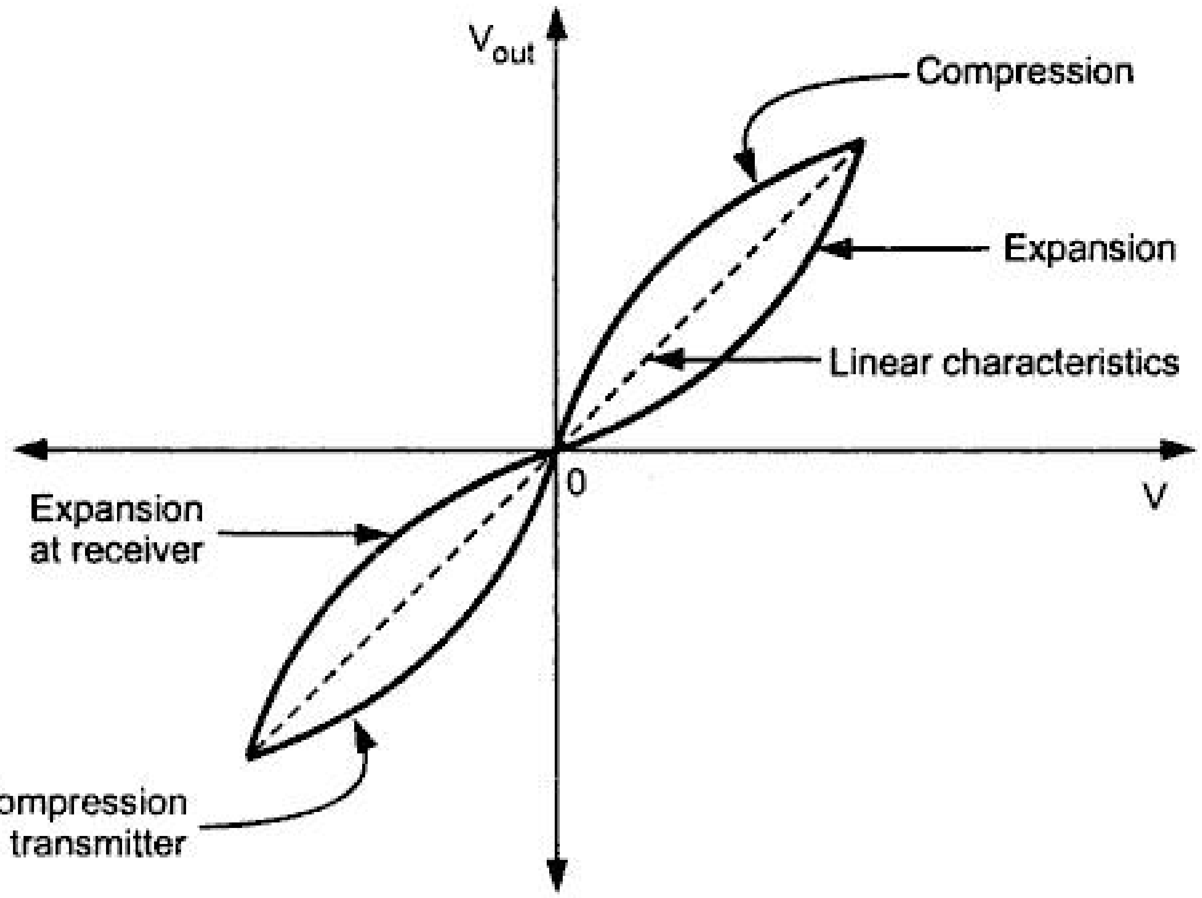

Fig. 1.8.9 Companding curves for PCM

1.8.6.4 μ - Law Companding for Speech Signals

Normally for speech and music signals a μ - law compression is used. This compression is defined by the following equation,

$$Z(x) = (\text{Sgn } x)\frac{ln(1+\mu|x|)}{ln(1+\mu)} \quad |x| \leq 1 \qquad \dots (1.8.52)$$

Fig. 1.8.10 shows the variation of signal to noise ratio with respect to signal level without companding and with companding.

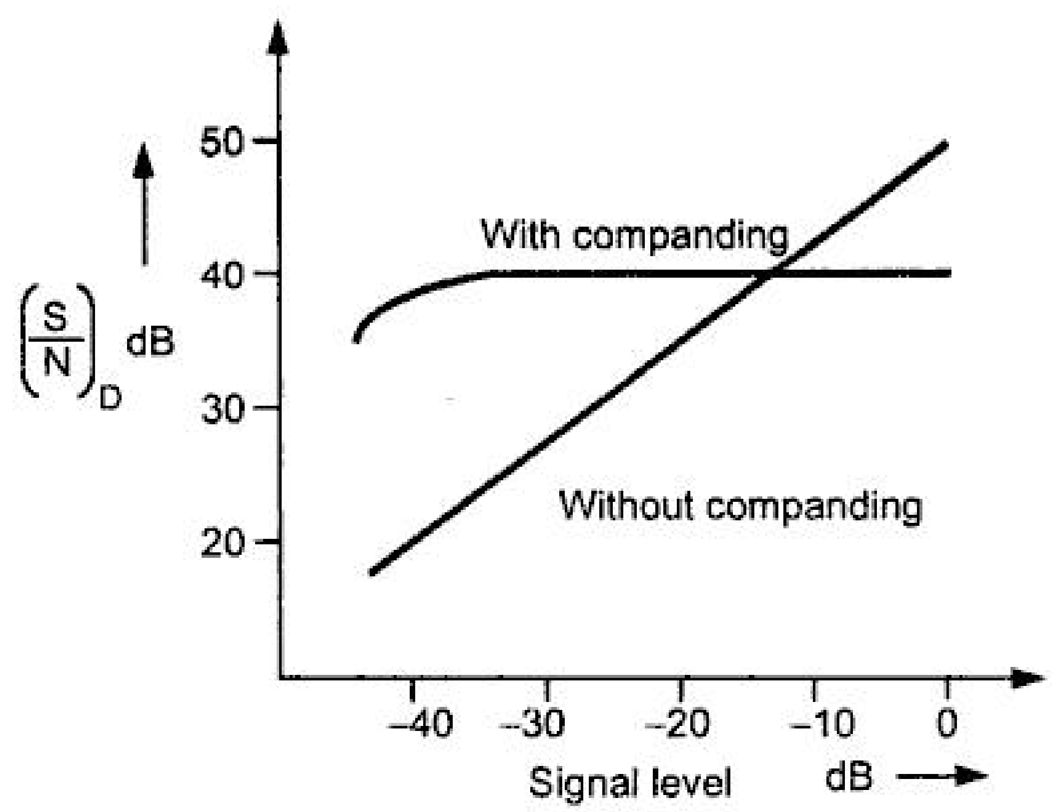

Fig. 1.8.10 PCM performance with μ - law companding

The desired signal to noise ratio is 42 dB. Hence rise in $\left(\frac{S}{N}\right)$ ratio is 42 – 30 = 12 dB

We know that $\left(\frac{S}{N}\right)$ ratio increases by 6 dB for 1 bit. Hence 2 bits are required to increase signal to noise ratio by 12 dB. Hence,

$$v = 10 + 2 = 12 \text{ bits are required.}$$

(ii) To obtain fractional increase in bandwidth

Bandwidth in PCM is given as,

$$B_T = \frac{1}{2} v f_s$$

$$\therefore \quad B_T \text{ (10 bits)} = \frac{1}{2} \times 10 \times f_s = 5 f_s$$

and $$B_T \text{ (12 bits)} = \frac{1}{2} \times 12 \times f_s = 6 f_s$$

$$\therefore \text{ Fractional increase in } B_T = \frac{6f_s - 5f_s}{5f_s} \times 100\% = 20\ \%$$

Example 1.8.15 : *A telephone signal with cutoff frequency of 4 kHz is digitized into 8 bit PCM, sampled at Nyquist rate. Calculate baseband transmission bandwidth and quantization $\frac{S}{N}$ ratio.*

Solution : Given data is,

$$W = 4 \text{ kHz}$$

$$v = 8 \text{ bits}$$

From equation 1.8.6 transmission bandwidth is given as,

$$B_T = vW = 4k \times 8 = 32 \text{ kHz}$$

Telephone signal is nonsinusoidal signal. Its signal to quantization noise ratio is given by equation 1.8.38 as,

$$\frac{S}{N} = 4.8 + 6v$$

$$= 4.8 + 6 \times 8 = 52.8 \text{ dB.}$$

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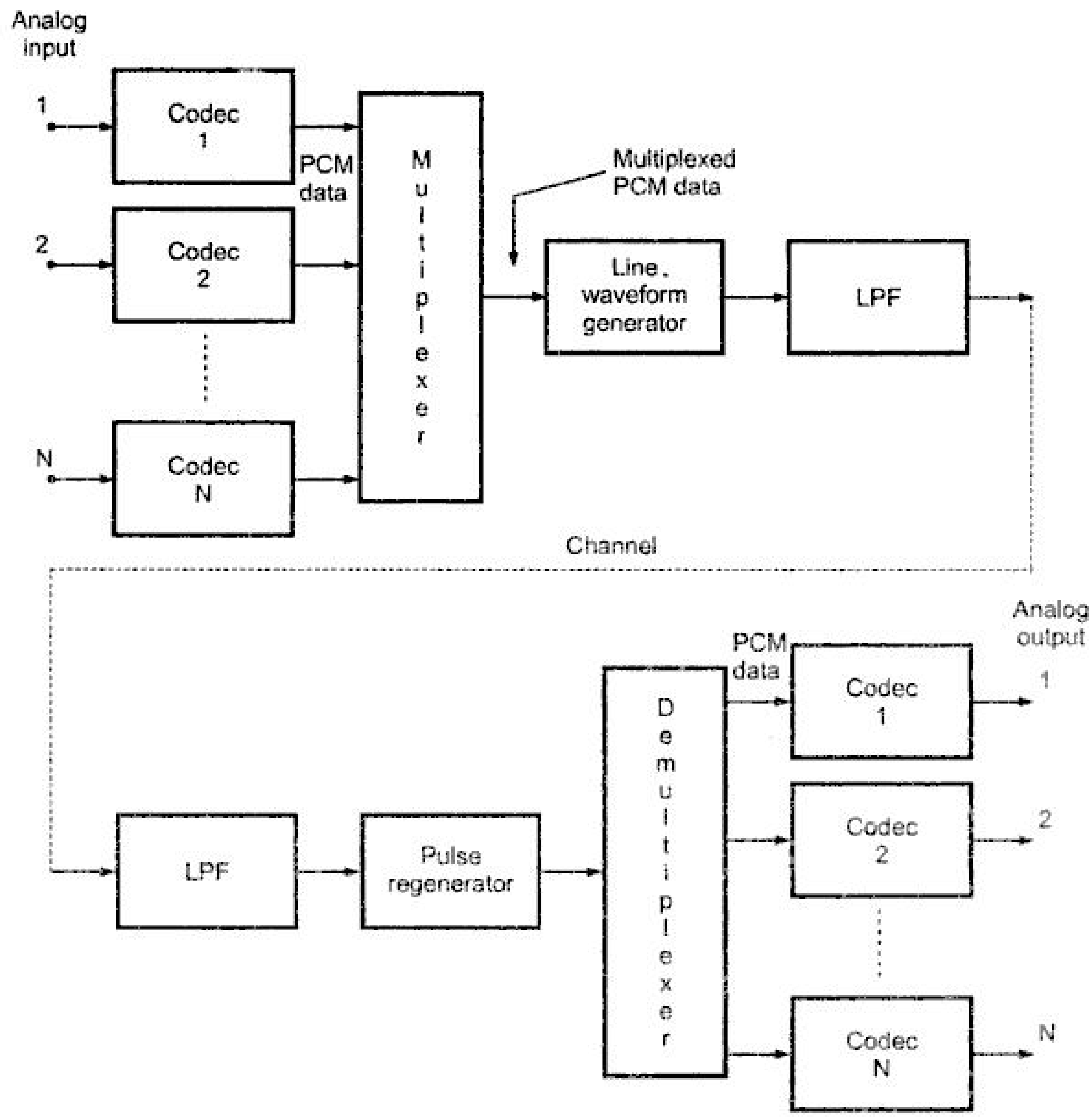

Fig. 1.9.2 TDM/PCM system

1.9.3.2 Multiple Channel Frame Alignment For TDM / PCM (T_1 System)

The multiple channel alignment is very important in TDM/PCM system. Fig. 1.9.3 shows the TDM frame format of most widely used T1 system.

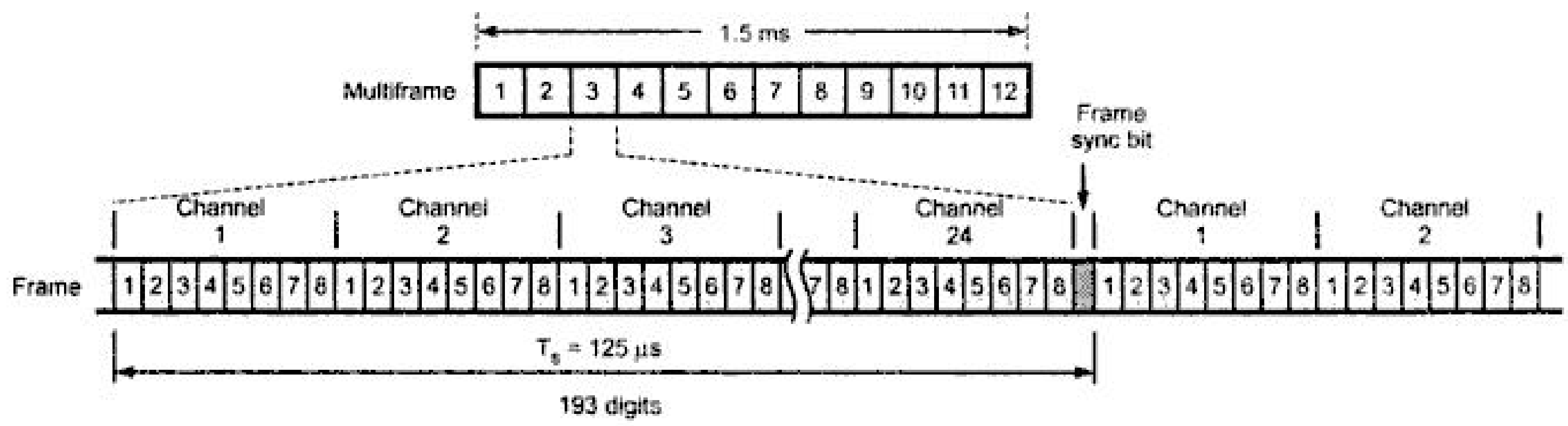

Fig. 1.9.3 Multiple channel frame alignment in T1 system

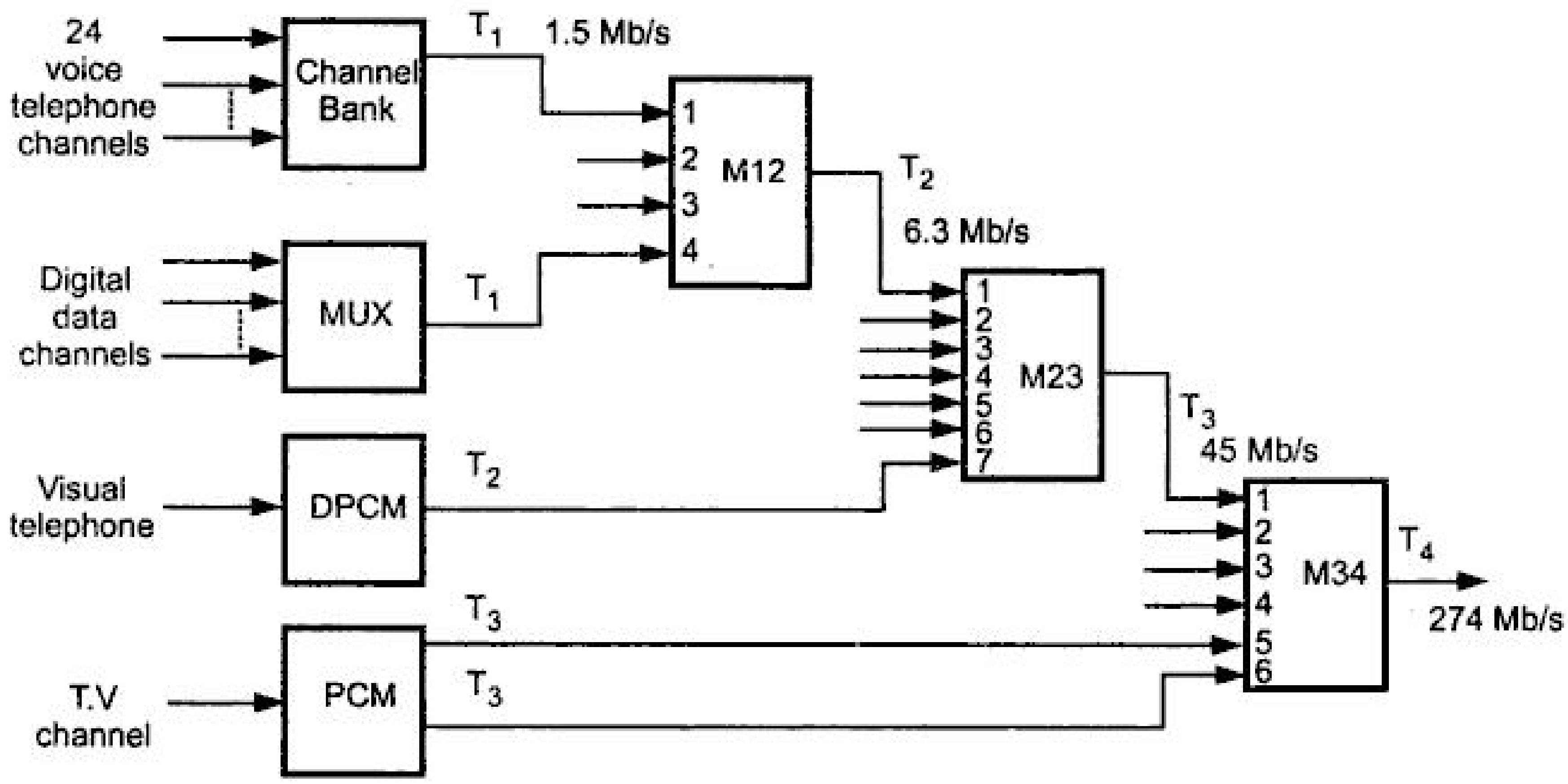

Fig. 1.9.4 Digital multiplexing of voice telephone channels, digital data, TV etc. for AT & T standard

Theory Questions

1. *Which are the types of digital multiplexers?*
2. *Explain the frame structure of T1 system in detail.*
3. *With the help of block diagram explain PCM/TDM system.*

1.10 Virtues, Limitation and Modifications of PCM

Advantages of PCM

(i) Effect of channel noise and interference is reduced.

(ii) PCM permits regeneration of pulses along the transmission path. This reduces noise interference.

(iii) The bandwidth and signal to noise ratio are related by exponential law.

(iv) Multiplexing of various PCM signals is easily possible.

(v) Encryption or decryption can be easily incorporated for security purpose.

Limitations of PCM

(i) PCM systems are complex compared to analog pulse modulation methods.

(ii) The channel bandwidth is also increased because of digital coding of analog pulses.

2

Delta Modulation

We have seen in PCM that, it transmits all the bits which are used to code the sample. Hence signaling rate and transmission channel bandwidth are large in PCM. To overcome this problem Delta Modulation is used.

2.1 Delta Modulation

2.1.1 Operating Principle of DM

Delta modulation transmits only one bit per sample. That is the present sample value is compared with the previous sample value and the indication,whether the amplitude is increased or decreased is sent. Input signal $x(t)$ is approximated to step signal by the delta modulator. This step size is fixed. The difference between the input signal $x(t)$ and staircase approximated signal confined to two levels, i.e. $+\delta$ and $-\delta$. If the difference is positive, then approximated signal is increased by one step i.e. 'δ'. If the difference is negative, then approximated signal is reduced by 'δ'. When the step is reduced, '0' is transmitted and if the step is increased, '1' is transmitted. Thus for each sample, only one binary bit is transmitted. Fig. 2.1.1 shows the analog signal $x(t)$ and its staircase approximated signal by the delta modulator.

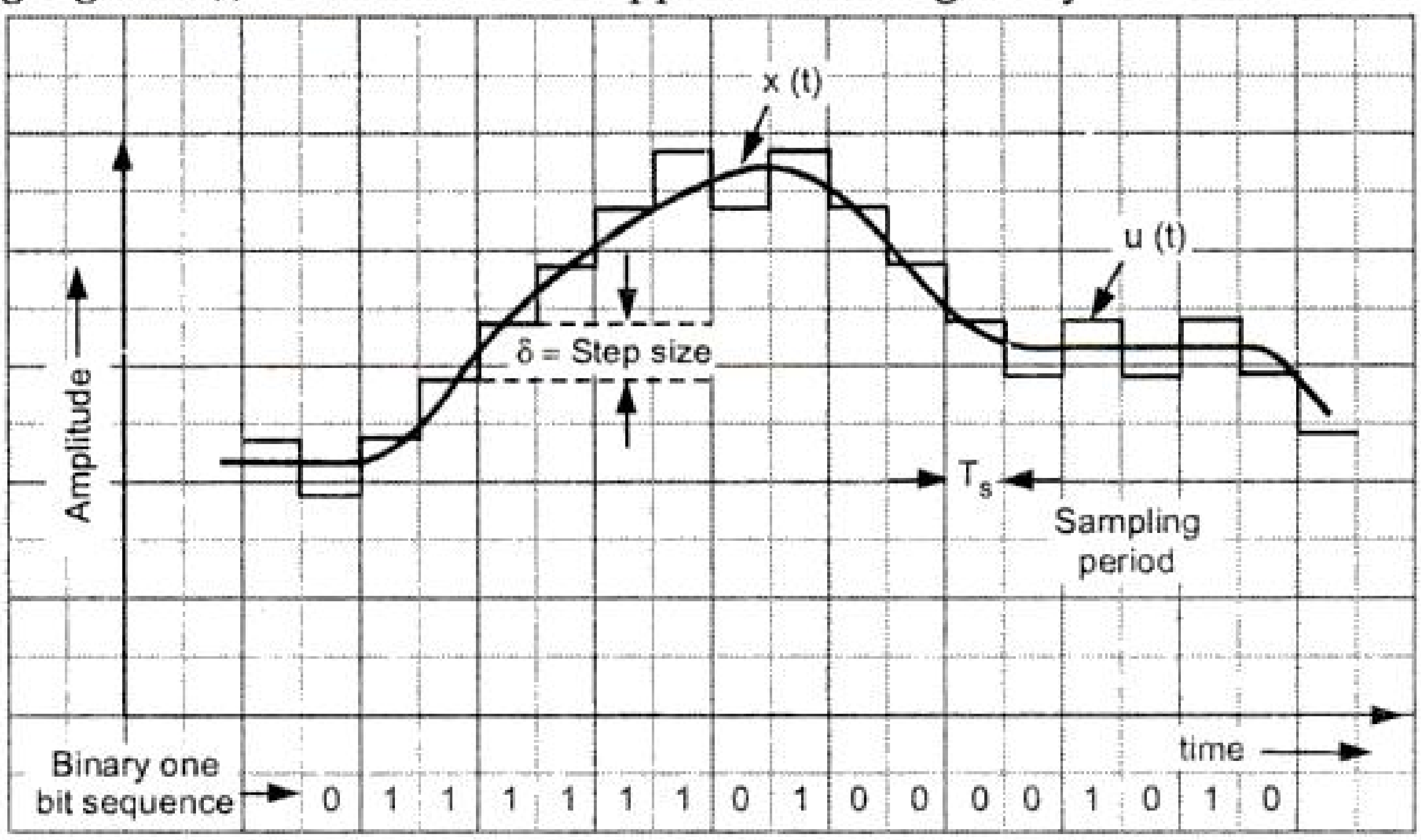

Fig. 2.1.1 Delta modulation waveform

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signal is changed by large amount (δ) because of large step size. Fig. 2.2.1 shows that when the input signal is almost flat, the staircase signal $u(t)$ keeps on oscillating by $\pm\delta$ around the signal. The error between the input and approximated signal is called *granular noise.* The solution to this problem is to make step size small.

Thus large step size is required to accommodate wide dynamic range of the input signal (to reduce slope overload distortion) and small steps are required to reduce granular noise. Adaptive delta modulation is the modification to overcome these errors.

Example 2.2.1 : *Using predictability theory, prove that transmission of encoded error signal (rather than encoded signal itself is sufficient for reasonable reconstruction of signal. With the help of block schematic suggest any one technique to transmit and receive encoded errors. What are the limitations and advantages of such techniques with reference to linear or uniform PCM ?*

Solution : Here the technique that uses predictibility theory is basically delta modulation. The output of the accumulator in DM transmitter is given by equation 2.1.5 as,

$$u(nT_s) = u[(n-1)T_s] + b(nT_s) \qquad \dots (2.2.1)$$

Here $$b(nT_s) = \pm\delta \text{ or } \delta \operatorname{sgn}[e(nT_s)]$$

Thus $b(nT_s)$ basically represents error signal. Sign of step size 'δ' depends upon whether $e(nT_s)$ is positive or negative.

Now we will show that the signal can be reconstructed only with the help of encoded error signal, i.e. $b(nT_s)$. The accumulator of Fig. 2.1.2(b) acts as a delta modulation receiver. $u(nT_s)$ is the output of accumulator. For simplicity let us drop T_s in equation 2.2.1 Then we get,

$$\text{u(n)} = u(n-1) + b(n) \qquad \dots (2.2.2)$$

Observe that this is recursive equation. Hence $u(n-1)$ can be calculated as,

$$u(n-1) = u(n-2) + b(n-1) \qquad \dots (2.2.3)$$

Hence equation 2.2.2 becomes,

$$u(n) = u(n-2) + b(n-1) + b(n) \qquad \dots (2.2.4)$$

From equation 2.2.3 we can calculate $u(n-2)$ as,

$$u(n-2) = u(n-3) + b(n-2)$$

Hence equation 2.2.4 becomes,

$$u(n) = u(n-3) + b(n-2) + b(n-1) + b(n)$$

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$\therefore$ Slope overload will not occur if,

$$A_m \leq \frac{\delta}{2\pi f_m T_s} \qquad \text{... (2.3.4)}$$

The maximum frequency in the signal is,

$$W = 3 \text{ kHz}$$

$$\therefore \quad \text{Nyquist rate} = 2W = 2 \times 3 \text{ kHz} = 6 \text{ kHz}$$

$$\text{Sampling frequency} = 5 \text{ times Nyquist rate}$$

$$f_s = 5 \times 6 \text{ kHz}$$

$$= 30 \text{ kHz}$$

$$\text{Sampling interval } T_s = \frac{1}{f_s} = \frac{1}{30 \times 10^3}$$

$$\text{Step size} \quad \delta = 250\, mV = 250 \times 10^{-3}$$

$$= 0.25\, V$$

$$\text{Given that} \quad f_m = 2 \text{ kHz} = 2 \times 10^3 \text{ Hz}$$

$\therefore$ Putting these values in equation 2.3.4.

$$A_m \leq \frac{0.25}{2\pi \times 2 \times 10^3 \times \frac{1}{30 \times 10^3}}$$

$$\therefore \quad A_m \leq 0.6 \text{ volts}$$

Example 2.3.3 : *With reference to delta modulation System shown in Fig. 2.3.3 show that the optimum step size*

$$k_{opt} = \frac{2\pi A}{f_s / f_m}$$

where A is amplitude of the sine wave m(t)

f_s is the sampling rate

f_m is the frequency of the sine wave.

For k = 4 mV and k = 60 mV, does the slope overload occurs ? If so, in which case? Given, m(t) = 0.1 sin $(2\pi \times 10^3 t)$

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Putting for A_m from equation 2.3.6,

$$P = \frac{\delta^2}{8\pi^2 f_m^2 T_s^2} \quad \text{... (2.3.7)}$$

This is an expression for signal power in delta modulation.

(ii) To obtain noise power

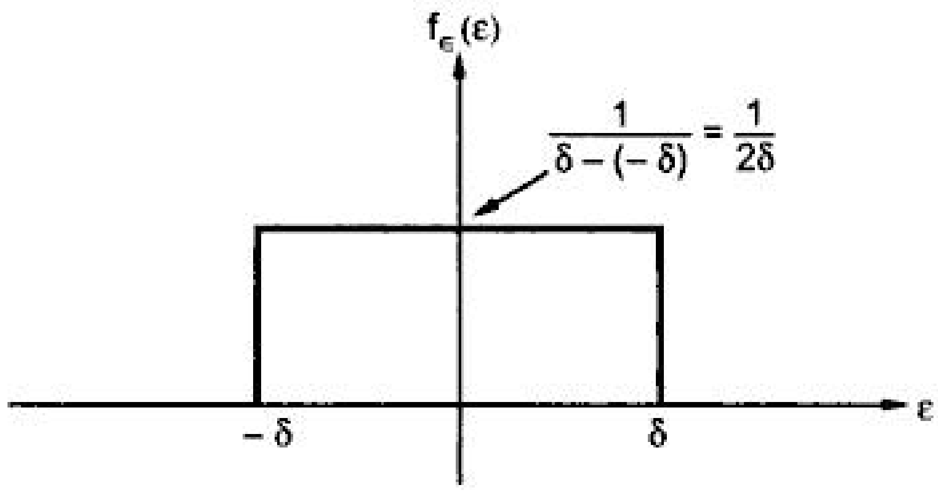

Fig. 2.3.4 Uniform distribution of quantization error

We know that the maximum quantization error in delta modulation is equal to step size 'δ'. Let the quantization error be uniformly distributed over an interval $[-\delta, \delta]$. This is shown in Fig. 2.3.4 From this figure the PDF of quantization error can be expressed as,

$$f_\epsilon(\varepsilon) = \begin{cases} 0 & for \quad \varepsilon < \delta \\ \frac{1}{2\delta} & for \quad -\delta < \varepsilon < \delta \\ 0 & for \quad \varepsilon > \delta \end{cases} \quad \text{....(2.3.8)}$$

The noise power is given as,

$$\text{Noise power} = \frac{V_{noise}^2}{R}$$

Here V_{noise}^2 is the mean square value of noise voltage. Since noise is defined by random variable 'ε' and PDF $f_\epsilon(\varepsilon)$, its mean square value is given as,

$$\text{mean square value} = E[\varepsilon^2] = \overline{\varepsilon^2}$$

mean square value is given as,

$$E[\varepsilon^2] = \int_{-\infty}^{\infty} \varepsilon^2 f_\epsilon(\varepsilon)\, d\varepsilon$$

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$$\therefore \qquad T_s = \frac{1}{f_s} = \frac{1}{200 \times 10^3} \text{ sec.}$$

(i) To obtain step size

From equation 2.3.4 we have,

$$A_m \le \frac{\delta}{2\pi f_m T_s}$$

Under this condition slope overload will not occur. From above equation step size will be,

$$\delta \ge 2\pi f_m T_s A_m$$

Putting values in above equation,

$$\delta \ge 2\pi \times 10{,}000 \times \frac{1}{200 \times 10^3} \times 0.5$$

$$\ge 0.157 \text{ V}$$

Thus the step size greater than 157 mV will prevent the slope overload.

(ii) To obtain signal to noise ratio

Signal to noise ratio of delta modulation system is given by equation 2.3.12 as,

$$\frac{S}{N} = \frac{3}{8\pi^2 W f_m^2 T_s^3}$$

This is post filtered signal to noise ratio. In this example value of 'W' is not given. Hence we will calculate signal to noise ratio from equation 2.3.7 and equation 2.3.10 as,

$$\frac{S}{N} = \frac{\dfrac{\delta^2}{8\pi^2 f_m^2 T_s^2}}{\dfrac{\delta^2}{3}}$$

$$= \frac{3}{8\pi^2 f_m^2 T_s^2}$$

Putting values in above equation.

$$\frac{S}{N} = \frac{3}{8\pi^2 \times (10{,}000)^2 \times \dfrac{1}{\left(200 \times 10^3\right)^2}}$$

$$= 15.2 = 11.8 \text{ dB.}$$

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$$\overline{X^2} = \int_{-1}^{1} x^2 f_X(x)dx = \int_{-1}^{1} x^2 \cdot \frac{1}{2} dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{3}$$

Normalized signal power $= \frac{\overline{X^2}}{R} = \overline{X^2}$ with R = 1

$$= \frac{1}{3} \text{ W}$$

Hence signal to noise ratio becomes,

$$\frac{S}{N} = \frac{\text{Signal power}}{\text{Noise power}} = \frac{\frac{1}{3}}{2.15 \times 10^{-3}} = 155$$

$$\text{or } \left(\frac{S}{N}\right)_{dB} = 10 \log_{10} 155 = 21.9 \text{ dB}$$

Theory Questions

1. *Explain delta modulation in detail suitable diagram. Explain ADM and compare its performance with DM.*
2. *What is slope overload distortion and granular noise in delta modulation and how it is removed in ADM ?*

Unsolved Example

1. *What is the maximum power that may be transmitted without slope overload distortion ?*

[Ans. : $\frac{\delta^2}{8\pi^2 f_m^2 T_s^2}$**]**

2.4 Comparison of Digital Pulse Modulation Methods

Table 2.4.1 shows the comparison of PCM, Differential PCM, Delta Modulation and Adaptive Delta Modulation. The comparison is done on the basis of various parameters like transmission bandwidth, quantization error, number of transmitter bits per sample etc.

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1. **Message source** : It emits the symbol at the rate of T seconds.
2. **Encoder** : It is signal transmission encoder. It produces the vector s_i made up of 'N' real elements. The vector s_i is unique for each set of 'M' symbols.
3. **Modulator** : It constructs the modulated carrier signal $s_i(t)$ of duration 'T' seconds for every symbol m_i. The signal $s_i(t)$ is energy signal.
4. **Channel** : The modulated signal $s_i(t)$ is transmitted over the communication channel.
   - The channel is assumed to be linear and of enough bandwidth to accommodate the signal $s_i(t)$.
   - The channel noise is white Gaussian of zero mean and psd of $\frac{N_0}{2}$.
5. **Detector** : It demodulates the received signal and obtains an estimate of the signal vector.
6. **Decoder** : The decoder obtains the estimate of symbol back from the signal vector. Here note that the detector and decoder combinely perform the reception of the transmitted signal. The effect of channel noise is minimized and correct estimate of symbol $\hat{m}$ is obtained.

3.2 Binary Phase Shift Keying (BPSK)

3.2.1 Principle of BPSK

- In binary phase shift keying (BPSK), binary symbol '1' and '0' modulate the phase of the carrier. Let the carrier be,

$$s(t) = A \cos(2\pi f_0 t) \qquad \text{... (3.2.1)}$$

'A' represents peak value of sinusoidal carrier. In the standard 1Ω load register, the power dissipated will be,

$$P = \frac{1}{2} A^2$$

$$\therefore \quad A = \sqrt{2P} \qquad \text{... (3.2.2)}$$

- When the symbol is changed, then the phase of the carrier is changed by 180 degrees (π radians).
- Consider for example,

$$\text{Symbol '1'} \Rightarrow s_1(t) = \sqrt{2P} \cos(2\pi f_0 t) \qquad \text{... (3.2.3)}$$

$$= b(t)\sqrt{\frac{P}{2}}\,[1 + \cos 2(2\pi f_0\, t + \theta)] \qquad \text{... (3.2.8)}$$

6) **Bit synchronizer and integrator :** The above signal is then applied to the bit synchronizer and integrator. The integrator integrates the signal over one bit period. The bit synchronizer takes care of starting and ending times of a bit.

- At the end of bit duration T_b, the bit synchronizer closes switch S_2 temperorily. This connects the output of an integrator to the decision device. It is equivalent to sampling the output of integrator.
- The synchronizer then opens switch S_2 and switch S_1 is closed temperorily. This resets the integrator voltage to zero. The integrator then integrates next bit.
- Let us assume that one bit period 'T_b' contains integral number of cycles of the carrier. That is the phase change occurs in the carrier only at zero crossing. This is shown in Fig. 3.2.1 (c). Thus BPSK waveform has full cycles of sinusoidal carrier.

To show that output of integrator depends upon transmitted bit

- In the k^{th} bit interval we can write output signal as,

$$s_o(k\,T_b) = b(k\,T_b)\sqrt{\frac{P}{2}} \int_{(k-1)\,T_b}^{k\,T_b} [1 + \cos 2(2\pi f_0\, t + \theta)]\; dt$$

from equation 3.2.8

The above equation gives the output of an interval for k^{th} bit. Therefore integration is performed from $(k-1)\,T_b$ to $k\,T_b$. Here T_b is the one bit period.

- We can write the above equation as,

$$s_o(k\,T_b) = b(k\,T_b)\sqrt{\frac{P}{2}} \left[\int_{(k-1)\,T_b}^{k\,T_b} 1\, dt + \int_{(k-1)\,T_b}^{k\,T_b} \cos 2(2\pi f_0\, t + \theta)\, dt \right]$$

Here $\int_{(k-1)\,T_b}^{k\,T_b} \cos 2(2\pi f_0\, t + \theta)\, dt = 0$, because average value of sinusoidal waveform is zero if integration is performed over full cycles. Therefore we can write above equation as,

$$s_o(k\,T_b) = b(k\,T_b)\sqrt{\frac{P}{2}} \int_{(k-1)\,T_b}^{k\,T_b} 1\, dt$$

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Interchannel Interference and ISI :

- Let's assume that BPSK signals are multiplexed with the help of different carrier frequencies for different baseband signals. Then at any frequency, the spectral components due to all the multiplexed channels will be present. This is because $S(f)$ as well as $S_{BPSK}(f)$ of every channel extends over all the frequency range.
- Therefore a BPSK receiver tuned to a particular carrier frequency will also receive frequency components due to other channels. This will make interference with the required channel signals and error probability will increase. This result is called *Interchannel Interference.*
- To avoid interchannel interference, the BPSK signal is passed through a filter.This filter attenuates the side lobes and passes only main lobe. Since side lobes are attenuated to high level, the interference is reduced. Because of this filtering the phase distortion takes place in the bipolar NRZ signal, i.e. $b(t)$. Therefore the individual bits (symbols) mix with adjacent bits (symbols) in the same channel. This effect is called *intersymbol interference* or ISI.
- The effect of ISI can be reduced to some extent by using equalizers at the receiver. Those equalizers have the reverse effect to that filter's adverse effects. Normally equalizers are also filter structures.

3.2.5 Geometrical Representation of BPSK Signals

We know that BPSK signal carries the information about two symbols. Those are symbol '1' and symbol '0'. We can represent BPSK signal geometrically to show those two symbols.

(i) From equation 3.2.6 we know that BPSK signal is given as,

$$s(t) = b(t)\sqrt{2P}\cos(2\pi f_0 t) \qquad \text{... (3.2.15)}$$

(ii) Let's rearrange the above equation as,

$$s(t) = b(t)\sqrt{P\,T_b} \cdot \sqrt{\frac{2}{T_b}}\cos(2\pi f_0 t) \qquad \text{... (3.2.16)}$$

(iii) Let $\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_0 t)$ represents an orthonormal carrier signal. Equation 3.2.14 also gives equation for carrier. It is slightly different than $\phi_1(t)$ defined here. Then we can write equation 3.2.16 as,

$$s(t) = b(t)\sqrt{P\,T_b}\,\phi_1(t) \qquad \text{... (3.2.17)}$$

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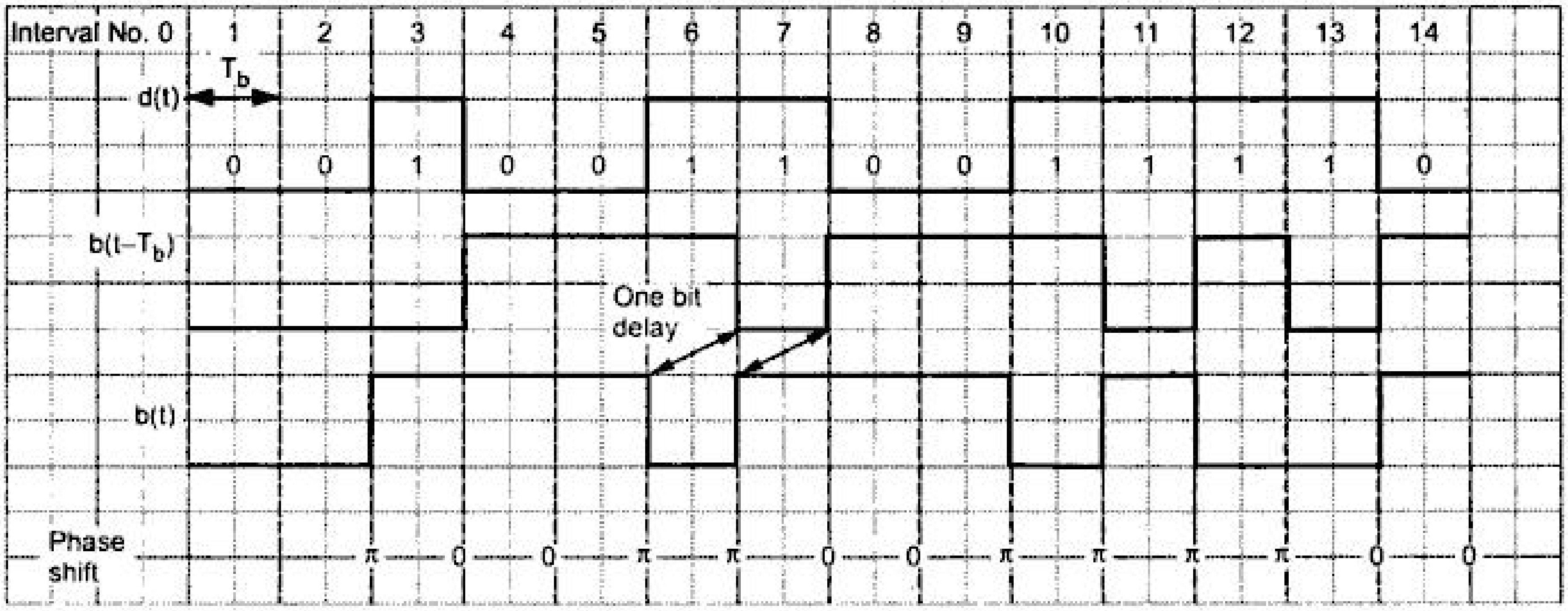

Fig. 3.3.2 DPSK waveforms

From the waveforms of Fig. 3.3.2 it is clear that $b(t-T_b)$ is the delayed version of $b(t)$ by one bit period T_b. The exclusive OR operation is satisfied in any interval i.e. in any interval $b(t)$ is given as,

$$b(t) = d(t) \oplus b(t-T_b) \qquad \text{... (3.3.1)}$$

While drawing the waveforms the value of $b(t-T_b)$ is not known initially in interval no. 1. Therefore it is assumed to be zero and then waveforms are drawn.

Important conclusions from the waveforms

1. Output sequence $b(t)$ changes level at the beginning of each interval in which $d(t)=1$ and it does not changes level when $d(t)=0$. Observe that $d(3)=1$, hence level of $b(3)$ is changed at the beginning of interval 3. Similarly in intervals 10, 11, 12 and 13 $d(t)=1$. Hence $b(t)$ is changed at the starting of these intervals. In interval 8 and 9 $d(t)=0$. Hence $b(t)$ is not changed in these intervals.

2. When $d(t)=0$, $\quad b(t)=b(t-T_b)$ and
   When $d(t)=1$, $\quad b(t)=\overline{b(t-T_b)}$

3. In interval no. 1. we has assumed $b(t-T_b)=0$ and we obtained the waveform as shown in Fig. 3.3.2. If we assume $b(t-T_b)=1$ in interval no. 1, then the waveform of $b(t)$ will be inverted. But still $b(t)$ changes the level at the beginning each interval in which $d(t)=1$.

4. The sequence $b(t)$ modulates sinusoidal carrier.

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If $$s_o(kT_b) = \begin{cases} -PT_b, \text{ then } d(t)=1 & \text{and} \\ +PT_b, \text{ then } d(t)=0 \end{cases}$$

3.3.2 Bandwidth of DPSK Signal

We know that one previous bit is used to decide the phase shift of next bit. Change in $b(t)$ occurs only if input bit is at level '1'. No change occurs if input bit is at level '0'.

Since one previous bit is always used to define the phase shift in next bit, the symbol can be said to have two bits. Therefore one symbol duration (T) is equivalent to two bits duration ($2T_b$).

i.e.

Symbol duration $T = 2T_b$ ... (3.3.11)

Bandwidth is given as,

$$BW = \frac{2}{T}$$

$$= \frac{1}{T_b}$$

or $$\boxed{BW = f_b}$$ ... (3.3.12)

Thus the minimum bandwidth in DPSK is equal to f_b ; i.e. maximum baseband signal frequency.

3.3.3 Advantages and Disadvantages of DPSK

DPSK has some advantages over BPSK, but at the same time it has some drawbacks.

Advantages :

1) DPSK does not need carrier at its receiver. Hence the complicated circuitry for generation of local carrier is avoided.

2) The bandwidth requirement of DPSK is reduced compared to that of BPSK.

Disadvantages :

1) The probability of error or bit error rate of DPSK is higher than that of BPSK.

2) Since DPSK uses two successive bits for its reception, error in the first bit creates error in the second bit. Hence error propagation in DPSK is more. Whereas in PSK single bit can go in error since detection of each bit is independent.

Step 2 : Demultiplexing into odd and even numbered sequences

The demultiplexer divides $b(t)$ into two separate bit streams of the odd numbered and even numbered bits. $b_e(t)$ represents even numbered sequence and $b_o(t)$ represents odd numbered sequence. The symbol duration of both of these odd and even numbered sequences is $2T_b$. Thus every symbol contains two bits. Fig. 3.4.2 (b) and (c) shows the waveforms of $b_e(t)$ and $b_o(t)$.

Observe that the first even bit occurs after the first odd bit. Therefore even numbered bit sequence $b_e(t)$ starts with the delay of one bit period due to first odd bit. Thus first symbol of $b_e(t)$ is delayed by one bit period 'T_b' with respect to first symbol of $b_o(t)$. This delay of T_b is called offset. Hence the name offset QPSK is given. This shows that the change in levels of $b_e(t)$ and $b_o(t)$ cannot occur at the same time because of offset or staggering.

Step 3 : Modulation of quadrature carriers

The bit stream $b_e(t)$ modulates carrier $\sqrt{P_s}\cos(2\pi f_0 t)$ and $b_o(t)$ modulates $\sqrt{P_s}\sin(2\pi f_0 t)$. These modulators are balanced modulator. The two carriers $\sqrt{P_s}\cos(2\pi f_0 t)$ and $\sqrt{P_s}\sin(2\pi f_0 t)$ are shown in Fig. 3.4.2 (d) and (e). These carriers are also called quadrature carriers. The two modulated signals are,

$$s_e(t) = b_e(t)\sqrt{P_s}\sin(2\pi f_0 t) \quad \text{... (3.4.1)}$$

and

$$s_o(t) = b_o(t)\sqrt{P_s}\cos(2\pi f_0 t) \quad \text{... (3.4.2)}$$

Thus $s_e(t)$ and $s_o(t)$ are basically BPSK signals and they are similar to equation 3.2.3 and equation 3.2.5. The only difference is that $T = 2T_b$ here. The value of $b_e(t)$ and $b_o(t)$ will be $+1V$ or $-1V$. Fig. 3.4.2 (f) and (g) shows the waveforms of $s_e(t)$ and $s_o(t)$.

Step 4 : Addition of modulated carriers

The adder of Fig. 3.4.1 adds these two signals $b_e(t)$ and $b_o(t)$. The output of the adder is OQPSK signal and it is given as,

$$\begin{aligned} s(t) &= s_o(t) + s_e(t) \\ &= b_o(t)\sqrt{P_s}\cos(2\pi f_0 t) + b_e(t)\sqrt{P_s}\sin(2\pi f_0 t) \quad \text{... (3.4.3)} \end{aligned}$$

Step 5 : QPSK signal and phase shift

Fig. 3.4.2 (h) shows the QPSK signal represented by above equation. In QPSK signal of Fig. 3.4.2 (h), if there is any phase change, it occurs at minimum duration of T_b. This is because the two signals $s_e(t)$ and $s_o(t)$ have an offset of 'T_b'. Because of this offset, the phase shift in QPSK signal is $\frac{\pi}{2}$. It is clear from the waveforms of Fig. 3.4.2 that $b_e(t)$ and $b_o(t)$ cannot change at the same time because of offset between them. Fig. 3.4.3 shows the phasor diagram of QPSK signal of equation 3.4.2.

To show that output of integrator depends upon respective bit sequence.

- Let's consider the product signal at the output of upper multiplier.

$$s(t)\sin(2\pi f_0 t) = b_o(t)\sqrt{P_s}\cos(2\pi f_0 t)\sin(2\pi f_0 t) + b_e(t)\sqrt{P_s}\sin^2(2\pi f_0 t) \quad \text{... (3.4.4)}$$

- This signal is integrated by the upper integrator in Fig. 3.4.4.

$$\therefore \int_{(2k-1)T_b}^{(2k+1)T_b} s(t)\sin(2\pi f_0 t)\,dt = b_o(t)\sqrt{P_s}\int_{(2k-1)T_b}^{(2k+1)T_b}\cos(2\pi f_0 t)\sin(2\pi f_0 t)\,dt$$

$$+ b_e(t)\sqrt{P_s}\int_{(2k-1)T_b}^{(2k+1)T_b}\sin^2(2\pi f_0 t)\,dt$$

Since $\frac{1}{2}\sin(2x) = \sin x\cdot\cos x$

and $\sin^2(x) = \frac{1}{2}[1-\cos(2x)]$

- Using the above two trigonometric identities in the above equation,

$$\int_{(2k-1)T_b}^{(2k+1)T_b} s(t)\sin(2\pi f_0 t)\,dt = \frac{b_o(t)\sqrt{P_s}}{2}\int_{(2k-1)T_b}^{(2k+1)T_b}\sin 4\pi f_0 t\,dt + \frac{b_e(t)\sqrt{P_s}}{2}\int_{(2k-1)T_b}^{(2k+1)T_b} 1\cdot dt$$

$$-\frac{b_e(t)\sqrt{P_s}}{2}\int_{(2k-1)T_b}^{(2k+1)T_b}\cos 4\pi f_0 t\,dt$$

- In the above equation, the first and third integration terms involves integration of sinusoidal carriers over two bit period. They have full (integral number of) cycles over two bit period and hence integration will be zero.

$$\int_{(2k-1)T_b}^{(2k+1)T_b} s(t)\sin(2\pi f_0 t)\,dt = \frac{b_e(t)\sqrt{P_s}}{2}[t]_{(2k-1)T_b}^{(2k+1)T_b}$$

$$= \frac{b_e(t)\sqrt{P_s}}{2}\times 2T_b$$

$$= b_e(t)\sqrt{P_s}\,T_b \quad \text{... (3.4.5)}$$

- Thus the upper integrator responds to even sequence only. Similarly we can obtain the output of lower integrator as $b_o(t)\sqrt{P_s}\,T_b$.

Thus the length of each signal point from origin is $\sqrt{E_s}$.

- We know that $b_e(t)$ and $b_o(t)$ represent two successive bits. There is an offset of 'T_b' between $b_e(t)$ and $b_o(t)$. Therefore $b_e(t)$ and $b_o(t)$ both cannot change their levels simultaneously. Therefore either $b_e(t)$ *or* $b_o(t)$ can change at a time.
- Let's say that $b_e(t) = b_o(t) = 1$ representing signal point 'A' in Fig. 3.4.6. In the next bit interval if $b_o(t) = -1$, then signal point will be 'D'. Otherwise if $b_e(t)$ changes its level (i.e. $b_e(t) = -1$), then next signal point will be 'B'. Thus from signal point 'A', then next signal points will be either 'D' or 'B'.

Distance between signal points :

Normally the ability to determine a bit without error is measured by the distance between two nearest possible signal points in the signal space. Such points differed in a single bit. For example signal points 'A' and 'B' are two nearest points since they differ by a single bit $b_e(t)$. As 'A' and 'B' becomes closer to each other, the possibility of error increases. Hence this distance should be as large as possible. This distance is denoted by 'd'. In Fig. 3.4.6, the distance between signal points 'A' and 'B' is given as,

$$d^2 = (\sqrt{E_s})^2 + (\sqrt{E_s})^2$$

$$\therefore \quad d = \sqrt{2E_s} \qquad \text{... (3.4.18)}$$

$$\text{or} \quad d = 2\sqrt{P_s T_b} = 2\sqrt{E_b} \qquad \text{... (3.4.19)}$$

Compare this distance with the distance of BPSK signals given by equation 3.2.20. This shows that the distance for QPSK is the same as that for BPSK. Since this distance represents noise immunity of the system, it shows that noise immunities of BPSK and QPSK are same.

3.4.3 Spectrum of QPSK Signal

Step 1 : PSD of NRZ waveform

The input sequence $b(t)$ is of bit duration T_b. It is NRZ bipolar waveform. In section 3.2.4 we have obtained the power spectral density of such waveform as,

$$S(f) = V_b^2 T_b \left[\frac{\sin(\pi f T_b)}{\pi f T_b}\right]^2 \qquad \text{from equation 3.2.12}$$

and $V_b = \sqrt{P_s}$, then above equation becomes,

$$S(f) = P_s T_b \left[\frac{\sin(\pi f T_b)}{\pi f T_b}\right]^2 \qquad \text{... (3.4.20)}$$

The above equation gives power spectral density of signal $b(t)$.

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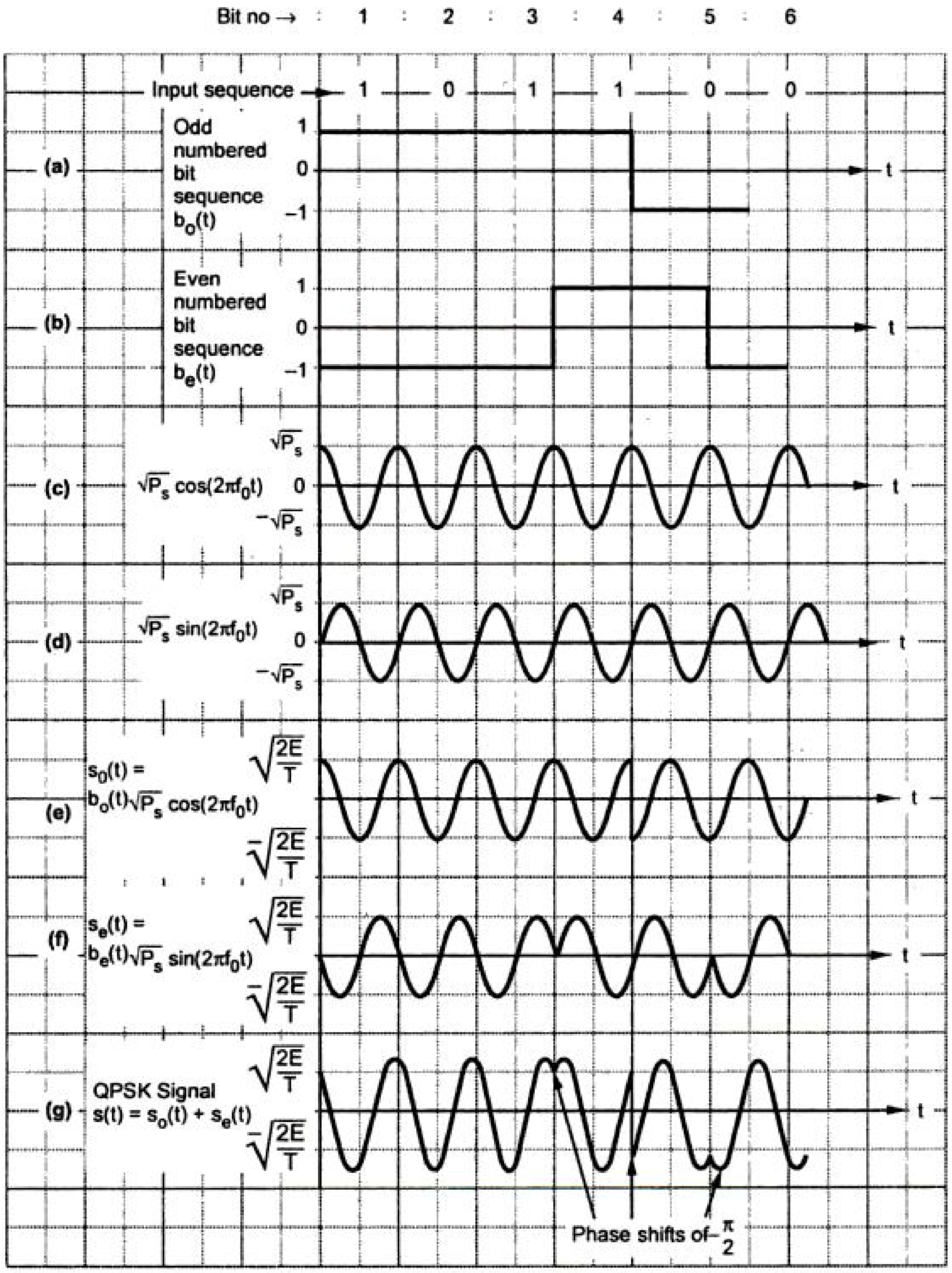

Fig. 3.4.8 QPSK waveform

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The above two equations are orthonormal waveforms. Fig. 3.5.1 shows the signal space diagram based on equation 3.5.6. The orthonormal carriers $\phi_1(t)$ and $\phi_2(t)$ form two axes. The signal points $S_0, S_1, S_2 \ldots. S_{m-1}$ are placed on the circumference of the circle. The signal points are equispaced with the phase shift of $\frac{2\pi}{M}$. The distance of each signal point from the origin is $\sqrt{P_s T_s}$.

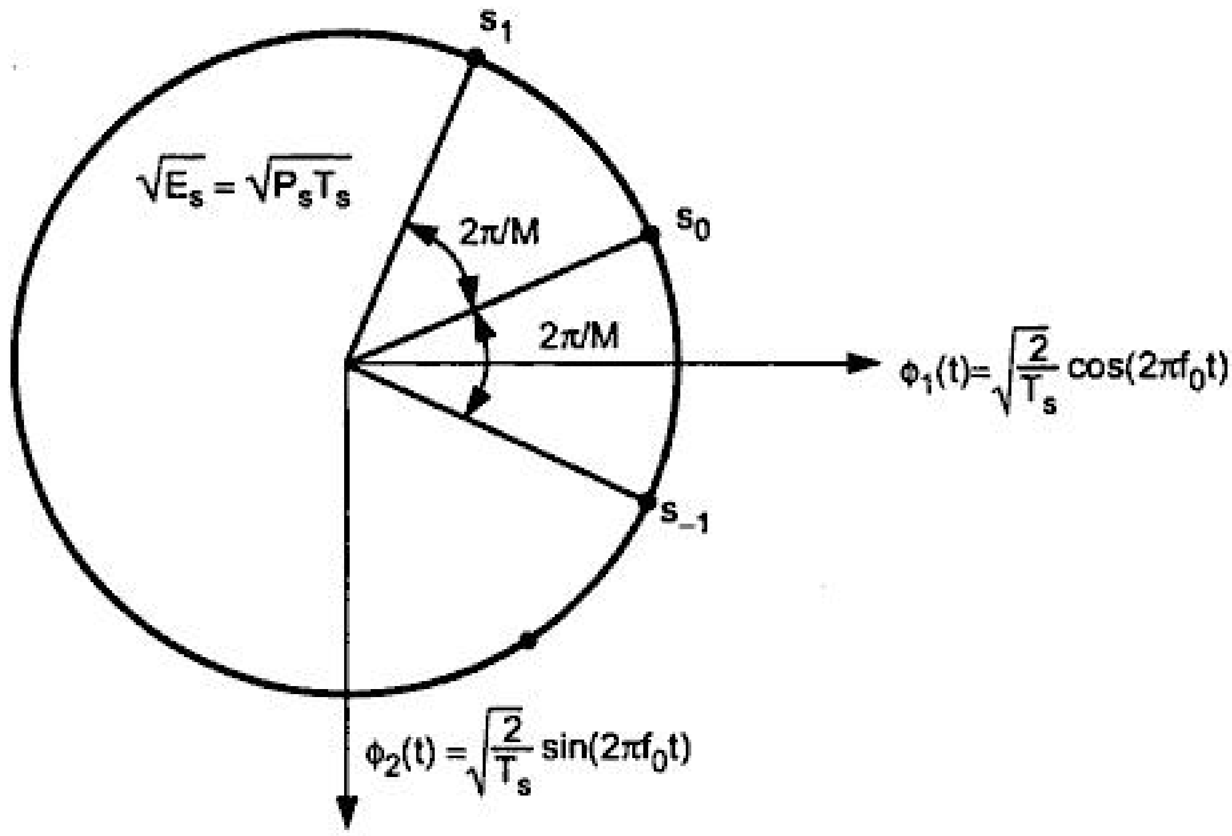

Fig. 3.5.1 Signal space diagram or geometrical representation of M-ary PSK signals

Here $\qquad P_s T_s = E_s$ (Symbol energy) $\qquad$ ... (3.5.9)

Thus we can say that QPSK is the special case of M-ary PSK with $M=4$. Then the signal space diagram of QPSK and 4-ary PSK will be similar

3.5.2 Power Spectral Density of M-ary PSK

PSK and QPSK are the special cases of M-ary PSK. The symbol duration for M-ary PSK is given by equation 3.5.2 as,

$$T_s = N T_b \qquad \ldots (3.5.10)$$

Here N is the number of input successive bits combined. The baseband power spectral density of QPSK is given as,

$$S_{B(QPSK)}(f) = 2P_s T_s \left[\frac{\sin(\pi f T_s)}{\pi f T_s}\right]^2 \qquad \text{from equation 3.4.23}$$

If we put $T_s = NT_b$ in above equation we will get power spectral density of M-ary PSK i.e.,

$$S_B(f) = 2P_s N T_b \left[\frac{\sin(\pi f N T_b)}{\pi f N T_b}\right]^2 \qquad \ldots (3.5.11)$$

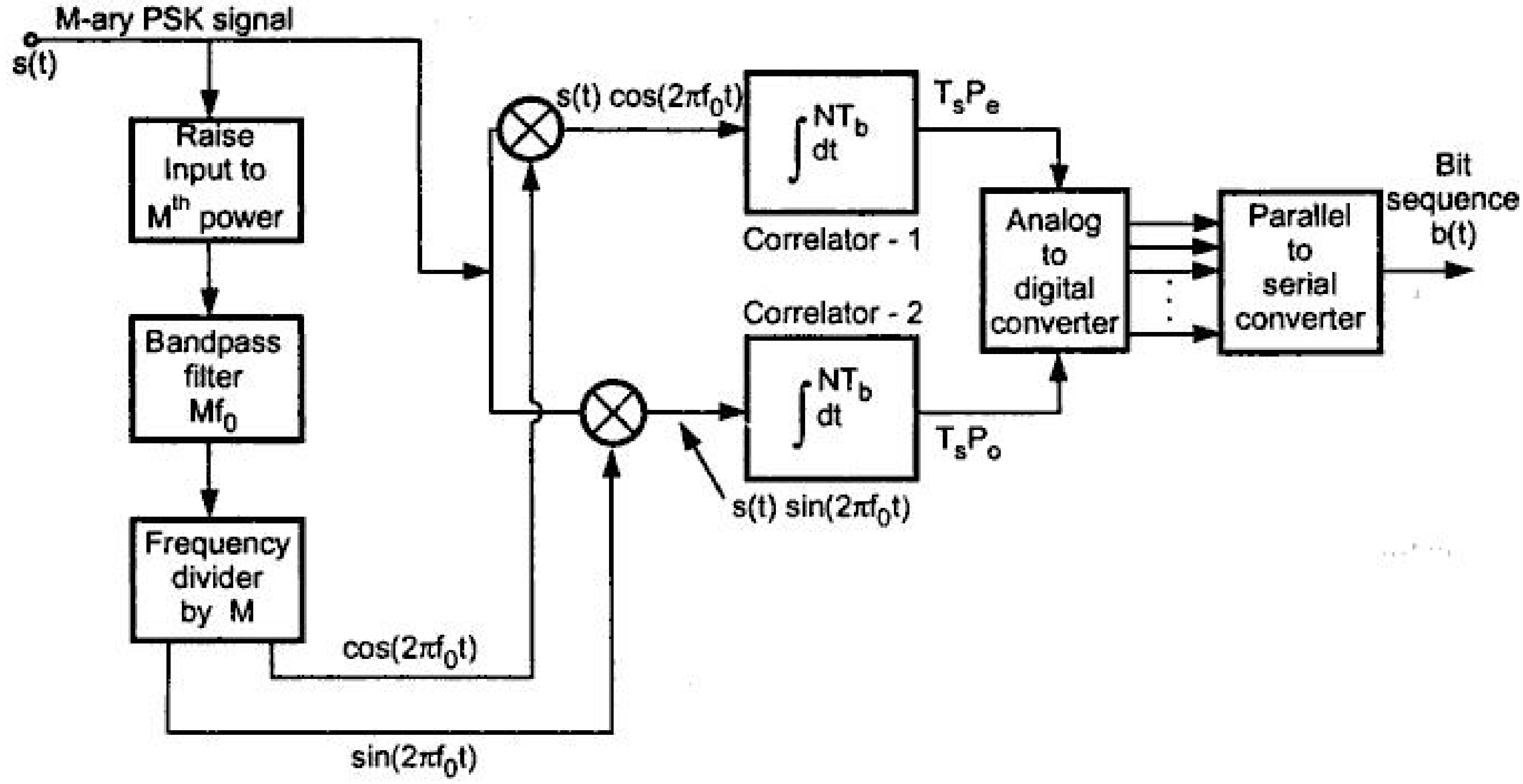

Fig. 3.5.5 M-ary PSK receiver

A/D converter, which reconstructs 'N' bit symbol. This 'N' bit symbol is given to the parallel to serial converter. It then generates the bit sequence $b(t)$.

Example 3.5.1 : *A 4-ary PSK has the transmitted waveforms,*

$$s_i(t) = 4\cos\left(2\pi f_c t + \frac{i\pi}{2}\right) \quad \text{... (3.5.15)}$$

$$i = 0, 1, 2, 3 \text{ and } 0 \le t \le T_s \text{ or } 0 \le t \le 2T_s$$

Draw its signal space diagram.

Solution : The M-ary PSK signal is represented by equation 3.5.3 as,

$$s(t) = \sqrt{2P_s}\cos(2\pi f_0 t + \phi_m)$$

and

$$\phi_m = (2m+1)\frac{\pi}{M} \quad \text{... (3.5.16)}$$

Comparing equation 3.5.14 and equation 3.5.15 we get,

$$\sqrt{2P_s} = 4 \quad \therefore 2P_s = 16 \quad \therefore P_s = 8 \quad \text{... (3.5.17)}$$

$$\phi_i = \phi_m = \frac{i\pi}{2}; i = 0, 1, 2, 3$$

$$\phi_0 = 0,\ \phi_1 = \frac{\pi}{2},\ \phi_2 = \pi,\ \phi_3 = \frac{3\pi}{2}$$

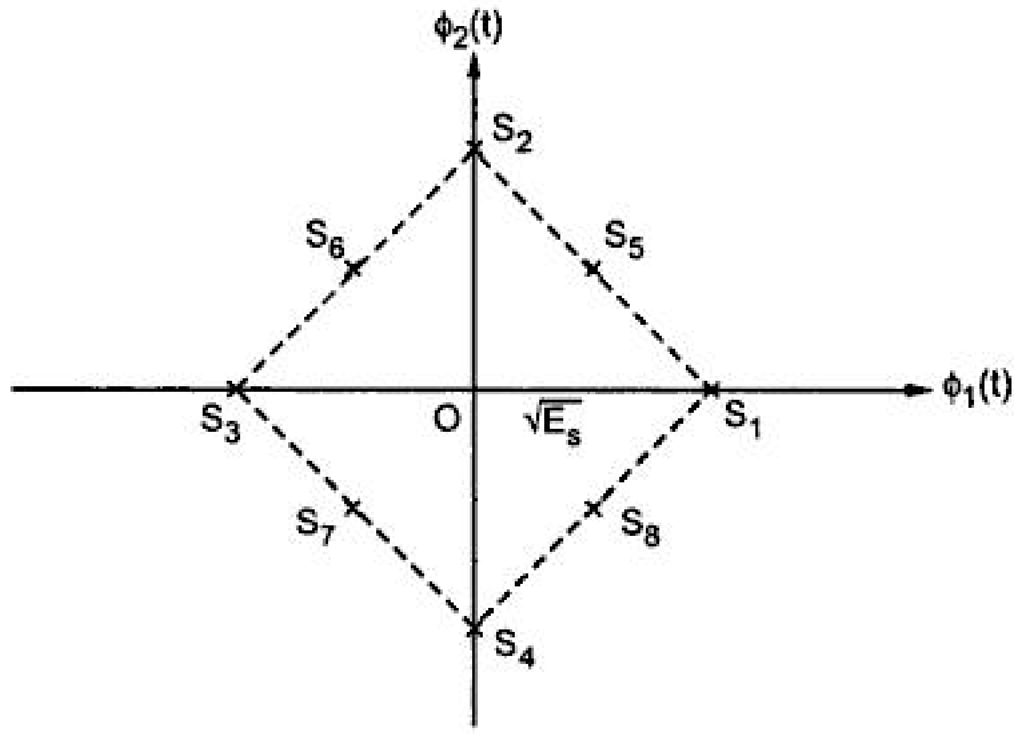

Fig. 3.5.8 Signal space diagram of two amplitude levels of PSK

Let us assume that other PSK level uses half amplitude. Then the remaining four symbols (S_5, S_6, S_7, S_8) will lie as shown in above figure. Thus S_5 lies exactly midway between S_1 and S_2. Thus the minimum distance will be distance between S_1 and S_5. We know that distance between S_1 and S_2 will be,

$$d_{12} = \sqrt{E_s + E_s} = \sqrt{2E_s}$$

Since S_5 is at the centre of S_1 and S_2, the distance between S_1 and S_5 will be,

$$d_{15} = \frac{d_{12}}{2} = \sqrt{\frac{E_s}{2}} = 0.707\sqrt{E_s}$$

In standard 8-level PSK, all the 8 symbols will lie on the circle of radius $\sqrt{E_s}$. Equation 3.5.14 gives the distance as,

$$d_{12} = 2\sqrt{E_s} \sin\frac{\pi}{M} = 2\sqrt{E_s} \sin\frac{\pi}{8} = 0.765\sqrt{E_s}$$

From above result, it is clear that, distance is more in case of standard PSK. As the distance between the signal points becomes more, the probability of error reduces. Hence it is clear from above results that probability of error is less in case of standard 8-level PSK.

Example 3.5.5 : *Derive an expression for the spectral spread of 16-ary PSK system.*

Solution : Power spectral density of M-ary PSK is given by equation 3.5.11 as,

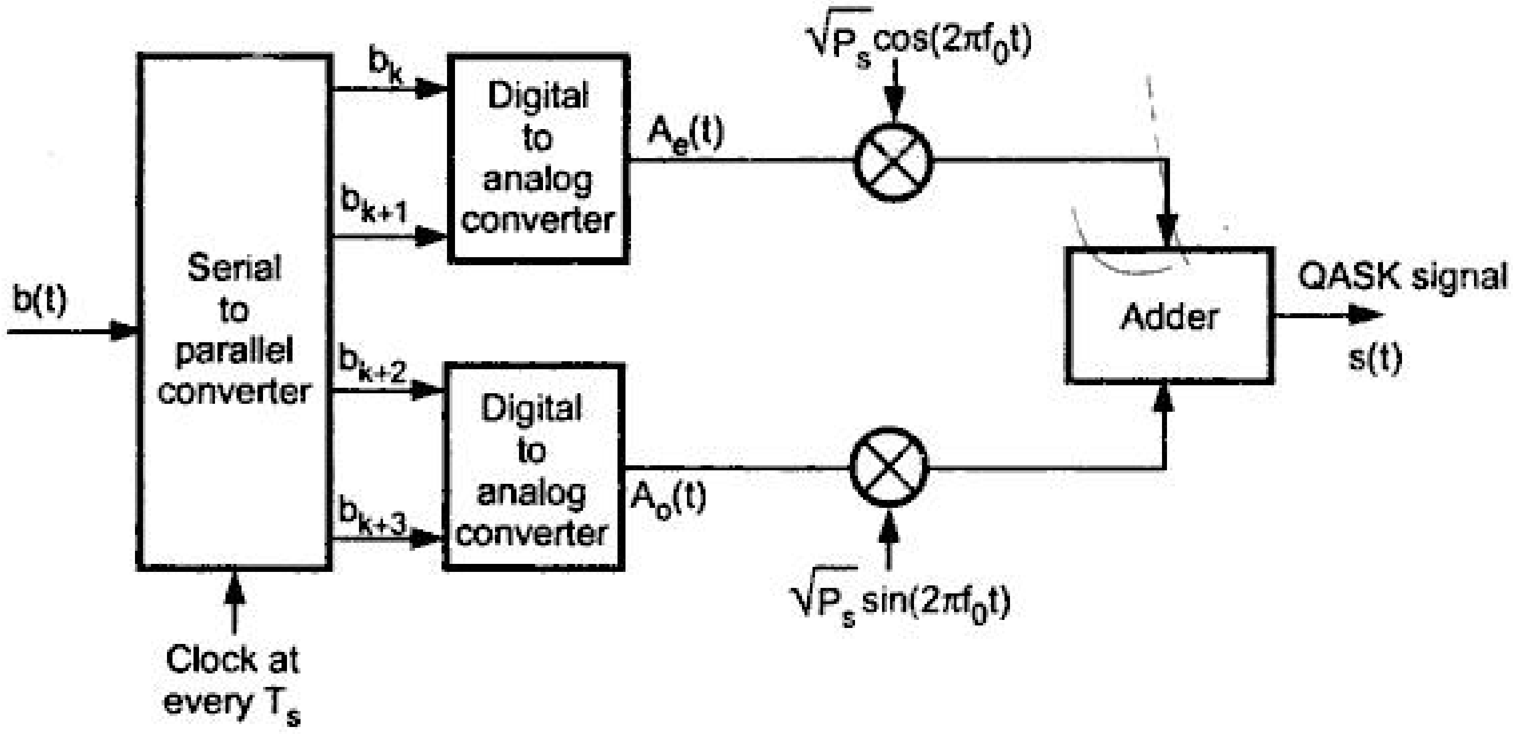

Fig. 3.6.2 Generation of QASK signal

digital to analog converter and b_{k+2} and b_{k+3} are applied to lower digital to analog converter. Depending upon two input bits, the output of digital to analog converter takes four output levels. Thus $A_e(t)$ and $A_o(t)$ takes 4 levels depending upon combination of two inputs bits. $A_e(t)$ modulates the carrier $\sqrt{P_s}\,cos\,(2\pi f_0\,t)$ and $A_o(t)$ modulates $\sqrt{P_s}\,sin\,(2\pi f_0\,t)$. The adder combines two signals to give QASK signal. It is given as,

$$s(t) = A_e(t)\sqrt{P_s}\,cos\,(2\pi f_0\,t) + A_o(t)\sqrt{P_s}\,sin\,(2\pi f_0\,t) \qquad \ldots (3.6.11\ (a))$$

If we compare the above equation with equation 3.6.11

We can write

$$A_e(t) \text{ and } A_o(t) = \pm\sqrt{0.2} \text{ or } \pm 3\sqrt{0.2} \qquad \ldots (3.6.12)$$

(depending upon input to D/A converter)

3.6.2.2 Receiver of QASK Signal

Fig. 3.6.3 shows the receiver of 16-QASK (4-bits QASK) system. The input signal $s(t)$ is raised to 4^{th} power. It then passed through a bandpass filter centered around the frequency $4f_0$ the signal is then divided in frequency by four. It gives a coherent carrier $cos\,(2\pi f_0\,t)$. Quadrature carrier $sin\,(2\pi f_0\,t)$ is produced by phase shifting of 90°. The inphase and quadrature coherent carriers are multiplied with QASK signal $s(t)$.

Since the amplitudes of $A_e(t)$ and $A_o(t)$ are bit constant and equal, let us check whether we can really recover the carrier correctly. The 4^{th} power QASK signal is,

$$s^4(t) = P_s^2\,[A_e(t)\,cos\,(2\pi f_0\,t) + A_o(t)\,sin\,(2\pi f_0\,t)]^4 \qquad \ldots (3.6.13)$$

3.7.1 BFSK Transmitter

From the table 3.7.1, we know that $P_H(t)$ is same as $b(t)$. And $P_L(t)$ is inverted version of $b(t)$. The block diagram of BFSK transmitter is shown in Fig. 3.7.1.

We know that input sequence $b(t)$ is same as $P_H(t)$. An inverter is added after $b(t)$ to get $P_L(t)$. $P_H(t)$ and $P_L(t)$ are unipolar signals. The level shifter converts the '+1' level to $\sqrt{P_s T_b}$. Zero level is unaffected. Thus the output of the level shifters will be either $\sqrt{P_s T_b}$ (if '+1') or zero (if input is zero). Further there are product modulators after level shifter. The two carrier signals $\phi_1(t)$ and $\phi_2(t)$ are used. $\phi_1(t)$ and $\phi_2(t)$ are orthogonal to each other. In one bit period of input signal (i.e. T_b), $\phi_1(t)$ or $\phi_2(t)$ have integral number of cycles.

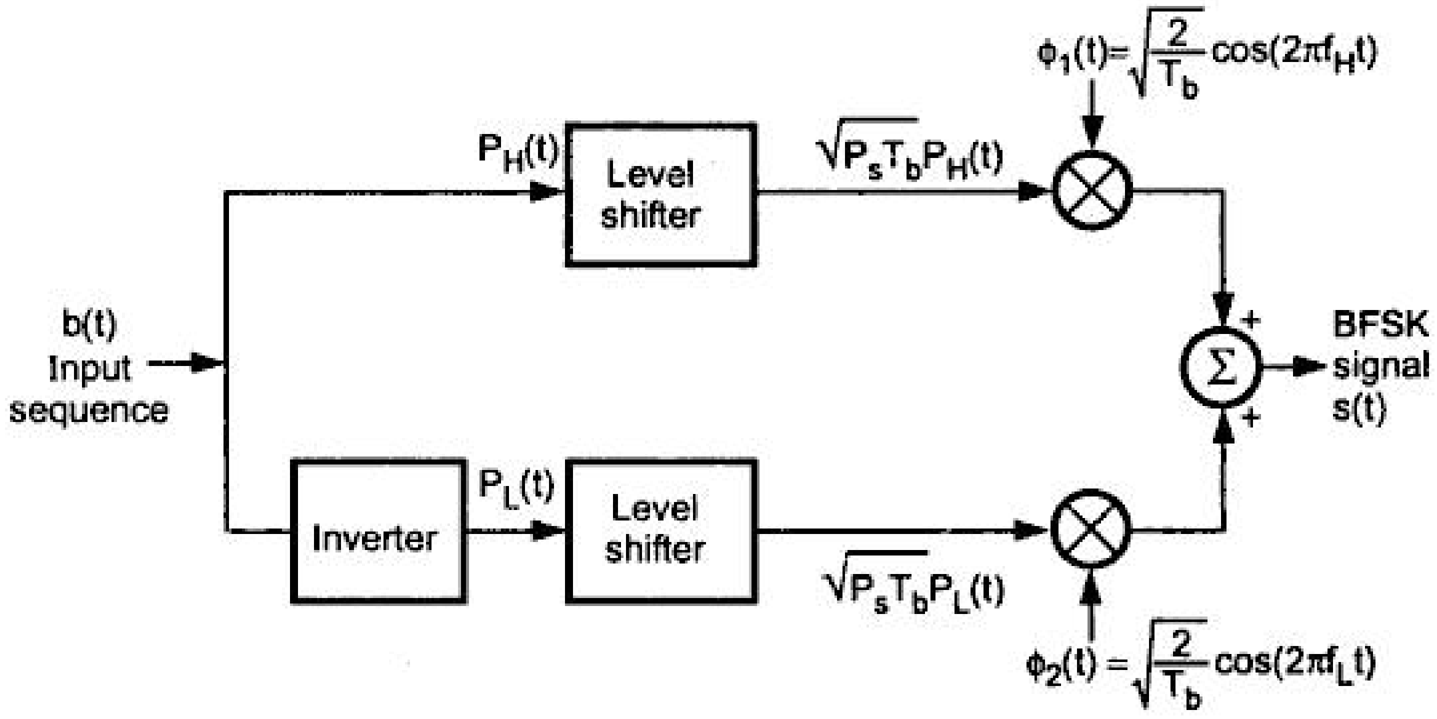

Fig. 3.7.1 Block diagram of BFSK transmitter

Therefore the modulated signal has continuous phase. Such BFSK signal is shown in Fig. 3.7.2. The adder then adds the two signals.

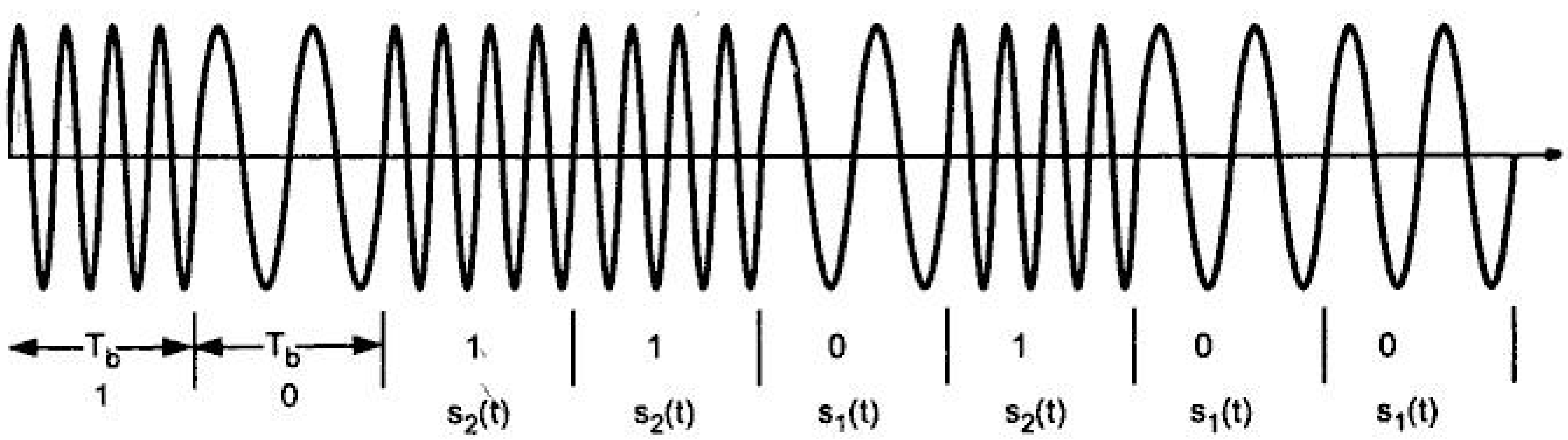

Fig. 3.7.2 BFSK signal

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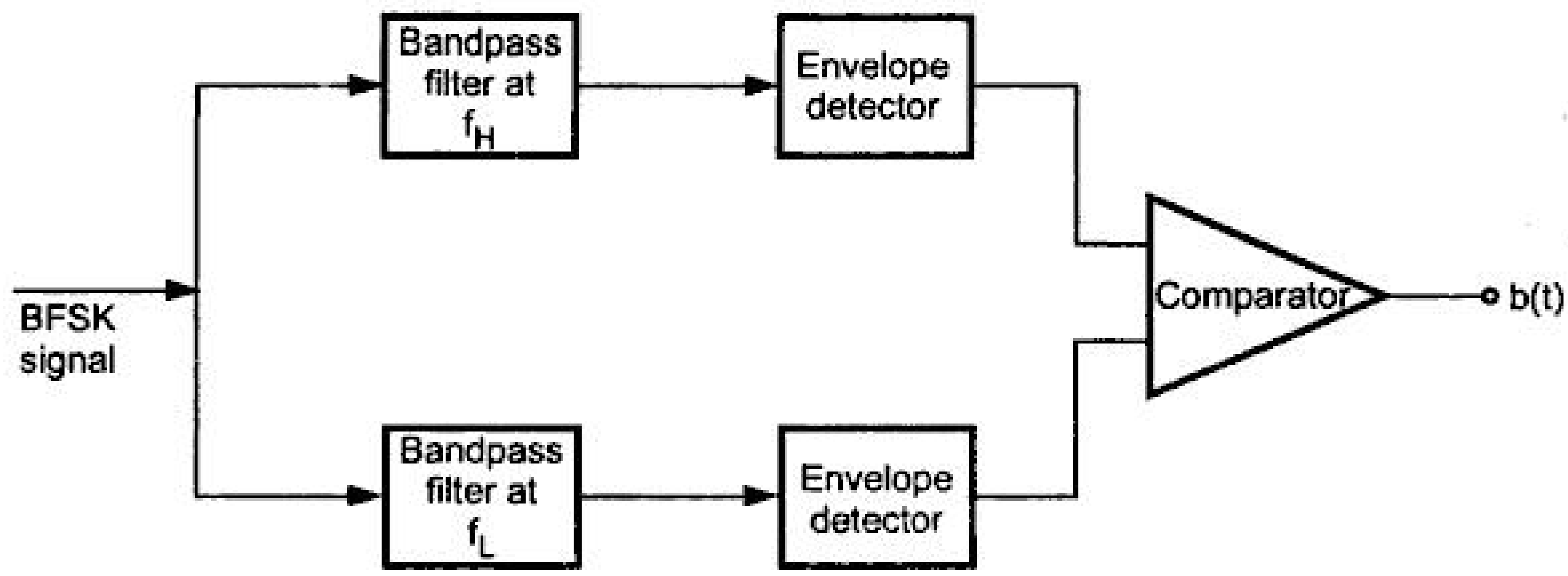

Fig. 3.7.5 Block diagram of BFSK receiver

The outputs of filters are applied to envelop detectors. The outputs of detectors are compared by the comparator. If unipolar comparator is used, then the output of comparator is the bit sequence $b(t)$.

3.7.5 Geometrical Representation of Orthogonal BFSK or Signal Space Representation of Orthogonal BFSK

Orthogonal carriers are used for M-ary PSK and QASK. The different signal points are represented geometrically in $\phi_1 \phi_2$ plane. For geometrical representation of BFSK signals such orthogonal carriers are required. From Fig. 3.7.1, we know that, two carriers $\phi_1(t)$ and $\phi_2(t)$ of two different frequencies f_H and f_L are used for modulation. To make $\phi_1(t)$ and $\phi_2(t)$ orthogonal, the frequencies f_H and f_L should be some integer multiple of base band frequency 'f_b'.

i.e. $$f_H = m f_b \qquad \text{... (3.7.14)}$$

and $$f_L = n f_b \qquad \text{... (3.7.15)}$$

Here $f_b = \frac{1}{T_b}$, then the carriers will be

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi m f_b t) \qquad \text{... (3.7.16)}$$

and $$\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi m f_b t) \qquad \text{... (3.7.17)}$$

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3.8.1.2 Receiver

Fig. 3.8.2 shows block diagram of M-ary FSK receiver. It is the extension of BFSK receiver of Fig. 3.8.1. The M-ary FSM signal is given to the set of 'M' bandpass filters. The center frequencies of those filters are $f_0, f_1, f_2, ...f_{M-1}$. These filters pass their particular frequency and alternate others. The envelope detectors outputs are applied to a decision device. The decision device produces its output depending upon the highest input. Depending upon the particular symbol, only one envelope detector will have higher output. The outputs of other detectors will be very low. The output of the decision device is given to 'N' bit analog to digital converter. The analog to digital converter output is the 'N' bit symbol in parallel. These bits are then converted to serial bit stream by parallel to serial converter. In some cases the bits appear in parallel. Then there is no need to use serial to parallel and parallel to serial converters.

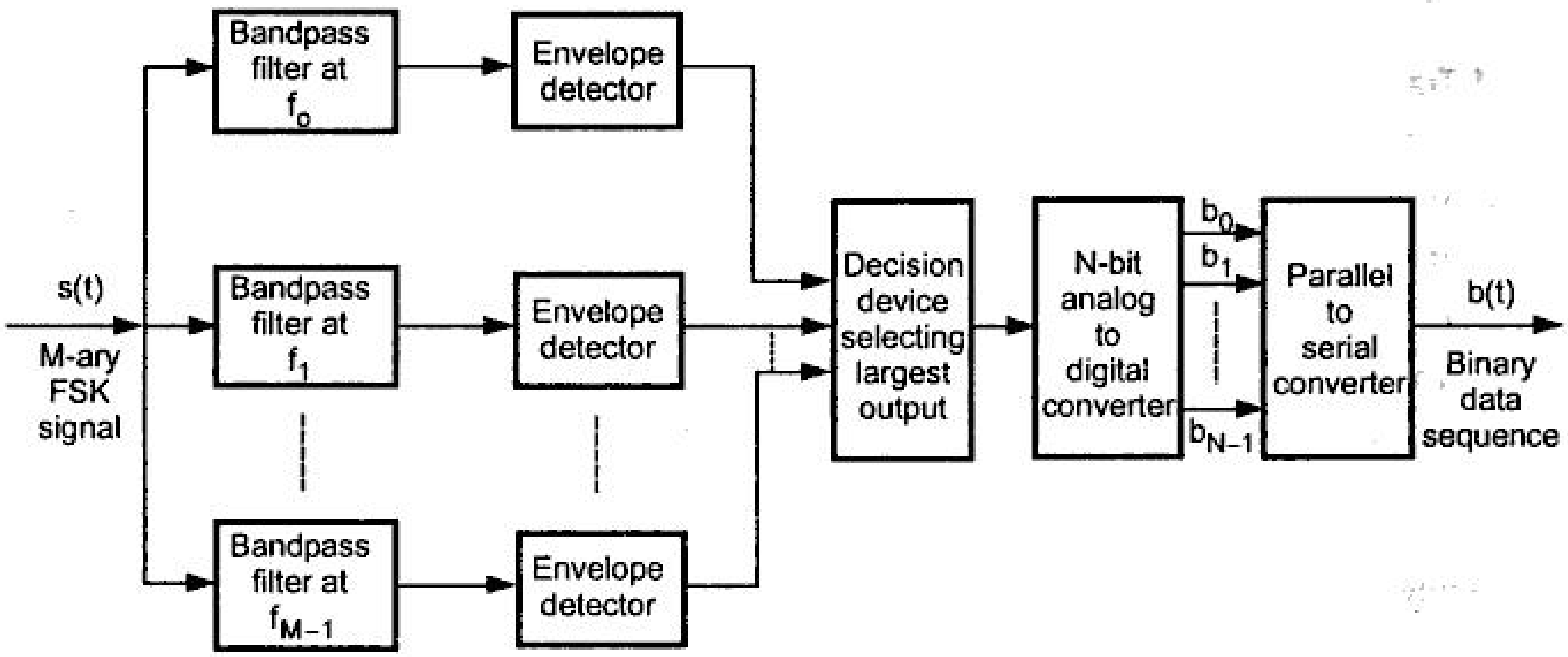

Fig. 3.8.2 Block diagram of M-ary FSK system

3.8.2 Power Spectral Density and Bandwidth of M-ary FSK

We know that for M symbol $f_0, f_1, f_2 \dots f_{m-1}$ frequencies are used for transmission. The probability of error is minimized by selecting those frequencies such that transmitted signals are mutually orthogonal. If those frequencies are selected as successive even harmonics of symbol frequency f_s, then transmitted signals will be orthogonal.

Let's say that the lowest carrier frequency f_0 is the k^{th} harmonic of symbol frequency i.e.,

$$f_0 = kf_s \qquad \dots (3.8.1)$$

Then the other frequencies will be,

$$f_1 = (k+2) f_s, f_2 = (k+4) f_s \dots \text{etc.} \qquad \dots (3.8.2)$$

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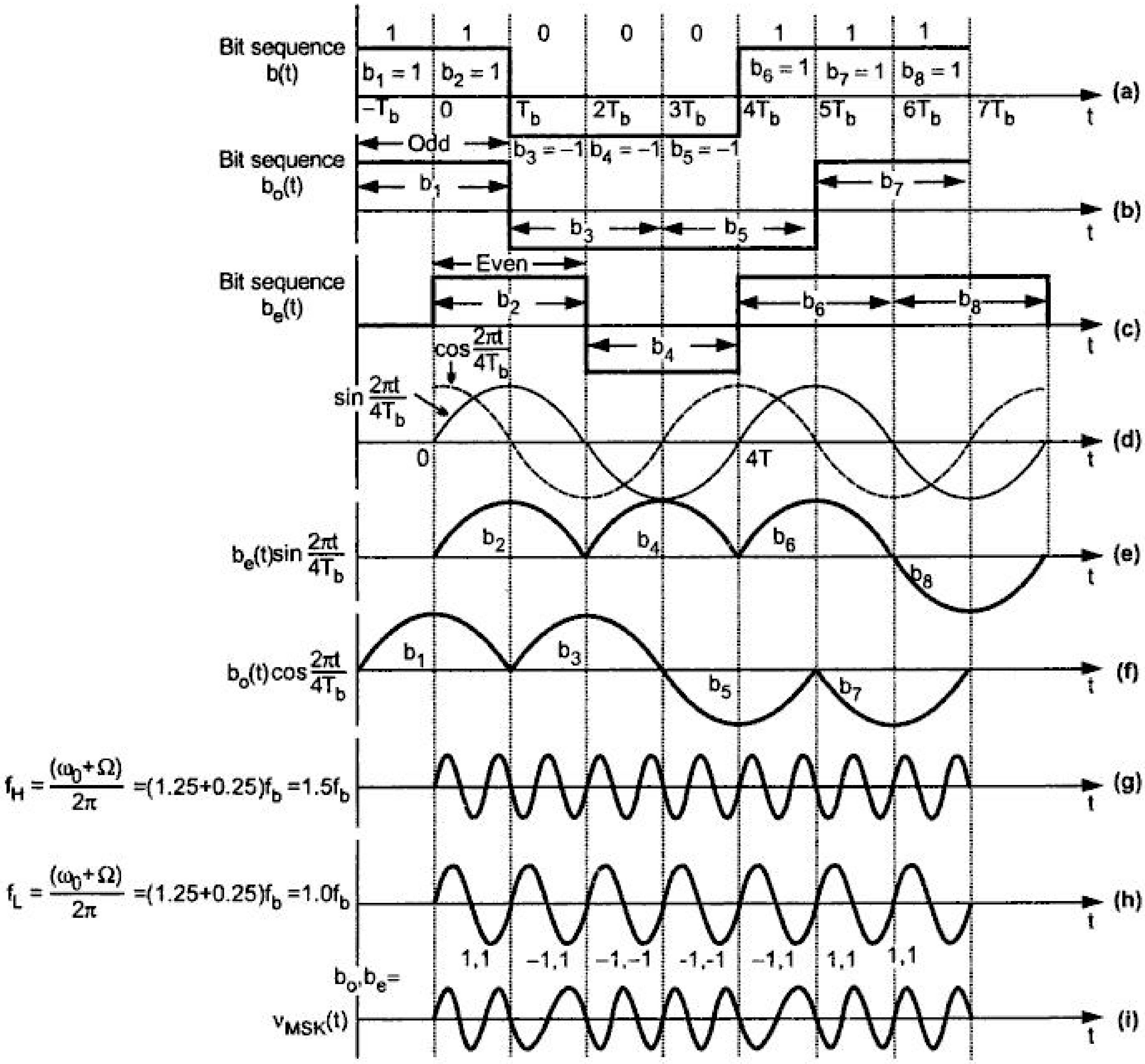

Fig. 3.9.1 MSK waveforms

(a) Bipolar NRZ waveform representing bit sequence

(b) Odd bit sequence waveforms $b_o(t)$

(c) Even bit sequence waveform $b_e(t)$

(d) Waveforms of frequency $\frac{f_b}{4}$ used for smoothing of $b_e(t)$ and $b_o(t)$

(e) Modulating waveform of even sequence

(f) Modulating waveform of odd sequence

(g) Waveform of frequency f_H

(h) Waveform of frequency f_L

(i) MSK waveform

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MSK is called "Shaped QPSK" ?

In QPSK, $b_o(t)$ and $b_e(t)$ directly multiply the carrier. Hence there are abrupt changes in phase (and hence amplitude) in the QPSK waveform. In MSK, two waveforms $b_e(t)\, sin(2\pi t / 4T_b)$ and $b_o(t)\, cos(2\pi t / 4T_b)$ are first generated as shown in Fig. 3.9.1 (e) and (f). These waveforms multiply the carriers. Thus $b_e(t)\, sin(2\pi t / 4T_b)$ and $b_o(t)\, cos(2\pi t / 4T_b)$ does not have abrupt changes in their amplitudes. Hence the multiplied carriers have no abrupt changes and they have continuous phase. The MSK waveform of Fig. 3.9.1 is drawn for m = 5. From equation 3.9.14, we can obtain the carrier frequency as,

$$f_0 = \frac{m}{4} f_b = \frac{5}{4} f_b = 1.25 f_b$$

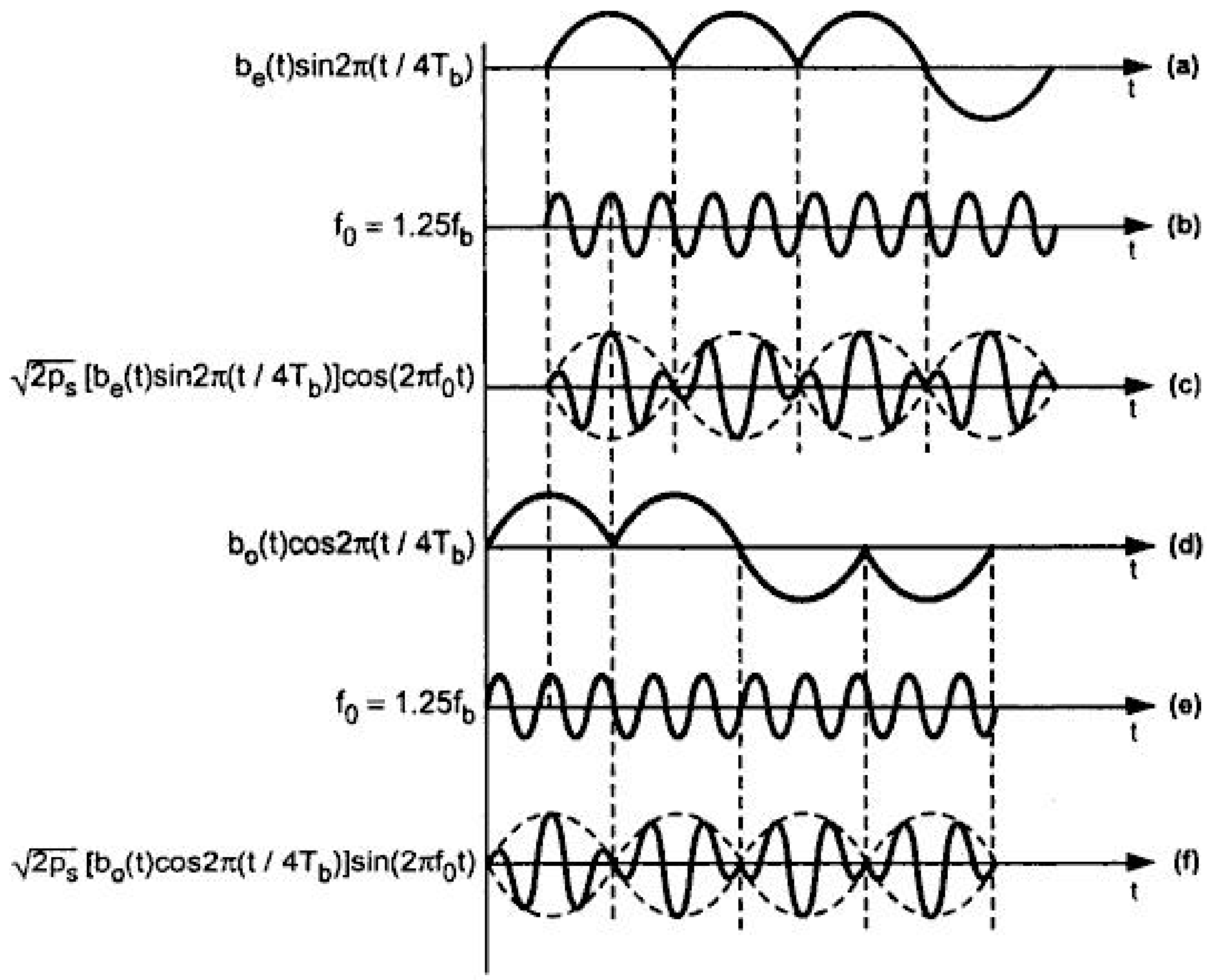

Fig. 3.9.2 MSK waveforms showing 'smoothing' effect on modulated carriers "shaped QPSK".

(a) & (d) smoothed modulating waveforms (Odd & Even sequences)

(b) & (e) Carrier $f_0 = 1.25 f_b$ $(m = 5)$

(c) & (f) Modulated carriers

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3.9.5 Advantages and Disadvantages of MSK as Compared to QPSK

From the discussion of MSK we can now compare the advantages of MSK over QPSK.

Advantages :

1. The MSK baseband waveforms are smoother compared to QPSK.
2. MSK signal have continuous phase in all the cases, whereas QPSK has abrupt phase shift of $\frac{\pi}{2}$ or π.
3. MSK waveform does not have amplitude variations, whereas QPSK signal have abrupt amplitude variations.
4. The main lobe of MSK is wider than that of QPSK. Main lobe of MSK contains around 99% of signal energy whereas QPSK main lobe contains around 90% signal energy.
5. Side lobes of MSK are smaller compared to that of QPSK. Hence interchannel interference is significantly large in QPSK.
6. To avoid interchannel interference due to sidelobes, QPSK needs bandpass filtering, where as it is not required in MSK.
7. Bandpass filtering changes the amplitude waveform of QPSK because of abrupt changes in phase. This problem doesnot exist in MSK.

The distance between signal points is same in QPSK as well as MSK. Hence the probability of error is also same. However there are few drawbacks of MSK.

Disadvantages :

1. The bandwidth requirement of MSK is 1.5 f_b, whereas it is f_b in QPSK. Actually this cannot be said serious drawback of MSK. Because power to bandwidth ratio of MSK is more. 99% of signal power can be transmitted within the bandwidth of 1.2 f_b in MSK. While QPSK needs around 8 f_b to transmit the same power.
2. The generation and detection of MSK is slightly complex. Because of incorrect synchronization, phase jitter can be present in MSK. This degrades the performance of MSK.

3.9.6 Gaussian MSK

Power spectra of MSK is given in Fig. 3.9.8. Observe that the the main lobe is wide. This makes MSK unsuitable for the applications where extremely narrow bandwidths and sharp cut-offs are required. Slow decay of MSK psd curve creates adjacent channel interference. Hence MSK cannot be used for multiuser communications. This problem can be overcome with Gaussian MSK. Fig. 3.9.8 shows the little modification.

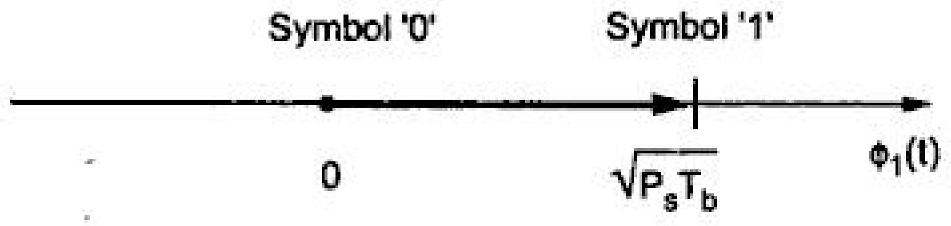

Fig. 3.10.2 Signal space diagram of ASK

Therefore the distance between the two signal points will be,

$$d = \sqrt{P_s T_b} = \sqrt{E_b} \qquad \dots (3.10.3)$$

3.10.2 Generator and Detector of ASK

3.10.2.1 ASK Generator

Fig. 3.10.3 shows the ASK generator. The input binary sequence is applied to the product modulator. The product modulator amplitude modulates the sinusoidal carrier. It passes the carrier when input bit is '1'. It blocks the carrier (i.e. zero output) when intput bit is '0'. The wavefrom of ASK is as shown in Fig. 3.10.1.

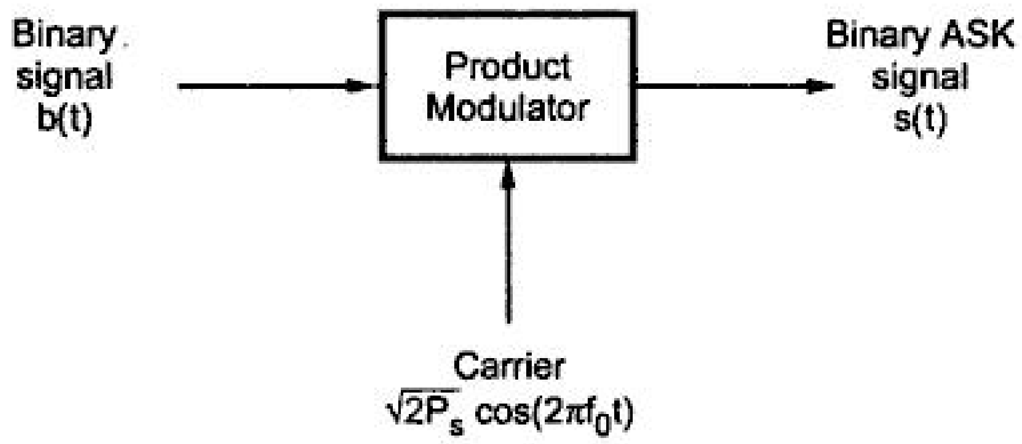

Fig. 3.10.3 Block diagram of ASK generator

3.10.2.2 ASK Detector

Fig. 3.10.4 shows the block diagram of coherent ASK detector. The ASK signal is applied to the correlator consisting of multiplier and integrator. The locally generated coherent carrier is applied to the multiplier. The output of multiplier is integrated over one bit period. The decision device takes the decision at the end of every bit period. It compares the output of integrator with the threshold. Decision is taken in favour of '1' when threshold is exceeded. Decision is taken as '0' if threshold is not exceeded.

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4 Data Transmission

The analog signal is converted to digital or binary waveform by means of waveform coding techniques. In first and second chapter we have seen such waveform coding techniques. They are PCM, DM, ADM, DPCM etc. This digital data is then converted to RZ, NRZ, AMI etc. type of signal waveforms. The digital (binary) signal then can be transmitted either using baseband transmission or using bandpass transmission.

In *bandpass transmission*, the digital signal modulates high frequency sinusoidal carrier. The analysis of such techniques we have seen in previous chapter. They are called digital modulation techniques. With the help of such techniques, it is possible to transmit data over long distances. In *baseband transmission*, the data is transmitted without modulation.

During the transmission of data over the channel, it is corrupted by noise. Hence at the receiver, the noisy signal is received. Therefore correct detection of the transmitted signal is difficult. For example consider the transmitted signal and received noisy signal as shown in Fig. 4.1 (a) and (b).

The received signal $\hat{x}(t)$ is a noisy signal at the receiver. Let us consider that, the detector checks $\hat{x}(t)$ at 'T' during every bit interval. In above figure observe that the decision in first interval will be correct i.e. symbol '1'. But in second interval, the decision will be '1' but it is wrong. At the time when detector checks $\hat{x}(t)$ [i.e. at $t = T$], noise pulse is detected and decision is taken in favour of '1'. But actually symbol '0' is transmitted in second interval as shown in Fig. 4.1(a). Thus errors are introduced because of noise. The detecting method of the baseband signal perform following jobs:

(i) The detection method should attenuate noise and amplify signal, i.e. it should improve signal to noise ratio of the received signal.

(ii) The detection method should check the received signal at the time instant in the bit interval when signal to noise ratio is maximum.

(iii) The detection should be performed with minimum error probability.

In this chapter we will study some methods for detection of digital signals. We will also compare these methods on the basis of their performances.

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Let us rearrange the above equation as,

$$\overline{n_0^2(t)} = \frac{N_0}{2} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{\left(\frac{xT}{\tau}\right)} \cdot \left(\frac{T}{\tau}\right)^2 \cdot \frac{1}{\pi\tau} dx$$

$$= \frac{N_0}{2} \cdot \frac{T^2}{\pi\tau^3} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{\left(\frac{xT}{\tau}\right)} \cdot dx$$

Let $\frac{xT}{\tau} = u$

$\therefore \ dx = \frac{\tau}{T} du$ and limits will be unchanged, therefore above equation becomes,

$$\overline{n_0^2(t)} = \frac{N_0}{2} \cdot \frac{T^2}{\pi\tau^3} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \cdot \frac{\tau}{T} du$$

$$= \frac{N_0 T}{2\pi \tau^2} \int_{-\infty}^{\infty} \left(\frac{\sin u}{u}\right)^2 du$$

Since the function $\frac{\sin u}{u}$ is squared, we can write above equation as,

$$\overline{n_0^2(t)} = \frac{N_0 T}{2\pi\tau^2} \cdot 2 \int_{0}^{\infty} \left(\frac{\sin u}{u}\right)^2 du$$

$$= \frac{N_0 T}{2\pi\tau^2} \cdot 2 \cdot \frac{\pi}{2} \quad \text{By equation C-46 in appendix 'C'.}$$

$$\therefore \quad \overline{n_0^2(t)} = \frac{N_0 T}{2\tau^2} \qquad \text{... (4.1.9)}$$

The above relation gives noise power at the output. We obtain the signal to noise power ratio at output of integrator as,

Signal to noise ratio, $\rho = \frac{\text{Signal power}}{\text{Noise power}}$

Putting the values of signal power from equation 4.1.3 and noise power from equation 4.1.9 we get,

$$\rho = \frac{\frac{A^2 T^2}{\tau^2}}{\frac{N_0 T}{2\tau^2}}$$

Thus as shown in Table 4.1.1, the error is introduced depending upon probability that $n_0(t)$ takes a particular value. These probabilities can be obtained from PDF of $n_0(t)$. We know that the probability density function (PDF) of the gaussian distributed function is given by standard relation as,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} \quad \text{... (4.1.15)}$$

Here $f_X(x)$ is the PDF of random function x.

m is the mean value and

σ is the standard deviation.

Here since we want to evaluate PDF for white gaussian noise we have,

$$x = n_0(t)$$

Since this noise has zero mean value, $m = 0$ equation 4.1.15 can be written as,

$$f_X(n_0(t)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \quad \text{... (4.1.16)}$$

The standard deviation σ is given as,

$$\sigma = [\text{mean square value} - \text{square of mean value}]^{\frac{1}{2}}$$

i.e.

$$\sigma_x = [\overline{X^2} - m_x^2]^{\frac{1}{2}}$$

Here mean square value $\overline{X^2} = \overline{n_0^2(t)} = \frac{N_0 T}{2\tau^2}$ ... from equation 4.1.9

And of mean value $m_x = 0$ for this noise.

$$\therefore \quad \sigma = [\overline{n_0^2(t)}]^{\frac{1}{2}} = \sqrt{\frac{N_0 T}{2\tau^2}}$$

Hence equation 4.1.16 can be written as,

$$f_X(n_0(t)) = \frac{1}{\sqrt{\frac{N_0 T}{2\tau^2}}\sqrt{2\pi}} \cdot e^{-[n_0(t)]^2/2\left(\frac{N_0 T}{2\tau^2}\right)}$$

Here note that $n_0(t)$ is the function like 'x'. It is random variable and we are evaluating its PDF. On simplifying above equation we get,

Solution : Given data :

psd of white noise, $\frac{N_0}{2} = 10^{-9}$ W/Hz

amplitude, A = $10 \times 10^{-3}\ V$

data rate = 10×10^3. Hence $T_b = \frac{1}{10 \times 10^3}$

Probability of occurrence of both the symbols is equal, i.e. 0.5.

(i) To obtain probability of error P_e

Equation 4.1.22 give probability of error of integrate and dump receiver as,

$$P_e = \frac{1}{2}\, erfc \sqrt{\frac{A^2 T}{N_0}}$$

Putting values in above equation,

$$P_e = \frac{1}{2}\, erfc \sqrt{\frac{\left(10 \times 10^{-3}\right)^2}{2 \times 10^{-9} \times 10 \times 10^3}}, \text{ Here } T = \frac{1}{10 \times 10^3}$$

$$= \frac{1}{2}\, erfc \sqrt{5}$$

(ii) To obtain 'A' for bit rate of 10 Mbits/sec

The probability of error is to be maintained same. i.e.,

$$\frac{1}{2}\, erfc \sqrt{5} = \frac{1}{2}\, erfc \sqrt{\frac{A^2}{2 \times 10^{-9} \times 10 \times 10^6}}, \text{ Here } T = \frac{1}{10 \times 10^6}$$

$$5 = \frac{A^2}{2 \times 10^{-9} \times 10 \times 10^6}$$

$$\therefore \quad A = \sqrt{0.1} = 0.3162 \text{ volts}$$

Thus the amplitude must be increased to 0.3162 volts to maintain same probability of error.

Theory Questions

1. *Explain how integrator is used to detect baseband digital signals. Derive the expression for signal to noise ratio of integrate and dump receiver.*
2. *Derive an expression for error probability of an integrator.*

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$$= \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 \, df \quad \text{... (4.3.13)}$$

Parseval's power theorem states that,

$$\int_{-\infty}^{\infty} |X(f)|^2 \, df = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{0}^{T} x^2(t)\, dt \quad \text{... (4.3.14)}$$

In the last integral we have taken limits from 0 to T since $x(t)$ exists from 0 to T only. We know that $x(t) = x_1(t) - x_2(t)$. Hence above equation becomes,

$$\int_{-\infty}^{\infty} |X(f)|^2 \, df = \int_{0}^{T} [x_1(t) - x_2(t)]^2 \, dt$$

$$= \int_{0}^{T} \left[x_1^2(t) + x_2^2(t) - 2x_1(t)\, x_2(t)\right] dt$$

$$= \int_{0}^{T} x_1^2(t)\, dt + \int_{0}^{T} x_2^2(t)\, dt - 2\int_{0}^{T} x_1(t)\, x_2(t)\, dt \quad \text{... (4.3.15)}$$

Here, $\int_{0}^{T} x_1^2(t)\, dt = E_1$ i.e. energy of $x_1(t)$ by standard relations.

and $\int_{0}^{T} x_2^2(t)\, dt = E_2$ i.e. energy of $x_2(t)$ by standard relations.

and $\int_{0}^{T} x_1(t)\, x_2(t) = E_{12}$ represents energy due to auto-correlation between $x_1(t)$ and $x_2(t)$. If we select $x_1(t) = -x_2(t)$, then the energies are equal. i.e.,

$$E_1 = E_2 = -E_{12} = E \quad \text{... (4.3.16)}$$

Putting these values in equation 4.3.15 we get,

$$\int_{-\infty}^{\infty} |X(f)|^2 \, df = [E + E - 2(-E)]$$

$$= 4E \quad \text{... (4.3.17)}$$

Putting the above value of $\int_{-\infty}^{\infty} |X(f)|^2 \, df$ in equation 4.3.13 we get,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0} \cdot 4E$$

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From this relation we can obtain $x_0(t)$ by taking inverse fourier transform. i.e.,

$$x_0(t) = \int_{-\infty}^{\infty} X_0(f)\, e^{j2\pi ft}\, df \qquad \text{By definition of IFT}$$

$$= \int_{-\infty}^{\infty} \frac{2k}{N_0} |X(f)|^2\, e^{-j2\pi fT} \cdot e^{j2\pi ft}\, df$$

Putting $X_0(f)$ from equation 4.3.27

$$= \frac{2k}{N_0} \int_{-\infty}^{\infty} |X(f)|^2\, e^{j2\pi f(t-T)}\, df$$

at $t = T, x_0(t)$ becomes,

$$x_0(T) = \frac{2k}{N_0} \int_{-\infty}^{\infty} |X(f)|^2\, e^{j2\pi f(T-T)}\, df$$

$$= \frac{2k}{N_0} \int_{-\infty}^{\infty} |X(f)|^2\, df \qquad \text{since} \quad e^{j2\pi f(T-T)} = e^0 = 1$$

By Rayleigh's energy theorem we know that

$$\int_{-\infty}^{\infty} |X(f)|^2\, df = \int_{-\infty}^{\infty} x^2(t)\, dt = E, \text{ hence above equation becomes,}$$

$$x_0(T) = \frac{2k}{N_0} \cdot E$$

Maximum value of $x_0(T)$ will result when $\frac{2k}{N_0} = 1.$

Hence, $\quad x_0(T) = E \qquad$ when $t = T$

Example 4.3.4 : *Show that the output signal of a matched filter is proportional to a shifted version of the auto-correlation function of the input signal to which the filter is matched.*

Solution : The output signal $x_0(t)$ is obtained from its spectrum $X_0(f)$ by taking inverse fourier transform. i.e.,

$$x_0(t) = \int_{-\infty}^{\infty} X_0(f)\, e^{j2\pi ft}\, df$$

Putting $X_0(f)$ from equation 4.3.27 we get,

$$x_0(t) = \int_{-\infty}^{\infty} \frac{2k}{N_0} |X(f)|^2\, e^{-j2\pi fT} \cdot e^{j2\pi ft}\, df$$

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As shown in column-I a dotted line is drawn between m_1 and m_2. This line makes two partitions. In upper partition there is only one message and its probability is $\frac{16}{32}$. Lower partition contains m_2 to m_8 and sum of their probabilities is also $\frac{16}{32}$.

Thus the partition is made such that sum of probabilities in both the partitions are almost equal. The messages in upper partition are assigned bit '0' and lower partition are assigned bit '1'. Those partitions are further subdivided into new partitions following the same rule. The partitioning is stopped when there is only one message in partition. Thus in column-I upper partition has only one message hence no further partition is possible. But lower partition of column-I is further subdivided in column-II.

In column-II, the dotted line is drawn between m_3 and m_4. Observe that in upper partition we have two messages m_2 and m_3. The sum of probabilities of m_2 and m_3 is $\frac{4}{32}+\frac{4}{32}=\frac{8}{32}$. The lower partition in column-II has messages m_4 to m_8. Their sum of probabilities is $\frac{1}{32}+\frac{1}{32}+\frac{2}{32}+\frac{2}{32}+\frac{2}{32}=\frac{8}{32}$. Thus the sum of probabilities in both the partitions is equal. The messages in upper partition are assigned '0' and lower partition are assigned '1'. Since both the partitions in column-II contains more than one message, they are further subdivided. This subdivision is shown in column-III.

This partitioning is continued till there is only one message in the partition. The partitioning process is self explanatory in columns-III, IV, V in Table 6.3.2. In the last column of the table codeword for the message and number of bits / message are shown. The codeword is obtained by reading the bits of a particular message rowwise through all columns. For example message m_1 has only one bit i.e. 0, message m_2 has three bits i.e. 1 0 0, message m_3 has also three bits i.e. 1 0 1, message m_8 has five bits i.e 1 1 1 1 1. This shows that, the message m_1 has highest probability hence it is coded using single bit i.e. '0'. As the probabilities of messages goes on decreasing, the bits in codeword increase.

We know by equation (5.4.6) that average information per message (entropy) is given as,

$$H = \sum_{k=1}^{M} p_k \log_2\left(\frac{1}{p_k}\right)$$

$$= p_1 \log_2\left(\frac{1}{p_1}\right)+p_2 \log_2\left(\frac{1}{p_2}\right)+....+p_8 \log_2\left(\frac{1}{p_8}\right)$$

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To obtain codeword for m_2 :

Similarly if we trace in the direction of arrows for message m_2, we obtain the sequence as 000 (Tracing path is not shown in Table 6.3.5). Reading from LSB side we get codeword for m_2 as 000 again. Table 6.3.5 shows the messages, their probabilities the sequence obtained by tracing and codeword obtained by reading from LSB to MSB.

Message	Probability	Digits obtained by tracing	Codeword obtained by reading digits of column-3 from LSB side	No. of digits
m_0	$p_0 = 0.4$	1	1	(1)
m_1	$p_1 = 0.2$	10	01	(2)
m_2	$p_2 = 0.2$	000	000	(3)
m_3	$p_3 = 0.1$	0100	0010	(4)
m_4	$p_4 = 0.1$	1100	0011	(4)

Table 6.3.5 Huffman Coding

Average information per message (entropy) is given as,

$$H = \sum_{k=1}^{M} p_k \log_2\left(\frac{1}{p_k}\right)$$

For five messages above equation can be expanded as,

$$H = p_0 \log_2\left(\frac{1}{p_0}\right) + p_1 \log_2\left(\frac{1}{p_1}\right) + p_2 \log_2\left(\frac{1}{p_2}\right) + p_3 \log_2\left(\frac{1}{p_3}\right) + p_4 \log_2\left(\frac{1}{p_4}\right)$$

Here we started from $k = 0$. Putting values of probabilities in above equation from Table 6.3.5 we get,

$$H = 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.2 \log_2\left(\frac{1}{0.2}\right) + 0.2 \log_2\left(\frac{1}{0.2}\right) + 0.1 \log_2\left(\frac{1}{0.1}\right) + 0.1 \log_2\left(\frac{1}{0.1}\right)$$

$$= 0.52877 + 0.46439 + 0.46439 + 0.33219 + 0.33219$$

$$= 2.12193 \text{ bits of information / message} \qquad \dots (6.3.8)$$

Now let us calculate the average number of binary digits (binits) per message. Since each message is coded with different number of binits, we should use their probabilities to calculate average number of binary digits (binits) per message. It is calculated as follows :

(ii) To obtain average codeword length :

Average codeword length can be calculated as,

$$\bar{N} = \sum_{k=0}^{4} p_k \, n_k$$

$$= 0.4\ (2) + 0.2\ (2) + 0.1\ (3) + 0.2\ (2) + 0.1\ (3)$$

$$= 2.2$$

(iii) To obtain variance of code :

Variance can be calculated as,

$$\sigma^2 = \sum_{k=0}^{4} p_k \left[n_k - \bar{N}\right]^2$$

$$= 0.4[2-2.2]^2 + 0.2[2-2.2]^2 + 0.1[3-2.2]^2$$

$$+ 0.2[2-2.2]^2 + 0.1[3-2.2]^2$$

$$= 0.16$$

(II) Placing combined symbol as low as possible :

(i) To obtain codewords :

Table 6.3.14 shows the listing of huffman coding algorithm. The combined symbol is placed as low as possible.

Symbol	Probability Stage - I	Stage - II	Stage - III	Stage - IV
s_0	0.4	0.4	0.4	0.6 0
s_1	0.2	0.2	0.4 0	0.4 1
s_3	0.2	0.2 0	0.2 1	
s_2	0.1 0	0.2 1		
s_4	0.1 1			

Table 6.3.14 : Huffman coding algorithm

(I) Symbol by symbol occurrence :

(i) To determine entropy of the source :

Entropy is given as,

$$H = \sum_{k=1}^{3} p_k \log_2\left(\frac{1}{p_k}\right)$$

$$= 0.45\log_2\left(\frac{1}{0.45}\right)+0.35\log_2\left(\frac{1}{0.35}\right)+0.2\log_2\left(\frac{1}{0.2}\right)$$

$$= 0.5184 + 0.5301 + 0.4643$$

$$= 1.5128$$

(ii) To determine Huffman code :

Table 6.3.18 lists the Huffman coding.

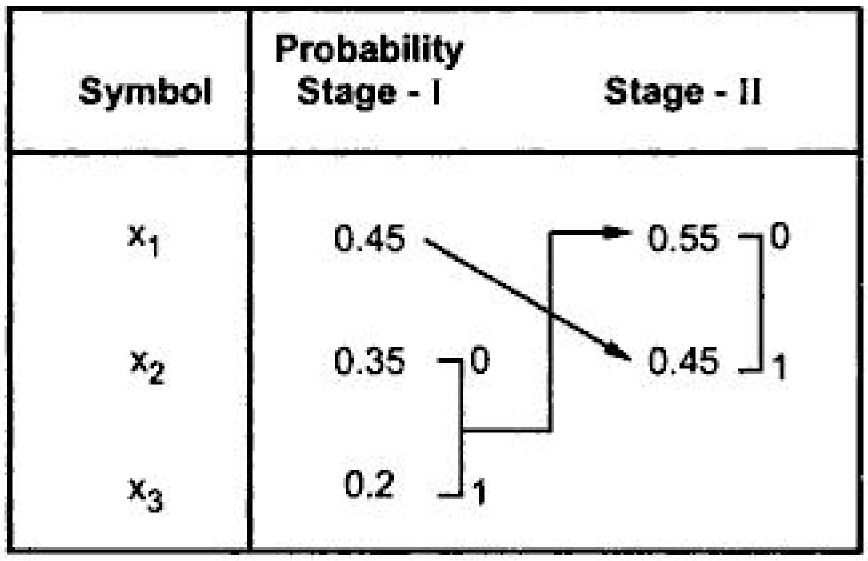

Table 6.3.18 : Huffman coding

Based on above table, codes are prepared as follows :

Symbol	Probability p_k	Digits obtained by tracing b_0 b_1	Huffman code b_1 b_0	Number of bits / symbol n_k
x_1	0.45	1	1	1
x_2	0.35	0 0	0 0	2
x_3	0.2	1 0	0 1	2

Table 6.3.19

(iii) To determine average codeword length $(\overline{N})$:

Average codeword length is given as,

$$\overline{N} = \sum_{k=1}^{3} p_k n_k = 0.45\ (1) + 0.35\ (2) + 0.2\ (2)$$

$$= 1.55$$

6.4.3 Tradeoff between Bandwidth and Signal to Noise Ratio

Channel capacity of the gaussian channel is given as,

$$C = B \log_2 \left(1+\frac{S}{N}\right) \quad \text{... (6.4.4)}$$

Above equation shows that the channel capacity depends on two factors :

i) Bandwidth (B) of the channel.

ii) Signal to noise ratio $\left(\frac{S}{N}\right)$.

Noiseless channel has infinite capacity :

If there is no noise in the channel, then $N=0$. Hence $\frac{S}{N}=\infty$. Such channel is called noiseless channel. Then capacity of such channel will be

$$C = B \log_2 (1+\infty)=\infty$$

Thus noiseless channel has infinite capacity.

Infinite bandwidth channel has limited capacity :

Now if bandwidth 'B' is infinite, the channel capacity is limited. This is because, as bandwidth increases, noise power (N) also increases. Noise power is given by equation 6.4.3 as,

$$N = N_0 B$$

Due to this increase in noise power, signal to noise (S / N) ratio decreases. Hence even if B appraches infinity, capacity does not approach infinity. As $B \rightarrow \infty$, capacity approaches an upper limit. This upper limit is given as,

$$C_\infty = \lim_{B\to\infty} C = 1.44 \frac{S}{N_0}$$

This equation is proved in next examples.

Example 6.4.1 : *This example explains tradeoff between 'B' and S / N.*

The data is to be transmitted at the rate of 10000 bits/sec over a channel having bandwidth B = 3000 Hz. Detrmine the signal to noise ratio required. If the bandwidth is increased to 10000 Hz, then determine the signal to noise ratio.

Solution : The data is to be transmitted at the rate of 10,000 bits/sec. Hence channel capacity must be at least 10000 bits/sec for errorfree transmission. Hence from equation 6.4.2,

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$$\therefore \qquad H = \frac{1}{2}\log_2\left(1+\frac{12}{\lambda^2}\cdot\frac{S}{N}\right) \text{ bits / message} \qquad \dots (6.4.14)$$

The bandwidth of the signal is 'B' Hz. Since the signal is sampled at the Nyquist rate, it will be,

$$\text{Sampling rate} = 2\times\text{maximum frequency of the signal}$$

$$= 2B \text{ samples / sec.}$$

Each sample represents one message. Hence the message rate will be '2B' messages per second. Thus,

$$\text{Message rate} \quad r = 2B \quad \text{messages / sec.} \qquad \dots (6.4.15)$$

Information Rate 'R' is given as,

$$R = rH$$

Here r is message rate and H is average amount of information putting values of r and H from equation 6.4.15 and equation 6.4.14 we get,

$$R = 2B \text{ messages / sec.} \times\frac{1}{2}\log_2\left(1+\frac{12}{\lambda^2}\cdot\frac{S}{N}\right) \text{bits/message}$$

$$= B\log_2\left(1+\frac{12}{\lambda^2}\cdot\frac{S}{N}\right) \text{ bits / sec.} \qquad \dots (6.4.16)$$

Shannon's theorem gives the lower limit on channel capacity with acceptable probability of error. It is given by equation 6.4.1 i.e.,

$$C \geq R$$

$$\therefore \qquad C \geq B\log_2\left(1+\frac{12}{\lambda^2}\cdot\frac{S}{N}\right)$$

Considering the lower limit on channel capacity, above equation will be,

$$C = B\log_2\left(1+\frac{12}{\lambda^2}\cdot\frac{S}{N}\right) \qquad \dots (6.4.17)$$

This is the required expression for channel capacity for given signal waveform.

The above equation will be identical to equation 6.4.2 if

$$\frac{12}{\lambda^2} = 1 \text{ or } \lambda = 3.5$$

$$\therefore \qquad \frac{S}{N} \geq 255$$

or
$$\left(\frac{S}{N}\right)_{dB} = 10 \log 255 = \mathbf{24\ dB}$$

iv) To determine bandwidth :

The given $\frac{S}{N}$ ratio is 20 dB. Hence,

$$\left(\frac{S}{N}\right)_{dB} = 10 \log \frac{S}{N}$$

$$20 = 10 \log \frac{S}{N}$$

$$\therefore \qquad \frac{S}{N} = 100 \text{ dB}$$

We know that for error-free transmission $R \leq C$. Hence,

$$R \leq C = B \log_2\left(1+\frac{S}{N}\right)$$

i.e.
$$R \leq B \log_2\left(1+\frac{S}{N}\right)$$

Here R = 80000 bits/sec. and $\frac{S}{N}$ = 100. The bandwidth can be obtained from above equation i.e.

$$80000 \leq B \log_2(1+100)$$

$$\leq B \log_2(101)$$

$$\therefore \qquad B \geq \frac{80000}{\log_2(101)} \qquad \therefore \quad B \geq \mathbf{11.974\ kHz.}$$

This is the bandwidth required for errorfree transmission.

Example 6.4.5 : *For an AWGN channel with 4 kHz bandwidth and noise power spectral density* $\frac{N_0}{2} = 10^{-12}$ *W/Hz, the signal power required at the receiver is 0.1 mW. Calculate capacity of this channel.*

Solution : From equation 6.4.2, the channel capacity is given as,

$$C = B \log_2\left(1+\frac{S}{N}\right) \text{ bits/sec.}$$

Here
$$B = 4000 \text{ Hz}$$

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Picture repetition rate = 32/sec.

$$\left(\frac{S}{N}\right)_{dB} = 30$$

(ii) To obtain the source symbol entropy (H) :

Source emits any one of the 16 brightness levels. Here M = 16. These levels are equiprobable. Hence entropy of such source is given as,

$$H = \log_2 M$$

$$= \log_2 16$$

$$= \mathbf{4} \textbf{ bits/symbol (level)}$$

(iii) To obtain symbol rate (r) :

Each picture consists of 2×10^6 picture elements. Such 32 pictures are transmitted per second. Hence number of picture elements per second will be,

$$r = 2 \times 10^6 \times 32 \quad \text{symbols/sec.}$$

$$= \mathbf{64 \times 10^6} \quad \textbf{symbols/sec.}$$

(iv) To calculate average information rate (R) :

Information rate of the source is given as,

$$R = rH = 64 \times 10^6 \times 4 \text{ bits/sec.}$$

$$= \mathbf{2.56 \times 10^8} \textbf{ bits/sec.}$$

This is the average rate of information conveyed by TV picture.

(v) To obtain the required bandwidth for $\frac{S}{N}$ = 30 dB :

We know that $\left(\frac{S}{N}\right)_{dB} = 10 \log_{10} \frac{S}{N}$

$$\therefore \quad 30 = 10 \log_{10} \frac{S}{N}$$

$$\therefore \quad \frac{S}{N} = 1000$$

Channel coding theorem states that information can be received without error if,

$$R \leq C$$

We have $R = 2.56 \times 10^8$ and $C = B \log_2\left(1 + \frac{S}{N}\right)$. Hence above equation becomes,

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7 Linear Block Codes

7.1 Introduction

- Errors are introduced in the data when it passes through the channel. The channel noise interferes the signal. The signal power is also reduced.
- Hence errors are introduced. In this chapter we will study various types of error detection and correction techniques.

7.1.1 Rationale for Coding and Types and Codes

- The transmission of the data over the channel depends upon two parameters. They are transmitted power and channel bandwidth. The power spectral density of channel noise and these two parameters determine signal to noise power ratio.
- The signal to noise power ratio determine the probability of error of the modulation scheme. For the given signal to noise ratio, the error probability can be reduced further by using coding techniques. The coding techniques also reduce signal to noise power ratio for fixed probability of error.

Fig. 7.1.1 shows the block diagram of the digital communication system which uses channel coding.

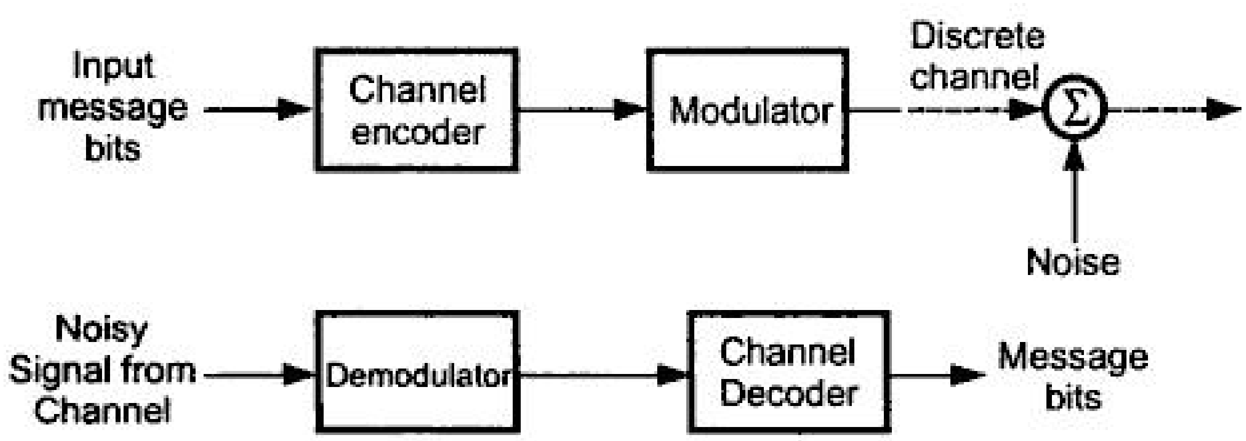

Fig. 7.1.1 Digital communication system with channel encoding

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$$= \frac{k}{2^k} \qquad \text{... (7.2.39)}$$

This shows that with increase in 'k', the code rate becomes very small.

Hadamard Matrix and Codewords :

There are some following important points as follows :

i) One code vector represented by hadamard matrix contains all zero elements. That is one row of hadamard matrix contains all zero elements.

ii) The other code vectors contain $\frac{n}{2}$ 1's and $\frac{n}{2}$ 0's. That is other rows of hadamard matrix contains half number of 1's and half number of 0's.

iii) Every code vector differs from other code vectors at $\frac{n}{2}$ places. This means every row of hadamard matrix differs with other rows at $\frac{n}{2}$ places (i.e. half number of places). Consider the hadamard matrix with single message bit i.e. $k = 1$. Hence,

$$n = 2^k = 2^1 = 2$$

Thus hadamard matrix for single message bit will be of $n \times n$ (i.e. 2×2) size. The first row will be all zero elements. This matrix is shown below.

$$H_2 = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\uparrow} \begin{matrix} \leftarrow \text{All zero row} \\ \end{matrix} \qquad \text{... (7.2.40)}$$

These elements satisfy the points

(ii) and (iii) mentioned above.

Here H_2 is 2×2 size hadamard matrix. Observe that the second row contains half number of elements as zero and half as 1's. We know that the codewords are the row of hadamard matrix. Here the codewords are 00 and 01. Consider the hadamard matrix for two message bits (i.e. $k = 2$). Then we have,

$$n = 2^k = 2^2 = 4$$

Thus the hadamard matrix will be of size 4×4. This matrix is shown below.

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & \overline{H}_2 \end{bmatrix} \qquad \text{... (7.2.41)}$$

Solution : (i) Consider (n, k) block code

For this code generator matrix and parity check matrix are defined as,

$$[G]_{k\times n} = \left[I_{k\times k} \middle| P_{k\times q}\right]_{k\times n} \quad \text{... (7.2.55)}$$

$$[H]_{q\times n} = \left[P^T_{q\times k} \middle| I_{q\times q}\right]_{q\times n} \quad \text{... (7.2.56)}$$

The generator matrix and parity check matrix satisfy following property,

$$[HG^T]_{q\times k} = \left[P^T_{q\times k} \middle| I_{q\times q}\right] \left[\frac{I_{k\times k}}{P^T_{q\times k}}\right]_{n\times k}$$

$$= \left[P^T \oplus P^T\right]_{q\times k} = 0 \quad \text{... (7.2.57)}$$

(ii) Consider (n, q) block code

We know that (n, q) code is dual code of (n, k) block code. The generator and parity check matrices of this (n, q) code can be written from equation 7.2.53 and equation 7.2.54 as,

$$[G_{dual}]_{q\times n} = \left[I_{q\times q} \middle| P_{q\times k}\right]_{q\times n} \quad \text{... (7.2.58)}$$

$$[H_{dual}]_{k\times n} = \left[P^T_{k\times q} \middle| I_{k\times k}\right]_{k\times n} \quad \text{... (7.2.59)}$$

Here we have written G_{dual} and H_{dual} so that they can be differentiated from G and H of (n, k) code.

Now let us check whether (n, q) code also satisfies the property of equation 7.2.52, i.e.,

$$[H_{dual}\, G^T_{dual}] = \left[P^T_{k\times q} \middle| I_{k\times k}\right]_{k\times n} \left[\frac{I_{q\times q}}{P^T_{k\times q}}\right]_{n\times q}$$

$$= \left[P^T \oplus P^T\right]_{k\times q} = 0$$

i.e. $[H_{dual}\, G^T_{dual}]_{k\times q} = [0]_{k\times q}$

Taking transpose of both the sides of above equation,

$$[H_{dual}\, G^T_{dual}]^T_{k\times q} = [0]^T_{k\times q}$$

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$$\therefore \quad 3 \geq s+1$$

$$\therefore \quad s \leq 2$$

Thus two errors will be detected

and $$d_{min} \geq 2t+1$$

$$\therefore \quad 3 \geq 2t+1$$

$$\therefore \quad t \leq 1$$

Thus one error will be corrected.

iv) To prepare the decoding table :

The parity check matrix (H) is given as,

$$H = [P^T \; : \; I_q]_{q \times n}$$

Hence transpose of above matrix becomes,

$$H^T = \begin{bmatrix} P \\ \cdots \\ I_q \end{bmatrix}_{n \times q}$$

From equation 7.2.65, above matrix will be,

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots (7.2.67)$$

The syndrome vector (S) can be calculated from error vector (E) and H^T by equation 7.2.30 as,

$$S = EH^T$$

Here E is the 1×6 size error vector. Let us calculate syndrome for 2nd bit in error. The E will be,

$$E = [0\ 1\ 0\ 0\ 0\ 0]$$

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Hence the matrix P will be,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The generator matrix is given as (equation 7.2.6)

$$G = [I_k : P_{k \times q}]_{k \times n}$$

$$\therefore \qquad G = \begin{bmatrix} 1 & 0 & 0 : 1 & 1 & 0 \\ 0 & 1 & 0 : 0 & 1 & 1 \\ 0 & 0 & 1 : 1 & 0 & 1 \end{bmatrix} \qquad \text{... (7.2.69)}$$

This is the required generator matrix.

ii) To obtain the codeword that begins with 101

Here the codeword begins with 101. This means first three bits of the codeword are 101. Length of the message bits is $k=3$. In systematic code, first 'k' bits of codeword are message bits. Hence first '3' bits in every codeword will be message bits. Thus 101 are message bits. i.e.,

$$M = [1\ 0\ 1]$$

This is (6, 3) code. The three check bits can be obtained by the equation,

$$C = MP$$

Putting appropriate matrices,

$$C = [1\ 0\ 1] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= [1 \oplus 0 \oplus 1 \quad 1 \oplus 0 \oplus 0 \quad 0 \oplus 0 \oplus 1] = [0\ 1\ 1]$$

Hence the code vector is,

$$X = (m_1\ m_2\ m_3\ C_1\ C_2\ C_3) = (1\ 0\ 1\ 0\ 1\ 1)$$

Thus the codeword that begins with 1 0 1 .... is X = 1 0 1 0 1 1.

iii) To decode 1 1 0 1 1 0 :

Let the received codeword be,

$$Y = 1\ 1\ 0\ 1\ 1\ 0$$

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We know that $(AB)^T = B^T A^T$. Hence above equation becomes,

$$(G^T)^T H^T = (0)^T$$

Here $(0)^T = 0$. Therefore above equation will be,

$$GH^T = 0$$

Thus the relation between G and H is,

$$HG^T = GH^T = 0$$

Example 7.2.11 : *The parity check bits of a (8,4) block code are generated by,*

$$C_5 = d_1 + d_2 + d_4$$
$$C_6 = d_1 + d_2 + d_3$$
$$C_7 = d_1 + d_3 + d_4$$
$$C_8 = d_2 + d_3 + d_4$$

where d_1, d_2, d_3 and d_4 are the message bits.

i) Find the generator matrix and the parity check matrix for this code.

ii) List all code vectors

iii) Find the errors detecting and correcting capabilities of this code.

iv) Show through an example that this code detects upto 3 errors.

Solution : i) To obtain the generator matrix and parity check matrix

We know that the check bits, message bits and parity matrix are related as,

$$[C_5\, C_6\, C_7\, C_8]_{1\times 4} = [d_1\, d_2\, d_3\, d_4]_{1\times 4} [P]_{4\times 4}$$

$$= [d_1\, d_2\, d_3\, d_4] \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}$$

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$$d_{\min} = [w(X)]_{\min} = 3$$

ii) Error correction and detection capabilities

$$d_{\min} \geq s+1$$

$$3 \geq s+1$$

$$\therefore \quad s \leq 2$$

Thus two errors will be detected.

and $$d_{\min} \geq 2t+1$$

$$3 \geq 2t+1$$

$$\therefore \quad t \leq 1$$

Thus one error will be corrected.

This is hamming code ($d_{\min} = 3$) and it always detects double errors and corrects single errors.

(iii) To obtain message bits, if Y = 1 0 1 1 0 1

We have to determine whether the received vector is a valid code vector. This can be done by calculating syndrome.

To obtain syndrome (S)

Syndrome vector is given as,

$$S = YH^T$$

We know that $$H^T = \begin{bmatrix} P \\ \cdots\cdots \\ I_q \end{bmatrix}_{n \times q}$$

Putting the value of P submatrix in above equation,

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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This codeword can be represented by a polynomial of degree less than or equal to $(n-1)$. i.e.,

$$X(p) = x_{n-1} p^{n-1} + x_{n-2} p^{n-2} + ... + x_1 p + x_0 \qquad ... (7.3.4)$$

Here $X(p)$ is the polynomial of degree $(n-1)$.

p is the arbitrary variable of the polynomial.

The power of 'p' represent the positions of the codeword bits. i.e.,

p^{n-1} represents MSB

p^0 represents LSB

p^1 represents second bit from LSB side.

Why to represent codewords by a polynomial ?

Polynomial representation is used due to following reasons :

i) These are algebraic codes. Hence algebraic operations such as addition, multiplication, division, subtraction etc. becomes very simple.

ii) Positions of the bits are represented with the help of powers of p in a polynomial.

7.3.3.1 Generation of Code vectors in Nonsystematic Form

Let $M = \{m_{k-1}, m_{k-2}, ... m_1, m_0\}$ be 'k' bits of message vector. Then it can be represented by the polynomial as,

$$M(p) = m_{k-1} p^{k-1} + m_{k-2} p^{k-2} + ... + m_1 p + m_0 \qquad ... (7.3.5)$$

Let $X(p)$ represent the codeword polynomial. It is given as,

$$\boxed{X(p) = M(p)\, G(p)} \qquad ... (7.3.6)$$

Here $G(p)$ is the *generating polynomial* of degree 'q'. For an (n, k) cyclic code, $q = n - k$ represent the number of parity bits. The generating polynomial is given as,

$$G(p) = p^q + g_{q-1} p^{q-1} + ... + g_1 p + 1 \qquad ... (7.3.7)$$

Here $g_{q-1}, g_{q-2}, ... g_1$ are the parity bits.

If M_1, M_2, M_3 .... etc are the other message vectors, then the corresponding code vectors can be calculated as,

$$X_1(p) = M_1(p)\, G(p)$$

$$X_2(p) = M_2(p)\, G(p)$$

$$X_3(p) = M_3(p) G(p) \text{ and so on} \qquad ... (7.3.8)$$

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This is the required cyclic code vectors in systematic form. The other code vectors can be obtained using the same procedure.

Table 7.3.2 lists all the systematic cyclic codes.

Sr.	Message bits				Systematic code vectors						
No.	M = m_3	m_2	m_1	m_0	X = m_3	m_2	m_1	m_0	c_2	c_1	c_0
1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	1	0	1	1
3	0	0	1	0	0	0	1	0	1	1	0
4	0	0	1	1	0	0	1	1	1	0	1
5	0	1	0	0	0	1	0	0	1	1	1
6	0	1	0	1	0	1	0	1	1	0	0
7	0	1	1	0	0	1	1	0	0	0	1
8	0	1	1	1	0	1	1	1	0	1	0
9	1	0	0	0	1	0	0	0	1	0	1
10	1	0	0	1	1	0	0	1	1	1	0
11	1	0	1	0	1	0	1	0	0	1	1
12	1	0	1	1	1	0	1	1	0	0	0
13	1	1	0	0	1	1	0	0	0	1	0
14	1	1	0	1	1	1	0	1	0	0	1
15	1	1	1	0	1	1	1	0	1	0	0
16	1	1	1	1	1	1	1	1	1	1	1

Table 7.3.2 Code vectors of a (7, 4) cyclic code for $G(p) = p^3 + p + 1$

We have obtained nonsystematic code vectors for the same generating polynomial in Example 7.3.1. They are listed in Table 7.3.1.

Example 7.3.3 : *An 'n' digit code polynomial* $X(p)$ *is obtained as,*

$$X(p) = C(p) + p^{(n-k)} M(p)$$

where $M(p)$ *represents message polynomial for k digit data and* $C(p)$ *is remainder polynomial obtained by dividing* $p^{(n-k)} M(p)$ *by proper generator polynomial* $G(p)$, *in modulo-2 sense. Prove that* $X(p)$ *represents a systematic cyclic code if* $G(p)$ *is the factor of* $p^n + 1$ *in modulo-2 sense.* **(March-2006, 8 Marks)**

Since $q=3$, there are '3' flip-flops in shift register to hold check bits $c_1\ c_2$ and c_0. Since $g_2=0$, its link is not connected. $g_1=1$, hence its link is connected. Now let's verify the operation of this encoder for message vector $M=(m_3\ m_2\ m_1\ m_0)=(1100)$. Table 7.3.3 shows the contents of shift registers before and after shifts.

Input message bit m	Register bit inputs before shift			Register bit outputs after shift		
	$r_2=r_2'$	$r_1=r_1'$	$r_0=r_0'$	$r_2'=r_1$	$r_1'=r_0\oplus r_2\oplus m$	$r_0'=r_2\oplus m$
–	0	0	0	0	0	0
1	0	0	0	0	$0\oplus 0\oplus 1=1$	$0\oplus 1=1$
1	0	1	1	1	$1\oplus 0\oplus 1=0$	$0\oplus 1=1$
0	1	0	1	0	$1\oplus 1\oplus 0=0$	$1\oplus 0=1$
0	0	0	1	0	$1\oplus 0\oplus 0=1$	$0\oplus 0=0$

Table 7.3.3 Shift register bits positions for input message M = (1100)

The above table shows that at the end of last message bit the register bit outputs are $r_2'=0$, $r_1'=1$ and $r_0'=0$. The feedback switch is opened and output switch is closed to check bits position. The check bits are then shifted to the transmitter. The check bits are shifted as $c_2=r_2'$, $c_1=r_1'$ and $c_0=r_0'$. The following table illustrates the shift operation of message and check bits. We know that the code vector is,

$$X = (m_3\ m_2\ m_1\ m_0\ c_2\ c_1\ c_0)=(1\ 1\ 0\ 0\ 0\ 1\ 0)$$

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2) The encoders and decoders for cyclic codes are simpler compared to noncyclic codes.

3) Cyclic codes also detect error burst that span many successive bits.

4) Cyclic codes have well defined mathematical structure. Hence very efficient decoding schemes are possible.

Inspite of these advantages cyclic codes also have some disadvantages.

Disadvantages :

1) The error detection in cyclic codes is simpler but error correction is little complicated since the combinational logic circuits in error detector are complex.

To avoid such complex circuits some special cyclic codes are used which are discussed next.

7.3.9 BCH Codes (Bose - Chaudhri - Hocquenghem Codes)

Features

- BCH codes are most extensive and powerful error correcting cyclic codes. The decoding of BCH codes is comparatively simpler.
- For any positive integer m and t (where $t<(2^m-1)/2$) there exists a BCH code with following parameters

Block length : $n=2^m-1$

Number of parity check bits : $n-k \leq mt$

Minimum distance : $d_{\min} \geq 2t+1$

- Each BCH code can detect and correct upto 't' random errors.
- BCH codes have flexibility in selection of block length and code rate.
- BCH codes are the best codes for block lengths upto few hundred bits.

The decoding schemes of BCH codes can be implemented on digital computer. Because of software implementation of decoding schemes they are quite flexible compared to hardware implementation of other schemes.

7.3.10 Reed-Soloman (RS) Codes

Features

- These are nonbinary BCH codes. The encoder of RS codes operate on multiple bits simultaneously.
- The (n, k) RS code takes the groups of m-bit symbols of the incoming binary data stream. It takes such 'k' number of symbols in one block. Then the encoder adds (n – k) redundant symbols to form the codeword of 'n' symbols.

$$x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3 \ldots\ldots\ldots\ldots x_{14} y_{14} z_{14} x_{15} y_{15} z_{15} \qquad \ldots (7.3.68)$$

This is interlaced sequence. Here we have interlaced three code vectors. Hence $\lambda = 3$. Therefore the interlaced code will be of dimension,

$$(\lambda n\ \lambda k) = (3\times 15,\ 3\times 8) = (45,\ 24)$$

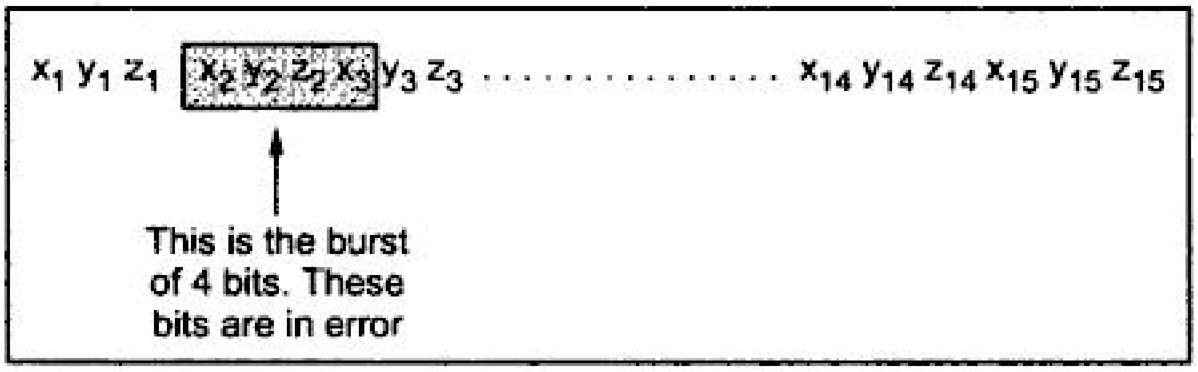
How burst errors are corrected by interlaced codes ?

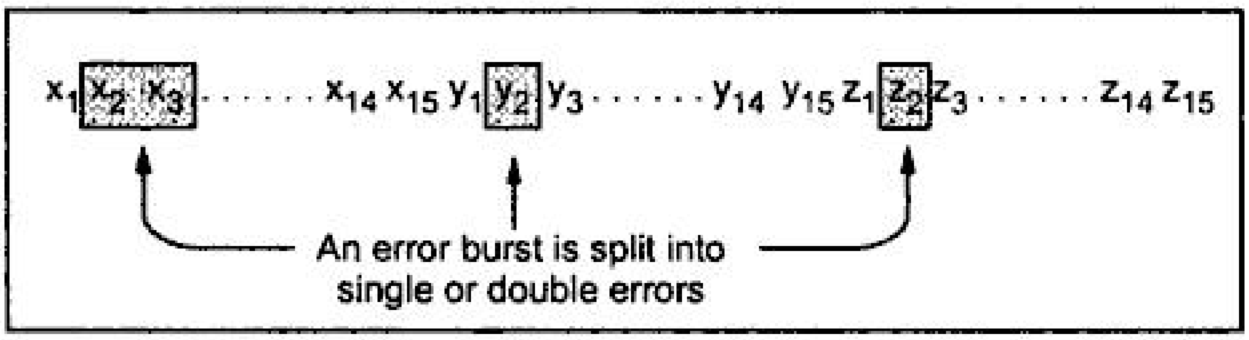
Consider the interlaced sequence of equation 7.3.68,

During transmission of the interlaced sequence, a burst of error takes place as shown below. Four successive bits are in error. At the receiver, when this sequence is received, it is converted to its noninterlaced natural form. That is given by equation 7.3.67. It is shown adjacent.

As shown above, the single burst of 4 digits is split into single or double errors.

Error detecting and correcting capabilities of $(\lambda n, \lambda k)$ code :





Let the (n,k) code corrects 't' digits. Then the interlaced code can correct any combination of 't' bursts of length λ or less.

Why cyclic codes are more suitable for burst error correction ?

If the code (n,k) is cyclic, then its interlaced version $(\lambda n, \lambda k)$ is also cyclic. If $G(p)$ is the generating polynomial of (n,k) code, then $G(p^{\lambda})$ is the generating polynomial of $(\lambda n, \lambda k)$ code.

Therefore encoding and decoding of interlaced code is also possible using shift registers. To obtain the decoder of interlaced code, each shift register stage of (n,k) cyclic code is replaced with λ stages without changing other connections. Because of all the above reasons, cyclic codes are more suitable for detecting and correcting error bursts.

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and $$x_2 = m \oplus m_2 \quad \text{... (8.1.2)}$$

The output switch first samples x_1 and then x_2. The shift register then shifts contents of m_1 to m_2 and contents of m to m_1. Next input bit is then taken and stored in m. Again x_1 and x_2 are generated according to this new combination of m, m_1 and m_2 (equation 8.1.1 and equation 8.1.2). The output switch then samples x_1 then x_2. Thus the output bit stream for successive input bits will be,

$$X = x_1 x_2 x_1 x_2 x_1 x_2 \ldots \text{ and so on} \quad \text{... (8.1.3)}$$

Here note that for every input message bit two encoded output bits x_1 and x_2 are transmitted. In other words, for a single message bit, the encoded code word is two bits i.e. for this convolutional encoder,

Number of message bits, k = 1

Number of encoded output bits for one message bit, n = 2

8.1.2 Code Rate of Convolutional Encoder

The code rate of this encoder is,

$$r = \frac{k}{n} = \frac{1}{2} \quad \text{... (8.1.4)}$$

In the encoder of Fig. 8.1.1, observe that whenever a particular message bit enters a shift register, it remains in the shift register for three shifts i.e.,

First shift $\rightarrow$ Message bit is entered in position 'm'.

Second shift $\rightarrow$ Message bit is shifted in position m_1.

Third shift $\rightarrow$ Message bit is shifted in position m_2.

And at the fourth shift the message bit is discarded or simply lost by overwriting. We know that x_1 and x_2 are combinations of m, m_1, m_2. Since a single message bit remains in m during first shift, in m_1 during second shift and in m_2 during third shift; it influences output x_1 and x_2 for 'three' successive shifts.

8.1.3 Constraint Length (K)

The constraint length of a convolution code is defined as the number of shifts over which a single message bit can influence the encoder output. It is expressed in terms of message bits.

---

* Some authors define constraint length as number of output bits influenced by a single message bit i.e.

Constraint length (k) = (n × M) bits ... (8.1.5)

where n = number of encoded output bits for every input bit

and M = number of storage elements in the shift register

For the encoder of Fig. 8.1.1 Constraint length = $2 \times 3 = 6$ bits.

$i = 6$ in equation 8.2.6, $x_6^{(1)} = g_0^{(1)} m_6 \oplus g_1^{(1)} m_5 \oplus g_2^{(1)} m_4$

$$= g_2^{(1)} m_4 \qquad \text{since } m_6 \text{ and } m_5 \text{ are not available}$$

$$= 1 \times 1$$

$$= 1$$

Thus the output of adder 1 is,

$$\boxed{x_1 = x_i^{(1)} = \{1\ 1\ 1\ 1\ 0\ 0\}}$$

To obtain output due to adder 2

Similarly from equation 8.2.2,

$$x_1 = x_i^{(2)} = \sum_{l=0}^{M} g_l^{(2)} m_{i-l}$$

And $\qquad m_{i-l} = 0$ for all $l > i$.

with $i = 0$ in above equation we get,

$$x_0^{(2)} = g_0^{(2)} m_0 = (1 \times 1) = 1 \quad \text{Here } g_0^{(2)} = 1 \text{ and } m_0 = 1$$

With $i = 1$, $\quad x_1^{(2)} = g_0^{(2)} m_1 \oplus g_1^{(1)} m_0$

$$= (1 \times 0) \oplus (0 \times 1)$$

$$= 0$$

With $i = 2$, $\quad x_2^{(2)} = g_0^{(2)} m_2 \oplus g_1^{(2)} m_1 \oplus g_2^{(2)} m_0$

$$= (1 \times 0) \oplus (0 \times 0) \oplus (1 \times 1)$$

$$= 1$$

With $i = 3$, $\quad x_3^{(2)} = g_0^{(2)} m_3 \oplus g_1^{(2)} m_2 \oplus g_2^{(2)} m_1$

$$= (1 \times 1) \oplus (0 \times 0) \oplus (1 \times 0)$$

$$= 1$$

With $i = 4$, $\quad x_4^{(2)} = g_0^{(2)} m_4 \oplus g_1^{(2)} m_3 \oplus g_2^{(2)} m_2$

$$= (1 \times 1) \oplus (0 \times 1) \oplus (1 \times 0)$$

$$= 1$$

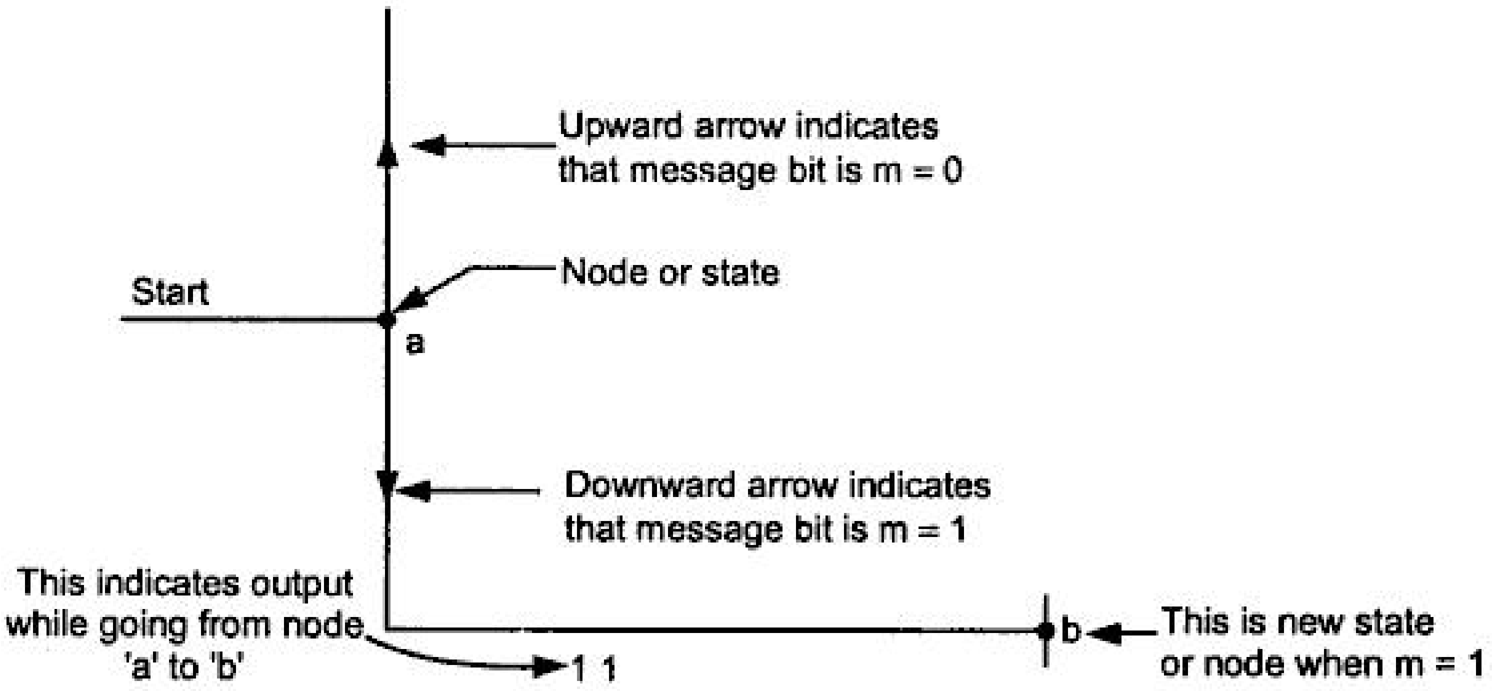
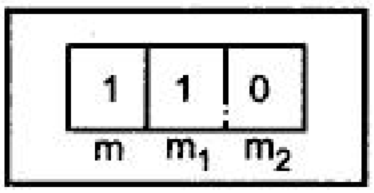


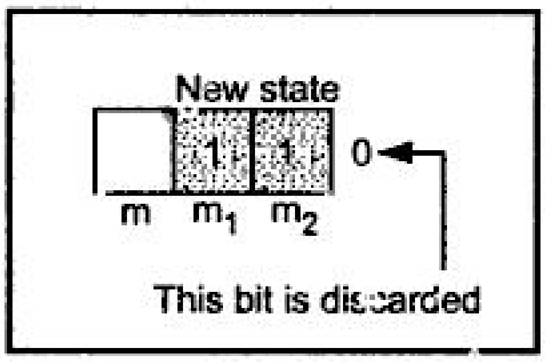
Fig. 8.3.2 Code tree from node 'a' to 'b'

Observe that if m = 1 we go downward from node 'a'. Otherwise if m = 0, we go upward from node 'a'. It can be verified that if m = 0 then next node (state) is 'a' only. Since m = 1 here we go downwards toward node b and output is 11 in this node (or state).

2) *When $m = 1$ i.e. second bit*

Now let the second message bit be 1. The contents of shift register with this input will be as shown below.





$$x_1 = 1 \oplus 1 \oplus 0 = 0$$

$$x_2 = 1 \oplus 0 = 1$$

These values of $x_1 x_2 = 01$ are then transmitted to output and register contents are shifted to right by one bit. The next state formed is as shown.

Thus the new state of the encoder is $m_2 m_1 = 11$ or 'd' and the outputs transmitted are $x_1 x_2 = 01$. Thus the encoder goes from state 'b' to state 'd' if input is '1' and transmitted output $x_1 x_2 = 01$. This operation is illustrated by Table 8.3.2 in second row. The last column of the table shows the code tree for those first and second input bits.

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For example consider that the encoder is in state 'a'. If input $m = 0$, then next state is same i.e. a (i.e. 00) with outputs $x_1 x_2 = 00$. This is shown by self loop at node 'a' in the state diagram. If input $m = 1$, then state diagram shows that next state is 'b' with outputs $x_1 x_2 = 11$

Comparison between code tree and trellis diagram :

Table 8.3.3 shows the comparison between code tree and trellis diagram as a graphic structures to generate and decode convolutional code.

Sr. No.	Code tree	Trellis diagram
1	Code tree indicates flow of the coded signal along the nodes of the tree.	Trellis diagram indicates transitions from current to next states.
2	Code tree is lengthy way of representing coding process.	Code trellis diagram is shorter or compact way of representing coding process.
3	Decoding is very simple using code tree.	Decoding is little complex using trellis diagram.
4	Code tree repeats after number of stages used in the encoder.	Trellis diagram repeats in every state. In steady state, trellis diagram has only stage.
5	Code tree is complex to implement in programming.	Trellis diagram is simpler to implement in programming.

Table 8.3.3 Comparison between code tree and trellis diagram

8.4 Decoding Methods of Convolutional Codes

These methods are used for decoding of convolutional codes. They are viterbi algorithm, sequential decoding and feedback decoding. Let's consider them in details in subsequent sections.

8.4.1 Viterbi Algorithm for Decoding of Convolutional Codes (Maximum Likelihood Decoding)

Let's represent the received signal by Y. Convolutional encoding operates continuously on input data. Hence there are no code vectors and blocks as such. Let's assume that the transmission error probability of symbols 1's and 0's is same. Let's define an integer variable metric as follows.

Metric :

It is the discrepancy between the received signal Y and the decoded signal at particular node. This metric can be added over few nodes for a particular path.

Surviving Path :

This is the path of the decoded signal with minimum metric.

the code vector is equal to minimum weight of the code vector. Therefore a free distance is equal to the minimum distance between the code vectors. Since minimum distance is equal to minimum weight of the code vector we can write,

$$\text{Free distance } (d_f) = \text{Minimum distance between code vectors}$$

$$= \text{Minimum weight of the code vectors}$$

i.e. $$d_f = [w(X)]_{min} \text{ and } X \text{ is non zero} \quad \text{... (8.4.3)}$$

Here $[w(X)]_{min}$ is the minimum weight of code vector. For convolutional coding free distance (d_f) represents the error control power.

Coding Gain :

Coding gain is used as a basis of comparison for different coding methods. To achieve the same bit error rate the coding gain is defined as,

$$A = \frac{\left(\frac{E_b}{N_0}\right) Encoded}{\left(\frac{E_b}{N_0}\right) coded} \quad \text{... (8.4.4)}$$

For convolutional coding the coding gain is given as,

$$A = \frac{r\, d_f}{2} \quad \text{... (8.4.5)}$$

here 'r' is the code rate and

d_f is the free distance.

Example 8.4.1 : *The figure below depicts a rate 1/2, constraint length N = 2 convolutional code encoder. Sketch the code tree for the same.*

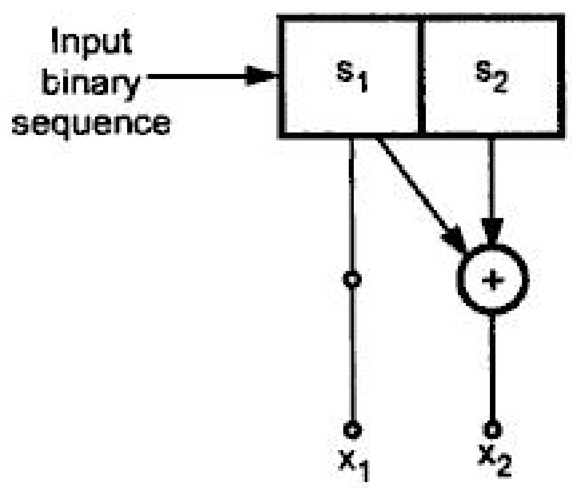

Fig. 8.4.6 Convolutional encoder

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b) Obtain message polynomial

The message polynomial becomes,

$$m=(10011) \Rightarrow m(p)=1+p^3+p^4$$

c) Output sequence due to g_1

Output of sequence g_1 is given as,

$$x_1(p)=g_1(p)m(p) = (1+p^2)(1+p^3+p^4)$$
$$= 1+p^2+p^3+p^4+p^5+p^6$$

Hence $\quad x_1 = (1011111)$

d) Output sequence due to g_2

Similarly output of g_2 is given as,

$$x_2(p)=g_2(p)m(p) = (1+p)(1+p^3+p^4)$$
$$= 1+p+p^3+p^5$$

Hence $\quad x_2 = (1\,1\,0101)$

e) Output sequence due to g_3

Output of g_3 is given as,

$$x_3(p)=g_3(p)m(p) = (1+p+p^2)(1+p^3+p^4)$$
$$= 1+p+p^2+p^3+p^6$$

Hence $\quad x_3 = (1111001)$

f) Multiplexing the sequences due to g_1, g_2 and g_3

The multiplexer will multiplex the bits of x_1, x_2 and x_3 as follows :

Output sequence = {1 1 1 0 1 1 1 0 1 1 1 1 1 0 0 1 1 0 1 0 1}

Note that x_2 contains only 6 output digits. Hence its 7th output digit is assumed zero in above multiplexed sequence.

If hardware is enhanced by adding shift registers and adders

a) Effect on generated output sequence

For each input message bit, three output bits are generated. There are three mod-2 adders in the encoder. Therefore three output bits are generated. *If mod-2 adders are increased, then output bits for every message bit are increased.* Therefore length of the coded sequence increases.

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d) To determine lowest order of 'D' in $T(D)$

Let us determine the polynomial of $T(D)$. i.e.,

$$
\begin{array}{r|l}
 & D^6 + 2D^8 + 4D^{10} + 8D^{12} \\
\hline
1 - 2D^2 & D^6 \\
 & \underline{D^6 - 2D^8} \\
 & \quad 2D^8 \\
 & \quad \underline{2D^8 - 4D^{10}} \\
 & \qquad 4D^{10} \\
 & \qquad \underline{4D^{10} - 8D^{12}} \\
 & \qquad\quad 8D^{12} \\
 & \qquad\quad \underline{8D^{12} - 16D^{14}} \\
 & \qquad\qquad 16D^{14} \quad \text{.... and so on}
\end{array}
$$

Thus,

$$T(D) = \frac{D^6}{1-2D^2} = D^6 + 2D^8 + 4D^{10} + 8D^{12} + ...$$

Above equation shows that first term is D^6. This means, there is single path of distance '6' between node 'a' and 'e'. This path is abce in Fig. 8.7.8. Second term in $T(D)$ is $2D^8$. This means there are two paths of distance 8 and so on.

Free distance (d_{free}) :

The free distance is given by lowest order of the term in $T(D)$. Here it is 6. Hence,

$$d_{free} = 6$$

d_{free} can be obtained by inspection of signal flow graph

Here we derived $T(D)$, then determine d_{free}. We know that d_{free} is the minimum distance path between nodes 'a' and 'e'. By looking at Fig. 8.7.8 we can say that the minimum distance path is a-b-c-e. Along this path the distances are D^3, D and D^2. They correspond to distances of 3, 1 ans 2 respectively. Hence the distance between 'a' and 'e' will be $3+1+2=6$. Other paths will have definitely more distances.

Hence,

$$d_{free} = 6$$

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Solution : i) To prepare code trellis :

First we will prepare the code trellis diagram. Let the states of the encoder be defined as follows :

$$s_2\, s_1 = 00, \quad \text{state 'a'}$$
$$s_2\, s_1 = 01, \quad \text{state 'b'}$$
$$s_2\, s_1 = 10, \quad \text{state 'c'}$$
$$s_2\, s_1 = 11, \quad \text{state 'd'}$$

A table is prepared that shows the state transitions, message input and output. This table is as follows :

Sr. No.	Current state $s_2\, s_1$	Input s	Outputs $x_1 = s \oplus s_1 \oplus s_2$	$x_2 = s \oplus s_2$	Next state $s_1\, s$
1	a = 0 0	0	0	0	0 0, i.e. a
		1	1	1	0 1, i.e. b
2	b = 0 1	0	1	0	1 0, i.e. c
		1	0	1	1 1, i.e. d
3	c = 1 0	0	1	1	0 0, i.e. a
		1	0	0	0 1, i.e. b
4	d = 1 1	0	0	1	1 0, i.e. c
		1	1	0	1 1, i.e. d

Table 8.7.4 : State transition table for encoder of Fig. 8.7.9

Observe that above table is similar to Table 8.7.3 in previous example. This is because the convolutional encoder of Fig. 8.7.13 and Fig. 8.7.9 are also similar. Note that the flip-flop for input is not shown in Fig. 8.7.13. But it does not affect the operation of the encoder. Hence code trellis diagram is similar to that of Fig. 8.7.11. It is shown below :

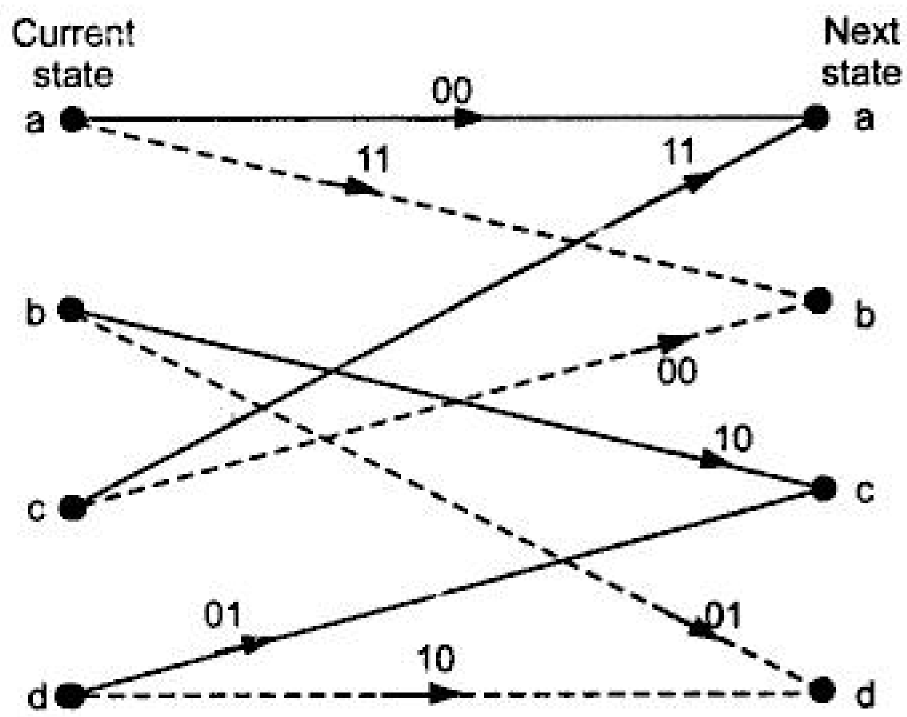

Fig. 8.7.14 Code trellis for encoder of Fig. 8.7.13

C Integration

Definition : $$\int f(x)\,dx = \lim_{\Delta x \to 0} \left\{ \sum_n [f(n\,\Delta\,x)]\,\Delta\,x \right\} \tag{C-1}$$

Integration Techniques :

1. Change in variable. Let $v = u(x)$:

$$\int_a^b f(x)\,dx = \int_{u(a)}^{u(b)} \left(\frac{f(x)}{dv\,/\,dx} \bigg|_{x = u^{-1}(v)} \right) dv \tag{C-2}$$

2. Integration by parts

$$\int u\,dv = uv - \int v\,du \tag{C-3}$$

Integral Tables :

Indefinite Integrals :

$$\int (a + bx)^n\,dx = \frac{(a + bx)^{n+1}}{b\,(n+1)},\ 0 < n \tag{C-4}$$

$$\int \frac{dx}{a + bx} = \frac{1}{b}\,ln\,|a + bx| \tag{C-5}$$

$$\int \frac{dx}{(a + bx)^n} = \frac{-1}{(n-1)\,b\,(a + bx)^{n-1}},\ 1 < n \tag{C-6}$$

$$\int \frac{dx}{c + bx + ax^2} = \begin{cases} \dfrac{2}{\sqrt{4ac - b^2}}\,tan^{-1}\left(\dfrac{2ax + b}{\sqrt{4ac - b^2}}\right),\ b^2 < 4ac \\ \dfrac{1}{\sqrt{b^2 - 4ac}}\,ln\left(\dfrac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right),\ b^2 > 4ac \\ \dfrac{-2}{2ax + b},\ b^2 = 4ac \end{cases} \tag{C-7}$$

$$\int \frac{x\,dx}{c + bx + ax^2} = \frac{1}{2a}\,ln\,|ax^2 + bx + c| - \frac{b}{2a}\int \frac{dx}{c + bx + ax^2} \tag{C-8}$$

$$\int \frac{dx}{a^2 + b^2\,x^2} = \frac{1}{ab}\,tan^{-1}\left(\frac{bx}{a}\right) \tag{C-9}$$

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\,ln\,(a^2 + x^2) \tag{C-10}$$

$$\int \frac{x^2\,dx}{a^2 + x^2} = x - a\,tan^{-1}\left(\frac{x}{a}\right) \tag{C-11}$$

$$f(\lambda) = \int_a^b \left[\lambda x_1^*(t) + x_2^*(t)\right] \left[\lambda x_1(t) + x_2(t)\right] dt$$

$$\therefore \quad f(\lambda) = \lambda^2 A + \lambda(B + B^*) + C \tag{E-9}$$

Here A, B and C are given by equation (E-6), (E-7) and (E-8).

$f(\lambda)$ is a quadratic equation in λ given by equation (E-9). Since its value is non-negative, coefficients of equation (E-9) are related as,

$$(B + B^*)^2 \leq 4AC \tag{E-10}$$

Putting the values of B, B^* and A, C in above equation we have,

$$\left[\int_a^b x_1^*(t)\, x_2(t)\, dt + \int_a^b x_1(t)\, x_2^*(t)\, dt\right]^2 \leq 4\int_a^b |x_1(t)|^2 dt \int_a^b |x_2(t)|^2\, dt \tag{E-11}$$

When both $x_1(t)$ and $x_2(t)$ are real,

we have

$$x_1^*(t)\, x_2(t) + x_1(t)\, x_2^*(t) = 2x_1(t)\, x_2(t) \tag{E-12}$$

This is the standard relation.

We can write LHS of equation (E-11) as,

$$\left[\int_a^b [x_1^*(t)\, x_2(t) + x_1(t)\, x_2^*(t)\, dt\right]^2 \leq 4\int_a^b |x_1(t)|^2\, dt \int_a^b |x_2(t)|^2\, dt$$

Putting equation (E-12) inside the integral in above equation we have,

$$\left[\int_a^b 2x_1(t)\, x_2(t)\, dt\right]^2 \leq 4\int_a^b |x_1(t)|^2\, dt \int_a^b |x_2(t)|^2\, dt$$

$$\therefore \quad 4\left[\int_a^b x_1(t)\, x_2(t)\, dt\right]^2 \leq 4\int_a^b |x_1(t)|^2\, dt \int_a^b |x_2(t)|^2\, dt$$

$$\therefore \quad \left|\int_a^b x_1(t)\, x_2(t)\, dt\right|^2 \leq \int_a^b |x_1(t)|^2\, dt \int_a^b |x_2(t)|^2\, dt$$

which is the required result same as equation (E-3) and has been proved.

❑❑❑

1

Pulse Digital Modulation

Q.1 *Establish the principles of flat-top sampling with neat schematics. Hence explain the phenomenon of aperture effect and equalization.* **(Nov.-2005, Set-1, 8 Marks)**

Ans. : Refer section 1.4.3.

Q.2 *Distinguish between natural sampling and flat top sampling with neat schematics, listing out their merits and demerits.* **(March-2006, Set-2, 8 Marks)**

Ans. : Refer table 1.2.1.

Q.3 *A signal $m(t) = 4\cos(60\pi t) + 2\cos(160\pi t) + \cos(280\pi t)$ is sampled at (i) 150 Hz, (ii) 75 Hz, (iii) 300 Hz. Find the frequency components of the signal that appear at the output of an ideal lowpass filter with cut-off at 290 Hz, in each case. What is the nyquist rate of sampling and nyquist interval for m(t) ?* **(March-2006, Set-2, 8 Marks)**

Ans. : **(i) f_s = 150 Hz**

$$m(t) = 4\cos(60\pi t) + 2\cos(160\pi t) + \cos(280\pi t)$$

$$= A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t) + A_3 \cos(2\pi f_3 t)$$

Hence $A_1 = 4 \quad f_1 = 30$ Hz

$A_2 = 2 \quad f_2 = 80$ Hz

$A_3 = 1 \quad f_3 = 140$ Hz

This signal will have components of

$\frac{A_1}{2}$ at $\pm f_1$ i.e. ± 30 Hz

$\frac{A_2}{2}$ at $\pm f_2$ i.e. ± 80 Hz

$\frac{A_3}{2}$ at $\pm f_3$ i.e. ± 140 Hz

When the signal is sampled at f_s 150 Hz, then the sampled signal $x_\delta(t)$ will have a spectrum whose components will be as follows :

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4

Data Transmission

Q.1 *A statistically independent sequence of equiprobable binary digits is transmitted over a channel having infinite bandwidth using the rectangular signalling wave shown in Fig. 1.*

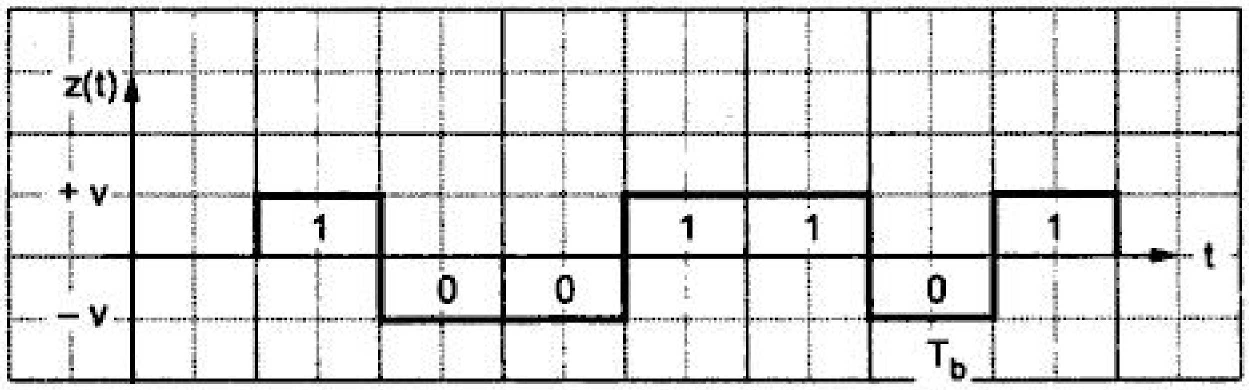

Fig. 1 Waveform of Q. 1

The bit rate is r_b and the channel noise is white Gaussian with a psd of $N_0/2$.

a) Derive the structure of an optimum receiver for this signaling scheme.

b) Derive an expression for the probability of error. **(May-2005, Set-1, 16 Marks)**

Ans. : a) Structure of optimum filter

Here the noise is white Gaussian with psd of $\frac{N_0}{2}$. Hence optimum filter is actually matched filter. The two symbols are 1 and 0. The shape of such signals is as shown in Fig. 2 (a) and (b). And combined signal is shown in Fig. 2 (c). The impulse response of the matched filter for this signal will be as shown in Fig. 2 (d) i.e.,

$$h(t) = \begin{cases} +2V & \text{for } 0 \le t \le T_b \\ 0 & \text{elsewhere} \end{cases}$$

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7 Linear Block Codes

Q.1 *Explain about the syndrome calculation, error correction and error detection in (n, k) cyclic codes.* **(June-2003, Set-1, 8 Marks; March-2006, Set-3, 8 Marks)**

Ans. : Refer sections 7.3.6 and 7.3.7.

Q.2 *What is an interlaced code ? Explain about its capability for correcting errors with suitable example.* **(June-2003, Set-3, 8 Marks;May/June-2004, Set-4, 8 Marks)**

Ans. : Refer section 7.3.15.

Q.3 *Briefly discuss the following error control techniques (a) Linear block codes (b) cyclic codes.* **(June-2003, Set-4, 8 Marks)**

Ans. : Refer sections 7.2, 7.3.1, 7.3.2, 7.3.3.

Q.4 *Find all the code vectors for a (6, 3) block code, whose generator matrix is given as,*

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and also find parity check matrix. Show how an error can be detected and corrected ? **(Nov.-2003, Set-2, 12 Marks;March-2006, Set-4, 12 Marks)**

Ans. : Refer example 7.2.1.

From above example,

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad P^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

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