Polynomials

Answers

November 30, 2019

Problem 1 Degrees, Leading Coefficients, and Constant Terms

Recall: The degree of a polynomial is the highest exponent of x with a non-zero coefficient. The leading coefficient is the coefficient of the highest exponent term. The constant term is the coefficient of the x^0 term, i.e. the monomial which does not depend on x (hence, constant).

Find the degree, leading coefficient, and constant term for each of the following polynomials.

Hint: The polynomials in this homework are not written in the same order as we did in class. Don't let that confuse you! The leading coefficient is always in the term with the highest exponent, regardless of what order it is written in.

1. $-12 + x + x^2$ Degree: $\boxed{2}$ Leading coefficient: $\boxed{1}$ Constant term: $\boxed{-12}$
2. $-x^2 + x^{11} - x^{17} + x^{26}$ Degree: $\boxed{26}$ Leading coefficient: $\boxed{1}$ Constant term: $\boxed{0}$

Problem 2 Long Division

Use long division to find the following quotients.

1. $\dfrac{-12 + x + x^2}{4 + x} = \boxed{-3 + x}$
2. $\dfrac{-x^2 + x^{11} - x^{17} + x^{26}}{-1 + x^9} = \boxed{x^2 + x^17}$

Problem 3 Factorization in $\mathbf{Q}[x]$ with Integer Coefficients

Define $\mathbf{Q}[x]$ to be the set of polynomials with rational coefficients. In $\mathbf{Q}[x]$, fully factor the following. Use the rational root theorem, which states that any rational root $\frac{m}{n}$ (in lowest terms) must have m be a factor the constant term and n be a factor of the leading coefficient. You may use the fact that any degree 2 or 3 rational polynomial with no rational root is irreducible.

1. $16 - x^2 = \boxed{(4 + x)(-4 + x)}$
2. $-x + 2x^2 = \boxed{x(-1 + 2x)}$

3. $2 + 5x + 3x^2 = \boxed{(2 + 3x)(1 + x)}$

4. $1 + 6x + 12x^2 + 8x^3 = \boxed{(1 + 4x + 4x^2)(1 + 2x)}$

5. $1 - 4x + 6x^2 - 4x^3 + x^4 = \boxed{(-1 + x)^4}$

6. $1 - x^4 = \boxed{(1 - x)(1 + x)(1 + x^2)}$

Problem 4 Factorization in $\mathbf{Q}[x]$

Define $\mathbf{Q}[x]$ to be the set of polynomials with rational coefficients. In $\mathbf{Q}[x]$, fully factor the following.

Hint: The rational root theorem requires all the coefficients to be integers. To get the polynomials to have integer coefficients, first factor out an appropriate fraction, e.g. $\frac{1}{2}x + \frac{3}{2} = \frac{1}{2}(x + 3)$. Then use the rational root theorem ignoring the fraction you factored out. Any root of $p(x)$ is still a root of $qp(x)$, where q is any non-zero rational number!

1. $\frac{1}{4} + x + x^2 = \boxed{\frac{1}{4}(1 + 2x)^2}$

2. $\frac{4}{9} - \frac{4}{9}x + \frac{1}{9}x^2 = \boxed{\frac{1}{9}(-2 + x)^2}$

Problem 5 Finite Geometric Series

1. Expand $\frac{1-x^2}{1-x}$ as a polynomial. Hint: Use long division. $\boxed{\frac{1-x^2}{1-x} = 1 + x}$

2. Expand $\frac{1-x^5}{1-x}$ as a polynomial. $\boxed{\frac{1-x^5}{1-x} = 1 + x + x^2 + x^3 + x^4}$

3. Define
$$\sum_{k=0}^{n} x^k := 1 + x + \cdots + x^n$$
(this is just a shorthand notation; you can read it as the "sum with k ranging from 0 to n of x^k). Find a fraction of the form $\frac{p(x)}{q(x)}$, where p and q are polynomials, such that
$$\frac{p(x)}{q(x)} = \sum_{k=0}^{n} x^k$$
$\boxed{\frac{1-x^{k+1}}{1-x}}$

4. Calculate
$$\sum_{k=0}^{20} 2^k := 1 + 2 + 2^2 + \cdots + 2^{20} = \boxed{2097151}$$
You may use the fact that $2^{21} = 2097152$.