BV CH5: DUALITY

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- **LAGRANGIAN**

Opt problem in standard form:

$$\begin{aligned} \text{minimize} \quad & f_0(x), & x \in \mathbb{R}^n \\ \text{subject to} \quad & f_i(x) \le 0, & i = 1 \cdots m \\ & h_i(x) = 0, & i = 1 \cdots p \end{aligned} \quad \Bigg\} \ (5.1)$$

do NOT assume convex!

$$D = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=0}^{p} \operatorname{dom} h_i$$

$$L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$$

$$\to L(x, \lambda, v) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x)$$

$$\operatorname{dom} L = D \times \mathbb{R}^m \times \mathbb{R}^p$$

- **LAGRANGE DUAL FUNCTION**

The Dual function finds the minimum value of the Lagrangian function over x.

$$g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}, \qquad \lambda \in \mathbb{R}^m, \quad v \in \mathbb{R}^p$$

$$g(\lambda, v) = \inf_{x \in D} L(x, \lambda, v) = \inf_{x \in D} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{m} v_i f_i(x) \right)$$

When the Lagrangian is unbounded below in xi, then the Dual function takes on the value -infinity.
The Dual function is concave, even when the problem is not convex.

The Dual functions yields a lower bound on the optimal value of the problem.

For $\lambda \ge 0$ and any v, $g(\lambda, v) \le p^*$

$$\tilde{x} \in D \quad f_i(\tilde{x}) \le 0, \quad h_i(\tilde{x}) \le 0, \quad \lambda \succeq 0$$

$$\sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} v_i h_i(\tilde{x}) \le 0$$

$$L(\tilde{x}, \lambda, v) = f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} v_i h_i(\tilde{x}) \le f_0(\tilde{x})$$

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x})$$

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x})$$