ADMM LASSO

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Original Lasso:

minimize: $\frac{1}{2}\|Ax-b\|_2^2 + \lambda\|x\|_1$

ADMM Standard Form:

minimize: $f(x)+g(z)$

subject to: $Ax+Bz=c$

ADMM Lasso:

minimize: $\frac{1}{2}\|Ax-b\|_2^2 + \lambda\|z\|_1$

subject to: $x-z=0$

$$L(x,y) = f(x) + y^T(x-z)$$

Augmented Lagrangian:

$$L_\rho(x,z,y) = \frac{1}{2}\|Ax-b\|_2^2 + \lambda\|z\|_1 + \rho y^T(x-z) + \frac{\rho}{2}\|x-z\|_2^2$$

Algorithm Steps:

$$\rho y^T(x-z)$$

$$x = \underset{x}{\operatorname{argmin}}\, L_\rho(x,z,y)$$

$$= \underset{x}{\operatorname{argmin}}\left(\frac{1}{2}\|Ax-b\|_2^2 + \lambda\|z\|_1 + \rho y^T(x-z) + \frac{\rho}{2}\|x-z\|_2^2\right)$$

$$= \underset{x}{\operatorname{argmin}}\left(\underbrace{\frac{1}{2}(Ax-b)^T(Ax-b)} + \lambda\|z\|_1 + \rho y^T x - \rho y^T z + \underbrace{\frac{\rho}{2}(x-z)^T(x-z)}\right)$$

$$= \frac{1}{2}(x^TA^T - b^T)(Ax-b)$$

$$= \frac{1}{2}(x^TA^TAx - x^TA^Tb - b^TAx + b^Tb)$$

$$= \frac{1}{2}(x^TA^TAx - 2x^TA^Tb + b^Tb)$$

$$= \frac{\rho}{2}(x^T - z^T)(x-z)$$

$$= \frac{\rho}{2}(x^Tx - x^Tz - z^Tx + z^Tz)$$

$$= \frac{\rho}{2}(x^Tx - 2x^Tz + z^Tz)$$

$$\nabla_x \frac{1}{2}(x^TA^TAx - 2x^TA^Tb + b^Tb)$$

$$= \frac{1}{2}(2A^TAx - 2A^Tb)$$

$$= A^TAx - A^Tb$$

$$\nabla_x \frac{\rho}{2}(x^Tx - 2x^Tz + z^Tz)$$

$$= \frac{\rho}{2}(2x - 2z)$$

$$= \rho x - \rho z$$

$$\nabla_x(\rho y^T x - \rho y^T z) = \rho y \qquad \nabla_x \lambda\|z\|_1 = 0$$

$$0 = \nabla_x L_\rho(x,z,y)$$

$$0 = A^TAx - A^Tb + \rho y + \rho x - \rho z$$

$$0 = x(A^TA + \rho I) - A^Tb + \rho(y-z)$$

$$x(A^TA + \rho I) = A^Tb - \rho(y-z)$$

$$x = (A^TA + \rho I)^{-1}(A^Tb + \rho(z-y))$$

$$z^k = \operatorname*{argmin}_z L(x,z,y)$$

$$= \operatorname*{argmin}_z \left(\cancelto{\nabla_z = 0}{\tfrac{1}{2}\|Ax-b\|_2^2} + \lambda\|z\|_1 + \rho y^T(x-z) + \tfrac{\rho}{2}\|x-z\|_2^2 \right)$$

$$= \operatorname*{argmin}_z \left(\lambda\|z\|_1 + \rho y^T(x-z) + \tfrac{\rho}{2}(x-z)^T(x-z) \right)$$

$$= \operatorname*{argmin}_z \left(\lambda\|z\|_1 + \rho y^Tx + \rho y^Tz + \tfrac{\rho}{2}(x^Tx - 2z^Tx + z^Tz) \right)$$

$$= \operatorname*{argmin}_z \left(\lambda\|z\|_1 + \rho y^Tx + \rho y^Tz + \tfrac{\rho}{2}x^Tx - \rho z^Tx + \tfrac{\rho}{2}z^Tz \right)$$

$$= \operatorname*{argmin}_z \left(\lambda\|z\|_1 + \tfrac{\rho}{2}(2y^Tx + 2y^Tz + x^Tx - 2z^Tx + z^Tz) \right)$$

$$= \operatorname*{argmin}_z \left(\lambda\|z\|_1 + \tfrac{\rho}{2}(x^Tx + 2y^Tx \quad) \right)$$