Modeling with particle-spring systems

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1 Exercise 1

The first exercise models two particles connected by a spring. The particles are assumed to be unaffected by gravity. The properties of the spring is as follows.

- $k_s = 10\,\frac{N}{m}$, spring constant.
- $L = 1\,m$, rest length of the spring.

1.1 Undamped system

The first part och the exercise models an undamped system. The displacement in x and y direction after simulating $T = 4\,s$:

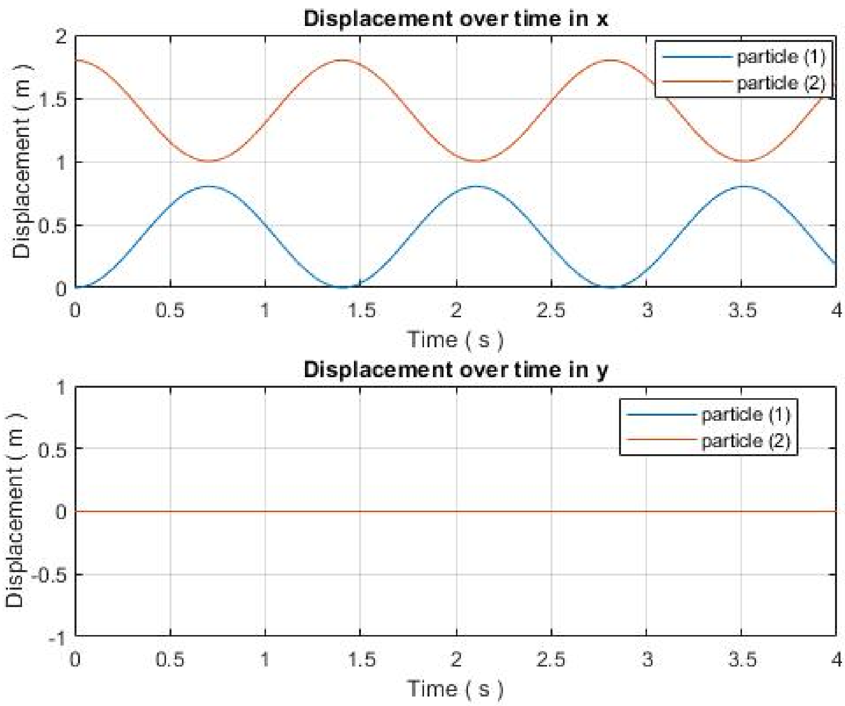

Figure 1: The displacement in the coordinates over time of two connected particles of equal mass.

As you can see in the figure above the particles oscillate continuously without any loss of amplitude. This is due to the fact that there is no damping in the spring. This simulation used a time step, dt, of 10^{-3}. Since there is no damping then the total energy is conserved and simply converted between kinetic and energy stored in the spring. This can be visualized by plotting the various energies of the system over time.

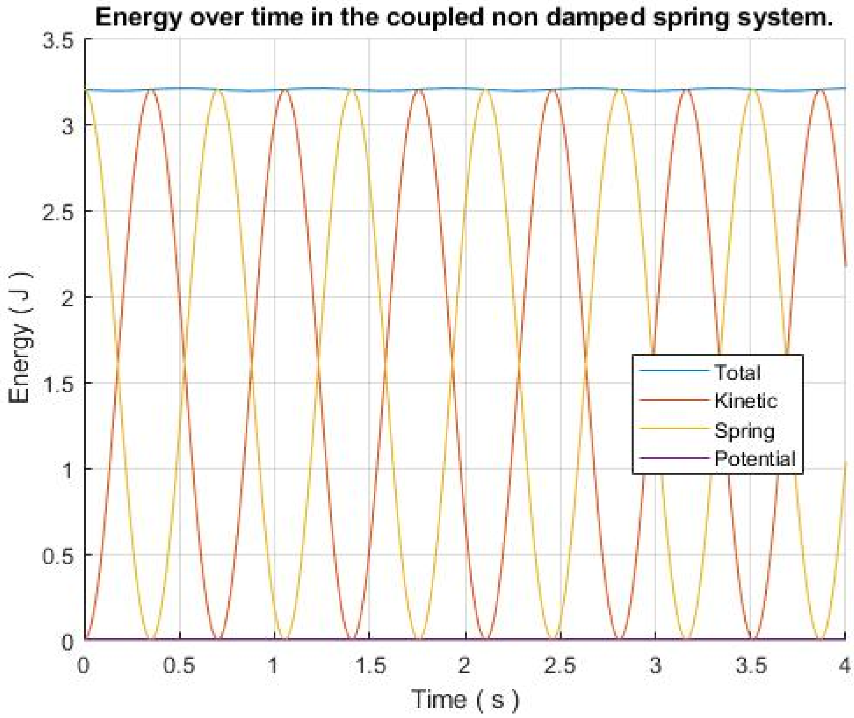

Figure 2: The energy of an undamped system.

From the figure there is a clear indication of the interaction between the kinetic and spring energies resulting in a constant total energy. The total energy is not exactly constant due to the finite precision of the time step, dt. But if dt is kept below 0.04 I find that the relative difference between each oscillation does note exceed 1%.

1.2 Damped system

The second part of the system utilizes damping in the spring to more realistically simulate the spring. The energy of this damped system decreases over time:

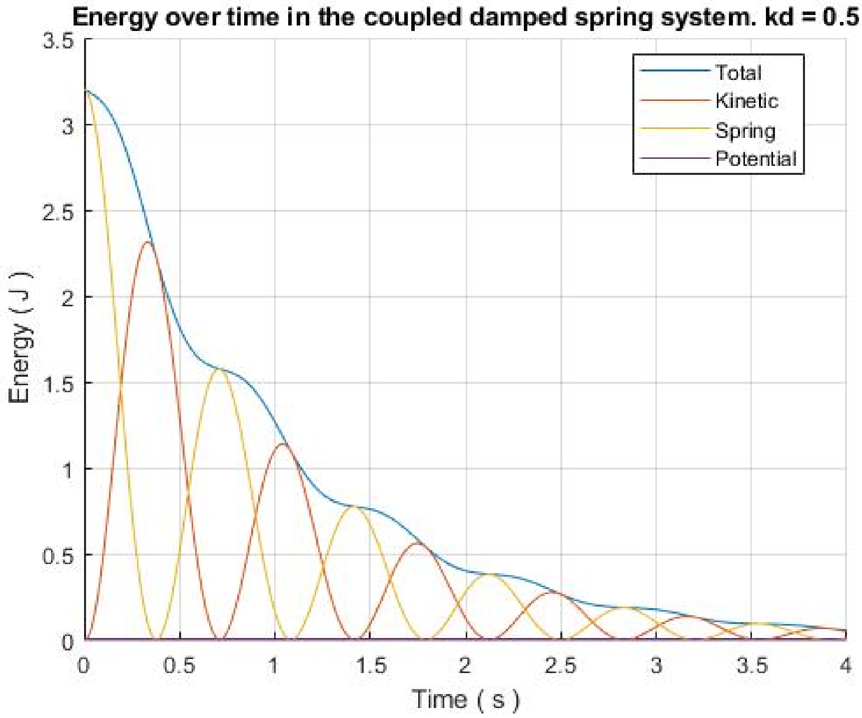

Figure 3: The energy of a damped system.

We can use the fact that the total energy is the sum of the kinetic and the spring, since the potential is set to zero, to find when the amplitude of the system no longer reaches 10% of the initial amplitude. We can analyze the to total energy of the system. Since the total energy E is given by:

$$E = \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) + \frac{1}{2}(k_s * (r - L)^2), \tag{1}$$

Where r is the current length of the spring. We can calculate the maximum length to which the spring can be extended given the current total energy. So, if the energy decreases to 1% of the initial energy then the amplitude to which the spring can extend has decreased by 10%. Resultingly we can utilize our calculation of the energy over time to find the time stamp where the energy has decrease to 1% of the initial energy. This happens after roughly 4.614 s, this result varies slightly depending on what time step being used. We also know that the analytical solution to a spring system of this sort can be described by the second order differential equation

$$\ddot{r} + 2(k_d/m)\dot{r} + 2(k_s/m)r = 0,\ r(0) = r_0,\ \dot{r}(0) = 0 \tag{2}$$

In this exercise we are given that $r_0 = 0.8$. Using this we can find the analytical solution using the characteristic equation. The analytical solution is given by:

$$r_0 e^{\frac{-k_d}{m}t} cos(\omega t) + \frac{r_0 k_d}{\omega} e^{\frac{-k_d}{m}t} sin(\omega t),\ \ \omega = \sqrt{\left|\frac{k_d^2}{m^2} - 2\frac{k_s}{m}\right|} \tag{3}$$

For this case this $\omega = \sqrt{\frac{79}{4}}$.

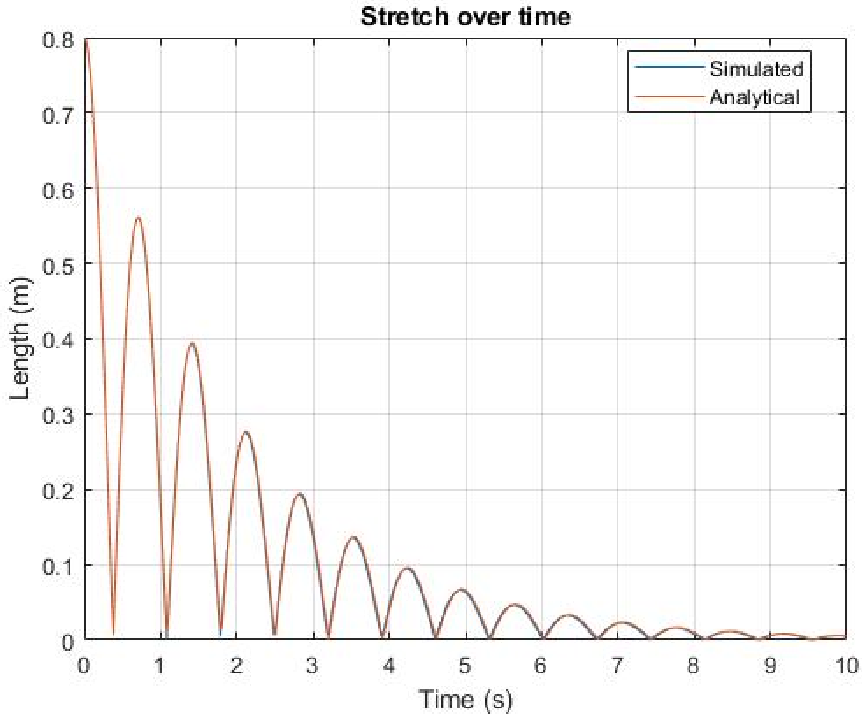

Figure 4: Comparison of the analytical and simulated amplitude of the string.

We can see that the simulated system matches the analytical extremely well.

1.3 Rotating System

If we set the initial velocity of the particles to $5\frac{m}{s}$ in opposite y direction in the undamped system obtain a rotating spring which carries angular momentum. Since there are no external forces then this angular momentum will be constant with respect to time.

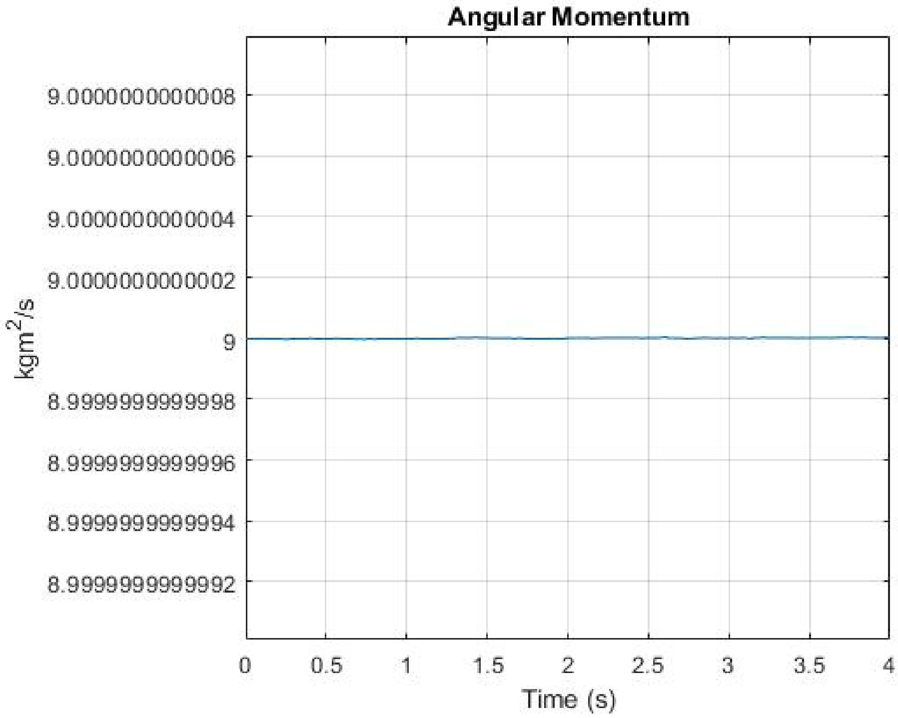

Figure 5: Angular momentum of the undamped rotating system.

We want to find the angular frequency, ω, of the spring. In order to calculate the frequency we first obtain the angular momentum $\mathcal{L}$ and the moment of inertia I, because they are related as:

$$\mathcal{L} = I\omega \tag{4}$$

We let the particles be approximated by point masses and we can then obtain the moment of inertia by:

$$I = r_1^2 * m_1 + r_2^2 * m_2$$

Where r_1 and r_2 are the distances from the the axis of rotation to each of the particles. But since $m_1 = m_2$ we know that $r_1 = r_2$. We then calculate the angular momentum and obtain the angular frequency.

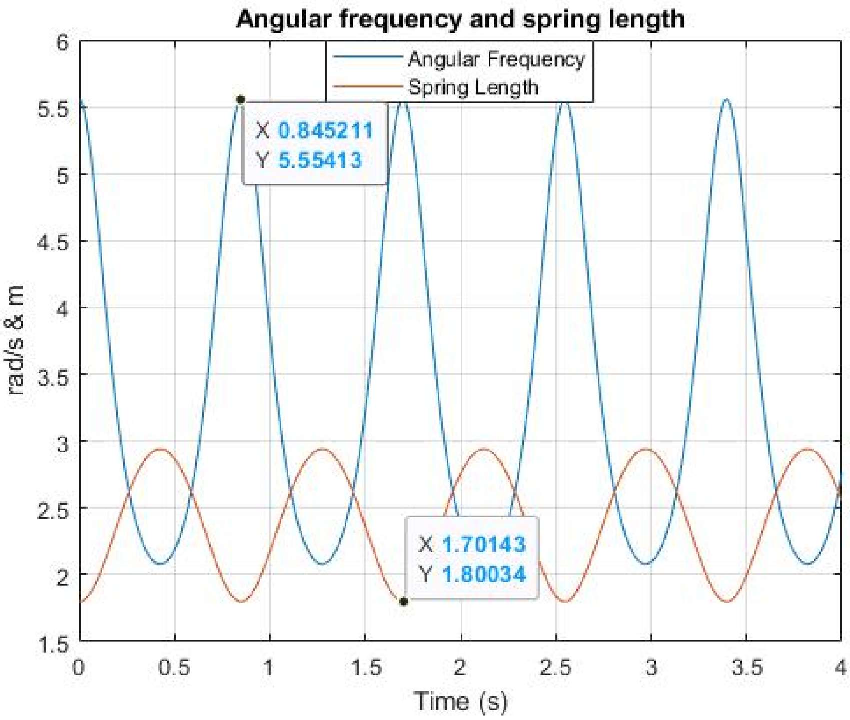

Figure 6: The angular frequency vs the length of the spring. For the spring with no damping and equal masses fixed at each end.

From the figure we also see that the spring does not oscillate around the rest length $L = 1.8$ as in figure (2) but rather around 2.4. This can be found numerically by the mean length of the spring. These are not equal since there is a centrifugal force acting outward to retain the circular motion.

2 Exercise 3

The exercise focused on creating a 2D grid of particles connected by strings. In the first part of the exercise there should be no damping of the springs separated by springs with rest length $L = 1m$ on the horizontal and vertical springs whilst diagonal connections should have $L = \sqrt{2}\,m$. Each spring has spring constant $k_s = 10\,\frac{N}{m}$. These particles are then given initial velocity over floor a made of somewhat randomly scattered circles. The the particles bounce which can be thought of as friction. Each particle should have mass $m = 1\,kg$. The gravitation constant $g = 1\,\frac{m}{s^2}$.

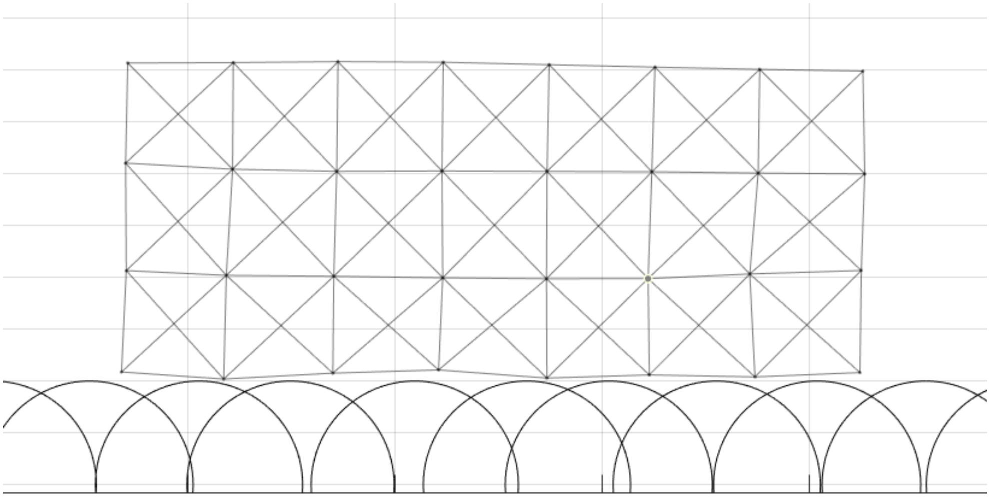

Figure 7: 2D grid of spring connected particles.

2.1 Energy

The grid bounces across the floor and continuously converts energy between; kinetic, potential and spring energy.

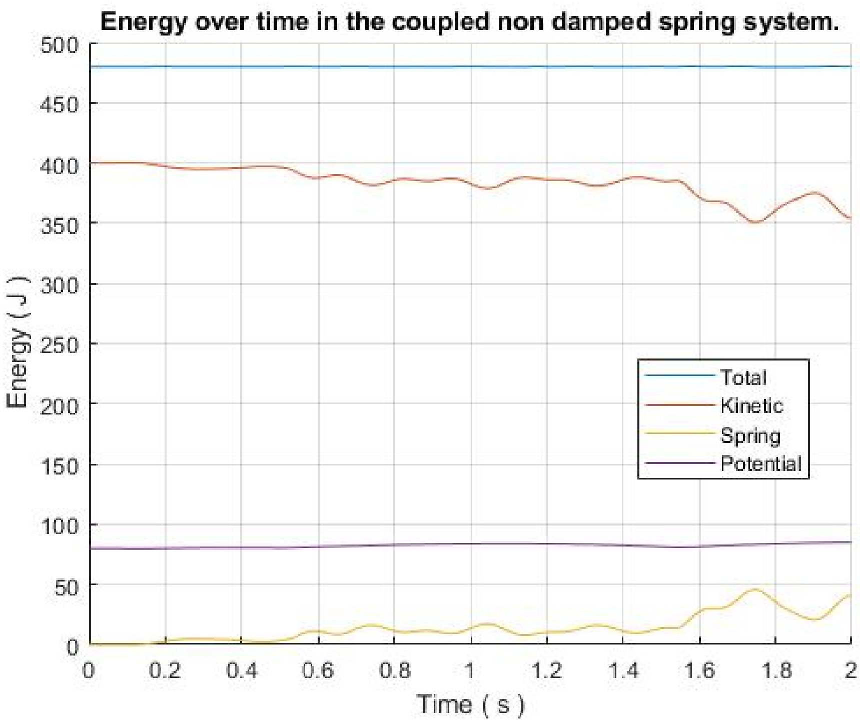

Figure 8: Energy of the 2D grid system. The initial velocity of the grid was $v = 5\ \frac{m}{s}$

As seen in the picture the total energy is conserved over time and the kinetic decreases whilst spring energy increases. This is due to the face that the bouncing grid is not subject to any damping.

If we focus on the velocity of the entire grid i.e. the velocity of the center of mass. We can see that the velocity decreases, not linearly but at every bounce with the surface. This is due to the face that the impact might send the colliding node in the opposite direction resulting in a loss of velocity in the initial direction.

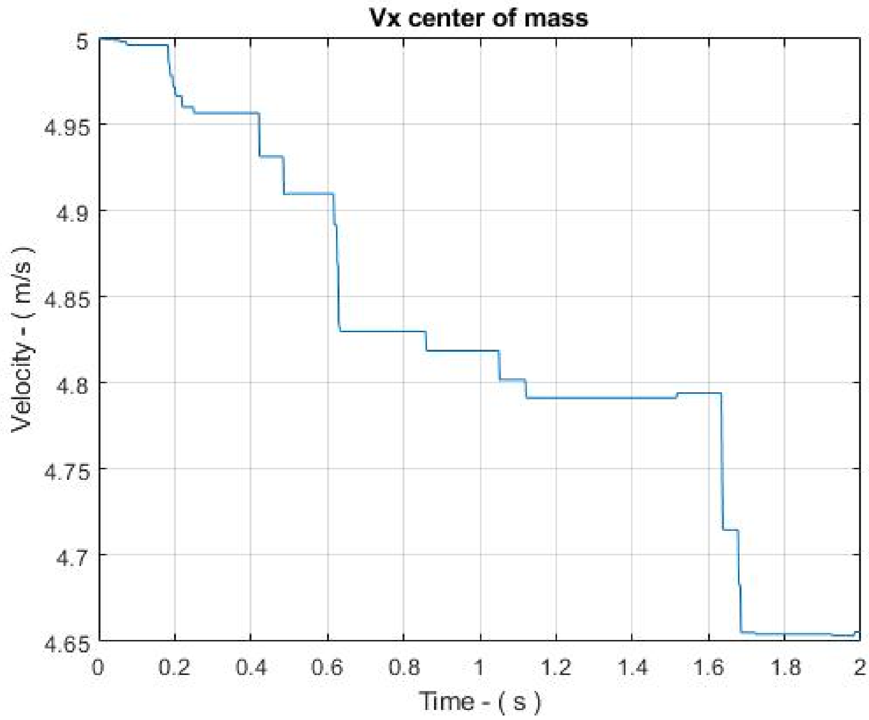

Figure 9: The initial velocity of the grid was $v = 5\,\frac{m}{s}$. The velocity in the x-direction decreases on impact.

2.2 Coefficient of Friction

To find the coefficient of friction, μ, of the floor we want to find the average force in the x-direction as if it was in contact with the floor during the entire time. This can be done by running multiple simulations for different initial velocities. During this exercise the initial velocities were; 3,5 and 7. The velocities were simulated 400 times each resulting in the following average μ's

v	μ
3	0.1918
5	0.1636
7	0.1638

Table 1: Estimations of the friction coefficients, μ, using 400 repetitions for each initial velocity v.

The friction seems to depend somewhat on velocity, this might be due to the fact that when the grid moves faster then the angle of impact will be more tangential as if it the grid is skipping on the circles which construct the floor. The coefficient of friction is determined by simulating a period of time when the grid does not overshoot the length of the floor. Then compare the initial velocity by the last. We know that

$$v(t) = v_0 + a_x t. \tag{5}$$

We want to find a_x since the acceleration is proportional to the coefficient of friction. We can estimate the average acceleration as follows:

$$a_x \approx \frac{v_0 + v(T)}{T}. \tag{6}$$

Where $v(T)$ is the final simulated velocity and T is the total simulation time. Since there is no other force but the gravitational force working on the grid then the acceleration must be due to the frictional force. Which we know can be calculated as:

$$\begin{gathered} F_\mu = ma_x = \mu mgcos(\theta),\ \theta = 0 \implies \\ \mu = a_x/g \implies \\ \mu \approx \frac{v_0 + v(T)}{Tg} \end{gathered} \tag{7}$$

The values in Table 1. were determined using equation 7.

3 Exercise 4

The fourth exercise is a extension of exercise 3, only adding damping $k_d = 50\ \frac{N}{m/s}$, a more compact floor and more gravity, $g = 10\ \frac{m}{s^2}$. This will result in a more rigid system since "heat" is lost causing vibrations in the springs to disappear. The energy of such a system will decrease over time since heat is lost.

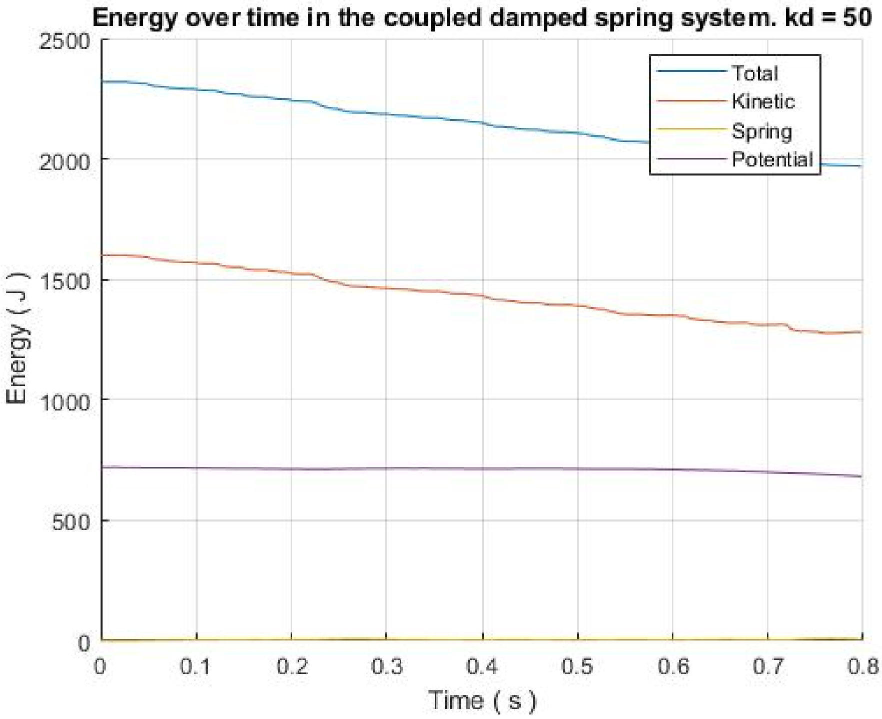

Figure 10: The energy of the spring system when using hard damping.

The tight springs doesn't retain energy very well due to the high damping resulting in a very low spring energy for this type of system. You can also see that the kinetic energy decreases rapidly due to the large amount of contact with the floor.

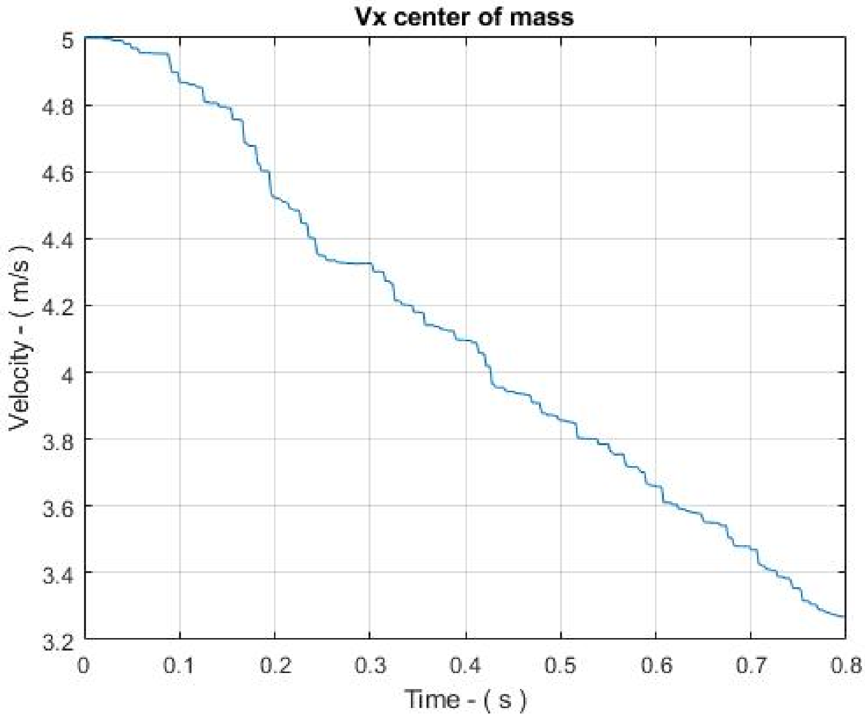

Figure 11: The initial velocity of the grid was $v = 5\,\frac{m}{s}$. The velocity in the x-direction decreases on impact.

Compared to figure (9) there is a more linear decrease over time. This is probably because there is more time of contact when the grid is more tight. This also causes the coefficient of friction become larger. Using the same method as explained under section 2.2 with 400 repetitions it was found that $\mu = 0.2053$. Which is significantly larger then the results from exercise 3 when using the same initial velocity.