Lecture 3

- **Gradient** $\Phi(x, y, z)$ is a continously differentiable function at the point $\bar{r} = (x, y, z)$

$$\bar{r} = x\bar{e}_x + y\bar{e}_y + z\bar{e}_z$$

Moving an infinitesimal distance results in:

$$\bar{r}' = (x + \Delta x)\bar{e}_x + (y + \Delta y)\bar{e}_y + (z + \Delta z)\bar{e}_z$$

The difference in the function value is:

$$\Delta\bar{r} = \bar{r}' - \bar{r} = \Delta x\bar{e}_x + \Delta y\bar{e}_y + \Delta z\bar{e}_z$$

As $\Delta x, \Delta y, \Delta z$ are infinitesimal, we can rewrite:

$$\Delta\bar{r} = \bar{r}' - \bar{r} = dx\bar{e}_x + dy\bar{e}_y + dz\bar{e}_z$$

The change in the function value is:

$$\Delta\Phi = \Phi(x + \Delta x, y + \Delta y, z + \Delta z) - \Phi(x, y, z \overset{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}}{\to} d\Phi(x, y, z)$$

Which is the same as:

$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

Which can be written with a dot product as:

$$d\Phi = \left(\frac{\partial\Phi}{\partial x}\bar{e}_x + \frac{\partial\Phi}{\partial y}\bar{e}_y + \frac{\partial\Phi}{\partial z}\bar{e}_z\right) \cdot (dx\bar{e}_x + dy\bar{e}_y + dz\bar{e}_z) = \nabla\Phi \cdot d\bar{r} = grad\,\Phi \cdot d\bar{r}$$

- **Directional Derivative** Write $d\bar{r}$ as:

$$d\bar{r} = ds\bar{e}$$

Where $\bar{e}$ is a unit vector in the direction of $d\bar{r}$, i.e.

$$\bar{e}_s = \frac{d\bar{r}}{|d\bar{r}|}$$

The directional derivative is:

$$\frac{d\Phi}{ds} = \nabla\Phi \cdot \bar{e}$$

This tells us how much the function Φ changes in the direction of $\bar{e}$.

Theorem:

- The vector $\nabla\Phi$ is the gradient of Φ and is the direction of maximum rate of change of Φ .

Proof

Let $\bar{e}$ be a unit vector in the direction of most rapid change of Φ.

$$\frac{d\Phi}{ds} = \nabla\Phi \cdot \bar{e} = |\nabla\Phi|\cos\theta$$

Where θ is the angle between $\nabla\Phi$ and $\bar{e}$. The derivative is maximum when $\cos\theta = 1$, since $|\cos\theta| \leq 1$ when $0 \leq \theta \leq \pi$. Therefore, $\nabla\Phi$ is the direction of maximum rate of change of Φ.

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Level Surfaces:

A Level surface is a surface on which a scalar field Φ is constant.

$$\Phi(\bar{r}_c) = C$$

For all points $\bar{r}_c$ on the surface. One surface for each value of C in the scalar field.

Theorem:

If Φ is a scalar field and has a max,min or saddle point at $\bar{r}$, then the gradient of Φ is 0 at $\bar{r}$.

$$\nabla\Phi(\bar{r}) = \bar{0}$$

Theorem:

If Φ is a scalar field, then the gradient of Φ is perpendicular to the level surfaces of Φ.

Proof:

Choosing two infinitesimal vectors $d\bar{r}$ and $d\bar{r}'$ such that then:

$$\Phi(\bar{r_p} + d\bar{r}) = \Phi(\bar{r_p}) \iff \Phi(\bar{r_p} + d\bar{r}') - \Phi(\bar{r_p}) = 0$$

But as showed earlier:

$$\Phi(\bar{r_p} + d\bar{r}') - \Phi(\bar{r_p}) = d\Phi$$

$$d\Phi = \nabla\Phi \cdot d\bar{r}' = 0$$

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Theorem:

The distance Δs between two iso-surfaces:

$$\Phi(\bar{p}) = C,\ \Phi(\bar{p}) = C + h$$

Can be approximated by:

$$\Delta s \approx \frac{h}{|\nabla\Phi(\bar{p})|}$$

Proof:

Let

$$\Phi = C,\ \Phi = C + h$$

Be the two iso-surfaces. Choose a vector $d\bar{r}$ such that $d\bar{r}\perp$ to the level surfaces. Then:

$$h = \Delta\Phi \approx d\Phi = \nabla\Phi \cdot d\bar{r} = |\nabla\Phi|\Delta s \implies \Delta s \approx \frac{h}{|\nabla\Phi|}$$

Since $d\bar{r}$ is perpendicular to the level surfaces, $|\nabla\Phi|$ is the distance between the two level surfaces.

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Example:

Find the normal to the surface $z = x^2 + y^2$ at the point $(1, 2, 5)$. This surface can be seen as a level surface $\Phi = 0$ for the scalar field $\Phi = x^2 + y^2 - z$. Then:

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial x}\bar{e}_x + \frac{\partial\Phi}{\partial y}\bar{e}_y + \frac{\partial\Phi}{\partial z}\bar{e}_z\right) = (2x\bar{e}_x + 2y\bar{e}_y - 1\bar{e}_z)$$

Evaluating at $(1, 2, 5)$:

$$\bar{n} = (2\bar{e}_x + 4\bar{e}_y - 1\bar{e}_z)$$

Normalizing:

$$\bar{n} = \frac{1}{\sqrt{21}}(2\bar{e}_x + 4\bar{e}_y - 1\bar{e}_z)$$

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Scalar Potential:

- If for a given vector filed $\bar{A}$ there is a scalar field Φ such that:

$$\bar{A} = \nabla\Phi$$

Then Φ is called the scalar potential of $\bar{A}$.

- If $\bar{A}$ has a potential Φ, then $\Phi + C$ is also a potential for $\bar{A}$.
- The potential is often defined with a minus sign e.g. the electrostatic potential:

$$\bar{E} = -\nabla V$$

Conditions for existence of a scalar potential:

- A continously differentiable vector field $\bar{A}$ has a scalar potential if and only if $\bar{A}$ satisfies:

$$\frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x} \tag{1}$$

$$\frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y} \tag{2}$$

$$\frac{\partial A_z}{\partial x} = \frac{\partial A_x}{\partial z} \tag{3}$$

Proof:

Assume that $\bar{A} = \nabla\Phi$. Then:

$$A_x = \frac{\partial \Phi}{\partial x}$$

$$A_y = \frac{\partial \Phi}{\partial y}$$

Using this we known that since $\bar{A}$ is continously differentiable:

$$\frac{\partial A_x}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial y} = \frac{\partial A_y}{\partial x}$$

The mixed partial derivatives of Φ are the same as the mixed partial derivatives of $\bar{A}$.

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Example: Check if there exists a scalar potential

Check if the vector field $\bar{A} = y\bar{e}_x - x\bar{e}_y$ has a scalar potential.

$$\frac{\partial A_x}{\partial y} = \frac{\partial}{\partial y}y = 1 \neq \frac{\partial A_y}{\partial x} = \frac{\partial}{\partial x}(-x) = -1$$

The vector field does not have a scalar potential since it does not satisfy equation 1.

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Example: Find the scalar potential

Find the scalar potential of the vector field $\bar{A} = 2y^2\bar{e}_x + (4xy + y^2z^2)\bar{e}_y + (\frac{2}{3}y^3z + z)\bar{e}_z$.

- Known that $\bar{A} = \nabla\Phi$. We known that each term of $\bar{A}$ corresponds to the partial derivative of Φ with respect to x, y or z.

$$A_x = \frac{\partial \Phi}{\partial x} = 2y^2 \overset{\int dx}{\Longrightarrow} \Phi = 2y^2x + F(y, z)$$

Using this we can find the other terms succequently:

$$A_y = \frac{\partial \Phi}{\partial y} = 4xy + \frac{\partial}{\partial y} F(y,z) = 4xy + y^2 z^2 \implies$$

Canceling $4xy$ and integrating with respect to y yields:

$$F(y,z) = \frac{1}{3} y^3 z^2 + G(z)$$

Replacing $F(y,z)$ in the equation for Φ:

$$\Phi = 2y^2 x + \frac{1}{3} y^3 z^2 + G(z)$$

Deriving with respect to z:

$$A_z = \frac{\partial \Phi}{\partial z} = \frac{2}{3} y^3 z + \frac{\partial}{\partial z} G(z) = \frac{2}{3} y^3 z + z \implies G(z) = \frac{1}{2} z^2 + C$$

Replacing $G(z)$ in the equation for Φ concludes the calculation.

$$\Phi = 2y^2 x + \frac{1}{3} y^3 z^2 + \frac{1}{2} z^2 + C$$

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