Lecture 3

Surface integrals:

Flux of a fluid.

- How much fluid is flowing through a surface? f is the voluime of fluid passing S per unit time.

$$x = |\bar{v}| \cdot \Delta t$$
$$F = \Delta V/\Delta t,\ \Delta V = S|\bar{v}|\Delta t$$
$$F = \frac{S|\bar{v}|\Delta t}{\Delta t} = S|\bar{v}|$$

- At an angle the perpendicular component of the velocity does not contribute to the flux.

- Note! To only obtain the parallel component of the velocity we use the dot product.

$$F = S \cdot \bar{v} \cdot \cos\theta = S\bar{v} \cdot \hat{n}$$

We could rewrite this as:

$$F = \bar{v} \cdot \bar{S}$$

Where $\bar{S}$ is the area vector of the surface which is perpendicular to the surface.

$$\bar{S} = S \cdot \hat{n}$$

The decomposion of $\bar{v}$ can be as:

$$\bar{v} = (v \cdot \hat{n})\hat{n} + v^{\perp},\ v^{\perp} \cdot \hat{n} = 0$$

For a general surface S the flux is:

$$\bar{F} \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \Delta\bar{S}_i \cdot \bar{v}_i$$
$$\bar{F} = \lim_{\Delta S \to 0} \sum_{i=1}^{n} \Delta\bar{S}_i \cdot \bar{v}_i = \int_S \bar{v} \cdot d\bar{S}$$

Which can be written as a dubble integral:

Flux of a fluid

$$\bar{F} = \iint \bar{v} \cdot d\bar{S} = \iint \bar{v} \cdot \hat{n} dS$$

1. Parametize the surface $S : \bar{r}(u,v)$
2. Express the field as a function of the parameters: $\bar{A} = \bar{A}(u,v)$
3. Express the surface elements $d\bar{S}$ as a function of the parameters: $d\bar{S} = d\bar{S}(u,v)$
4. Performe the double integral: $\bar{F} = \iint \bar{v} \cdot d\bar{S}$

In order to express the surface elements we need to use the cross product between:

$$\Delta r_1 = \bar{r}(u + \Delta u, v) - \bar{r}(u,v)$$
$$\Delta r_2 = \bar{r}(u, v + \Delta v) - \bar{r}(u,v)$$

Then using the cross product we obtain the surface element:

$$\Delta S = \Delta r_1 \times \Delta r_2$$

As Δu and Δv are infinitesimal we can write:

$$\Delta r_1 = \frac{\bar{r}(u + \Delta u, v) - \bar{r}(u,v)}{\Delta u} \cdot \Delta u = d\bar{r}_1 = \frac{\partial \bar{r}}{\partial u} \cdot du$$
$$\Delta r_2 = \frac{\bar{r}(u, v + \Delta v) - \bar{r}(u,v)}{\Delta v} \cdot \Delta v = d\bar{r}_2 = \frac{\partial \bar{r}}{\partial v} \cdot dv$$

Then to obatin the surface element we use the cross product since the cross product of two vectors generates a vector which size is the area of the parallelogram formed by the two vectors, remember Linear Algebra!

Surface element

The surface element from a two dimensional surface is:

$$d\bar{S} = d\bar{r_u} \times d\bar{r_v} = \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} dudv$$

Ex. Compute the flux of the vector field $\bar{A}$

$$\bar{A} = yz^2\bar{e}_x$$

Through the surface S:

$$\bar{r}(x,y) = \begin{cases} x = y^2 + z^2 \\ 0 < y < 1 \\ 0 < z < 1 \end{cases}$$
$$\bar{r} = (u^2 + v^2)\bar{e}_x + u\bar{e}_y + v\bar{e}_z$$

With the following limits:

$$u = 0 \to 1,\ \ v = 0 \to 1$$

Derivative with respecpt to the paramters u and v:

$$\frac{\partial \bar{r}}{\partial v} = 2v\bar{e}_x + \bar{e}_z$$
$$\frac{\partial \bar{r}}{\partial u} = 2u\bar{e}_x + \bar{e}_y$$

Which gives the following surface element using the formula provided above:

$$d\bar{S} = (2u\bar{e}_x + \bar{e}_y) \times (2v\bar{e}_x + \bar{e}_z)dudv$$

Resulting in:

$$d\bar{S} = (\bar{e}_x - 2u\bar{e}_y - 2v\bar{e}_z)\, dudv$$

The vector field is given in terms of u and v:

$$\bar{A} = yz^2\bar{e}_x = uv^2\bar{e}_x$$

Then the flux is:

$$\bar{F} = \iint d\bar{S} \cdot \bar{A} = \int d\bar{S} \cdot uv^2\bar{e}_x$$
$$\bar{F} = \iint (\bar{e}_x - 2u\bar{e}_y - 2v\bar{e}_z) \cdot uv^2\bar{e}_x dudv$$

Since $\bar{e}_x \cdot \bar{e}_x = 1$ and $\bar{e}_x \cdot \bar{e}_y = \bar{e}_x \cdot \bar{e}_z = 0$ we can the integral as:

$$\bar{F} = \int_0^1 \int_0^1 uv^2 dudv = \frac{1}{2}\frac{1}{3}$$

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Independent of choice of parametrization: Proof: Consider two parametrizations $\bar{r}(u,v)$ and $\bar{r}(s,t)$.

The surface element in terms of u and v:

$$d\bar{S} = \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} dudv$$

We known that the flux in terms of u and v is:

$$\bar{F} = \iint \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \cdot \bar{A}\, dudv$$

If we let $s = s(u,v)$ and $t = t(u,v)$ then via the chain rule:

$$d\bar{S} = \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} dudv = \left(\frac{\partial \bar{r}}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial \bar{r}}{\partial t}\frac{\partial t}{\partial u}\right) \times \left(\frac{\partial \bar{r}}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial \bar{r}}{\partial t}\frac{\partial t}{\partial v}\right) dudv$$

The terms with the same derivative are parallell and does therefore not affect the cross product. Then the flux integral can be written as:

$$\bar{F} = \iint d\bar{S} \cdot \bar{A} = \iint \frac{\partial \bar{r}}{\partial s} \times \frac{\partial \bar{r}}{\partial t} \cdot det(J)dudv \cdot \bar{A}$$

Where $det(J)$ is the determinant of the Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{bmatrix}$$

Which give the following result:

$$det(J) = \frac{\partial s}{\partial u}\frac{\partial t}{\partial v} - \frac{\partial s}{\partial v}\frac{\partial t}{\partial u} = \frac{\partial(s,t)}{\partial(u,v)}$$

Using knowledge from multivariable calculus we can rewrite:

$$dudv\frac{\partial(s,t)}{\partial(u,v)} = dsdt$$

This means the at function can be written as:

$$\bar{F} = \iint dsdt\frac{\partial \bar{r}}{\partial s} \times \frac{\partial \bar{r}}{\partial t}$$

Thereby the flux integral is independent of the parametrization.

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Post break

There are differnt types of surface integrals:

- $\int_S dS\Phi(\bar{r})$ where Φ is a scalar function
- $\int_S d\bar{S}\Phi(\bar{r})$ where Φ is a scalar function
- $\int_S d\bar{S} \cdot \bar{A}(\bar{r})$ where $\bar{A}$ is a vector field
- $\int_S d\bar{S} \times \bar{A}(\bar{r})$ where $\bar{A}$ is a vector field Note that:

$$dS = |d\bar{S}| = \left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right| dudv$$

The surface area of a rotation paraboloid?

- Note surface area can be obtained by the surface integral of the normal vector field.

The surface S is given by:

$$\bar{r}(u,v) = \begin{cases} x^2 + y^2 \leq 1 \\ z = x^2 + y^2 \end{cases}$$

The surface is given by:

$$\int_S dS\Phi(\bar{r}),\ \Phi(\bar{r}) = 1$$

It is easier to work with cylindrical coordinates:

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = \rho^2 \end{cases}$$

If we call $u = \rho$ and $v = \phi$ we get:

$$\bar{r}(u,v) = \rho\bar{e}_\rho + z\bar{e}_z = \rho\bar{e}_\rho + \rho^2\bar{e}_z$$

Returning to cartesian coordinates we gete:

$$\bar{r}(u,v) = \rho\cos\phi\bar{e}_x + \rho\sin\phi\bar{e}_y + \rho^2\bar{e}_z$$

Finding the surface element by finding the partial derivatives:

$$\frac{\partial \bar{r}}{\partial \rho} = \cos\phi\bar{e}_x + \sin\phi\bar{e}_y + 2\rho\bar{e}_z = \bar{e}_\rho + 2\rho\bar{e}_z$$
$$\frac{\partial \bar{r}}{\partial \phi} = -\rho\sin\phi\bar{e}_x + \rho\cos\phi\bar{e}_y$$

Then the absolute surface element is:

$$|d\bar{S}| = \left|\frac{\partial \bar{r}}{\partial \rho} \times \frac{\partial \bar{r}}{\partial \phi}\right| = |\rho\bar{e}_z - 2\rho^2\bar{e}_\rho| = \sqrt{4\rho^4 + \rho^2} = (\sqrt{4\rho^2 + 1})\rho$$

The area is then:

$$\int_S dS = \int_0^{2\pi} d\phi \int_0^1 (\sqrt{4\rho^2 + 1})\rho d\rho = \frac{\pi}{6}\left(5^{\frac{3}{2}} - 1\right)$$

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The force on the paraboloid if it is filled wih a fluid of constant density.

$$p = \rho \cdot g \cdot z$$

Choose units such that $g \cdot \rho = 1$. Then the pressure is gien by:

$$p = 1 - z$$

Let the surface S be the paraboloid:

$$S = \begin{cases} \rho \leq 1 \\ z = \rho^2 \end{cases}$$

The parametization $\bar{r}$ is given by:

$$\bar{r} = \rho\bar{e}_\rho + \rho^2\bar{e}_z$$

Then the surface element $d\bar{S}$ is given by:

$$\frac{\partial \bar{r}}{\partial \rho} \times \frac{\partial \bar{r}}{\partial \phi} d\rho d\phi = (\rho\bar{e}_z - 2\rho^2\bar{e}_\phi)d\rho d\phi$$

Then the force is given by:

$$\bar{F} = -\int_S pd\bar{S} = -\int_0^1 d\rho \int_0^{2\pi} \left(\rho\bar{e}_z - 2\rho^2\bar{e}_\rho\right)\left(1 - \rho^2\right) d\phi$$

Since $\bar{e}_\rho$ depends on ϕ we can integrate over ϕ first using

$$\bar{e}_\rho = \cos\phi\bar{e}_x + \sin\phi\bar{e}_y$$

And by using symmetry we get:

$$\int_0^{2\pi} \bar{e}_\rho d\phi = 0$$

Since when we integrate over ϕ we get:

$$\int_0^{2\pi} \bar{e}_\rho d\phi = \int_0^{2\pi} \cos(\phi)\bar{e}_x + \sin(\phi)\bar{e}_y d\phi = 0\bar{e}_x + 0\bar{e}_y = 0$$

Then the force is given by: Then we get:

$$\bar{F} = -2\pi\bar{e}_z \int_0^1 \rho(1 - \rho^2)d\rho = -\frac{\pi}{2}\bar{e}_z$$

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Compute the out of a sphere of radius R when $\bar{A}$ is given by:

$$\bar{A} = -\left(\frac{1}{r^2} + \frac{\lambda}{r}\right) e^{-\lambda R}\bar{e_r}$$

A visualization of the sphere:

Through the surface S:

$$d\bar{S} = r^2\sin(\theta)d\theta\, d\phi\bar{e_r}$$
$$\frac{\partial \bar{r}}{\partial \theta} = r\frac{\partial}{\partial \theta}\bar{e_r} = r\bar{e_\theta}$$
$$\frac{\partial \bar{r}}{\partial \phi} = r\frac{\partial}{\partial \phi}\bar{e_r} = r\sin\theta\bar{e}_\phi$$

Using the cross product to obtain the surface element gives:

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Then the flux is given by:

$$F = \int_S d\bar{S} \cdot \bar{A} = \int_S d\bar{S} \cdot \bar{A} = \int_S r^2\sin(\theta)d\theta d\phi\bar{e}_\phi \cdot \left(-\left(\frac{1}{r^2} + \frac{\lambda}{r}\right) e^{-\lambda R}\bar{e_r}\right)$$

But since we are on the surface of the sphere we can use the fact that $r = R$ and that $\bar{e}_r$ is a unit vector in the direction of $\bar{r}$. Then we get:

$$F = -\int_0^{(} \pi) sin(\theta)d\theta \int_0^{2\pi} d\phi\left(\frac{1}{R^2} + \frac{\lambda}{R}\right) e^{-\lambda R} = -4\pi\left(\frac{1}{R^2} + \frac{\lambda}{R}\right) e^{-\lambda R}$$