Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Your turn: num_neg – tail recursive

```
# let num_neg list =
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =







  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
   match list with [] ->
     | (x :: xs) ->



 in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->



  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
        num_neg_aux xs  ?

  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
        num_neg_aux xs
          (if x < 0 then 1 + num_neg
           else num_neg)
  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
         num_neg_aux xs
           (if x < 0 then 1 + num_neg
            else num_neg)
  in num_neg_aux list  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
   match list with [] -> curr_neg
     | (x :: xs) ->
       num_neg_aux xs
         (if x < 0 then 1 + num_neg
          else num_neg)
  in num_neg_aux list 0
```

Tail Recursion - length

- How can we write length with tail recursion?

```
let length list =
    let rec length_aux list acc_length =
        match list
         with [ ] -> acc_length
           | (x::xs) ->
             length_aux xs (1 + acc_length)
  in length_aux list 0
```

accumulated value

initial acc value

combing operation

length, fold_left

```
let length list =
  fold_left
   (fun acc -> fun x -> 1 + acc) // comb op
   0  // initial accumulator cell value
   list
```

Your turn: num_neg, fold_left

```
let num_neg list =
   fold_left
    ?  // comb op



    ?  // initial accumulator cell value
    ?
```

Your turn: num_neg, fold_left

```
let num_neg list =
   fold_left
    ?  // comb op



    0  // initial accumulator cell value
    ?
```

Your turn: num_neg, fold_left

```
let num_neg list =
   fold_left
    (fun cur_neg -> fun x ->
      if x < 0 then 1 + num_neg else num_neg)
        // comb op
    0  // initial accumulator cell value
    ?
```

Your turn: num_neg, fold_left

```
let num_neg list =
  fold_left
   (fun cur_neg -> fun x ->
     if x < 0 then 1 + num_neg else num_neg)
       // comb op
   0  // initial accumulator cell value
   list
```

Extra Material

poor_rev – forward recursive

```
# let rec poor_rev list =
    match list with [] -> []
      | (x :: xs) -> poor_rev xs @ [x]
```

Tail Recursion - Example

```
# let rec rev_aux list revlist =
  match list with [ ] -> revlist
  | x :: xs -> rev_aux xs (x::revlist);;
val rev_aux : 'a list -> 'a list -> 'a list = <fun>

# let rev list = rev_aux list [ ];;
val rev : 'a list -> 'a list = <fun>
```

- What is its running time?

Comparison

- poor_rev [1;2;3] =
- (poor_rev [2;3]) @ [1] =
- ((poor_rev [3]) @ [2]) @ [1] =
- (((poor_rev [ ]) @ [3]) @ [2]) @ [1] =
- (([ ] @ [3]) @ [2]) @ [1]) =
- ([3] @ [2]) @ [1] =
- (3:: ([ ] @ [2])) @ [1] =
- [3;2] @ [1] =
- 3 :: ([2] @ [1]) =
- 3 :: (2:: ([ ] @ [1])) = [3; 2; 1]

Comparison

- rev [1;2;3] =
- rev_aux [1;2;3] [ ] =
- rev_aux [2;3] [1] =
- rev_aux [3] [2;1] =
- rev_aux [ ] [3;2;1] = [3;2;1]

Folding - Tail Recursion

```
# let rev list =
      fold_left
        (fun l -> fun x -> x :: l)    //comb op
        []                 //accumulator cell
        list
```

End of Extra Material

Folding

```
# let rec fold_left f a list = match list
  with [] -> a | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
   <fun>
```

fold_left f a $[x_1; x_2;\ldots;x_n]$ = $f(\ldots(f\ (f\ a\ x_1)\ x_2)\ldots)x_n$

```
# let rec fold_right f list b = match list
  with [ ] -> b | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
   <fun>
```

fold_right f $[x_1; x_2;\ldots;x_n]$ b = $f\ x_1(f\ x_2\ (\ldots(f\ x_n\ b)\ldots))$

Folding

- Can replace recursion by fold_right in any forward primitive recursive definition
  - Primitive recursive means here it only recurses on immediate subcomponents of recursive data structure
- Can replace recursion by fold_left in any tail primitive recursive definition

Extra Material

How long will it take?

- Remember the big-O notation from CS 225 and CS 374
- Question: given input of size *n*, how long to generate output?
- Express output time in terms of input size, omit constants and take biggest power

How long will it take?

Common big-O times:

- Constant time $O(1)$
  - input size doesn't matter
- Linear time $O(n)$
  - double input $\Rightarrow$ double time
- Quadratic time $O(n^2)$
  - double input $\Rightarrow$ quadruple time
- Exponential time $O(2^n)$
  - increment input $\Rightarrow$ double time

Linear Time

- Expect most list operations to take linear time $O(n)$
- Each step of the recursion can be done in constant time
- Each step makes only one recursive call
- List example: multList, append
- Integer example: factorial

Quadratic Time

- Each step of the recursion takes time proportional to input
- Each step of the recursion makes only one recursive call.
- List example:

```
# let rec poor_rev list = match list
  with [] -> []
    | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

Exponential running time

- Poor worst-case running times on input of any size
- Each step of recursion takes constant time
- Each recursion makes two recursive calls
- Easy to write naïve code that is exponential for functions that can be linear

Exponential running time

```
# let rec slow n =
     if n <= 1
    then 1
    else 1+slow (n-1) + slow(n-2);;
val slow : int -> int = <fun>
# List.map slow [1;2;3;4;5;6;7;8;9];;
- : int list = [1; 3; 5; 9; 15; 25; 41; 67;
  109]
```

Recall: Tail Recursion

- A recursive program is tail recursive if all recursive calls are tail calls
- Tail recursive programs may be optimized to be implemented as loops, thus removing the function call overhead for the recursive calls
- Tail recursion generally requires extra “accumulator” arguments to pass partial results
  - May require an auxiliary function

Terminology

- Available: A function call that can be executed by the current expression
- The fastest way to be unavailable is to be guarded by an abstraction (anonymous function, lambda lifted).
  - if (h x) then f x else (x + g x)
  - if (h x) then (fun x -> f x) else (g (x + x))

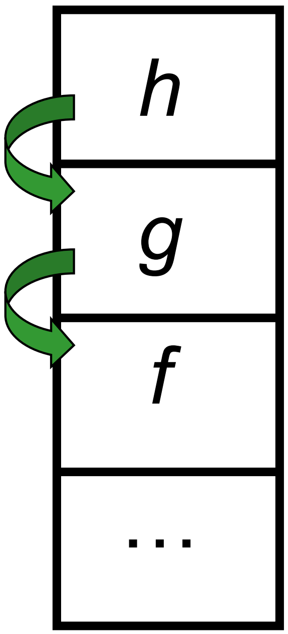
Not available

Terminology

- Tail Position: A subexpression s of expressions e, which is available and such that if evaluated, will be taken as the value of e
  - if (x>3) then x + 2 else x - 4
  - let x = 5 in x + 4
- Tail Call: A function call that occurs in tail position
  - if (h x) then f x else (x ± g x)

An Important Optimization

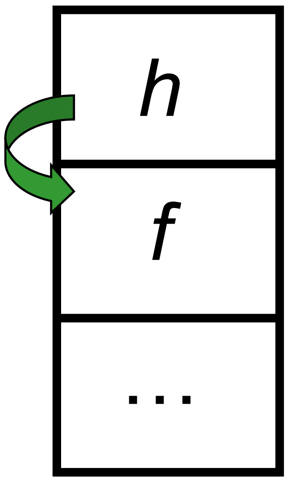
Normal call



- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?

An Important Optimization

Tail call



- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?
- Then *h* can return directly to *f* instead of *g*

End of Extra Material

Continuations

- A programming technique for all forms of “non-local” control flow:
  - non-local jumps
  - exceptions
  - general conversion of non-tail calls to tail calls
- Essentially it’s a higher-order function version of GOTO

Continuations

- Idea: Use functions to represent the control flow of a program
- Method: Each procedure takes a function as an extra argument to which to pass its result; outer procedure "returns" no result
- Function receiving the result called a continuation
- Continuation acts as "accumulator" for work still to be done

Continuation Passing Style

- Writing procedures such that all procedure calls take a continuation to which to give (pass) the result, and return no result, is called continuation passing style (CPS)

Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.
- A formalization of non-local control flow in denotational semantics
- Possible intermediate state in compiling functional code

Why CPS?

- Makes order of evaluation explicitly clear
- Allocates variables (to become registers) for each step of computation
- Essentially converts functional programs into imperative ones
  - Major step for compiling to assembly or byte code
- Tail recursion (and forward recursion) easily identified

Other Uses for Continuations

- CPS designed to preserve order of evaluation
- Continuations used to express order of evaluation
- Can be used to change order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

Example

- Simple reporting continuation:

```
# let report x = (print_int x; print_newline( ) );;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```
# let addk (a, b) k = k (a + b);;
val addk : int * int -> (int -> ’a) -> ’a = <fun>
# addk (22, 20) report;;
2
- : unit = ()
```

Simple Functions Taking Continuations

- Given a primitive operation, can convert it to pass its result forward to a continuation
- Examples:

```
# let subk (x, y) k = k(x - y);;
val subk : int * int -> (int -> 'a) -> 'a = <fun>
# let eqk (x, y) k = k(x = y);;
val eqk : 'a * 'a -> (bool -> 'b) -> 'b = <fun>
# let timesk (x, y) k = k(x * y);;
val timesk : int * int -> (int -> 'a) -> 'a = <fun>
```

Nesting Continuations

```
# let add_triple (x, y, z) = (x + y) + z;;
val add_triple : int * int * int -> int = <fun>
# let add_triple (x,y,z)=let p = x + y in p + z;;
val add_triple : int * int * int -> int = <fun>
# let add_triple_k (x, y, z) k =
   addk (x, y) (fun p -> addk (p, z) k );;
val add_triple_k: int * int * int -> (int -> 'a) ->
   'a = <fun>
```

add_three: a different order

- # let add_triple (x, y, z) = x + (y + z);;
- How do we write add_triple_k to use a different order?

- let add_triple_k (x, y, z) k =

add_three: a different order

- # let add_triple (x, y, z) = x + (y + z);;
- How do we write add_triple_k to use a different order?

- let add_triple_k (x, y, z) k =

```
addk (y,z) (fun r -> addk(x,r) k)
```

Recursive Functions

- Recall:

```
# let rec factorial n =
    if n = 0 then 1 else n * factorial (n - 1);;
  val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```

Terms

- A function is in Direct Style when it returns its result back to the caller.
- A function is in Continuation Passing Style when it, and every function call in it, passes its result to another function.
- Instead of returning the result to the caller, we pass it forward to another function giving the computation after the call.

Recursive Functions

```
# let rec factorial n =
    let b = (n = 0) in (* First computation *)
    if b then 1 (* Returned value *)
    else let s = n – 1 in (* Second computation *)
         let r = factorial s in  (* Third computation *)
         n * r (* Returned value *) ;;
val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```

Recursive Functions

```
# let rec factorialk n k =
   eqk (n, 0)
   (fun b ->  (* First computation *)
    if b then k 1 (* Passed value *)
    else subk (n, 1)  (* Second computation *)
    (fun s -> factorialk s  (* Third computation *)
      (fun r -> timesk (n, r) k))) (* Passed value *)
val factorialk : int -> (int -> 'a) -> 'a = <fun>
# factorialk 5 report;;
120
- : unit = ()
```

Recursive Functions

- To make recursive call, must build intermediate continuation to
  - take recursive value: r
  - build it to final result: n * r
  - And pass it to final continuation:
  -   times (n, r) k = k (n * r)

Recursive Functions

```
# let rec factorialk n k =
   eqk (n, 0)
   (fun b ->  (* First computation *)
    if b then k 1 (* Passed value *)
    else subk (n, 1)  (* Second computation *)
    (fun s -> factorialk s  (* Third computation *)
      (fun r -> timesk (n, r) k))) (* Passed value *)
val factorialk : int -> (int -> ‘a) -> ‘a = <fun>
# factorialk 5 report;;
120
- : unit = ()
```

Example: CPS for length

```
let rec length list = match list with [] -> 0
    | (a :: bs) -> 1 + length bs
```

What is the let-expanded version of this?