Some notes about SM

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目录

1 系综理论

microstate

$$\rho(q,p,t)\,dq\,dp \equiv \dot{\rho}(q_1,\ldots,q_{3N},p_1,\ldots,p_{3N})\prod_{i=1}^{3N} dq_i\,dp_i$$

1.1 Liouvile's theorem

Liouvile's theorem

$$\frac{\partial}{\partial t}\int_{\omega}\rho dpdq$$

$$\rho=|\psi|^2 \quad j=\frac{\langle\vec{p}\rangle}{m} \quad \frac{\partial\rho}{\partial t}+\nabla\cdot\vec{j}=0$$

$$\frac{\partial}{\partial t}\int_{\omega}\rho dpdq=-\oint\rho\,\vec{v}\cdot\hat{n}d\sigma=-\int_{\omega}v(\rho\vec{v})dpdq\Rightarrow\int_{\omega}\left[\frac{\partial p}{\partial t}+\nabla\cdot(\rho\vec{v})\right]dpdq=0$$

$$\nabla(\rho\vec{v})=\sum_{i=1}^{3N}\left(\frac{\partial}{\partial q_i}(\rho\dot{q}_i)+\frac{\partial}{\partial p_i}(\rho\dot{p}_i)\right)=\sum_{i=1}^{3N}\left(\frac{\partial\rho}{\partial q_i}\dot{q}_i+\frac{\partial\rho}{\partial p_i}\dot{p}_i\right)+\rho\sum_{i=1}^{3N}\left(\frac{\partial\dot{q}_i}{\partial q_i}+\frac{\partial\dot{p}_i}{\partial p_i}\right)$$

$$\Longrightarrow\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+\sum_{i=1}^{3N}\left(\frac{\partial\rho}{\partial q_i}\dot{q}_i+\frac{\partial\rho}{\partial p_i}\dot{p}_i\right)=0$$

$$\frac{\partial\rho}{\partial t}=\sum_{i=1}^{3N}\left(\frac{\partial\rho}{\partial p_i}\frac{\partial H}{\partial q_i}-\frac{\partial\rho}{\partial q_i}\frac{\partial H}{\partial p_i}\right)=-\{\rho,H\}$$

如何计算系统内某个物理量的平均?

$$\frac{d\langle O\rangle}{dt}=\int dpdp\frac{\partial\rho(p,q,t)}{\partial t}O(p,q)=\sum_{i=1}^{3N}dpdqO\left(\frac{\partial\rho}{\partial p_i}\frac{\partial H}{\partial q_i}-\frac{\partial\rho}{\partial q_i}\frac{\partial H}{\partial p_i}\right)$$

$$=-\sum_{i=1}^{3N}\int dpdq\rho\left[\left(\frac{\partial O}{\partial p_i}\frac{\partial H}{\partial q_i}-\frac{\partial O}{\partial q_i}\frac{\partial H}{\partial p_i}\right)+O\left(\frac{\partial^2 H}{\partial p_i\partial q_i}-\frac{\partial^2 H}{\partial q_i\partial p_i}\right)\right]$$

$$=-\int dpdq\rho\{H,O\}=\langle\{O,H\}\rangle$$

Probability : $\frac{\partial\rho}{\partial t}=0$ possible solution: $\rho(p,q)=\rho(H(p,q))$

notation : $\frac{\mathrm{d}\rho}{\mathrm{d}t}=0$恒成立，但是对于平衡状态下，有$\frac{\partial\rho}{\partial t}=0=-\{\rho,H\}=\sum_{i=1}^{3N}\left(\frac{\partial\rho}{\partial p_i}\frac{\partial H}{\partial q_i}-\frac{\partial p}{\partial q_i}\frac{\partial H}{\partial p_i}\right)$.

$\{\rho,H\}=\rho'(H)\{H,H\}=0$

1.2 微正则系综

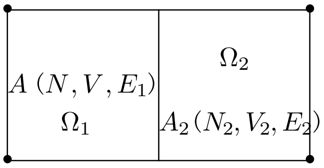
对于一个孤立系统

$\rho(p,q)=\text{constant. if } E\leqslant H(p,q)\leqslant E+\Delta$

$\rho(p,q)=0,\text{otherwise}$

↑Microcanonical ensemble.

相空间的体积：$\Gamma=\int\mathrm{d}q\mathrm{d}p\rho(H(p,q))$ $\quad S=k_B\ln\Gamma$



$$\Omega\left(E^{(0)}\right)=\sum_{E_1}\Omega_1(E_1)\,\Omega_2(E_2)\text{（分别选取最多的状态数）}$$

$$\frac{\partial\Omega_1(E_1)}{\partial E_1}\Big|_{E_1=\overline{E_1}}\Omega_2(\overline{E_2})+\Omega_1(\overline{E_1})\left(\frac{\partial\Omega_2(E_2)}{\partial E_2}\right)\Big|_{E_2=\overline{E_2}}=0$$

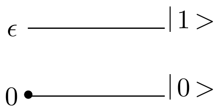
$$\Rightarrow\frac{\partial\ln\Omega_1(E_1)}{\partial E_1}\Big|_{E_1=\overline{E_1}}=\frac{\partial\ln\Omega_2}{\partial E_2}\Big|_{E_2=\overline{E_2}}$$

$$\beta\equiv\frac{\partial\ln\Omega(N,V,E)}{\partial E}\Big|_{N,V,E=\bar{E}}\quad\text{Equilibrium: }\beta_1=\beta_2$$

$$\left(\frac{\partial S}{\partial E}\right)\Big|_{N,V}=\frac{1}{T}\Rightarrow S=k_B\ln\Omega$$

$$\frac{\partial S_1}{\partial E_1}=\frac{\partial S_2}{\partial E_2}$$

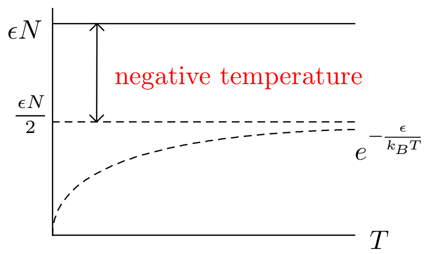
1.3 Two level System

impurity 

$E=\epsilon N_1$. 概率$P(\{n_i\})=\frac{\delta_{\epsilon\sum_i n_i,E}}{\Omega(E,N)}$, $n_i=0,1$。$\Omega(E,N)=\frac{N!}{(N-N_1)!\,N_1!}$

$$S(E,N)=k_B\ln(\Omega)=k_B(N(1-\ln N)-\ldots)\approx \mathrm{N}k_B\left[\left(\frac{E}{N\epsilon}\right)\ln\left(\frac{E}{N\epsilon}\right)+\left(1-\frac{E}{N\epsilon}\right)\ln\left(1-\frac{E}{N\epsilon}\right)\right]$$

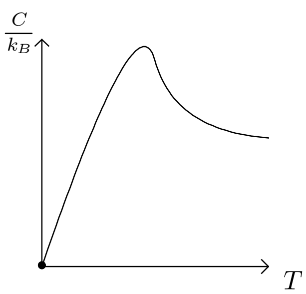
$$\frac{1}{T}=\frac{\partial S}{\partial E}\Big|_N=-\frac{k_B}{\epsilon}\ln\left(\frac{E}{N\epsilon-E}\right)\Longrightarrow E(T)=\frac{N\epsilon}{\exp\left(\frac{\epsilon}{k_BT}\right)+1}$$



负温度指随着温度的升高，熵反而下降。

计算指标Heat capacity

$$C=\frac{\mathrm{d}E}{\mathrm{d}T}$$



$p(n_1)=\sum_{\{n_2,...,n_i\}} p(\{n_i\})=\frac{\Omega(E-n_1\epsilon,N-1)}{\Omega(E,N)}\quad p(0)=\frac{1}{1+e^{-\frac{\epsilon}{k_BT}}},p(1)=\frac{e^{-\frac{\epsilon}{k_BT}}}{1+e^{-\frac{\epsilon}{k_BT}}}$

$$H=\sum_{i=1}^{N}\frac{\vec{p}_i^2}{2m}\qquad E\leqslant H\leqslant E+\Delta$$

$$\Omega(E)=\frac{1}{h^{3N}}\int_{E\leqslant H\leqslant E+\Delta}\mathrm{d}q_1...\mathrm{d}q_{3N}\mathrm{d}p_1...\mathrm{d}p_{3N}$$

球体的计算$\sum(E)=\frac{1}{h^{3N}}\int_{H\leqslant E}\mathrm{d}q_1...\mathrm{d}q_{3N}\mathrm{d}p_1...\mathrm{d}p_{3N}$

由于p,q与H,E无关，可以直接去d，积分得到6N维单位球的体积

$\Sigma=\frac{1}{h^{3N}}V^N\frac{(2\pi\mathrm{mE})^{3/2}}{(3N/2)!}$

$\Omega=\frac{\partial\Sigma}{\partial E}\Delta E=\frac{3N}{2}\frac{\Delta E}{E}\sum(E)$

$$\Omega(E)=\frac{\partial\Sigma}{\partial E}\Delta E$$

1.4 正则系综

能量相同的态出现的概率相同。

$$\rho(\mu_s)=\frac{1}{Z}e^{-\beta E_s}$$

$$Z=\sum_{\{\mu_s\}}e^{-\beta E_s}$$

$Z=\sum_{E_s}e^{-F(E_s)/k_BT}$, 作泰勒展开。

1.4.1 正则系综与热力学函数

自由能$F(E)=E-TS(E)$,

$$\frac{\partial S}{\partial E}|_{E=\bar{E}}=\frac{1}{T}$$

$$\frac{\partial F}{\partial E}|_{E=\bar{E}}=1-T\frac{\partial S}{\partial E}|_{E=\bar{E}}=0$$

$$\frac{\partial^2F}{\partial E^2}|_{E=\bar{E}}=-T\left(\frac{\partial^2S}{\partial E^2}\right)|_{E=\bar{E}}=\frac{1}{T}\frac{\partial T}{\partial E}|_{E=\bar{E}}=\frac{1}{TC_V}$$

$Z=\sum_{E_s}e^{-F(E_s)/k_BT}=e^{-\beta F(\bar{E})}\sqrt{2\pi k_BT^2C_V}$

$\sqrt{2\pi k_BT^2C_V}$ 在热力学极限下不重要

$$E=F+TS=F-T\frac{\partial F}{\partial T}=-T^2\frac{\partial}{\partial T}\left(\frac{F}{T}\right)=\frac{1}{k_B}\frac{\partial}{\partial\beta}\left(\frac{F}{T}\right)$$

$E=-\frac{\partial}{\partial\beta}\ln Z\quad F=-k_BT\ln Z$

连续情形。

$$Z=\frac{1}{h^{3N}N!}\int \mathrm{d}p\mathrm{d}qe^{-\beta H(p,q)}$$

$$S=-k_B\sum_{\{\mu_s\}}\rho(\mu_s)\ln\rho(\mu_s)$$

two-level system

$E=\epsilon\sum_{i=1}^{N}n_i \qquad \rho(\{n_i\})=\frac{1}{Z}e^{-\beta\epsilon\sum_{i=1}^{N}n_i}$

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$Z=\sum_{\{n_i\}}e^{-\beta\epsilon\sum_{i=1}n_i}=\sum_{\{n_i\}}\prod_i e^{-\beta\epsilon n_i}=\left(\sum_{n_1=0}^{1}e^{-\beta\epsilon n_1}\right)\left(\sum_{n_2=0}^{1}e^{-\beta\epsilon n_2}\right)\ldots\left(\sum_{n_N=0}^{1}e^{-\beta\epsilon n_N}\right)=(1+e^{-\beta\epsilon})^N$

$$\begin{aligned}F &= -Nk_BT\ln(1+e^{-\beta\epsilon})\\ \mathrm{S} &= -\frac{\partial F}{\partial T}|_N\\ &= \mathrm{Nk}_B\ln(1+e^{-\beta\epsilon})+\mathrm{Nk}_BT\left(\frac{\epsilon}{k_BT^2}\right)\frac{e^{-\beta\epsilon}}{1+e^{-\beta\epsilon}}\\ E &= \frac{N\epsilon e^{-\beta\epsilon}}{1+e^{-\beta\epsilon}}\end{aligned}$$

$$\rho_1(n_1)=\frac{e^{-\beta\epsilon n_i}}{1+e^{-\beta\epsilon}}$$

理想气体 Ideal gas

$$Z=\int\frac{1}{N!}\prod_{i=1}^{N}\frac{\mathrm{d}^3q_i\mathrm{d}^3p_i}{h^3}e^{-\beta\sum_{i=1}^{N}\frac{\vec{p}_i}{2m}}$$

$$\text{单个粒子}Z_1=\int\frac{\mathrm{d}^3q\mathrm{d}^3p}{h^3}e^{-\beta\frac{\vec{p}_i}{2m}}=\frac{V}{h^3}\int e^{-\beta\frac{\vec{p}_i}{2m}}\mathrm{d}^3p=\frac{V}{h^3}\left(\frac{2\pi m}{\beta}\right)^{3/2}$$

$$n\text{个粒子}Z_N=\int\frac{1}{N!}\prod_{i=1}^{N}\frac{\mathrm{d}^3q_i\mathrm{d}^3p_i}{h^3}e^{-\beta\sum_{i=1}^{N}\frac{\vec{p}_i{}^2}{2m}}=\frac{1}{N!}Z_i^N=\frac{1}{N!}V^N\left(\frac{2\pi m}{\beta h^2}\right)^{3N/2}=\frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N$$

thermal wavelength $\lambda=\frac{h}{\sqrt{2\pi mk_BT}}$

$$F\approx k_BT\left[\frac{3N}{2}\ln\left(\frac{2\pi mk_BT}{h^2}\right)+N\ln\frac{V}{N}+N\right]$$

$$\left(\frac{\partial F}{\partial V}\right)_{T,N}=-p\Rightarrow -k_BT\frac{N}{V}\Rightarrow pV=\mathrm{Nk}_BT$$

$$\left(\frac{\partial F}{\partial T}\right)_{V,\mu}=-S\Rightarrow S=k_B\left[\frac{3N}{2}\ln\left(\frac{2\pi mk_BT}{h^2}\right)+N\ln\frac{V}{N}+N\right]+k_B\frac{3N}{2}$$

Kardar

$$F=E-TS\rightarrow G=E-TS+pV$$

$$Z(N,T,p)=\int_0^\infty \mathrm{d}V\int\frac{1}{N!}\prod_{i=1}^N\frac{\mathrm{d}^3q_i\mathrm{d}^3p_i}{h^3}e^{-\beta\left[\sum\limits_{i=1}^N\frac{\vec{p_i}^2}{2m}+pV\right]}$$

$$G=E-TS+pV=-k_BT\ln Z\approx Nk_BT\left[\ln P-\frac{5}{2}\ln(k_BT)+\frac{3}{2}\ln\left(\frac{h^2}{2\pi m}\right)\right]$$

$$\mathrm{d}G=-S\mathrm{d}T+V\mathrm{d}p+\mu\mathrm{d}N$$

$$V=\frac{\partial G}{\partial p}|_{T,N}=\frac{Nk_BT}{p}$$

$$Z=\mathrm{Tr}(e^{-\beta\hat{H}})=\sum_n\langle n|e^{-\beta\hat{H}}|n\rangle$$

$$Z=\sum_{\{\ldots\}}e^{-\beta\left(\sum\limits_o\frac{\vec{p}^2}{2m}-\vec{B}\cdot\vec{M}\right)}\text{先不管第一部分}$$

$$Z=\sum_{\{\sigma_i\}}e^{\beta B\mu_i\sum\limits_{i=1}^N\sigma_i}\quad\sigma_i=\pm1$$

$$Z=[e^{\beta B\mu_0}+e^{-\beta B\mu_0}]^N=[2\cosh(\beta B\mu_0)]^N$$

$$G=-k_BT\ln\mathrm{Z}$$

$$M=-\frac{\partial G}{\partial B}$$

$$\chi(T)=\frac{\partial N}{\partial B}|_{B=0}=\frac{N\mu_0^2}{k_BT}$$

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$$Z(N,T,P)=\int_0^\infty \mathrm{d}V\int\frac{1}{N!}\prod_{i=1}^N\frac{\mathrm{d}^3q_i\mathrm{d}^3p_i}{h^3}e^{-\beta\left(\sum\frac{\vec{p_i}}{2m}+\mathrm{pV}\right)}$$

$\int_0^\infty\mathrm{d}V$是为了把引入的变量V通过积分积掉V的影响。

$$Z=\int\frac{1}{N!}\prod_{i=1}^N\frac{\mathrm{d}^3q_i\mathrm{d}^3p_i}{h^3}e^{-\beta E_s}$$

$$Z=\sum_{\{\mu_s\}}e^{-\beta E_s}$$

1.5 巨正则系综

Grand canonial ensemble (系综的粒子数可以改变的巨正则系综)（T,V,μ确定）

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N$$

$\rho(\mu_s)$

把剩下的空间和系综当作是一个整体，所有的粒子数为N，

$$\begin{aligned}
\rho(\mu_s) &= \frac{\boldsymbol{\Omega}_R(E-E_s,N-N_s)}{\Omega_{R+s}(E,N)}\\
&\approx \frac{1}{\Omega_{R+S}(E,N)}\exp\left[\frac{1}{k_B}\left(S_B(E,N)-\left(\frac{\partial S_B}{\partial E}\right)_{V,N}E_s-\left(\frac{\partial S_R}{}\right)\right)\right]\\
&= \mathrm{const}\times\exp\left[\frac{1}{k_B}\left(-\left(\frac{\partial S_R}{\partial E}\right)_{V,N}E_s-\left(\frac{\partial S_R}{N}\right)_{E,V}N_S\right)\right]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\rho(\mu_s) &\propto e^{-\beta(E_s-\mu N_s)}\\
\rho(\mu_s) &= \frac{1}{Z}e^{-\beta(E_s-\mu N_s)}
\end{aligned}$$

$$Z = \sum_{\{\mu_s\}} e^{-\beta(E_s-\mu N_s)} = \sum_{E_s,N_s}\frac{\Omega(E_s,N_s)e^{-\beta(E_s-\mu N_s)}}{e^{\frac{S}{k_B}}}$$

$$\rho(E_s,N_s) = \frac{1}{Z}\Omega(E_s,N_s)e^{-\beta(E_s-\mu N_s)} = \frac{1}{Z}\exp\left[\frac{1}{k_B}S(E,N_s)-\beta(E_s-\mu N_s)\right] = \frac{1}{Z}e^{-\beta\Phi(T,\mu,V)}$$

$$\Phi(T,\mu,N) = \bar{E} - \mathrm{TS} - \mu\bar{N}$$

$$\bar{E} - \mu\bar{N} = \Phi + \mathrm{TS} = \Phi - T\frac{\partial\Phi}{\partial T} = -T^2\frac{\partial}{\partial T}\left(\frac{\Phi}{T}\right) = \frac{1}{k_B}\frac{\partial}{\partial\beta}\left(\frac{\Phi}{T}\right) = -\frac{\partial}{\partial\beta}\ln Z$$

根据定义有 $\bar{E} - \mu\bar{N} = \frac{1}{Z}\sum_{\{\mu_s\}}(E_s-\mu N_s)e^{-\beta(E_s-\mu N_s)}$

由 $\Phi(T,\mu,N) = E - \mathrm{TS} - \mu N \quad \mathrm{d}\Phi = -S\mathrm{d}T - p\mathrm{d}V - N\mathrm{d}\mu$

$\Rightarrow\Phi = -k_B\mathrm{TlnZ}$

$$S = -\frac{1}{T}\frac{\partial}{\partial\beta}\ln Z + k_B\ln Z$$

$$S = -k_B\sum_{\{\mu_s\}}\rho(\mu_s)\ln\rho(\mu_s)$$

巨正则系综的配分函数为

$$Z = \sum_{N_S=0}^{\infty}(e^{\beta\mu})^{N_s}Z_c(T,N_s,V) = \sum_{N_s=0}^{\infty}Z^{N_s}Z_c$$

1.6 系综理论的一些应用

1.6.1 范德瓦尔斯气体

考虑各个粒子之间存在相互作用

Cumulant expansion

$$G(k)=\langle e^{ikY}\rangle=\int \mathrm{d}y e^{iky}P(y)=\sum_{r=0}^{\infty}\frac{(ik)^r}{r!}\langle Y^r\rangle$$

$\ln G(k)=\sum_{r=1}^{\infty}\frac{(ik)^r}{r!}\Xi_r$

$$\begin{aligned}\Xi_1&=\langle Y\rangle\\ \Xi_2&=\langle Y^2\rangle-\langle Y\rangle^2\\ \Xi_3&=\langle Y^3\rangle-3\langle Y^2\rangle\langle Y\rangle+2\langle Y\rangle^3\end{aligned}$$

范德瓦尔斯气体 状态方程

$$H=\sum\frac{p_i^2}{2m}+\sum_{i<j}v_{\mathrm{ij}}\qquad v_{\mathrm{ij}}\text{为相互作用项}$$

$v_{\mathrm{ij}}=v(|\vec{r}_i-\vec{r}_j|)$

由于粒子之间存在吸引力，气体的压强会略有减小，差别与体积的平方成反比

$$V_{\mathrm{eff}}=V-b\qquad b\approx\frac{4\pi}{3}N\left(\frac{r_0}{2}\right)^3$$

$$p=p_{\mathrm{kinetic}}-\frac{a}{V^2}$$

$$p_{\mathrm{kinetic}}V_{\mathrm{eff}}=RT$$

$$\text{范德瓦尔斯气体方程：}(V-b)\left(p+\frac{a}{V^2}\right)=RT$$

Viral expamsion

$$p+\frac{a}{V^2}=\frac{RT}{V}\left(1-\frac{b}{V}\right)^{-1}-\frac{a}{V^2}$$

$$\begin{aligned}\Rightarrow\frac{\mathrm{pV}}{\mathrm{RT}}&=\left(1-\frac{b}{V}\right)^{-1}-\frac{a}{RTV}\\ &=1+\frac{1}{V}\left(b-\frac{a}{RT}\right)+\left(\frac{b}{V}\right)^2+\left(\frac{b}{V}\right)^3\cdots\\ &=1+\frac{C_2}{V}+\frac{C_3}{V^2}+\cdots\end{aligned}$$

Cluster expansion

$$Z_N(V,T)=\frac{1}{N!h^{3N}}\int_i\prod_i \mathrm{d}^3q_i\mathrm{d}^3p_i\exp\left(-\beta\sum_{i=1}^{N}\frac{\vec{p_i}^2}{2m}-\beta\sum_{i<j}v_{ij}\right)=\frac{1}{N!\lambda^{3N}}Q_N(V,T)$$

$$Q_N(V,T)=\int\prod_i \mathrm{d}^3r_i\,e^{-\beta\sum_{i<j}u_{ij}}$$

$$\text{近似：}f_{ij}=e^{-\beta u_{ij}}-1$$

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$G(k)=\langle e^{-\mathrm{kY}}\rangle$

$$H=\sum\frac{\vec{p}_i^2}{2m}+U(q_1,\ldots,q_N)$$

$$\begin{aligned}Z(T,V,N) &= \frac{1}{N!}\int\prod_{i=1}^{N}\left(\frac{\mathrm{d}^3p_i\mathrm{d}^3q_i}{h^3}\right)\exp\left(-\beta\sum_i\frac{\vec{p}_i^2}{2m}\right)\exp(-\beta U(q_1,\ldots,q_N))\\ &= Z_0\langle\exp(-\beta U(q_1,\ldots,q_N))\rangle^0\\ \ln Z &= \ln Z_0+\sum_{l=1}^{\infty}\frac{(-\beta)^l}{l!}\langle U^l\rangle_c^0\end{aligned}$$

集团展开

$$H=\sum_{i=1}^{N}\frac{\vec{p}_i^2}{2m}+\sum_{i<j}v_{ij}(|\vec{r}_i-\vec{r}_j|)$$

体积温度恒定的情况下

$$Z_N(V,T)=\frac{1}{N!h^{3N}}\int\mathrm{d}^{3N}p\mathrm{d}^{3N}r\exp\left(-\beta\sum\frac{\vec{p}_i^2}{2m}-\beta\sum_{i<j}v_{ij}\right)=\frac{1}{N!\lambda^{3N}}Q_N(V,T)$$

$$\begin{aligned}Q_N(V,T) &= \int\mathrm{d}^{3N}r\exp\left(-\beta\sum_{i<j}v_{ij}\right)\\ &= \int\mathrm{d}^{3N}r\prod_{i<j}(1+f_{ij})\\ \text{其中} &\text{，}\quad f_{ij}=e^{-\beta v_{ij}}-1\\ Q_N(V,T) &= \int\mathrm{d}^3r_1\ldots\mathrm{d}^3r_N\Big[1+\sum f_{\mathrm{ij}}+\sum f_{\mathrm{ij}}f_{\mathrm{ii}}+\cdots+\Big]\\ &= \sum_{\{m_l\}}S(\{m_l\})\end{aligned}$$

Cluster expansion for a classical gas

Cluster integral

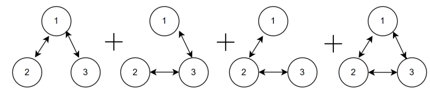
$$\text{定义：}b_l(V,T)=\frac{1}{l!\lambda^{3l-3}V}(\mathrm{sum\,of\,all\,possible}\,l-\mathrm{cluster})\text{（就是所有可能的相互作用）}$$

$$b_1=\frac{1}{V}[①]=\frac{1}{V}\int\mathrm{d}^3r=1$$

$$\begin{aligned}b_2 &= \frac{1}{2!\lambda^3V}[①-②]\\ &= \frac{1}{2\lambda^3V}\int f_{12}\mathrm{d}^3r_1\mathrm{d}^3r_2\\ &= \frac{1}{2\lambda^3}\int f_{12}\mathrm{d}^3r_{12}\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\pi}{\lambda^3}\int_0^\infty f(r_{12})\, r_{12}^2 \mathrm{d}r_{12} \\
&= \frac{2\pi}{\lambda^3}\int_0^\infty (e^{-\beta_1 v_{12}}-1)\, r_{12}^2 \mathrm{d}r_{12}
\end{aligned}
$$

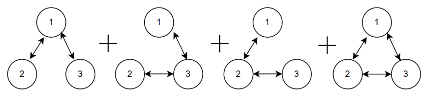
$$
b_3 = \frac{1}{3!\lambda^6 V}\left[\quad\right]
$$



$$
\begin{aligned}
&= \frac{1}{6\lambda^6 V}\int (f_{12}f_{13}+f_{13}f_{23}+f_{12}f_{23}+f_{12}f_{23}f_{13})\,\mathrm{d}^3r_1\mathrm{d}^3r_2\mathrm{d}^3r_3 \\
&= \frac{1}{6\lambda^6 V}\left(\int 3V\,(f_{12}f_{23}\mathrm{d}^3r_{12}\mathrm{d}^3r_{23})+V\int f_{12}f_{23}f_{13}\mathrm{d}^3r_{12}\mathrm{d}^3r_{13}\right)
\end{aligned}
$$

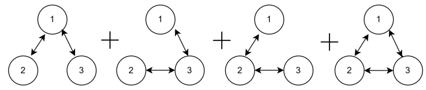
$\sum_{l=1}^{N} l\cdot m_l = N$, A given set of integers $\{m_l\}$ satisfying the equation, however, dose not uniquely sperify a gragh. (l:参与相互作用的粒子个数，m_l 相互作用的个数)

$$
\{[\bigcirc]....[\bigcirc]\,(m_1\text{个})\}\,\{[\bigcirc\!\leftrightarrow\!\bigcirc]...[\bigcirc\!\leftrightarrow\!\bigcirc]\,(m_2\text{个})\}\,\left\{\quad (m_3\text{个})\right\}
$$



每个位置可以填充粒子

$$
S\{m_l\} = \sum_p [\bigcirc]^{m_1}[\bigcirc\!\leftrightarrow\!\bigcirc]^{m_2}\left[\quad\right]^{m_3}
$$



$$
\begin{aligned}
&= \frac{N!}{(1!)^{m_1}(2!)^{m_2}(3!)^{m_3}}\times\frac{1}{m_1!m_2!m_3!\ldots}\times(1!\mathrm{V}\mathrm{b}_1)^{m_1}(2!\lambda^3\mathrm{V}\mathrm{b}_2)\,(3!\lambda^6\mathrm{V}\mathrm{b}_3)^{m_3}\ldots \\
&= N\,!\prod_{l=1}^{N}\frac{(V\lambda^{3l-3}bl)^{m_l}}{ml!} \\
&= N!\lambda^{3N}\prod_{l=1}^{N}\frac{1}{ml!}\left(\frac{Vbl}{\lambda^3}\right)^{m_l} \\
&= \sum_{\{m_l\}} N!\lambda^{3N}\prod_{l=1}^{N}\frac{1}{ml!}\left(\frac{Vb_l}{\lambda^3}\right)^{m_l}
\end{aligned}
$$

$$
Z_N(V,T)=\frac{1}{N!}\frac{1}{\lambda^{3N}}Q=\sum_{\{m_l\}}\prod_{l=1}^{N}\frac{1}{m_l}\left(\frac{V}{\lambda^3}b_l\right)^{m_l}
$$

若为巨正则

$$
\begin{aligned}
Z_G(Z,V,T) &= \sum_{N=0}^{\infty} Z^N Z_N(V,T) \\
&= \sum_{N=0}^{\infty} Z^N \sum_{\{m_l\}}{}'\prod_{l=1}^{\infty}\frac{1}{m_l!}\left(\frac{V}{\lambda^3}b_l\right)^{m_l} \\
&= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\left[\frac{1}{m_1!}\left(\frac{V}{\lambda^3}z^1b_1\right)^{m_1}\times\frac{1}{m_2}\left(\frac{V}{\lambda^3}z^2b_2\right)^{m_2}\cdots\right] \\
&= \prod_{l=1}^{\infty}\left(\sum_{m_l=0}^{\infty}\frac{1}{m\lambda!}\left(\frac{V}{\lambda^3}z^lb_l\right)^{m_l}\right) \\
&= \prod_{l=1}^{\infty} e^{b_l z^l \frac{V}{\lambda^3}}
\end{aligned}
$$

$$= \exp\left(\sum_{l=1}^{\infty} b_l z^l \frac{V}{\lambda^3}\right)$$

$$\therefore \frac{1}{V}\ln Z_G = \frac{1}{\lambda^3}\sum_{l=1}^{\infty} b_l z^l \qquad \mathrm{d}\Phi = -S\mathrm{d}T - p\mathrm{d}V - N\mathrm{d}\mu$$

$$\Phi = -k_B T \ln Z \qquad p = -\frac{\partial \Phi}{\partial V} = -\frac{\partial}{\partial V}\mathrm{lnZ}$$

$$\therefore \frac{P}{kT} = -\frac{\partial}{\partial V}\ln Z_G = \frac{1}{\lambda^3}\sum_{l=1}^{\infty} Z^l b_l$$

$$\because N = -\frac{\partial \Phi}{\partial \mu} \qquad Z = e^{\beta\mu} \qquad \frac{\partial Z}{\partial \mu} = Z \cdot \beta$$

$$\therefore \frac{N}{V} = -\frac{1}{V}\frac{\partial \Phi}{\partial \mu} = -\frac{1}{V}\frac{\partial \Phi}{\partial Z}\frac{\partial Z}{\partial \mu} = \frac{\beta Z}{V}\frac{\partial}{\partial Z}k_B T\,\mathrm{lnZ}_G = \frac{Z}{V}\frac{\partial}{\partial Z}\mathrm{lnZ}_G = \frac{Z}{V}\frac{\partial}{\partial Z}\frac{V}{\lambda^3}\sum_{l=1}^{\infty} Z^l b_l = \frac{1}{\lambda^3}\sum_{l=1}^{\infty} l Z^l b_l$$

$$\Rightarrow \left\{\begin{array}{c} \dfrac{p}{k_B T} = \dfrac{1}{\lambda^3}\displaystyle\sum_{l=1}^{\infty} b_l Z^l \\ \dfrac{N}{V} = \dfrac{1}{\lambda^3}\displaystyle\sum_{l=1}^{\infty} l b_l Z^l \end{array}\right\} \qquad \text{记}\boldsymbol{v} = \frac{V}{N}$$

$$\Rightarrow \frac{p\boldsymbol{v}}{k_B T} = \sum_{l=1}^{\infty} a_l(T)\left(\frac{\lambda^3}{\boldsymbol{v}}\right)^{l-1} \Rightarrow \frac{p\boldsymbol{v}}{k_B T} = \sum a_l(T)\left(\sum_{n=0}^{\infty} n\,\bar{b}_n Z^n\right)^{l-1}$$

$$\text{联立}\left\{\begin{array}{c} \dfrac{p\boldsymbol{v}}{k_B T} = \displaystyle\sum a_l(T)\left(\sum_{n=0}^{\infty} n\,\bar{b}_n Z^n\right)^{l-1} \\ \dfrac{p\boldsymbol{v}}{k_B T} = \dfrac{\sum_{i=1}^{\infty}\bar{b}_l Z^l}{\sum_{i=1}^{\infty} l\bar{b}_l Z^l} \end{array}\right\}$$

根据Z的次数相同匹配a 与$\bar{b}$ 之间的关系

$$\begin{aligned} a_1 &= \bar{b_1} = 1 \\ a_2 &= -\bar{b_2} \\ a_3 &= 4\bar{b}_2^2 - 2\bar{b}_3 \\ \cdots &\ \cdots\ \cdots \end{aligned}$$

$$\begin{aligned} \frac{p\boldsymbol{v}}{k_B T} &= 1 + a_2\left(\frac{\lambda^3}{2}\right) = 1 - \bar{b}_2\left(\frac{\lambda^3}{v}\right) = 1 - \frac{1}{2v}\int f_{12}\mathrm{d}^3 r_{12} = 1 - \frac{2\pi}{v}\int_0^{\infty} f_{12} r^2 \mathrm{d}r \\ &= 1 + \frac{b}{v} - \frac{a}{v k_B T} \end{aligned}$$

$$\Rightarrow \left[p + \left(\frac{N}{V}\right)^2 a\right] = N k_B T\left[\frac{1}{V} + \frac{N}{V^2}b\right] \Rightarrow \left[p + \left(\frac{N}{V}\right)^2 a\right](V - Nb) \approx N k_B T$$

2023.11.21

2 Quantum statistics 量子统计

当粒子的尺度与波长相近时，考虑量子效应（$\lambda=\frac{h}{p}$）

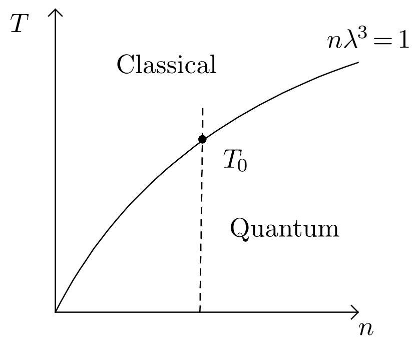
$$\frac{p^2}{2m}=\frac{3}{2}k_BT\,(\text{能量均分原理})\Longrightarrow\lambda_0=\frac{h}{\sqrt{3mk_BT}}$$

温度低→动量小→波长大→波长与粒子的间距接近，量子效应不可忽略。

热波长$\lambda=\frac{h}{\sqrt{2\pi mk_BT}}$

$$n=\frac{N}{V}\qquad n\lambda^3\ll 1\text{: classical region}\quad \lambda^3\text{是为了量纲统一}$$

$$n\lambda^3\approx 1\text{: quantum region}$$



密度一定时，温度超过T_0则转变为经典情形

量子力学中考虑全同粒子(Identical particles)

$$P\Psi(\vec{r}_1,\vec{r}_2)=\Psi(\vec{r}_2,\vec{r}_1)=e^{i\theta}\Psi(\vec{r}_1,\vec{r}_2)\quad P^2=I\quad [P,H]=0$$

$$\begin{aligned}H(P\Psi)&=PH\Psi\\&=PE\Psi\\&=E(P\Psi)\end{aligned}$$

$$P=e^{i\theta}\quad \theta=0\text{时对应玻色子}\quad \theta=\pi\text{对应费米子}\quad \Psi(\vec{r}_1,\vec{r}_2)=\pm\Psi(\vec{r}_2,\vec{r}_1)\,(\text{交换对称/反对称})$$

$$\Psi(\vec{r}_1,\vec{r}_2)=f(\vec{r}_1)\,g(\vec{r}_2)\pm f(\vec{r}_2)\,g(\vec{r}_1)$$

当两个相离比较远的时候，玻色子和费米子近似为玻尔兹曼统计。

2.1 Occupation numbers 占据数表象

$$\int \mathrm{d}^3 r\, u_\alpha^*(\vec{r})\, u_\beta(\vec{r}) = \delta_{\alpha\beta}$$

$$\Psi(\vec{r}_1, \ldots, \vec{r}_N) = \frac{1}{\sqrt{N!}} \sum_P \delta_p P[u_{\alpha_1}(\vec{r}_1) \cdots u_{\alpha_N}(\vec{r}_N)]$$

$$\sum_{\alpha_1, \alpha_2, \ldots} C(\alpha_1, \alpha_2, \ldots)\, u_{\alpha_1}(\vec{r}_1) \cdots u_{\alpha_N}(\vec{r}_N)$$

$$\text{Fermions} = \frac{1}{\sqrt{N!}} \begin{vmatrix} u_1(\vec{r}_1) & u_1(\vec{r}_2) & . \;.\; . & u_1(\vec{r}_N) \\ u_2(\vec{r}_1) & u_2(\vec{r}_2) & & u_2(\vec{r}_N) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ u_N(\vec{r}_1) & u_N(\vec{r}_2) & . \;.\; . & u_N(\vec{r}_N) \end{vmatrix}$$

slater行列式？

$$\text{每个态可以占据的粒子数} n_\alpha = \left\{ \begin{array}{cc} 0，1，2，3，\ldots，\infty & \text{for bosons} \\ 0，1 & \text{for fermions} \end{array} \right\}$$

$|n_1, n_2, \ldots, n_N>$ 表示第一个态所有粒子数n_1, 第二个态所占据的粒子数 . . . , 第n个态. . .,这就是占据数表象。

$$\begin{aligned} \sum_\alpha n_\alpha &= N \\ u_R(\vec{r}) &= \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}} \\ \sum_{\vec{k}} &\rightarrow \frac{V}{(2\pi)^3} \int \mathrm{d}^3 k = \int \frac{\mathrm{d}^3 r \mathrm{d}^3 P}{h^3} \end{aligned}$$

2.2 Incoherent Superposition of States

$$\Psi = \sum_n C_n(X, t)\, \psi_n$$

$$\|\Psi\|^2 \rightarrow \sum_{m,n} C_m^*(X, t)\, C_n(X, t)\, \psi_m^*(x)\, \psi_n(x) \qquad C_n = r_n e^{i\phi_n}$$

$$C_m^*(X, t)\, C_n(X, t) \rightarrow e^{i(\phi_m - \phi_n)} r_m r_n$$

2.3 Density matrix 密度矩阵

pure state $\rightarrow |\psi>$ 可以表示出来的态$\hat{A}$, $\langle A \rangle_\psi = \langle \psi | \hat{A} | \psi \rangle = \mathrm{Tr}(\rho \hat{A})$,$\rho = |\psi><\psi|$

$A_{mn} = \langle \phi_m | \hat{A} | \phi_n \rangle$,$P_{mn} = \langle \phi_m | \psi \rangle \langle \psi | \phi_n \rangle$

$$\begin{aligned} \mathrm{Tr}(\rho \hat{A}) &= \sum_n \langle n | \psi \rangle \langle \psi | \hat{A} | n \rangle \\ &= \sum_n \langle \psi | \hat{A} | n \rangle \langle n | \psi \rangle \\ &= \langle \psi | \hat{A} | \psi \rangle \end{aligned}$$

$$\mathrm{Tr}\, \rho = 1$$

$$\mathrm{Tr}\, \rho^2 < 1$$

$$\rho^2 = |\psi><\psi||\psi><\psi| = \rho$$

2023.12.1

量子统计 离散费米气体 与外界磁场的的相互作用

2.4 纯态与混态

纯态：

$\rho = |\psi><\psi| \quad \langle \hat{A} \rangle_\psi = \langle \psi | \hat{A} | \psi \rangle = \mathrm{Tr}\ \rho \hat{A}$ 纯态的特征：$\mathrm{Tr}\ \rho^2 = 1$

$\Leftarrow \sum_n \langle n|\psi\rangle\langle\psi|n> = \sum_n |\psi><\psi|$

系综的系统不处于纯态，则为混态

混态（Mixed state）：

无法写出波函数，但有混态的密度矩阵。

已知：$|\psi_1>$ with propability $P_1, \ldots, |\psi_i> \cdots\cdots P_i$ 其中，$\langle\psi_i|\psi_j\rangle = \delta_{ij}$; $P_1 + \cdots + P_N = 1$ ψ_i为纯态

混态的密度矩阵为：$\rho = \sum_i P_i |\psi_i><\psi_i|$

例如自旋（上或下）：$|\psi> = \frac{|\uparrow> + e^{i\theta}|\downarrow>}{\sqrt{2}}$, 两个态的概率为50%，但不知道态函数。

$$\text{密度矩阵} = \begin{pmatrix} <\uparrow|\rho|\uparrow> & <\uparrow|\rho|\downarrow> \\ <\downarrow|\rho|\uparrow> & <\downarrow|\rho|\downarrow> \end{pmatrix}, \text{其中}\rho = \sum_i P_i |\psi_i><\psi_i| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

混态中物理量的平均值计算$\langle \hat{A} \rangle = \sum_i P_i \langle\psi_i|\hat{A}|\psi_i\rangle = \mathrm{Tr}\ \rho\hat{A}$

$\mathrm{Tr}\ \rho\hat{A} = \sum_n \left\langle n \middle| \sum_i P_i \middle| \psi_i \right\rangle \langle\psi_i|\hat{A}|n\rangle = \sum_n\sum_i P_i \langle\psi_i|\hat{A}|n\rangle\langle n|\psi_i\rangle = \sum_i P_i \langle\psi_i|\hat{A}|\psi_i\rangle$

注意到$\mathrm{Tr}\ \rho^2 < 1$, 以下进行证明：(ρ为实数，$|\psi>$为厄密共轭)

$$\rho^2 = \sum_i \sum_j P_i P_j |\psi_i><\psi_i|\psi_j><\psi_j|$$

$$\mathrm{Tr}\ \rho^2 = \sum_n \sum_{i,j} P_i P_j <n|\psi_i><\psi_i|\psi_j><\psi_j|n> = \sum_n \sum_{i,j} P_i P_j <\psi_i|\psi_j><\psi_j|n><n|\psi_i>$$

$$\mathrm{Tr}\ \rho^2 = \sum_{i,j} P_i P_j |<\psi_j|\psi_i>|^2 = \sum_{i,j} P_i P_j |<\psi_j|\psi_i>|^2 \leqslant \left(\sum_i P_i\right)\left(\sum_j P_j\right) = 1$$

定理. *Liouville equation --Quantum Version*

$$i\hbar\frac{\partial}{\partial t}|\psi(t)> = H|\psi(t)> \qquad -i\hbar\frac{\partial}{\partial t}\langle\psi(t)| = \langle\psi(t)|H$$

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\rho &= i\hbar\sum_i P_i\left(\frac{\partial|\psi_i\rangle}{\partial t}\langle\psi_i| + |\psi_i\rangle\frac{\partial\langle\psi_i|}{\partial t}\right) \\ &= \sum_i P_i(H|\psi_i\rangle\langle\psi_i| - |\psi_i\rangle\langle\psi_i|H) \\ &= [H, \rho] \end{aligned}$$

equilibrium state:

$$\frac{\partial}{\partial t}\rho=0 \qquad \Rightarrow [H,\rho]=0 \quad \Rightarrow \rho=f(H)$$

2.5 Ensembles in Quantum statistical Mechanics.

Microcanonical（E,N,V确定），all wavefunctions $|\psi_s>, s=1,2,3,\ldots,\Omega$

$$\text{密度矩阵}\,\rho=\sum_{s=1}^{N}\frac{1}{\Omega}|\psi_s\rangle\langle\psi_s|$$

Canonical (T,N,V确定),

$$H|\psi_s>=E_s|\psi_s>$$

$$\text{经典}\quad \rho_s=\frac{e^{-\beta E_s}}{Z}\qquad Z=\sum_s e^{-\beta E_s}$$

$$\hat{\rho}=\sum_s\frac{e^{-\beta E_s}}{Z}|\psi_s\rangle\langle\psi_s|=\sum_s\frac{1}{Z}e^{-\beta\hat{H}}|\psi_s\rangle\langle\psi_s|=\frac{1}{Z}e^{-\beta\hat{H}}$$

$$Z=\mathrm{Tr}\,e^{-\beta\hat{H}}$$

Grand canonical (T,μ,V确定)

$$\hat{\rho}=\sum_s\frac{1}{Z}e^{-\beta(E_s-\mu N_s)}|\psi_s\rangle\langle\psi_s|=\frac{1}{Z}e^{-\beta(\hat{H}-\mu\hat{N})}$$

$$Z=\mathrm{Tr}\,e^{-\beta(\hat{H}-\mu\hat{N})}$$

定义. 冯诺依曼熵 *Von Neumomn entropy*

$$S=-\mathrm{Tr}\,(\rho\ln\rho)$$

$$\rho=\begin{pmatrix}P_1 & & & 0\\ & P_2 & & \\ & & \ldots & \\ 0 & & & P_N\end{pmatrix}$$

$$S=-\sum_s P_s\ln P_s$$

例. A free particle in a box

$$\hat{H}=-\frac{\hbar^2}{2m}\nabla^2=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

$$\phi_E(r)=\frac{1}{L^{\frac{3}{2}}}e^{i\vec{K}\cdot\vec{r}}\qquad \vec{k}\equiv(k_x,k_y,k_z)=\frac{2\pi}{L}(n_x,n_y,n_z)$$

$\phi_E(\vec{r})$为根据边界条件的波函数，而$\phi_E(\vec{r})=\langle E|r\rangle$，中$E$为抽象的波函数，

内积表示抽象的波函数在坐标表象下的波函数。

$$
\begin{aligned}
\langle\vec{r}\,|\,e^{-\beta\hat{H}}\,|\,\vec{r}'\,\rangle &= \left\langle\vec{r}\left|\sum_{E'}|E'\rangle\langle E'|e^{-\beta\hat{H}}\sum_{E}|E\rangle\langle E|\right|\vec{r}'\right\rangle = \sum_{E}\langle\vec{r}\,|\,\vec{E}\rangle e^{-\beta E}\langle\vec{E}\,|\,\vec{r}'\rangle \\
&= \sum_{E} e^{-\beta E}\phi_E(\vec{r})\,\phi'_E(\vec{r}') \\
&= \frac{1}{L^3}\sum_k \exp\left[-\frac{\beta\hbar^2}{2m}k^2 + i\vec{k}\cdot(\vec{r}-\vec{r}')\right] \\
&= \frac{1}{(2\pi)^3}\int\exp\left[-\frac{\beta\hbar^2}{2m}k^2 + i\vec{k}\cdot(\vec{r}-\vec{r}')\right]\mathrm{d}k \\
&= \left(\frac{m}{2\pi\beta\hbar^2}\right)^{3/2}\exp\left[-\frac{m}{2\beta\hbar^2}|\vec{r}-\vec{r}'\,|^2\right] \\
&= \frac{1}{V}e^{-\frac{m}{2\beta\hbar^2}|\vec{r}-\vec{r}'\,|^2} \quad \Leftrightarrow \text{高斯波包}
\end{aligned}
$$

波包的展宽类似于热波长λ

$$
\mathrm{Tr}\,(e^{-\beta\hat{H}}) = V\left(\frac{m}{2\pi\beta\hbar^2}\right)^{3/2} = \int\langle\vec{r}\,|\,e^{-\beta\hat{H}}\,|\,\vec{r}\rangle\mathrm{d}^3r
$$

2023.12.5

$Z = V\left(\frac{m}{2\pi\beta\hbar^2}\right)^{\frac{3}{2}}$

$\langle\hat{H}\rangle = \mathrm{Tr}\,(\hat{H}\rho) = \mathrm{Tr}\,(\rho\hat{H}) = \frac{\mathrm{Tr}\,(\hat{\mathrm{H}}e^{-\beta\hat{H}})}{\mathrm{Tr}\,e^{-\beta\hat{H}}} = -\frac{\partial}{\partial\beta}\ln(\mathrm{Tr}\,e^{-\beta\hat{H}}) = -\frac{\partial}{\partial\beta}\ln Z = -\frac{3}{2}\frac{1}{\beta} = \frac{3}{2}k_BT$

例. 2.5.1　Vibrational modes 震动

$$
Z = \int\frac{\mathrm{d}p\mathrm{d}q}{h}e^{-\beta\left(\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2\right)} = \frac{k_BT}{\hbar\omega} \qquad \text{经典情况下}
$$

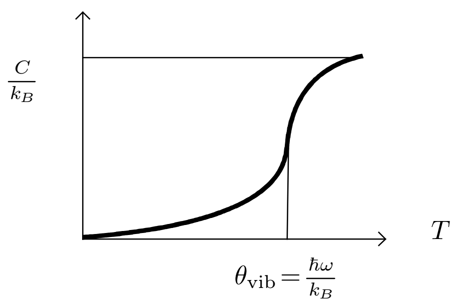
$$
\langle H\rangle = -\frac{\partial}{\partial\beta}\ln Z = \frac{1}{\beta} = k_BT
$$

$$
\text{量子的情形下：}\; H = \hbar\omega\left(n+\frac{1}{2}\right) = \hbar\omega\left(a^+a+\frac{1}{2}\right)
$$

$$
Z = \sum_{n=0}^{\infty}e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}\xrightarrow[T\to\infty]{}\frac{1}{\beta\hbar\omega} = \frac{k_BT}{\hbar\omega}
$$

$$
E = \langle H\rangle = -\frac{\partial}{\partial\beta}\ln Z = \frac{\hbar\omega}{2} + \hbar\omega\frac{1}{1-e^{\beta\hbar\omega}}
$$

$$
C = \frac{\partial E}{\partial T} = k_B\left(\frac{\hbar\omega}{k_BT}\right)^2\frac{e^{\beta\hbar\omega}}{(1-e^{\beta\hbar\omega})^2}
$$



例. Rotational modes

$$\mathcal{L}=\frac{I}{2}(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)\quad P_\theta=\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\quad P_\phi=\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=I\sin^2\theta\dot{\phi}$$

$$H=\frac{1}{2I}\left(P_\theta^2+\frac{P_\phi^2}{\sin^2\theta}\right)=\frac{\vec{L}^2}{2I}$$ [1]

$$Z=\frac{1}{\hbar^2}\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\int_{-\infty}^{\infty}\mathrm{d}P_\theta\int_{-\infty}^{\infty}\mathrm{d}P_\phi\, e^{-\beta H}=\frac{2Ik_BT}{\hbar^2}$$

$$\langle H\rangle=k_BT$$

量子 Rotational modes

$$Z=\sum_{l=0}^{\infty}e^{-\frac{\beta\hbar^2 l(l+1)}{2I}}(2l+1)\qquad \theta_{\text{rotational}}=\frac{\hbar^2}{2Ik_B}$$

when $T\gg\theta_{\text{rot}}$, $T\to\infty$ $\quad \lim\limits_{T\to\infty}Z=\int_0^\infty \mathrm{d}x\ (2x+1)\,e^{-\frac{\theta_{\text{rot}}x(x+1)}{T}}=\frac{T}{\theta_{\text{rot}}}$

2.6 量子系综的应用

Microcanonical ensemble: ideal gas

$$N=n_1+n_2+\cdots=\sum_i n_i\qquad \sum_i E_in_i=E$$

Ferminous: n_i个粒子，g_i个态中，每个态中只能有一个费米子

$$\omega_i=\binom{g_i}{n_i}=\frac{g!}{n_i!(g_i-n_i!)}$$

Bosons:

$$\omega_i=\binom{n_i+g_i-1}{g_i}=\frac{(n_i+g_i-1)!}{n_i!(g_i-1)!}$$

$$\Omega(E,V,N)=\begin{cases}\sum_{\{n_i\}}{}'\prod_i\frac{g!}{n_i!(g_i-n_i!)} & \text{fermions}\\ \sum_{\{n_i\}}{}'\prod_i\frac{(n_i+g_i-1)!}{n_i!(g_i-1)!} & \text{bosons}\end{cases}$$

In the limits $n_i/g_i\ll 1$, 能量相同的态很多，对应温度较高，态的个数远大于粒子个数。

$$\prod_i\frac{g!}{n_i!(g_i-n_i!)}=\prod_i\frac{g_i(g_i-1)\cdots(g_i-n_i+1)}{n_i!}\simeq\prod_i\frac{g_i^{n_i}}{n_i!}$$

$$\prod_i\frac{(n_i+g_i-1)!}{n_i!(g_i-1)!}=\prod_i\frac{(n_i+g_i-1)(n_i+g_i-2)\cdots g_i}{n_i!}=\prod_i\frac{g_i^{n_i}}{n_i!}$$

1. 补充：$\vec{L}=m\vec{r}\times\vec{v}=mr^2(\dot{\theta}\hat{\phi}-\dot{\phi}\sin\theta\hat{\theta})\quad L=I\omega\quad E=\frac{1}{2}I\omega^2=\frac{L^2}{2I}$

最概然分布

定义. $\sum_{\{n_i\}}{}'\prod_i \frac{(n_i+g_i-1)!}{n_i!(g_i-1)!} = \ln\mathcal{W}(\{n_i\})$

$$\delta\ln\mathcal{W}(\{n_i\}) - \left[\alpha\sum_i \delta n_i + \beta\sum_i E_i\delta n_i\right] = 0$$

$$\ln\mathcal{W}(\{n_i\}) \approx \left\{\begin{array}{cc} \sum_i g_i\ln g_i - n_i\ln_i - (g_i-n_i)\ln(g_i-n_i) & \text{fermions} \\ \sum_i (n_i+g_i-1)\ln(n_i+g_i-1) - n_i\ln n_i - (g_i-1)\ln(g_i-1) & \text{bosons} \end{array}\right\}$$

$$\begin{array}{lll} \text{Fermions} & \sum_i [-\ln n_i + \ln(g_i-n_i) - \alpha - \beta E_i]\delta n_i = 0 & \Rightarrow \frac{\bar{n_i}}{g_i} = \frac{1}{e^{\alpha+\beta E_i}+1} \\ \text{bosons} & \sum_i [-\ln n_i + \ln(g_i+n_i) - \alpha - \beta E_i]\delta n_i = 0 & \Rightarrow \frac{\bar{n_i}}{g_i} = \frac{1}{e^{\alpha+\beta E_i}-1} \\ \text{Boltzmann} & & \frac{\bar{n_i}}{g_i} = \frac{1}{e^{\alpha+\beta E}} \end{array}$$

巨正则系综（理想气体）

$\langle O\rangle = \mathrm{Tr}\,\hat{\rho}\hat{O}$

密度算符$\hat{\rho} = e^{\beta(\mu\hat{N}-\hat{H})}$

$Z = \mathrm{Tr}\,e^{\beta(\mu\hat{N}-\hat{H})} = \sum_{\{\mu_s\}} e^{\beta(\mu\hat{N}_s-E_s)} = \sum_{N_s=0}^{\infty}(e^{\beta\mu})^{N_s}Z_{N_s} = \sum_{N=0}^{\infty}\sum_{\{n_i\}}{}' e^{\beta\sum_i(\mu-E_i)n_i}$

其中$Z_{N_s} = \sum_{\{n_i\}}{}' e^{-\beta\sum_i n_iE_i}$, $\sum_i n_i = N_s$

$$Z = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\prod_i e^{\beta(\mu-E_i)n_i} = \prod_i^N\left[\sum_{n_i} e^{\beta(\mu-E_i)n_i}\right]$$

$$\left\{\begin{array}{lll} \text{对于费米子} & n_i = 0/1 & Z = \prod_i^N [1+e^{\beta(\mu-E_i)}] \\ \text{对于玻色子} & n_i = 0,1,\ldots & Z = \prod_i \frac{1}{1-e^{\beta(\mu-E_i)}} \end{array}\right\} \Rightarrow Z = \prod_i (1-\zeta e^{-\beta(E_i-\mu)})^{-\zeta}$$

其中，$\zeta = -1 \Leftrightarrow$ fermions　　$\zeta = +1 \Leftrightarrow$ bosons

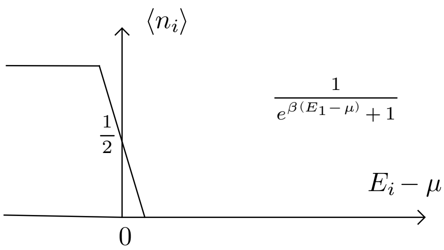
Occupation number布居数

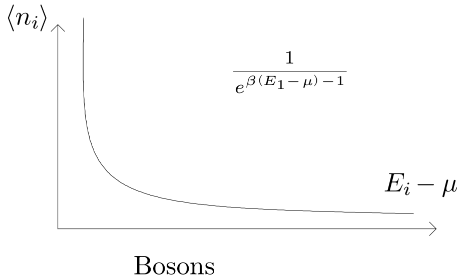
巨正则系综中的粒子数不确定，但可以计算平均值

$$\langle N\rangle = -\left(\frac{\partial\Phi}{\partial\mu}\right)_{V,T} = \frac{1}{\beta}\frac{\partial\ln Z}{\partial\mu} = \sum_i \frac{e^{-\beta(E_i-\mu)}}{1-\zeta e^{-\beta(E_i-\mu)}} = \sum_i \frac{1}{e^{\beta(E_i-\mu)}-\zeta}\text{（考试）}$$

在高温极限下，均退化为经典分布

$\langle E\rangle = \sum_i\langle n_i\rangle E_i = \sum_i \frac{E_i}{e^{\beta(E_i-\mu)}-\zeta}$　$\langle E\rangle - \mu\langle N\rangle = -\frac{\partial}{\partial\mu}\ln Z$





$$\sum_{\vec{k}} \to V\int \frac{\mathrm{d}^3 k}{(2\pi)^3} = \frac{V}{h^3}\int \mathrm{d}^3 p = \frac{1}{h^3}\int \mathrm{d}^3 p \mathrm{d}^3 q$$

$$\ln Z = -\zeta \sum_i \ln\left(1-\zeta e^{\beta\left(\mu-\frac{\hbar^2 k^2}{2m}\right)}\right)$$

因为$p=-\left(\frac{\partial \Phi}{\partial V}\right)_{T,\mu}$

$$\frac{p}{k_B T} = \frac{\ln Z}{V} = -\zeta \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \ln\left(1-\zeta e^{\beta\left(\mu-\frac{\hbar^2 k^2}{2m}\right)}\right)$$

$$\frac{N}{V} = g\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{e^{\beta\left(\frac{\hbar^2 k^2}{2m}-\mu\right)}-\zeta}$$ [2]

$$\frac{E}{V} = g\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\hbar^2 k^2/2m}{e^{\beta(\hbar^2 k^2/2m-\mu)}-\zeta}$$

$$x = \beta\hbar^2 k^2/2m \Rightarrow k = \frac{\sqrt{2m k_B T}}{\hbar}\sqrt{x} = \frac{2\sqrt{\pi x}}{\lambda} \qquad \mathrm{d}k = \frac{\sqrt{\pi}}{\lambda}\frac{\mathrm{d}x}{x}$$

$$\begin{aligned}
\frac{p}{k_B T} &= -\zeta g\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \ln\left(1-\zeta e^{\beta\left(\mu-\frac{\hbar^2 k^2}{2m}\right)}\right)\\
&= -\zeta \frac{g}{(2\pi)^3}\int \frac{\sqrt{\pi}}{\lambda}\frac{\mathrm{d}x}{\sqrt{x}}\frac{4\pi x}{\lambda^2}(4\pi)\ln(1-\zeta z e^{-s})\\
&= \frac{4g}{3\sqrt{\pi}}\frac{2}{\lambda^3}\int_0^\infty \mathrm{d}x\, x^{\frac{1}{2}} \ln(1-\zeta z e^{-x})\\
&= \frac{4g}{3\sqrt{\pi}}\frac{1}{\lambda^3}\int_0^\infty \mathrm{d}x \frac{x^{\frac{3}{2}} z e^{-x}}{1-\zeta z e^{-x}}\\
&= \frac{4g}{3\sqrt{\pi}}\frac{1}{\lambda^3}\int_0^\infty \mathrm{d}x \frac{x^{\frac{3}{2}}}{z^{-1}e^x-\zeta}
\end{aligned}$$

分别代回$\frac{N}{V}$, $\frac{E}{V}$

$$\frac{N}{V} = \frac{2g}{\sqrt{\pi}}\frac{1}{\lambda^3}\int_0^\infty \mathrm{d}x \frac{x^{\frac{1}{2}}}{z^{-1}e^x-\zeta}$$

$$\frac{E}{V} = \frac{2g}{\sqrt{\pi}}\frac{1}{\lambda^3}\int_0^\infty \mathrm{d}x \frac{x^{\frac{3}{2}}}{z^{-1}e^x-\zeta}$$

2. $g=2s+1$ (好像是简并度), λ为热波长$\lambda \equiv \frac{h}{\sqrt{2\pi m k_B T}}$

$$g_v(z)=\frac{1}{\Gamma(v)}\int_0^\infty \mathrm{d}x\frac{x^{v-1}}{z^{-1}e^x-1}\qquad\qquad f_v=\frac{1}{\Gamma(v)}\int_0^\infty \mathrm{d}x\frac{x^{v-1}}{z^{-1}e^x+1}$$

$$\frac{p}{k_BT}=\begin{Bmatrix}\frac{g}{\lambda^3}g_{\frac{5}{2}}(z)\\ \frac{g}{\lambda^3}f_{\frac{5}{2}}(z)\end{Bmatrix}\qquad \frac{N}{V}=\begin{Bmatrix}\frac{g}{\lambda^3}g_{\frac{3}{2}}(z)\\ \frac{g}{\lambda^3}f_{\frac{3}{2}}(z)\end{Bmatrix}\qquad \frac{E}{k_BTV}=\begin{Bmatrix}\frac{3}{2}\frac{g}{\lambda^3}g_{\frac{5}{2}}(z) & \text{bosons}\\ \frac{3}{2}\frac{g}{\lambda^3}f_{\frac{5}{2}}(z) & \text{fermions}\end{Bmatrix}\Rightarrow E=\frac{3}{2}pV \quad (1)$$

重点背记。

Classical limit : j

$$\begin{Bmatrix}g_v(z)\\ f_v(z)\end{Bmatrix}\Rightarrow\frac{1}{\Gamma(v)}\int_0^\infty \mathrm{d}x\,x^{v-1}e^{-x}z\sum_{k'=0}^{\infty}(\pm1)^{k'}e^{-xk'}z^{k'}=\sum_{k=1}^{\infty}\frac{(\pm1)^{k+1}z^k}{k^v}$$

$$\frac{N}{V}=\frac{g}{\lambda^3}\left(z+\zeta\frac{z^2}{2^{\frac{3}{2}}}+O(z^3)\right)\Rightarrow z+\zeta\frac{z^2}{2^{\frac{3}{2}}}+\mathrm{O}(z^3)=\frac{\lambda^3}{g}\frac{N}{V}\equiv\epsilon$$

而我们既可以将z 的不同次方展开，还可以反向按照ϵ的不同次方展开。assume $z=\epsilon+A_1\epsilon^2+\cdots$

$$\epsilon+A_1\epsilon^2+\zeta\frac{\epsilon^2}{z^{\frac{3}{2}}}+\cdots=\epsilon\quad\Rightarrow A_1=-\zeta\frac{1}{z^{\frac{3}{2}}}\Rightarrow z=\frac{\lambda^3}{g}\frac{N}{V}-\zeta\frac{1}{2^{\frac{3}{2}}}\left(\frac{\lambda^3N}{gV}\right)^2$$

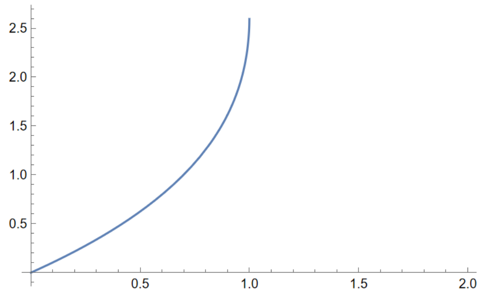
$$\begin{aligned}\frac{p}{k_BT}&=\frac{g}{\lambda^3}\left(z+3\frac{z^2}{2^{\frac{5}{2}}}+\cdots\right)\\ &=\frac{g}{\lambda^3}\left[\frac{\lambda^3}{g}\frac{N}{V}-\zeta\frac{1}{2^{\frac{3}{2}}}\left(\frac{\lambda^3}{g}\frac{N}{V}\right)+\zeta\frac{1}{2^{\frac{5}{2}}}\left(\frac{\lambda^3}{g}\frac{N}{V}\right)^2+\cdots\right]\\ \Rightarrow pV&=Nk_BT\left[1-\zeta\frac{1}{2^{\frac{5}{2}}}\left(\frac{N\lambda^3}{Vg}\right)+\cdots\right]\end{aligned}$$

多对数函数：

$$g_v(z)=\frac{1}{\Gamma(v)}\int_0^\infty \mathrm{d}x\frac{x^{v-1}}{z^{-1}e^x-1}\equiv\mathrm{PolyLog}[v,z]\qquad f_v=\frac{1}{\Gamma(v)}\int_0^\infty \mathrm{d}x\frac{x^{v-1}}{z^{-1}e^x+1}=-\mathrm{PolyLog}[v,-z]$$

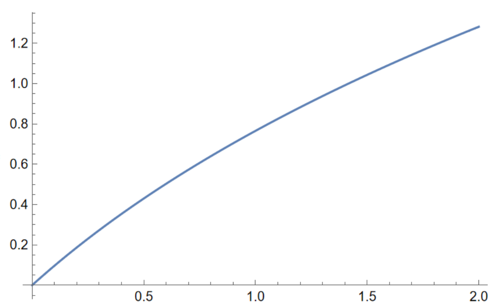
```
In[17]:= Plot[PolyLog[3/2, z], {z, 0, 2}]
         绘图  多对数函数
```

Out[17]=




```
In[25]:= Plot[-PolyLog[3/2, -z], {z, 0, 2}]
         绘图   多对数函数
```

Out[25]=



Ground states $T=0$,对于量子力学而言，电子全部处于基态。$\varepsilon_k=\frac{\hbar^2k^2}{2m}$

$$\frac{1}{e^{\beta(E_i-\mu)}+1}\xrightarrow{\beta\rightarrow\infty}\left\{\begin{array}{ll}1 & \text{if } E_i<\mu\equiv E_F\\ 0 & \text{if } E_i>\mu\equiv E_F\end{array}\right\}$$

$$E_F=\frac{p_F^2}{2m} \qquad\qquad p_F=\hbar k_F$$

$$\begin{aligned}N &= g\sum_{p\leqslant p_F}1\\ &= g\frac{V}{(2\pi\hbar)^3}\int \mathrm{d}^3p\,\delta(p_F-p)\\ &= \frac{gVp_{_F}^2}{6\pi^2\hbar^3}\end{aligned}$$

$$p_{_F}=\hbar\left(\frac{6\pi^2}{g}\frac{N}{V}\right)^{\frac{1}{3}} \qquad\qquad E_{_F}=\frac{\hbar^2}{2m}\left(\frac{6\pi^2}{g}\frac{N}{V}\right)^{\frac{2}{3}}$$

Ground-state energy

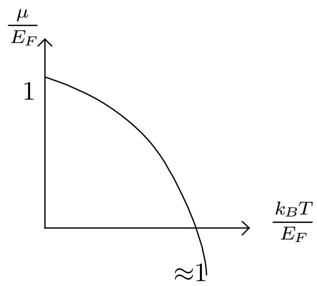
$$E=\frac{gV}{6\pi^2\hbar^3}\int \mathrm{d}p^3\frac{p^2}{2m}\delta(p_F-p)=\frac{gVp_{_F}^5}{20\pi^2\hbar^3m}$$

费米简并压 $$p=\frac{2}{3}\frac{E}{V}=\frac{gVp_{_F}^5}{30\pi^2\hbar^3m}=\frac{2}{5}\frac{N}{V}E_F\neq 0=\frac{\hbar^2}{5m}\left(\frac{6\pi^2}{g}\right)^{\frac{2}{3}}\left(\frac{N}{V}\right)^{\frac{5}{3}}$$

费米温度：$T_F=\dfrac{E_F}{k_B}$ 当温度远小于费米温度时，可以等效为$0K$

温度较低但不为0时。

The limit of complete degeneracy,$Z\gg 1$



$$\begin{aligned}f_\upsilon(z) &= \frac{1}{\Gamma(\upsilon)}\int_0^\infty \mathrm{d}x\frac{x^{\upsilon-1}}{z^{-1}e^x+1}\\ x=\xi+\ln Z &\Longrightarrow \frac{1}{\Gamma(\nu)}\int_{-\ln z}^\infty \mathrm{d}\xi\frac{(\xi+\ln z)^{\nu-1}}{e^\xi+1}\\ &= \frac{1}{\Gamma(\nu)}\int_{-\ln Z}^\infty \frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}(\xi+\ln Z)^\nu\frac{1}{e^\xi+1}\\ &= \frac{1}{\Gamma(\nu+1)}\int_{-\infty}^\infty\left(1+\frac{\xi}{\ln Z}\right)^\nu\frac{e^\xi}{(e^\xi+1)^2}\mathrm{d}\xi\end{aligned}$$

$$f_\nu(z)=\frac{(\ln Z)^\nu}{\Gamma(\nu+1)}\int_{-\infty}^{\infty}\left(\underset{A}{1}+\underset{B}{\frac{v\xi}{\ln Z}}+\underset{C}{\frac{\nu(\nu-1)\xi^2}{2(\ln z)^2}}+\cdots\right)\frac{e^\xi}{(e^\xi+1)^2}\mathrm{d}\xi$$

$$A=\frac{(\ln Z)^\upsilon}{\Gamma(\nu+1)}\int_{-\infty}^{\infty}\frac{e^\xi}{(e^\xi+1)^2}\mathrm{d}\xi=\frac{(\ln z)^\upsilon}{\Gamma(\nu+1)}\times 1$$

B部分奇函数积分，为0

$$C=\int_{-\infty}^{\infty}\frac{\nu(\nu-1)\boldsymbol{\xi^2}}{2(\ln z)^2}\times\frac{\boldsymbol{e^\xi}}{\boldsymbol{(e^\xi+1)^2}}\mathrm{d}\xi=\frac{\upsilon(\upsilon-1)}{2(\ln z)^2}\underset{?}{\zeta(2)}=\frac{\pi^2}{3}\frac{\nu(\nu-1)\xi^2}{2(\ln z)^2}$$

其中，$\zeta(n)$ 为Riemann zeta funciton，上式中使用$\zeta(2)$存疑，因为mathmatica中的四个函数似乎都不是这个函数,没有$\zeta(2)=\frac{\pi^2}{3}$的。但对于加粗部分积分，确实是$\frac{\pi^2}{3}$

$$f_\nu(z)\approx\frac{(\ln z)^\nu}{\Gamma(\nu+1)}\left[1+\frac{\pi^2}{3}\frac{\nu(\nu-1)}{2(\ln z)^2}+\cdots\right]$$

$$\begin{aligned}\frac{N}{V}&=\frac{g}{\lambda^3}\frac{(\ln Z)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left[1+\frac{\pi^2}{3}\frac{3}{8(\ln Z)^2}+\cdots\right]\\ \frac{\lambda^3}{g}\frac{N}{V}\Gamma\left(\frac{5}{2}\right)&=(\ln Z)^{\frac{3}{2}}\left[1+\frac{\pi^2}{8(\ln Z)^2}+\cdots\right]\end{aligned}$$

而$Z=e^{\beta\mu}$

$$\text{仅仅取第一项}(\beta\mu)^{\frac{3}{2}}=\frac{3}{4}\sqrt{\pi}\frac{N}{gV}\left(\frac{h}{\sqrt{2\pi m k_BT}}\right)^3=\frac{N}{gV}\frac{3h^2\beta^{\frac{3}{2}}}{(2m)^{\frac{3}{2}}4\pi}$$

$$\Rightarrow\mu=\frac{\hbar^2}{2m}\left(\frac{6\pi}{g}\frac{N}{V}\right)^{\frac{2}{3}}=E_F$$

$$\text{取到第二项乃至以后：}(E_F)^{\frac{3}{2}}=\mu^{\frac{3}{2}}\left[1+\frac{\pi^2}{8}\left(\frac{k_BT}{\mu}\right)^2+\cdots\right]$$

2.7 Ideal fermi gas

huang 14 16

$\frac{p}{k_BT}=\frac{g}{\lambda^3}f_{\frac{5}{2}}(z)$ $\qquad$ $\frac{N}{V}=\frac{g}{\lambda^3}f_{\frac{3}{2}}(z)$ $\qquad$ $\frac{E}{k_BTV}=\frac{3}{2}\frac{g}{\lambda^3}f_{\frac{5}{2}}(z)$

$E=\frac{3}{2}pV$

1. $\frac{N}{V}\approx\frac{g}{\lambda^3}\frac{(\ln Z)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left[1+\frac{\pi^2}{8}\frac{1}{(\ln Z)^2}\right]$

2. $E_F=\mu\left[1+\frac{2}{3}\frac{\pi^2}{8}\left(\frac{k_BT}{\mu}\right)^2\right]$ $\qquad$ $\mu=E_F\left(1-\frac{\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2\right)$

3. $\frac{E}{k_BTV}=\frac{3}{2}\frac{g}{\lambda^3}\frac{(\ln Z)^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)}\left[1+\frac{\pi^3}{3}\frac{15}{8(\ln Z)^2}+\cdots\right]$

$\Rightarrow \frac{E}{V} = \frac{3}{5}\frac{N}{V} E_F \left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{E_F}\right) + \cdots \right]$

$C_V = \left(\frac{\partial E}{\partial T}\right)_V \approx N k_B \frac{\pi^2}{2}\frac{T}{T_F}$

4. Gibbs - Duhe, relations

$$E(\lambda S, \lambda V, \lambda N) = \lambda E(S, V, N)$$

$$\frac{\partial E}{\partial \lambda S} S + \frac{\partial E}{\partial \lambda V} V + \frac{\partial E}{\partial \lambda N} N \,|_{\lambda=1} = E$$

$$\mathrm{d}E = \left(\frac{\partial E}{\partial S}\right)_{V,N} \mathrm{d}S + \left(\frac{\partial E}{\partial V}\right)_{S,N} \mathrm{d}V + \left(\frac{\partial E}{\partial N}\right)_{S,V} \mathrm{d}N$$

$$E = TS - pV + \mu N$$

$$S = \frac{1}{T}(E + pV - \mu N) = k_B N \frac{\pi^2}{2}\frac{T}{T_F} (\text{和} C_V \text{相等})$$

2.7.1 朗道铁磁相变理论

$H = \frac{(\vec{p} - e\vec{A})^2}{2m}$Landau diamagnetism

$H = \frac{p^2}{2m} - \mu_0 \vec{\sigma} \cdot \vec{B}$

$$Z_n = \sum_{\{n_p^+, n_p^-\}}{}' \exp(-\beta E_n)$$

Pauli paramagnetism $E_n = \sum_p (\varepsilon_{p,+1} n_{p,+1} - \varepsilon_{p,-1} n_{p,-1}) \quad \varepsilon_{p,\pm 1} = \frac{p^2}{2m} \mp \mu_0 B$

记$\sum_p n_p^+ = N_+ \quad \sum_p n_p^- = N_-$,$N_- + N_+ = N$

$$\begin{aligned} Z_n &= \sum_{N_+=0}^{N} e^{\beta\mu_0 B(2N_+ - N)} \sum_{\{n_p^+\}}{}'' e^{-\beta \sum_p \frac{p^2}{2m} n_p^+} \sum_{\{n_p^-\}}{}'' e^{-\beta \sum_p \frac{p^2}{2m} n_p^-} \\ &= e^{-\beta\mu_0 BN} \sum_{N_+=0}^{N} e^{2\beta\mu_0 BN_+} Z_{N_+}^{(0)} Z_{N-N_+}^{(0)} \end{aligned}$$

为了去$\sum$ ， 我们可以选择取极大分布的情况代替所有情况求和。

$$Z_N^{(0)} = \sum_{\sum_p n_p = N} e^{-\beta \sum \frac{p^2}{2m} n_p} \equiv e^{-\beta F(N)}$$

$$\Rightarrow \frac{1}{N} \ln Z_N = -\beta\mu_0 B + \frac{1}{N} \ln \sum_{N_+=0}^{N} e^{2\beta\mu_0 BN_+ - \beta F(N_+) - \beta F(N_-)}$$

$$\frac{1}{N} \ln Z_N = \beta f(\bar{N}_+) + O\left(\frac{1}{N} \ln N\right)$$

$$f(\bar{N}_+) = \max_{N_+} [f(N_+)]$$

$$f(N_+) = \mu_0 B \left(\frac{2N_+}{N} - 1\right) - \frac{1}{N}[F(N_+) - F(N - N_+)]$$

为了得到最概然的分布情况，我们需要取$f(N_+)$的最大值。

$$\frac{\partial f(N_+)}{\partial N_+}|_{N_+=\bar{N}_+}=0\Rightarrow 2\mu_0 B-\left[\frac{\partial F(N')}{\partial N'}\right]_{N'=\bar{N}_+}-\left[\frac{\partial F(N-N')}{\partial N'}\right]_{N'=\bar{N}_+}=0$$

$$\Rightarrow\mu^{(0)}(\bar{N}_+)-\mu^{(0)}(N-\bar{N}_+)=2\mu_0 B$$

$$\Rightarrow\mu^{(0)}\left(\frac{1+r}{2}N\right)-\mu^{(0)}\left(\frac{1-r}{2}N\right)=2\mu_0 B$$

磁化强度$M=\mu_0(\bar{N}_+-\bar{N}_-)=\mu_0 Nr$, $r=\frac{2\bar{N}_+}{N}-1=\frac{\bar{N}_+-\bar{N}_-}{N}$.

$B=0$时，r 也应该很小。当 r 很小时，r 可以近似为

$$r\simeq\frac{2\mu_0 B}{\left(\frac{\partial\mu^{(0)}(x_N)}{\partial x}\right)_{x=\frac{1}{2}}}$$

$$x=\frac{1}{V}\frac{\partial M}{\partial B}=\frac{2\mu_0^2 N/V}{\left(\frac{\partial\mu^{(0)}(x_N)}{\partial x}\right)_{x=\frac{1}{2}}}$$

For $T\to 0$

$$\mu^{(0)}(xN)|_{x=\frac{1}{2}}\equiv\frac{\hbar^2}{2m}\left(\frac{6\pi^2}{g}\frac{xN}{V}\right)$$

$$\Rightarrow\frac{\partial\mu^{(0)}}{\partial x}|_{xN}=\frac{2^{\frac{4}{3}}}{3}\left(\frac{3N}{4\pi V}\right)^{\frac{2}{3}}\frac{h^2}{2m}$$

$$E_F=\frac{\hbar^2}{2m}\left(\frac{6\pi^2}{2}\frac{N}{V}\right)^{\frac{2}{3}}$$

$$\Rightarrow^{(1)}x|_{T\to 0}\equiv x_0=\frac{3\mu_0^2 N}{2E_F V}$$

$$^{(2)}x=x_0\left(1-\frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2+\cdots\right)$$

$$\mu=E_F\left[1-\frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2+\cdots\right],\frac{\partial\mu(xN)}{\partial x}=\frac{\partial E_F(xN)}{\partial x}+\frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2\frac{\partial E_F(x_N)}{\partial x}$$

For $T\to\infty$

$$\frac{N}{V}=\frac{g}{\lambda^3}f_{\frac{3}{2}}(z)=\frac{g}{\lambda^3}\left(z-\frac{z^2}{2^{3/2}}+\cdots\right)\Rightarrow z=\frac{\lambda^3}{g}\frac{N}{V}+\frac{1}{2^{3/2}}\left(\frac{\lambda^3}{g}\frac{N}{V}\right)+\cdots$$

$$\Rightarrow\mu^{(0)}(N)\simeq k_B T\ln\left(\frac{N\lambda^3}{V}\right)\quad\Rightarrow x_\infty=\frac{\mu_0^2 N/V}{k_B T}\quad(1)$$

$$x=x_\infty\left(1-\frac{\lambda^3 N/V}{2^{5/2}}\right)$$

$$\frac{\partial\mu^{(0)}}{\partial x}|_{x=\frac{1}{2}}=2k_B T$$

2.8 理想玻色系统

2.8.1 光子

photons 光子 $\left(n+\frac{1}{2}\right)\hbar\omega \quad \hbar\omega = h\nu = \hbar c k = \hbar c\frac{2\pi}{\lambda} = \frac{hc}{\lambda}$

$$H = \frac{1}{2}\sum_{\vec{k},\alpha}\left[|\tilde{p}_{\vec{k},\alpha}|^2 + \omega_\alpha^2(\vec{k})|\tilde{u}_\alpha(\vec{k})|^2\right]$$

$$H^q = \sum_{\vec{k},\alpha}\hbar c k\left(n_\alpha(\vec{k}) + \frac{1}{2}\right)$$

Energy $\hbar\omega$. Momentum $\hbar\vec{k}$. Potential vector $\vec{\epsilon}$, $|\vec{\epsilon}| = 1$, $\vec{k}\cdot\vec{\epsilon} = 0$

$$Z = \sum_{\{n_\alpha(\vec{k})\}} e^{-\beta H_q} = \sum_{\{n_\alpha(\vec{k})\}}\prod_{k,\alpha} e^{\left[-\beta\hbar\omega(\vec{k})\left(n_\alpha(\vec{k})+\frac{1}{2}\right)\right]} = \prod_{\vec{k},\alpha}\frac{e^{-\beta\hbar c k/2}}{1-e^{-\beta\hbar c k}}$$

$$Z = \prod_{\vec{k},\alpha}\left[\frac{1}{1-e^{-\beta\hbar c k}}\right] \text{(不考虑零点能)}$$

$$\langle n_\alpha(k)\rangle = -\frac{1}{\beta}\frac{\partial}{\partial(\hbar\omega)}\ln Z = \frac{1}{e^{\beta\hbar\omega}-1}$$

$$E = \langle H^q\rangle = \sum_{\vec{k},\alpha}\hbar c k\left(\frac{e^{-\beta\hbar c k}}{1-e^{-\beta\hbar c k}} + \frac{1}{2}\right)$$

$$\langle H^q\rangle = -\frac{\partial}{\partial\beta}\ln Z = V E_0 + \frac{2V}{(2\pi)^3}\int d^3\vec{k}\,\frac{\hbar c k}{e^{\beta\hbar c k}-1}$$

$$F = -k_B T\ln Z, p = -\frac{\partial F}{\partial V} = p_0 + \frac{1}{3}\frac{E}{V}$$

2023.12.29

能量流密度(黑体辐射):

$$\phi = \langle c_\perp\rangle\frac{E}{V}$$

φ: escaping energy flux per unit area and per unit time

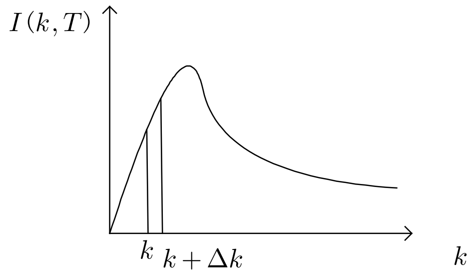
$$\langle c_\perp\rangle = c\times\frac{1}{4\pi}\int_0^{\frac{\pi}{2}} 2\pi\sin\theta\, d\theta\cdot\cos\theta = \frac{c}{4}$$

$$\begin{aligned}\phi &= \frac{cE}{4V}\\ &= \frac{1}{4}c\cdot\frac{1}{V}\frac{\pi^2}{15}V\left(\frac{k_B T}{\hbar c}\right)^3 k_B T = \frac{\pi^2}{60}\frac{k_B^4 T^4}{\hbar^3 c^2} = \sigma T^4\end{aligned}$$

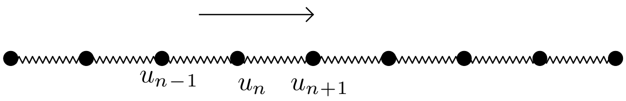
$$\frac{E}{V} = \int dk\,\varepsilon(k,T)$$

$$\varepsilon(k,T) = \frac{\hbar c}{\pi^2}\frac{k^3}{e^{\beta\hbar\omega}-1}$$

$$I(k,T) = \frac{c}{4}\varepsilon(k,t)\ \text{表示}k\sim k+\Delta k\text{之间的能量流}$$



2.8.2 Phonon.声子



$$C\left(u_{n+1}-u_n\right)-C\left(u_n-u_{n-1}\right)=m\frac{\mathrm{d}^2u_n}{\mathrm{d}t^2}$$

$$\text{形式解}u_n=\sum_k Q_k e^{ikna}$$

$$2C\left(\cos\left(ka\right)-1\right)Q_k=m\frac{\mathrm{d}^2Q_k}{\mathrm{d}t^2}$$

$$Q_k=A_k e^{i\omega_k t}$$

$$\omega_k=\sqrt{\frac{2C}{m}\left(1-\cos ka\right)}=\sqrt{\frac{4C}{m}}\left|\sin\frac{\mathrm{ka}}{2}\right|$$

Quantum version

$$H=\sum_{i=1}^{N}\frac{p_i}{2m}+\frac{1}{2}m\omega^2\sum_{i=j+1}\left(x_i-x_j\right)^2$$

这里使用$\sum\limits_{i=j+1}$是为了保证i,j相邻且不计算两边。

定义

$$Q_k = \frac{1}{\sqrt{N}}\sum_l e^{ikal}x_l, \qquad \prod_k = \frac{1}{\sqrt{N}}\sum_l e^{-ikal}p_l$$

$$[x_l, p_m] = i\hbar\delta_{l,m} \Rightarrow \left[Q_k, \prod_{k'}\right] = i\hbar\delta_{k,k'}$$

$$[Q_k, Q_{k'}] = [\Pi_k, \Pi_{k'}] = 0$$

$$\sum_l x_l, x_{l+m} = \sum_k Q_k Q_{-k} e^{iamk}, \sum_l p_l^2 = \sum_k \Pi_k \Pi_{k'}$$

$$\frac{1}{2}m\omega^2\sum_j (x_j - x_{j+1})^2 = \frac{1}{2}m\omega^2\sum_k Q_k Q_{-k}(2 - e^{ika} - e^{-ika}) = \frac{1}{2}\sum_k m\omega_k^2 Q_k Q_{-k}$$

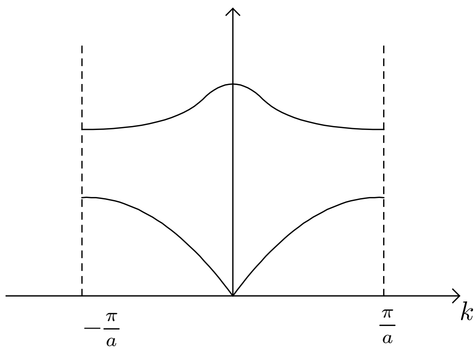
$$\omega_k = \sqrt{2\omega^2(1 - \cos ka)} = 2\omega\left|\sin\frac{ka}{2}\right|$$

$$H = \frac{1}{2m}\sum_k (\Pi_k \Pi_{-k} + m^2\omega_k^2 Q_k Q_{-k})$$

$$b_k = \sqrt{\frac{m\omega}{2\hbar}}\left(Q_k + \frac{i}{m\omega_k}\Pi_k\right) \qquad [b_k, b_{k'}^+] = \delta_{k,k'} \qquad [b_k, b_{k'}] = [b_k^+, b_{k'}^+] = 0$$

$$Q_k = \sqrt{\frac{\hbar}{2m\omega_k}}(b_k^+ + b_{-k}) \quad \Pi_k = i\sqrt{\frac{\hbar m\omega_k}{2}}(b_k^+ - b_{-k})$$

$$\Rightarrow H = \sum_k \hbar\omega_k\left(b_k^+ b_k + \frac{1}{2}\right)$$



$\omega(\vec{k}) = ck$

Einstein model: all oscillators are assumed to have the same frequency ω_E

震荡的频率全相同

$$\text{total Energy} \qquad E = 3N\frac{\hbar\omega_E}{e^{\beta\hbar\omega} - 1}$$

$$C_v = \frac{\mathrm{d}E}{\mathrm{d}T} = 3NK_B\left(\frac{T_E}{T}\right)^2 \frac{e^{-T_E/T}}{(1 - e^{-T_E/T})^2}$$

高温情况下：$C_v = 3\mathrm{NK}_B$

Debye model:

$\omega(\vec{k}) = ck$

$f(\omega)\,\mathrm{d}\omega=\frac{3V}{(2\pi)^3}4\pi k^2\mathrm{d}k$ ，3是指三种震动模式。$f(\omega)=\frac{1}{c}\frac{3V}{(2\pi^2)}k^2$

$[\omega,\omega+\mathrm{d}\omega]$之间的频率可以与$[k,k+\mathrm{d}k]$范围内的频率

对于含有N个粒子的系统，最多只有3N个震动模式。

$$\int_0^{\omega_m}f(\omega)\,\mathrm{d}\omega=3N\ \text{截断条件（}\omega\text{为截止频率）}$$

$\Rightarrow\omega_m=c\left(\frac{6\pi^2N}{V}\right)^{\frac{1}{3}}=ck_m$——Debye 频率, $\lambda_m=\frac{2\pi c}{\omega_m}=\left(\frac{4\pi}{3}\frac{V}{N}\right)^{\frac{1}{3}}$

$E(\{n_i\})=\sum_{i=1}^{3N}\hbar\omega_i\cdot n_i$, $Z=\sum_{\{n_i\}}e^{-\beta E(\{n_i\})}=\prod_{i=1}^{3N}\frac{1}{1-e^{-\beta\hbar\omega_i}}$

$$\Rightarrow\langle n_i\rangle=-\frac{1}{\beta}\frac{\partial}{\partial(\hbar\omega_i)}\ln Z=\frac{1}{e^{\beta\hbar\omega}-1}$$

$$E=-\frac{\partial}{\partial\beta}\ln Z=\sum_{i=1}^{3N}\frac{\hbar\omega_i}{e^{\beta\hbar\omega_i}-1}=\frac{3V}{2\pi^2c^3}\int_0^{\omega_m}\omega^2\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

$$\frac{E}{N}=\frac{9(k_BT)^4}{(\hbar\omega)^3}\int_0^{\beta\hbar\omega_m}\mathrm{d}t\frac{t^3}{e^t-1}$$

Debye function

$$D\equiv\frac{3}{x^3}\int_0^x\mathrm{d}t\frac{t^3}{e^t-1}=\left\{\begin{array}{ll}1-\frac{3}{8}x+\frac{1}{20}x^2+\cdots & x\ll1\\ \frac{\pi^4}{5x^3}+O(e^{-x}) & x\gg1\end{array}\right\}$$

$$\Rightarrow\frac{E}{N}=3k_BTD(\lambda)=\left\{\begin{array}{ll}3k_BT\left(1-\frac{3}{8}\frac{T_D}{T}\right) & T\gg T_D\\ 3k_BT\left[\frac{\pi^4}{5}\left(\frac{T}{T_D}\right)^2+\cdots\right] & T\ll T_D\end{array}\right.$$

$$\frac{C_V}{Nk_B}=3D(\lambda)+3T\frac{\mathrm{d}D(\lambda)}{\mathrm{d}T}=\left\{\begin{array}{ll}3\left[1-\frac{1}{20}\left(\frac{T_D}{T}\right)^2+\cdots\right] & T\gg T_D\\ \frac{12\pi^4}{5}\left(\frac{T}{T_D}\right)^3+\cdots & T\ll T_D\end{array}\right.$$

2.8.3 玻色-爱因斯坦凝聚

Bose - Einstein Condensation是一种宏观量子效应

色散关系不同导致$f(\omega)\,\mathrm{d}\omega=\frac{3V}{(2\pi)^3}4\pi k^2\mathrm{d}k$对应的转换关系也不同（$\frac{\mathrm{d}\omega}{\mathrm{d}k}$不同）。

以下讨论的是理想玻色气体（不考虑粒子间的相互作用）

每种模式的平均粒子数为$\frac{1}{e^{\beta(E_i-\mu)}-1}$(粒子数一定要大于零，因此$E_i$一定大于等于$\mu$,for any i)，$\Rightarrow\mu\leqslant0$ ($\mu\leqslant0$是从这里为了满足自洽引入的)。而$0\leqslant z=e^{\beta\mu}\leqslant1$.

由

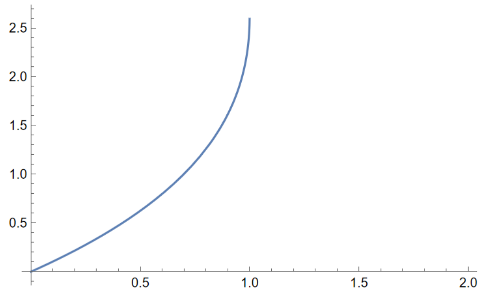
$$\text{粒子数密度：}\frac{N}{V}=\frac{g}{\lambda^3}g_{\frac{3}{2}}(z)$$

$$g_{\frac{3}{2}}(z)=\frac{2}{\sqrt{\pi}}\int_0^\infty\mathrm{d}x\frac{x^{\frac{1}{2}}}{z^{-1}e^x-1}=\sum_{k=1}^\infty\frac{z^k}{k^{\frac{3}{2}}}$$

$g_{\frac{3}{2}}(z)=\mathrm{PolyLog}\left[\frac{3}{2},z\right]$,$z=1$时，$g_{\frac{3}{2}}(1)\approx2.612$

```
In[17]:= Plot[PolyLog[3/2, z], {z, 0, 2}]
```
绘图 多对数函数

Out[17]=



而$\lambda=\frac{h}{\sqrt{2\pi m k_B T}}$，$g_{\frac{3}{2}}\leqslant g_{\frac{3}{2}}(1)$，$\frac{N}{V}\leqslant\frac{g}{\lambda^3}\xi\left(\frac{3}{2}\right)\sim T^{\frac{3}{2}}$，当温度有高温降低到一定温度时，$\frac{N}{V}\leqslant T^{\frac{3}{2}}$一定要满足，但是$\frac{N}{V}$不可能无穷的小，在低温下不太现实。

$$\frac{N}{V}=\int\frac{\mathrm{d}k^3}{(2\pi)^3}\frac{g}{e^{\beta(\hbar^2k^2/2m-\mu)}-1}=\frac{g}{\hbar^3}\frac{m^{\frac{3}{2}}}{\sqrt{2}\pi^2}\int_0^\infty\mathrm{d}E\frac{\sqrt{E}}{e^{\beta(E-\mu)}-1}$$

$$N=\sum_i\frac{1}{e^{\beta(E_i-\mu)}-1}=\sum_i\frac{1}{z^{-1}e^{\beta E_i}-1}=\frac{1}{z^{-1}-1}+\sum_i{}'\frac{1}{z^{-1}e^{\beta\hbar\omega}-1}=\frac{1}{z^{-1}+1}+\frac{g}{\lambda^3}Vg_{\frac{3}{2}}(z)$$

Critical temperature

$\frac{N}{V}=\frac{1}{\lambda^3}g_{\frac{3}{2}}(1)\qquad\lambda_c=\frac{h}{\sqrt{2\pi m k_B T}}$

2024.1.5

Bose-Einstein Condensation

理想玻色气体，无相互作用，低温条件产生玻色爱因斯坦凝聚

存在相互作用时，会产生超流动性。

$\frac{1}{e^{\beta(E,-\mu)}-1}\quad\longrightarrow\mu\leqslant 0\Rightarrow 0\leqslant z\leq 1$

状态方程：$\frac{N}{V}=\frac{g}{\lambda^3}g_{\frac{3}{2}}(z)\qquad\frac{N}{V}\leqslant\frac{g}{\lambda^3}\frac{\xi\left(\frac{3}{2}\right)}{g_{\frac{3}{2}}(1)}\sim T^{\frac{3}{2}}$，温度低的时候不等式总会有不成立的情况发生。

$g=1$，$g_{\frac{3}{2}\max}(z)=g(1)$

$$\text{不成立的临界温度}T_C\text{，}\frac{N}{V}=\frac{1}{\lambda^3}g_{\frac{3}{2}}(1)$$

$$\lambda_C=\frac{h}{\sqrt{2\pi m k_B T_C}}\Rightarrow\lambda_C=\left(\frac{g_{\frac{3}{2}}(1)V}{N}\right)^{\frac{1}{3}}$$

$$\Rightarrow k_B T_c=\frac{2\pi\frac{\hbar^2}{m}}{\left[Vg_{\frac{3}{2}}(1)/N\right]^{\frac{2}{3}}}$$

$$\begin{aligned}N&=\sum_i\frac{1}{e^{\beta(E_i-\mu)}-1}=\frac{V}{\hbar^3}\frac{m^{\frac{3}{2}}}{\sqrt{2}\pi^2}\int_0^\infty\mathrm{d}E\frac{\sqrt{E}}{e^{\beta(E_i-\mu)}-1}\\&=\frac{1}{z^{-1}-1}+\sum_i{}'\mathrm{d}E\frac{1}{e^{\beta(E_i-\mu)}}\\&=\underbrace{\frac{1}{z^{-1}-1}}_{N_0}+\frac{V}{\lambda^3}g_{\frac{3}{2}}(z)\end{aligned}$$

设凝聚的粒子数为N_0,下面求解凝聚的粒子数所占总粒子数的比例

$$N = \frac{1}{z^{-1}-1} + N\left(\frac{T}{T_C}\right)^{\frac{3}{2}} \frac{g_{\frac{3}{2}}(z)}{g_{\frac{3}{2}}(1)}$$

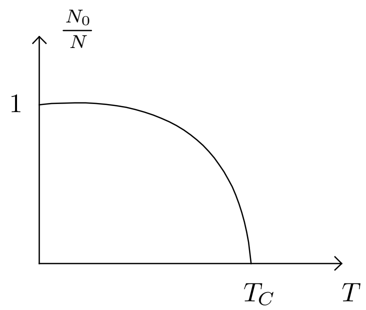
$$N_0 \Rightarrow z = \frac{N_0}{N_0+1}$$

将$\frac{V}{\lambda^3} g_{\frac{3}{2}}(z)$代换出来$N$. [3]

1. $T > T_C$, $z < 1$对应高温情况，不产生凝聚只要z 不接近 1 ,$\frac{\mathbf{1}}{\boldsymbol{z}^{\mathbf{-1}}\mathbf{-1}}$就可以看作是一个小的常量，可以忽略不计。

2. $T < T_C$, $z = 1 - O\left(\frac{1}{N}\right)$， z接近1 ，$N\left(\frac{T}{T_C}\right)^{\frac{3}{2}} \frac{g_{\frac{3}{2}}(z)}{g_{\frac{3}{2}}(1)} \approx N\left(\frac{T}{T_C}\right)^{\frac{3}{2}} \times 1$。

   $N = N_0 + N\left(\frac{T}{T_C}\right)^{\frac{3}{2}}$, $\frac{N_0}{N} = \left\{ \begin{array}{cc} 0, & T > T_C \\ 1 - \left(\frac{T}{T_C}\right)^{\frac{3}{2}}, & T < T_C \end{array} \right\}$



状态方程

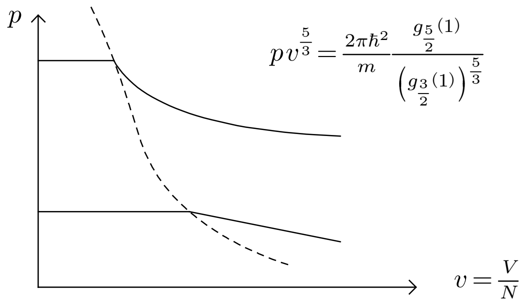
$Z = \prod_i \frac{1}{1 - z e^{-\beta E_i}}$

$$\begin{aligned} \frac{pV}{k_B T} &= \ln Z = -\sum_i \ln\,(1 - z e^{-\beta E_i}) \\ &= -V \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \ln\left(1 - z e^{-\beta \frac{\hbar^2 k^2}{2m}}\right) - \ln(1-z) \\ &= \frac{V}{\lambda^3} g_{\frac{5}{2}}(z) - \ln(1-z) \end{aligned}$$

$-\frac{1}{V}\ln(1-z) = \frac{1}{V}\ln(N_0+1)$,

$$\frac{p}{k_B T} = \frac{1}{\lambda^3} g_{\frac{5}{2}}(z) = \left\{ \begin{array}{l} \frac{1}{\lambda^3} g_{\frac{5}{2}}(z), \quad T > T_C \\ \frac{1}{\lambda^3} g_{\frac{5}{2}}(1), \quad T < T_C \end{array} \right\}$$

3. $\frac{V}{\lambda^3} g_{\frac{3}{2}}(z) = \frac{V}{\lambda_C^3} \frac{\lambda_C^3}{\lambda^3} \frac{g_{\frac{3}{2}}(z)}{g_{\frac{3}{2}}(1)} g_{\frac{3}{2}}(1)$

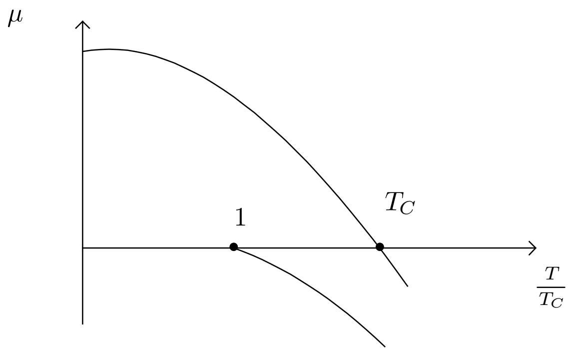


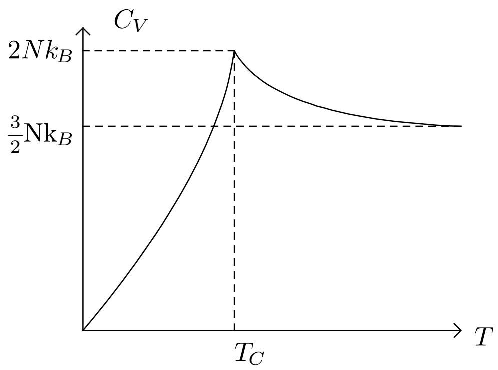
$$\frac{E}{k_BTV} = \frac{3}{2}\frac{1}{\lambda^3}g_{\frac{5}{2}}(z) \Rightarrow \frac{E}{N} = \frac{3}{2N}pV$$

$$\frac{C_V}{Nk_B} = \frac{3V}{2Nk_B}\left(\frac{\partial p}{\partial T}\right)_{V,N} = \begin{Bmatrix} \frac{15}{4}\frac{\frac{V}{N}}{\lambda^3}g_{\frac{5}{2}}(z) - \frac{9}{4}\frac{g_{\frac{3}{2}}(z)}{g_{\frac{1}{2}}(z)} & (T > T_C) \\ \frac{15}{4}\frac{V/N}{\lambda^3}g_{\frac{5}{2}}(1) & (T < T_C) \end{Bmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}g_v(z) = \frac{1}{z}g_{v-1}(z)$$

$$\begin{Bmatrix} \left(\frac{\mathrm{d}N}{\mathrm{d}T}\right)_{V,N} = 0 \\ \frac{N}{V} = \frac{1}{\lambda^3}g_{\frac{3}{2}}(z) \end{Bmatrix} \Rightarrow \frac{3}{2T}\frac{g_{\frac{3}{2}}(z)}{\lambda^3} + \frac{1}{\lambda^3}\frac{1}{z}g_{\frac{1}{2}}(z)\left(\frac{\partial z}{\partial T}\right)_{V,N} = 0 \text{ 考试不要求}$$





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