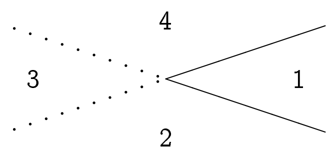
From these small cases, and after a little thought, we realize that a bent line is like two straight lines except that regions merge when the "two" lines don't extend past their intersection point.

. . . and a little afterthought. . .



Regions 2, 3, and 4, which would be distinct with two lines, become a single region when there's a bent line; we lose two regions. However, if we arrange things properly — the zig point must lie "beyond" the intersections with the other lines — that's all we lose; that is, we lose only two regions per line. Thus

Exercise 18 has the details.

$$\begin{aligned} Z_n \;=\; L_{2n} - 2n \;&=\; 2n(2n+1)/2 + 1 - 2n \\ &=\; 2n^2 - n + 1\,, \qquad \text{for } n \geqslant 0. \end{aligned} \tag{1.7}$$

Comparing the closed forms (1.6) and (1.7), we find that for large n,

$$\begin{aligned} L_n \;&\sim\; \tfrac{1}{2}n^2\,, \\ Z_n \;&\sim\; 2n^2\,; \end{aligned}$$

so we get about four times as many regions with bent lines as with straight lines. (In later chapters we'll be discussing how to analyze the approximate behavior of integer functions when n is large. The '$\sim$' symbol is defined in Section 9.1.)

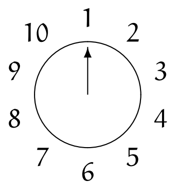
1.3 THE JOSEPHUS PROBLEM

Our final introductory example is a variant of an ancient problem named for Flavius Josephus, a famous historian of the first century. Legend has it that Josephus wouldn't have lived to become famous without his mathematical talents. During the Jewish–Roman war, he was among a band of 41 Jewish rebels trapped in a cave by the Romans. Preferring suicide to capture, the rebels decided to form a circle and, proceeding around it, to kill every third remaining person until no one was left. But Josephus, along with an unindicted co-conspirator, wanted none of this suicide nonsense; so he quickly calculated where he and his friend should stand in the vicious circle.

(Ahrens [5, vol. 2] and Herstein and Kaplansky [187] discuss the interesting history of this problem. Josephus himself [197] is a bit vague.)

. . . thereby saving his tale for us to hear.

In our variation, we start with n people numbered 1 to n around a circle, and we eliminate every *second* remaining person until only one survives. For example, here's the starting configuration for $n = 10$:



The elimination order is 2, 4, 6, 8, 10, 3, 7, 1, 9, so 5 survives. The problem: Determine the survivor's number, $J(n)$.

Here's a case where $n = 0$ makes no sense.

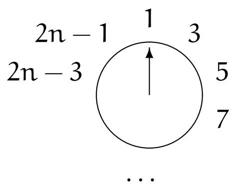
We just saw that $J(10) = 5$. We might conjecture that $J(n) = n/2$ when n is even; and the case $n = 2$ supports the conjecture: $J(2) = 1$. But a few other small cases dissuade us — the conjecture fails for $n = 4$ and $n = 6$.

n	1	2	3	4	5	6
$J(n)$	1	1	3	1	3	5

Even so, a bad guess isn't a waste of time, because it gets us involved in the problem.

It's back to the drawing board; let's try to make a better guess. Hmmm ... $J(n)$ always seems to be odd. And in fact, there's a good reason for this: The first trip around the circle eliminates all the even numbers. Furthermore, if n itself is an even number, we arrive at a situation similar to what we began with, except that there are only half as many people, and their numbers have changed.

So let's suppose that we have $2n$ people originally. After the first go-round, we're left with



and 3 will be the next to go. This is just like starting out with n people, except that each person's number has been doubled and decreased by 1. That is,

This is the tricky part: We have $J(2n) =$ newnumber$(J(n))$, where newnumber$(k) = 2k - 1$.

$$J(2n) = 2J(n) - 1, \qquad \text{for } n \geqslant 1.$$

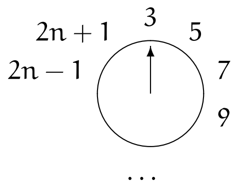
We can now go quickly to large n. For example, we know that $J(10) = 5$, so

$$J(20) = 2J(10) - 1 = 2\cdot 5 - 1 = 9.$$

Similarly $J(40) = 17$, and we can deduce that $J(5\cdot 2^m) = 2^{m+1} + 1$.

But what about the odd case? With $2n+1$ people, it turns out that person number 1 is wiped out just after person number $2n$, and we're left with

Odd case? Hey, leave my brother out of it.



Again we almost have the original situation with n people, but this time their numbers are doubled and *increased* by 1. Thus

$$J(2n+1) \;=\; 2J(n)+1\,, \qquad \text{for } n \geqslant 1.$$

Combining these equations with $J(1)=1$ gives us a recurrence that defines J in all cases:

$$\begin{aligned} J(1) &= 1\,; \\ J(2n) &= 2J(n)-1\,, \qquad \text{for } n \geqslant 1; \\ J(2n+1) &= 2J(n)+1\,, \qquad \text{for } n \geqslant 1. \end{aligned} \tag{1.8}$$

Instead of getting $J(n)$ from $J(n-1)$, this recurrence is much more "efficient," because it reduces n by a factor of 2 or more each time it's applied. We could compute $J(1000000)$, say, with only 19 applications of (1.8). But still, we seek a closed form, because that will be even quicker and more informative. After all, this is a matter of life or death.

Our recurrence makes it possible to build a table of small values very quickly. Perhaps we'll be able to spot a pattern and guess the answer.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Voilà! It seems we can group by powers of 2 (marked by vertical lines in the table); $J(n)$ is always 1 at the beginning of a group and it increases by 2 within a group. So if we write n in the form $n = 2^m + l$, where 2^m is the largest power of 2 not exceeding n and where l is what's left, the solution to our recurrence seems to be

$$J(2^m + l) \;=\; 2l+1\,, \qquad \text{for } m \geqslant 0 \text{ and } 0 \leqslant l < 2^m. \tag{1.9}$$

(Notice that if $2^m \leqslant n < 2^{m+1}$, the remainder $l = n - 2^m$ satisfies $0 \leqslant l < 2^{m+1} - 2^m = 2^m$.)

We must now prove (1.9). As in the past we use induction, but this time the induction is on m. When $m = 0$ we must have $l = 0$; thus the basis of

But there's a simpler way! The key fact is that $J(2^m) = 1$ for all m, and this follows immediately from our first equation,

$J(2n) = 2J(n) - 1.$

Hence we know that the first person will survive whenever n is a power of 2. And in the general case, when $n = 2^m + l$, the number of people is reduced to a power of 2 after there have been l executions. The first remaining person at this point, the survivor, is number $2l + 1$.

(1.9) reduces to $J(1) = 1$, which is true. The induction step has two parts, depending on whether l is even or odd. If $m > 0$ and $2^m + l = 2n$, then l is even and

$$J(2^m + l) \;=\; 2J(2^{m-1} + l/2) - 1 \;=\; 2(2l/2 + 1) - 1 \;=\; 2l + 1\,,$$

by (1.8) and the induction hypothesis; this is exactly what we want. A similar proof works in the odd case, when $2^m + l = 2n + 1$. We might also note that (1.8) implies the relation

$$J(2n + 1) - J(2n) \;=\; 2.$$

Either way, the induction is complete and (1.9) is established.

To illustrate solution (1.9), let's compute $J(100)$. In this case we have $100 = 2^6 + 36$, so $J(100) = 2 \cdot 36 + 1 = 73$.

Now that we've done the hard stuff (solved the problem) we seek the soft: Every solution to a problem can be generalized so that it applies to a wider class of problems. Once we've learned a technique, it's instructive to look at it closely and see how far we can go with it. Hence, for the rest of this section, we will examine the solution (1.9) and explore some generalizations of the recurrence (1.8). These explorations will uncover the structure that underlies all such problems.

Powers of 2 played an important role in our finding the solution, so it's natural to look at the radix 2 representations of n and $J(n)$. Suppose n's binary expansion is

$$n \;=\; (b_m\, b_{m-1} \ldots b_1\, b_0)_2\,;$$

that is,

$$n \;=\; b_m 2^m \;+\; b_{m-1} 2^{m-1} \;+\; \cdots \;+\; b_1 2 \;+\; b_0\,,$$

where each b_i is either 0 or 1 and where the leading bit b_m is 1. Recalling that $n = 2^m + l$, we have, successively,

$$\begin{aligned}
n \;&=\; (1\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)_2\,,\\
l \;&=\; (0\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)_2\,,\\
2l \;&=\; (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, 0)_2\,,\\
2l + 1 \;&=\; (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, 1)_2\,,\\
J(n) \;&=\; (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, b_m)_2\,.
\end{aligned}$$

(The last step follows because $J(n) = 2l + 1$ and because $b_m = 1$.) We have proved that

$$J\big((b_m\, b_{m-1} \ldots b_1\, b_0)_2\big) \;=\; (b_{m-1} \ldots b_1\, b_0\, b_m)_2\,; \tag{1.10}$$

that is, in the lingo of computer programming, we get $J(n)$ from n by doing a one-bit cyclic shift left! Magic. For example, if $n = 100 = (1100100)_2$ then $J(n) = J\big((1100100)_2\big) = (1001001)_2$, which is $64 + 8 + 1 = 73$. If we had been working all along in binary notation, we probably would have spotted this pattern immediately.

If we start with n and iterate the J function $m+1$ times, we're doing $m+1$ one-bit cyclic shifts; so, since n is an $(m+1)$-bit number, we might expect to end up with n again. But this doesn't quite work. For instance if $n = 13$ we have $J\big((1101)_2\big) = (1011)_2$, but then $J\big((1011)_2\big) = (111)_2$ and the process breaks down; the 0 disappears when it becomes the leading bit. In fact, $J(n)$ must always be $\leqslant n$ by definition, since $J(n)$ is the survivor's number; hence if $J(n) < n$ we can never get back up to n by continuing to iterate.

("Iteration" here means applying a function to itself.)

Repeated application of J produces a sequence of decreasing values that eventually reach a "fixed point," where $J(n) = n$. The cyclic shift property makes it easy to see what that fixed point will be: Iterating the function enough times will always produce a pattern of all 1's whose value is $2^{\nu(n)} - 1$, where $\nu(n)$ is the number of 1 bits in the binary representation of n. Thus, since $\nu(13) = 3$, we have

$$\overbrace{J(J(\ldots J}^{\text{2 or more J's}}(13)\ldots)) \;=\; 2^3 - 1 \;=\; 7\,;$$

similarly

$$\overbrace{J(J(\ldots J}^{\text{8 or more}}((101101101101011)_2)\ldots)) \;=\; 2^{10} - 1 \;=\; 1023\,.$$

Curious, but true.

Curiously enough, if M is a compact C^∞ n-manifold ($n > 1$), there exists a differentiable immersion of M into $\mathbf{R}^{2n-\nu(n)}$ but not necessarily into $\mathbf{R}^{2n-\nu(n)-1}$. I wonder if Josephus was secretly a topologist?

Let's return briefly to our first guess, that $J(n) = n/2$ when n is even. This is obviously not true in general, but we can now determine exactly when it *is* true:

$$\begin{aligned} J(n) &= n/2\,,\\ 2l+1 &= (2^m + l)/2\,,\\ l &= \tfrac{1}{3}(2^m - 2)\,. \end{aligned}$$

If this number $l = \frac{1}{3}(2^m - 2)$ is an integer, then $n = 2^m + l$ will be a solution, because l will be less than 2^m. It's not hard to verify that $2^m - 2$ is a multiple of 3 when m is odd, but not when m is even. (We will study such things in Chapter 4.) Therefore there are infinitely many solutions to the equation