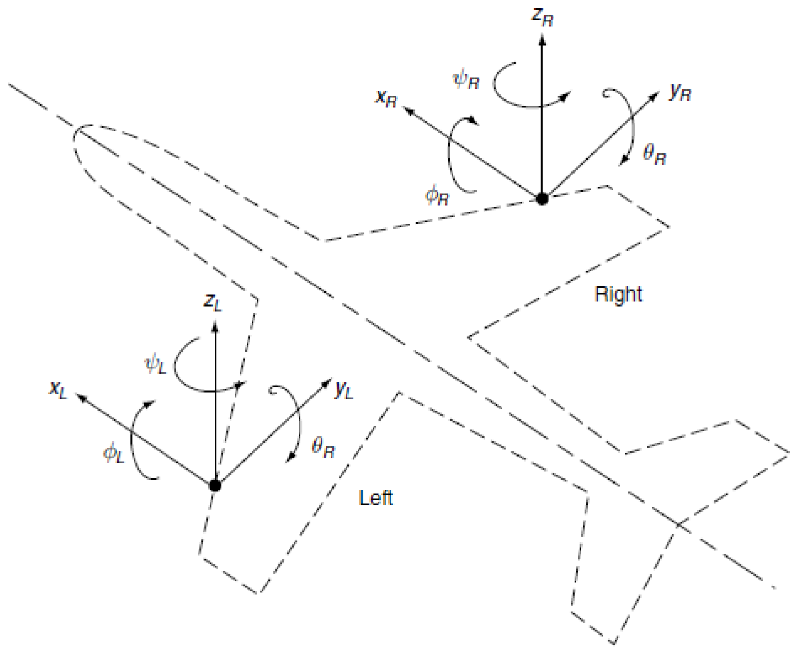
ETS de Ingeniería. Universidad de Sevilla

Examples: Structural Dynamics for Aircraft Structures

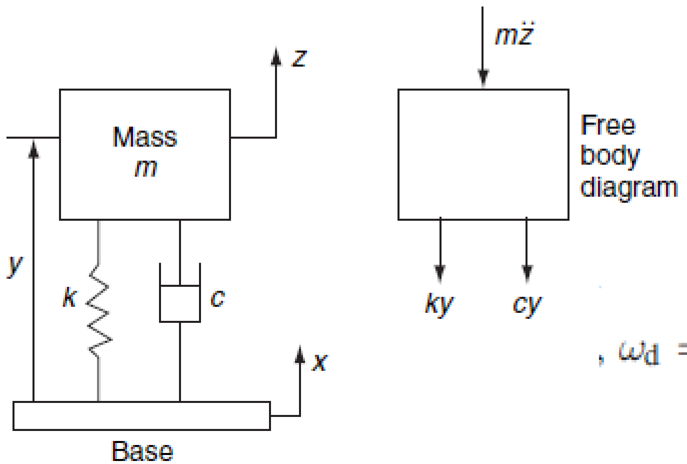


Estructuras Aeronáuticas
Grado en Ingeniería Aeroespacial

Summary

Example 1

The system shown in Fig. 2.1, but without the applied force, F, has the following properties: $m = 1$ kg; $k = 10\,000$ N/m and $c = 40$ N/m/s. Plot the time history of z for the initial conditions: $z = 0.1$ m, $\dot{z} = 0$, at $t = 0$.



$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{1}} = 100\,\mathrm{rad/s}$$

$$\gamma = \frac{c}{c_c} = \frac{c}{2m\omega_n} = 40/(2 \times 1 \times 100) = 0.2$$

$$, \omega_d = \omega_n\sqrt{(1-\gamma^2)} = 100\sqrt{(1-0.2^2)} = 98.0\,\mathrm{rad/s}.$$

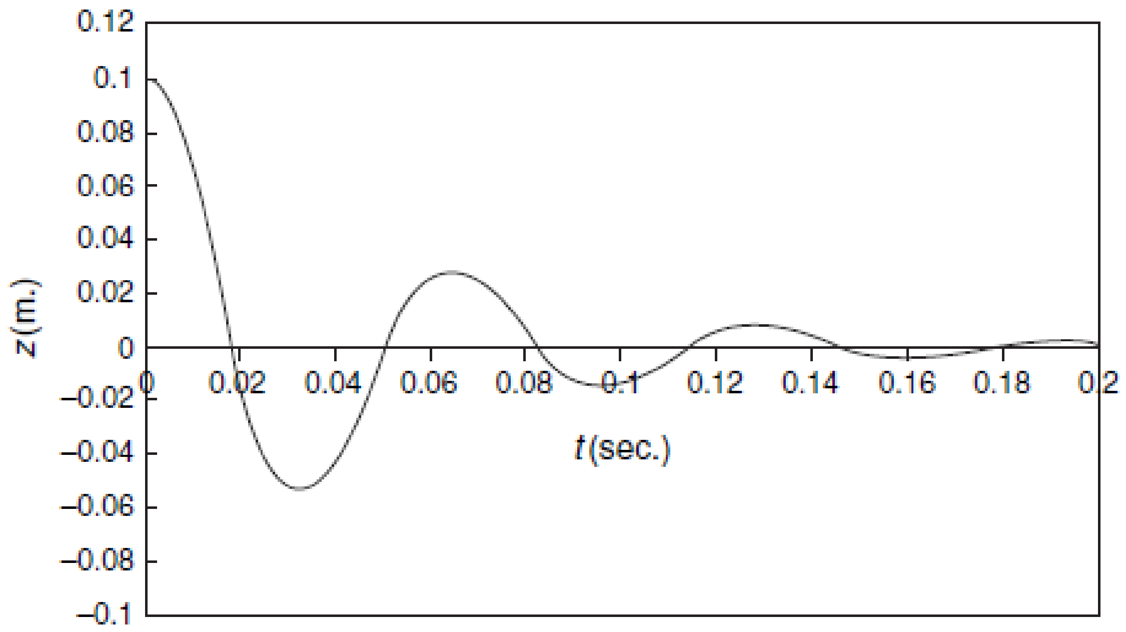
$$z = e^{-\gamma\omega_n t}(A \cos\omega_d t + B \sin\omega_d t)$$

$\gamma < 1$,

$$\dot{z} = e^{-\gamma\omega_n t}[(B\omega_d - A\gamma\omega_n)\cos\omega_d t - (A\omega_d + B\gamma\omega_n)\sin\omega_d t]$$

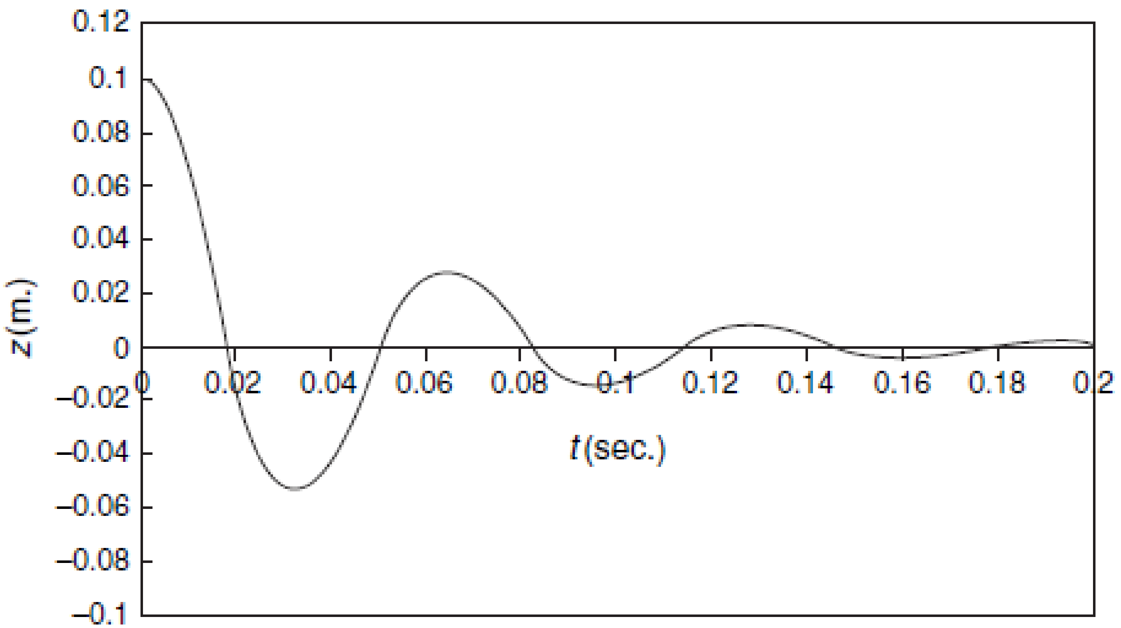
Example 1

$$z = e^{-20t}(0.1 \cos 98.0t + 0.0204 \sin 98.0t)$$



Example 1

$$z = e^{-20t}(0.1 \cos 98.0t + 0.0204 \sin 98.0t)$$



Example 2

Derive an expression for the displacement response of a single-DOF system excited by a force of the form $F = P \sin \omega t$. Assume that the initial conditions are $z = \dot{z} = 0$ at $t = 0$, and that the non-dimensional damping coefficient, γ, is less than 1.

$$m\ddot{z} + c\dot{z} + kz = P \sin \omega t$$

$$\ddot{z} + 2\gamma\omega_n\dot{z} + \omega_n^2 z = (P/m) \sin \omega t$$

Homogenous

$$z = e^{-\gamma\omega_n t}(A \cos \omega_d t + B \sin \omega_d t)$$

Particular

$$z = C \cos \omega t + D \sin \omega t$$

$$\dot{z} = -\omega C \sin \omega t + \omega D \cos \omega t$$

$$\ddot{z} = -\omega^2 C \cos \omega t - \omega^2 D \sin \omega t$$

Example 2

$$-\omega^2 C\cos\omega t - \omega^2 D\sin\omega t - 2\gamma\omega_n\omega C\sin\omega t + 2\gamma\omega_n\omega D\cos\omega t + \omega_n^2 C\cos\omega t + \omega_n^2 D\sin\omega t = (P/m)\sin\omega t$$

$$-\omega^2 D - 2\gamma\omega_n\omega C + \omega_n^2 D = P/m$$

$$-\omega^2 C + 2\gamma\omega_n\omega D + \omega_n^2 C = 0$$

$$C = \frac{P(-2\gamma\omega_n\omega)}{m\left[(\omega_n^2-\omega^2)^2 + (2\gamma\omega_n\omega)^2\right]}$$

$$D = \frac{P(\omega_n^2-\omega^2)}{m\left[(\omega_n^2-\omega^2)^2 + (2\gamma\omega_n\omega)^2\right]}$$

Example 2

$$z = \mathrm{e}^{-\gamma\omega_\mathrm{n}t}(A\cos\omega_\mathrm{d}t + B\sin\omega_\mathrm{d}t) + C\cos\omega t + D\sin\omega t$$

$$\dot{z} = \mathrm{e}^{-\gamma\omega_\mathrm{n}t}[(B\omega_\mathrm{d} - A\gamma\omega_\mathrm{n})\cos\omega_\mathrm{d}t - (A\omega_\mathrm{d} + B\gamma\omega_\mathrm{n})\sin\omega_\mathrm{d}t] - C\omega\sin\omega t + D\,\omega\cos\omega t$$

$$A = -C \quad = \frac{P(2\gamma\omega_\mathrm{n}\omega)}{m\left[\left(\omega_\mathrm{n}^2 - \omega^2\right)^2 + (2\gamma\omega_\mathrm{n}\omega)^2\right]}$$

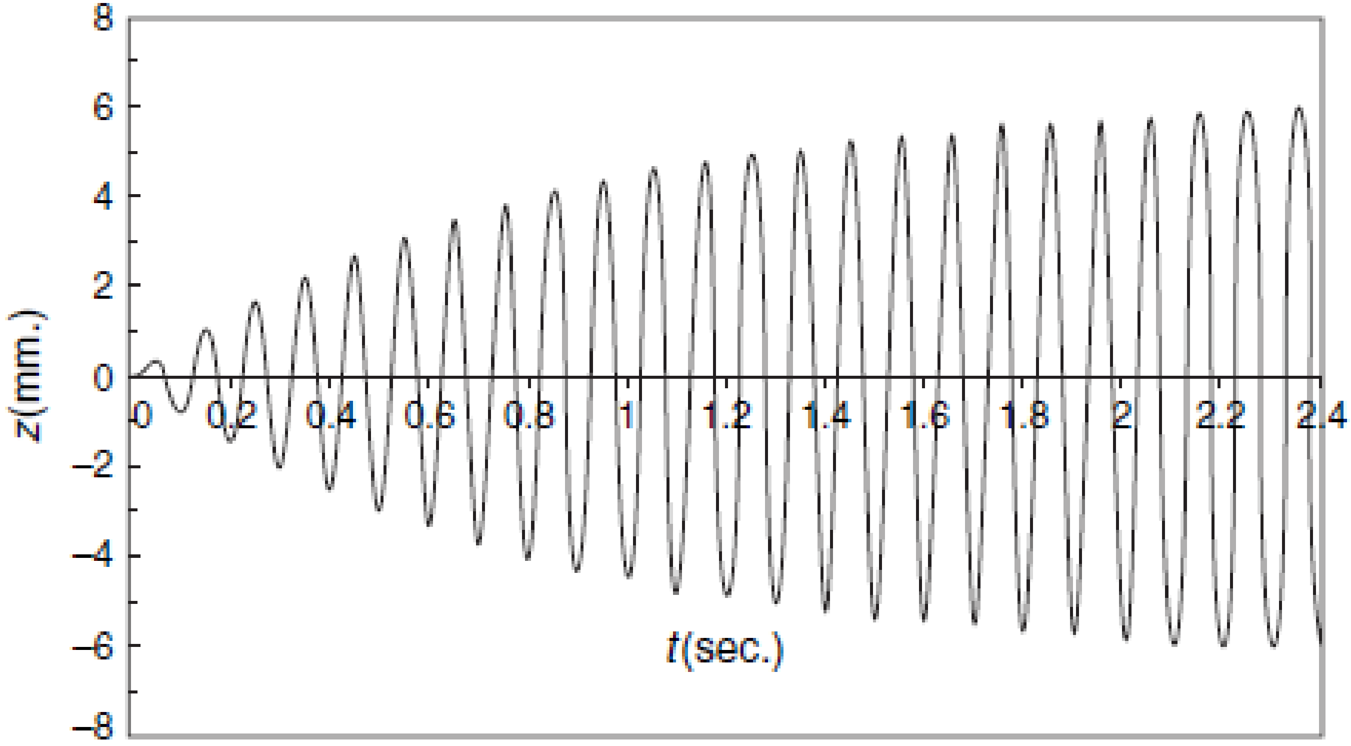
$$B = \frac{-(C\gamma\omega_\mathrm{n} + D\omega)}{\omega_\mathrm{d}} = \frac{P\omega\left[\left(\omega^2 - \omega_\mathrm{n}^2\right) + \left(2\gamma^2\omega_\mathrm{n}^2\right)\right]}{m\omega_\mathrm{d}\left[\left(\omega_\mathrm{n}^2 - \omega^2\right)^2 + (2\gamma\omega_\mathrm{n}\omega)^2\right]}$$

Example 3

Plot the displacement response of the system shown in Fig. 2.1, for the two cases given in Table 2.2, commenting on the results.

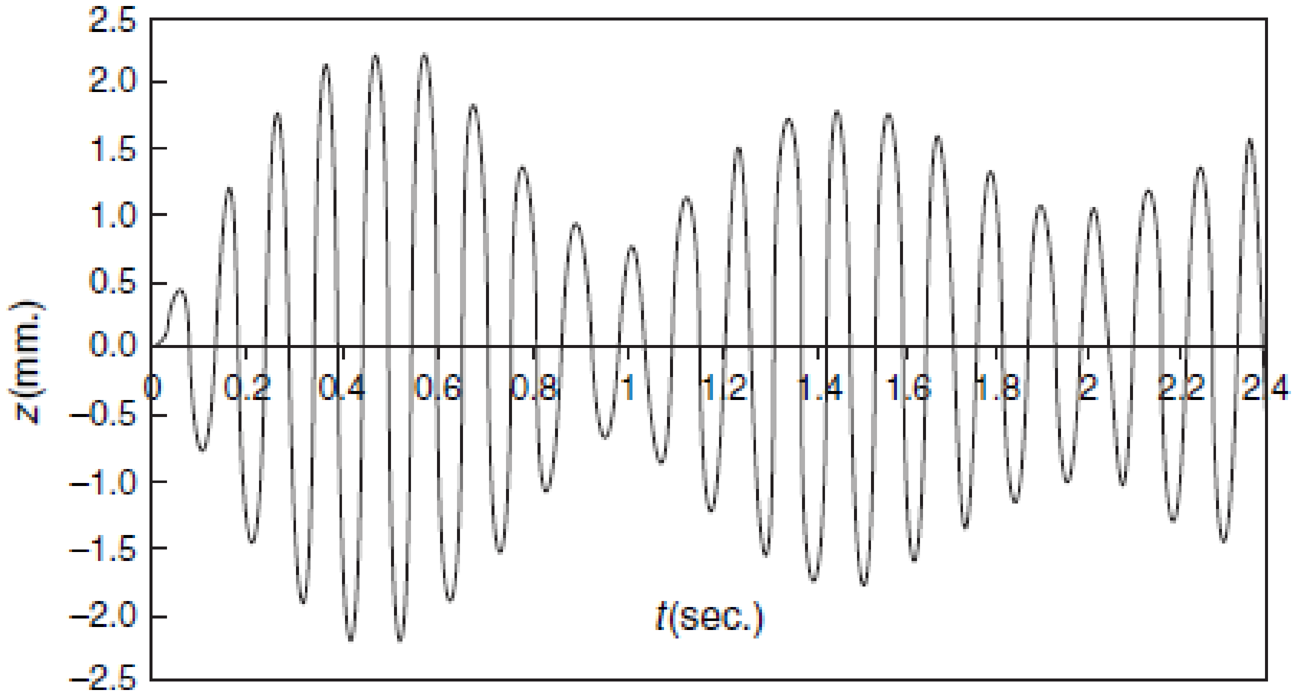
				Forcing frequency		Undamped nat. frequency		Viscous damping coeff.
Case	Forcing function	Peak force P (N)	Mass m (kg)	f (Hz)	ω (rad/s)	f_n (Hz)	ω_n (rad/s)	γ
(a)	$P \sin \omega t$	100	100	10	62.83	10	62.83	0.02
(b)	$P \sin \omega t$	100	100	9	56.55	10	62.83	0.01

Example 3



Response of a single-DOF system to a sinusoidal force at the natural frequency

Example 3



Response of a single-DOF system to sinusoidal excitation close to the natural frequency, showing beating.

Example 4: Laplace Method

External Force

$$m\ddot{z} + c\dot{z} + kz = F,$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$c_c = 2m\omega_n$$

$$\gamma = \frac{c}{c_c}$$

$$\ddot{z} + 2\gamma\omega_n\dot{z} + \omega_n^2 z = F/m$$

Base Motion

$$m\ddot{y} + c\dot{y} + ky = -m\ddot{x}$$

$$\ddot{y} + 2\gamma\omega_n\dot{y} + \omega_n^2 y = -\ddot{x}$$

Example 4: Laplace Method

External Force

$$(s^2 + 2\gamma\omega_n s + \omega_n^2)\underline{z} = \underline{F}/m$$

$$\left(\frac{\underline{z}}{\underline{F}}\right) = \frac{1}{m} \cdot \frac{1}{(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

$$\left(\frac{\underline{\dot{z}}}{\underline{F}}\right) = \frac{s}{m} \cdot \frac{1}{(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

$$\left(\frac{\underline{\ddot{z}}}{\underline{F}}\right) = \frac{s^2}{m} \cdot \frac{1}{(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

Base Motion

$$\left(\frac{\underline{y}}{\underline{x}}\right) = \frac{-s^2}{(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

$$\left(\frac{\underline{y}}{\underline{\dot{x}}}\right) = \left(\frac{\underline{\dot{y}}}{\underline{\ddot{x}}}\right) = \frac{-s}{(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

$$\left(\frac{\underline{y}}{\underline{\ddot{x}}}\right) = \frac{-1}{(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

Example 4: Laplace Method

$$f(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$f(s) = L[f(t)],$$

$$L[z(t)] = L[F(t)] \times \left(\frac{\underline{z}}{\underline{F}}\right)$$

A Short Table of Laplace Transforms

$f(t)$	$F(s)$	Notes
$\delta(t)$	1	Unit impulse or Dirac function.
$H(t)$	$\frac{1}{s}$	Unit step or Heaviside function.
e^{-at}	$\frac{1}{s+a}$	
te^{-at}	$\frac{1}{(s+a)^2}$	
t^n	$\frac{n!}{s^{(n+1)}}$	n must be a positive integer. $n! =$ factorial n
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	
$e^{-at}(1-at)$	$\frac{s}{(s+a)^2}$	
$1-\cos \omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$	
$\frac{1}{\omega_d} e^{-\gamma\omega_n t} \sin \omega_d t$	$\frac{1}{s^2+2\gamma\omega_n s+\omega_n^2}$	
where $\omega_d = \omega_n\sqrt{1-\gamma^2}$		

Example 4: Laplace Method

Use the Laplace transform method to solve Example 2.3, i.e. to find an expression for the displacement response, $z(t)$, of the system shown in Fig. 2.1 when the force, $F(t)$, consists of a positive step, of magnitude P, applied at $t = 0$, with the initial conditions: $z = \dot{z} = 0$ at $t = 0$.

$$L[z(t)] = L[F(t)] \times \left(\frac{\underline{z}}{\underline{F}}\right)$$

$$L[F(t)] = \frac{P}{s}$$

$$\left(\frac{\underline{z}}{\underline{F}}\right) = \frac{1}{m} \cdot \frac{1}{(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

Example 4: Laplace Method

$$L[z(t)] = \frac{P}{s} \times \frac{1}{m} \cdot \frac{1}{\left(s^2 + 2\gamma\omega_\mathrm{n} s + \omega_\mathrm{n}^2\right)} = \frac{P}{m}\left[\frac{1}{s} \cdot \frac{1}{\left(s^2 + 2\gamma\omega_\mathrm{n} s + \omega_\mathrm{n}^2\right)}\right]$$

$$\left[\frac{1}{s} \cdot \frac{1}{\left(s^2 + 2\gamma\omega_\mathrm{n} s + \omega_\mathrm{n}^2\right)}\right]$$

$$\left[\frac{1}{s} \cdot \frac{1}{\left(s^2 + 2\gamma\omega_\mathrm{n} s + \omega_\mathrm{n}^2\right)}\right] \equiv \frac{A}{s} + \frac{Bs + C}{\left(s^2 + 2\gamma\omega_\mathrm{n} s + \omega_\mathrm{n}^2\right)}$$

$$1 \equiv A\left(s^2 + 2\gamma\omega_\mathrm{n} s + \omega_\mathrm{n}^2\right) + Bs^2 + Cs$$

Example 4: Laplace Method

Equating constants:	$1 = A\omega_n^2$	or	$A = \frac{1}{\omega_n^2}$
Equating coeffs of s:	$0 = 2A\gamma\omega_n + C$	or	$C = -\frac{1}{\omega_n^2}(2\gamma\omega_n)$
Equating coeffs of s^2:	$0 = A + B$	so	$B = -\frac{1}{\omega_n^2}$.

$$\frac{1}{\omega_n^2}\left[\frac{1}{s} - \frac{s + 2\gamma\omega_n}{(s^2 + 2\gamma\omega_n s + \omega_n^2)}\right]$$

$$\frac{1}{\omega_n^2}\left[\frac{1}{s} - \frac{s + \gamma\omega_n}{(s + \gamma\omega_n)^2 + \omega_d^2} - \frac{\left[\frac{\gamma}{\sqrt{1-\gamma^2}}\right]\omega_d}{(s + \gamma\omega_n)^2 + \omega_d^2}\right]$$

$$(s^2 + 2\gamma\omega_n s + \omega_n^2)$$

$$(s + \gamma\omega_n)^2 + \omega_d^2 \text{ where } \omega_d^2 = \omega_n^2(1 - \gamma^2),$$

$$z = \frac{P}{m\omega_n^2}\left[1 - e^{-\gamma\omega_n t}\left(\cos\,\omega_d t + \frac{\gamma}{\sqrt{(1-\gamma^2)}}\sin\,\omega_d t\right)\right]$$

Example 5a: Convolution Integral

Find the unit impulse response $h(t)$ of the single-DOF system represented by Eq. (3.2), assuming that the damping is (a) zero and (b) non-zero, but less than critical.

Case (a): with zero damping:

$$z = \frac{1}{m\omega_n} \sin \omega_n t = h(t)$$

Case (b): with non-zero damping, but less than critical:

$$z = \frac{1}{m\omega_d} \left(e^{-\gamma\omega_n t} \sin\omega_d t \right) = h(t)$$

Example 5b: Convolution Integral

Use the convolution integral to find the response of the single-DOF system represented by Eq. (3.2), assuming that the damping is zero, when a step force of magnitude $\overline{F}$ is applied at $t = 0$, if the initial displacement, z, and the initial velocity, $\dot{z}$, are both zero at $t = 0$.

$$h(t) = \frac{1}{m\omega_n} \sin \omega_n t \qquad \begin{array}{ll} F(\tau) = 0 & \tau < 0 \\ F(\tau) = \overline{F} & 0 < \tau < \infty \end{array}$$

$$\sin(-\theta) = -\sin\theta, \qquad \cos(-\theta) = \cos\theta):$$

$$z(t) = \int_0^t F(\tau)\cdot h(t-\tau)\mathrm{d}\tau = \frac{\overline{F}}{m\omega_\mathrm{n}} \int_0^t \sin\,\omega_\mathrm{n}(t-\tau)\mathrm{d}\tau = \frac{\overline{F}}{m\omega_\mathrm{n}} \int_0^t \sin(-\omega_\mathrm{n}\tau + \omega_\mathrm{n} t)\mathrm{d}\tau$$

$$= \frac{\overline{F}}{m\omega_\mathrm{n}} \int_0^t -\sin(\omega_\mathrm{n}\tau - \omega_\mathrm{n} t)\mathrm{d}\tau = \frac{\overline{F}}{m\omega_\mathrm{n}^2}\,{}_0^t[\cos(\omega_\mathrm{n}\tau - \omega_\mathrm{n} t)]$$

$$z(t) = \frac{\overline{F}}{m\omega_\mathrm{n}^2}(1-\cos\,\omega_\mathrm{n} t) = \frac{\overline{F}}{k}(1-\cos\,\omega_\mathrm{n} t)$$

Examples: Structural Dynamics for Aircraft Structures

Transient Responses of the System: $1/(s^2+2\gamma\omega_n s+\omega_n^2)$ $(\gamma<1)$

Input function F					
Unit impulse $\delta(t)$	Unit step $H(t)$	Unit ramp t	$\sin\omega t$	$\cos\omega t$	Response functions
0	$\frac{1}{\omega_n^2}$	$\frac{-2\gamma}{\omega_n^3}$	0	0	Unit step $H(t)$
0	0	$\frac{1}{\omega_n^2}$	0	0	Unit ramp t
0	0	0	$\frac{\omega_n^2-\omega^2}{(\omega_n^2-\omega^2)^2+(2\gamma\omega_n\omega)^2}$	$\frac{2\gamma\omega_n\omega}{(\omega_n^2-\omega^2)^2+(2\gamma\omega_n\omega)^2}$	$\sin\omega t$
0	0	0	$\frac{-2\gamma\omega_n\omega}{(\omega_n^2-\omega^2)^2+(2\gamma\omega_n\omega)^2}$	$\frac{\omega_n^2-\omega^2}{(\omega_n^2-\omega^2)^2+(2\gamma\omega_n\omega)^2}$	$\cos\omega t$
$\frac{1}{\omega_d}$	$\frac{-\gamma}{\omega_n\omega_d}$	$\frac{2\gamma^2-1}{\omega_n^2\omega_d}$	$\frac{\omega\left[(\omega^2-\omega_n^2)+2\gamma^2\omega_n^2\right]}{\omega_d\left[(\omega_n^2-\omega^2)^2+(2\gamma\omega_n\omega)^2\right]}$	$\frac{-\gamma\omega_n(\omega_n^2+\omega^2)}{\omega_d\left[(\omega_n^2-\omega^2)^2+(2\gamma\omega_n\omega)^2\right]}$	$e^{-\gamma\omega_n t}\sin\omega_d t$
0	$\frac{-1}{\omega_n^2}$	$\frac{2\gamma}{\omega_n^3}$	$\frac{2\gamma\omega_n\omega}{(\omega_n^2-\omega^2)^2+(2\gamma\omega_n\omega)^2}$	$\frac{\omega^2-\omega_n^2}{(\omega_n^2-\omega^2)^2+(2\gamma\omega_n\omega)^2}$	$e^{-\gamma\omega_n t}\cos\omega_d t$

ω = forcing frequency (rad/s); ω_n = undamped natural frequency (rad/s), $\omega_n=\sqrt{k/m}$; ω_d = damped natural frequency (rad/s), $\omega_d=\omega_n\sqrt{1-\gamma^2}$; γ = non-dimensional viscous damping coefficient.

Example 6

Use Table 3.1 to find the displacement response of the system shown in Fig. 2.1 and represented by Eq. (3.2), when,

(a) a step force of magnitude a force units is applied at $t = 0$ and the initial conditions are $z = 0$ and $\dot{z} = 0$ at $t = 0$.

(b) a ramp force, starting from zero and growing linearly at the rate of b force units/second, is applied at $t = 0$, with the same initial conditions, $z = 0$ and $\dot{z} = 0$ at $t = 0$.

Assume that the non-dimensional damping coefficient, γ, is less than unity.

Example 6

Case (a), step input:

$$z = \frac{a}{m}\left[\left(\frac{1}{\omega_n^2}\right)H(t) + \left(\frac{-\gamma}{\omega_n\omega_d}\right)e^{-\gamma\omega_n t}\sin\,\omega_d t + \left(\frac{-1}{\omega_n^2}\right)e^{-\gamma\omega_n t}\cos\,\omega_d t\right]$$

$$z = \frac{a}{m\omega_n^2}\left[1 - e^{-\gamma\omega_n t}\left(\cos\,\omega_d t + \frac{\gamma}{\sqrt{(1-\gamma^2)}}\sin\,\omega_d t\right)\right]$$

Case (b), ramp input:

$$z = \frac{b}{m}\left[\left(\frac{-2\gamma}{\omega_n^3}\right)H(t) + \left(\frac{1}{\omega_n^2}\right)t + \left(\frac{2\gamma^2-1}{\omega_n^2\omega_d}\right)e^{-\gamma\omega_n t}\sin\omega_d t + \left(\frac{2\gamma}{\omega_n^3}\right)e^{-\gamma\omega_n t}\cos\,\omega_d t\right]$$

$$z = \frac{b}{m\omega_n^2}\left[t - \frac{2\gamma}{\omega_n}(1 - e^{-\gamma\omega_n t}\cos\omega_d t) + \left(\frac{2\gamma^2-1}{\omega_d}\right)e^{-\gamma\omega_n t}\sin\,\omega_d t\right]$$

Example 7

Write expressions for:
(i) the complex receptance, $\underline{z}/\underline{F}$; (ii) the complex mobility, $\underline{\dot{z}}/\underline{F}$; (iii) the complex inertance, $\underline{\ddot{z}}/\underline{F}$, of a single-DOF system with direct force excitation, having the following parameters:
mass, $m = 1$ kg;
undamped natural frequency, $f_n = 10$ Hz;
viscous damping coefficient, $\gamma = 0.05$.

Example 7

The complex receptance, $\underline{z}/\underline{F}$, is given by

$$\frac{\underline{z}}{\underline{F}} = \frac{(1-\Omega^2) - i(2\gamma\Omega)}{m\omega_n^2\left[(1-\Omega^2)^2 + (2\gamma\Omega)^2\right]}$$

where

$$\Omega = \frac{f}{f_n} = \frac{\omega}{\omega_n}$$

f = forcing frequency in Hz. f = 0–30 Hz
f_n = undamped natural frequency in Hz = 10 Hz
ω = forcing frequency in rad/s
ω_n = undamped natural frequency in rad/s = 20π
m = mass = 1 kg
γ = non-dimensional viscous damping coefficient = 0.05.

Example 7

The complex mobility, $\dot{\underline{z}}/\underline{F}$, is given by multiplying the complex receptance $\underline{z}/\underline{F}$ by $\mathrm{i}\omega$:

$$\frac{\dot{\underline{z}}}{\underline{F}} = \frac{\mathrm{i}\omega\left[\left(1-\Omega^2\right)-\mathrm{i}(2\gamma\Omega)\right]}{m\omega_{\mathrm{n}}^2\left[\left(1-\Omega^2\right)^2+(2\gamma\Omega)^2\right]} = \frac{2\pi f\left[(2\gamma\Omega)+\mathrm{i}\left(1-\Omega^2\right)\right]}{m\omega_{\mathrm{n}}^2\left[\left(1-\Omega^2\right)^2+(2\gamma\Omega)^2\right]}$$

The complex inertance, $\ddot{\underline{z}}/\underline{F}$, is given by multiplying the mobility by $\mathrm{i}\omega$ or the receptance $\underline{z}/\underline{F}$ by $(\mathrm{i}\omega)^2 = -\omega^2$:

$$\frac{\ddot{\underline{z}}}{\underline{F}} = \frac{-\omega^2\left[\left(1-\Omega^2\right)-\mathrm{i}(2\gamma\Omega)\right]}{m\omega_{\mathrm{n}}^2\left[\left(1-\Omega^2\right)^2+(2\gamma\Omega)^2\right]} = \frac{-\Omega^2\left[\left(1-\Omega^2\right)-\mathrm{i}(2\gamma\Omega)\right]}{m\left[\left(1-\Omega^2\right)^2+(2\gamma\Omega)^2\right]}$$

Example 8: Continuous Systems

A helicopter manufacturer is developing a code of practice for the installation of hydraulic service pipes, and wishes to comply with a 'minimum standard of integrity' vibration standard, which defines, among other requirements, a sinusoidal acceleration level of $\pm 5.0\,g$ from 50 to 500 Hz, at the pipe's attachments to the aircraft structure. One investigation considers straight lengths of steel pipe, installed with spacing, L, between support centers, as shown in Fig. 8.1(a). Only the fundamental vibration mode of the pipe is considered, and this is taken as the exact first bending mode of a simply supported, uniform beam. It may be assumed that a separate investigation has shown the response in higher order modes to be negligible.

Note: The system of units used in this example is the 'British' lbf inch system, in which the acceleration due to gravity, g, is 386 in./s^2, and mass is expressed in lb in^{-1}s^2, a unit sometimes known as the 'mug'.

The properties of the pipe are as follows:

E = Young's modulus for the material $= 30 \times 10^6$ lbf/in.^{2}

D = outer diameter = 0.3125 in.

d = inner diameter = 0.2625 in.

I = second moment of area of cross-section, $= (\pi/64)\,(D^4 - d^4) = 0.2350 \times 10^{-3}$ in.^{4}

μ = total mass of pipe per inch $= 0.0214 \times 10^{-3}$ mug/in. (including contained fluid).

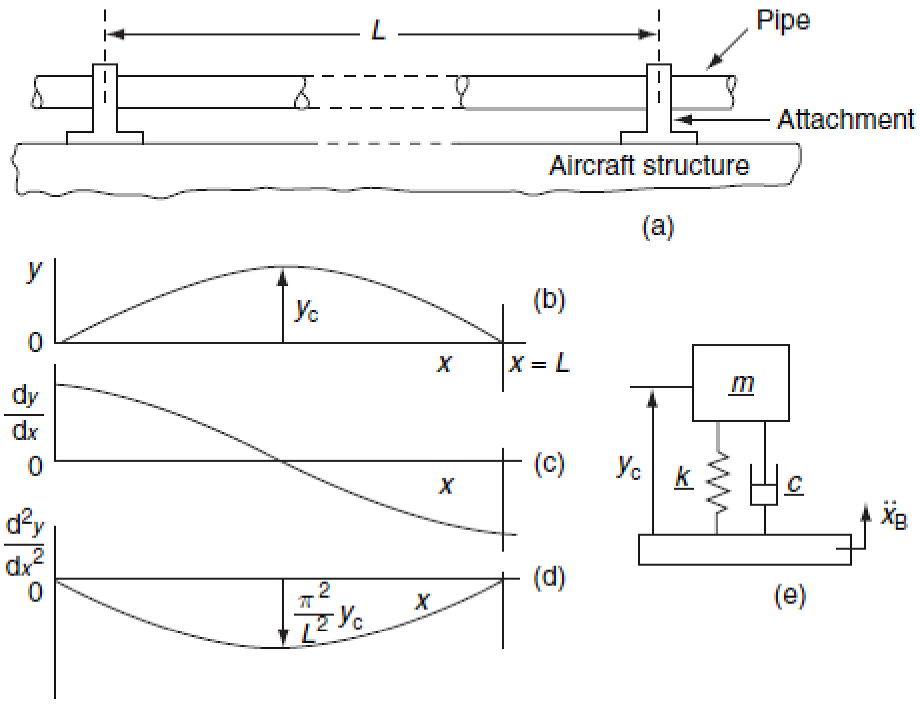
The non-dimensional viscous damping coefficient, γ, can be taken as 0.02 of critical.

Example 8: Continuous Systems

Find

(a) The vertical single-peak displacement at the center of the pipe span, relative to the supporting structure, if $L = 17$ in.;

(b) The maximum oscillatory stress in the pipe.



Example 8: Continuous Systems

Table 8.1
Natural Frequencies and Mode Shapes for Uniform Beams in Bending

End Conditions	Characteristic equation and roots $\beta_i L$	A	B	C	D
Simply-supported (Pinned-pinned)	$\sin \beta_i L = 0$ $\beta_1 L = \pi$ $\beta_2 L = 2\pi$ $\beta_3 L = 3\pi$ $\beta_4 L = 4\pi$	1	0	0	0
Free-free	$\cos \beta_i L \cdot \cosh \beta_i L = 1$ $\beta_1 L = 4.73004$ $\beta_2 L = 7.85321$ $\beta_3 L = 10.99561$ $\beta_4 L = 14.13717$	1	$\dfrac{\sin \beta_i L - \sinh \beta_i L}{\cosh \beta_i L - \cos \beta_i L}$	1	$\dfrac{\sin \beta_i L - \sinh \beta_i L}{\cosh \beta_i L - \cos \beta_i L}$
Fixed-fixed	$\cos \beta_i L \cdot \cosh \beta_i L = 1$ $\beta_1 L = 4.73004$ $\beta_2 L = 7.85321$ $\beta_3 L = 10.99561$ $\beta_4 L = 14.13717$	−1	$-\dfrac{\sinh \beta_i L - \sin \beta_i L}{\cos \beta_i L - \cosh \beta_i L}$	1	$\dfrac{\sinh \beta_i L - \sin \beta_i L}{\cos \beta_i L - \cosh \beta_i L}$

Example 8: Continuous Systems

Natural Frequencies and Mode Shapes for Uniform Beams in Bending

End Conditions	Characteristic equation and roots $\beta_i L$	A	B	C	D
Cantilever (Fixed-free) (x is measured from the fixed end)	$\cos\beta_i L \cdot \cosh\beta_i L = -1$ $\beta_1 L = 1.87510$ $\beta_2 L = 4.69409$ $\beta_3 L = 7.85475$ $\beta_4 L = 10.99554$	1	$-\dfrac{\sin\beta_i L + \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}$	-1	$\dfrac{\sin\beta_i L + \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}$
Fixed-pinned (x is measured from the fixed end)	$\tan\beta_i L - \tanh\beta_i L = 0$ $\beta_1 L = 3.92660$ $\beta_2 L = 7.06858$ $\beta_3 L = 10.21017$ $\beta_4 L = 13.35177$	1	$-\dfrac{\sin\beta_i L - \sinh\beta_i L}{\cos\beta_i L - \cosh\beta_i L}$	-1	$\dfrac{\sin\beta_i L - \sinh\beta_i L}{\cos\beta_i L - \cosh\beta_i L}$

Notes

In all cases the natural frequencies, in rad/s, are $\omega_i = \beta_i^2\sqrt{EI/\mu}$, where values of $\beta_i L$, where L is the length of the beam, corresponding to the first four non-zero natural frequencies, are given in the second column of the table. The mode shape is given by:

$$y_i = A\,\sin\beta_i x + B\,\cos\beta_i x + C\,\sinh\beta_i x + D\,\cosh\beta_i x$$

where A, B, C and D can be found from the table. The mode shapes are not normalized to any particular amplitude. Note that the free–free beam has zero frequency rigid modes.

Example 8: Continuous Systems

$$\omega_1 = \beta_1^2 \sqrt{\frac{EI}{\mu}} \quad \text{and} \quad \beta_1 L = \pi \qquad \omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\mu}} \qquad \omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\mu}}$$

$$y = y_c \sin\left(\frac{\pi x}{L}\right) \quad 0 < x < L$$

$$\frac{dy}{dx} = y_c \frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right); \quad 0 < x < L$$

$$\frac{d^2y}{dx^2} = y_c \frac{\pi^2}{L^2}\left[-\sin\left(\frac{\pi x}{L}\right)\right] \quad 0 < x < L$$

Example 8: Continuous Systems

$$T = \frac{1}{2}\mu \int_0^L \dot{y}^2 \mathrm{d}x$$

$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\mu}},$$

$$\dot{y} = \dot{y}_c \sin\frac{\pi x}{L}$$

$$\boxed{\underline{k} = \underline{m}\omega_1^2 = \frac{1}{2}EI\frac{\pi^4}{L^3}}$$

$$T = \frac{1}{2}\mu\, \dot{y}_c^2 \int_0^L \sin^2\frac{\pi x}{L}\mathrm{d}x = \frac{1}{4}\mu L\, \dot{y}_c^2$$

$$T = \frac{1}{2}\underline{m}\dot{y}_c^2$$

$$T = \frac{1}{4}\mu L \dot{y}_c^2 = \frac{1}{2}\underline{m}\dot{y}_c^2$$

$$\boxed{\underline{m} = \frac{1}{2}\mu L}$$

$$\underline{m}\ddot{y}_c + \underline{c}\dot{y}_c + \underline{k}y_c = -\underline{m}\ddot{x}_B$$

$$\frac{|y_c|}{|x_B|} = \frac{\Omega^2}{\sqrt{(1-\Omega^2)^2 + (2\gamma\Omega)^2}}$$

where

$$\Omega = \frac{\omega}{\omega_1} = \frac{f}{f_1};$$

Example 8: Continuous Systems

$$\frac{|y_c|}{|\ddot{x}_B|} = \frac{1}{\omega^2} \cdot \left(\frac{|y_c|}{|x_B|}\right)$$

$$\frac{|y_c|}{|\ddot{x}_B|} = \frac{\Omega^2}{\omega^2\sqrt{(1-\Omega^2)^2 + (2\gamma\Omega)^2}} = \frac{1}{\omega_1^2\sqrt{(1-\Omega^2)^2 + (2\gamma\Omega)^2}}$$

$$\boxed{\Omega = 1. \qquad |y_c|_{MAX} = \frac{|\ddot{x}_B|}{2\gamma\omega_1^2}}$$

Example 8: Continuous Systems

$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\mu}} = \frac{\pi^2}{17^2}\sqrt{\frac{(30\times 10^6)\times(0.2350\times 10^{-3})}{(0.0214\times 10^{-3})}}$$

$$= 619.8\,\mathrm{rad/s} \quad (\text{or } f_1 = 98.6\,\mathrm{Hz}).$$

$$|\ddot{x}_{\mathrm{B}}| = (5\times 386)\,\mathrm{in./s^2} = 1930\,\mathrm{in./s^2}$$

$$|y_{\mathrm{c}}|_{\mathrm{MAX}} = \frac{|\ddot{x}_{\mathrm{B}}|}{2\gamma\omega_1^2} = \frac{1930}{2\times 0.02\times(619.8)^2} = 0.126\,\mathrm{in.}$$

Example 8: Continuous Systems

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y_c \frac{\pi^2}{L^2}\left[-\sin\left(\frac{\pi x}{L}\right)\right].$$

$$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_c = y_c \frac{\pi^2}{L^2}\left(-\sin\frac{\pi}{2}\right) = -y_c \frac{\pi^2}{L^2}$$

$M_c = EI(\mathrm{d}^2 y/\mathrm{d}x^2)_c$ a

$$s_c = M_c \frac{D}{2I} = \frac{DE}{2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_c = -\frac{\pi^2 DE}{2L^2} y_c$$

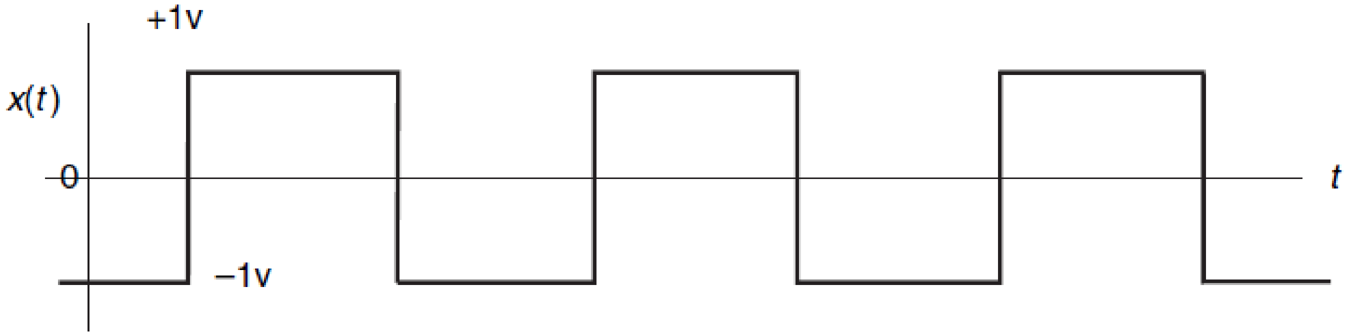
$$|s_c|_{MAX} = \frac{\pi^2 DE}{2L^2}|y_c|_{MAX}$$

, $D = 0.3125$ in.; $E = 30 \times 10^6$ lbf/in.^{2}, $L = 17$ in. and $|y_c|_{MAX} = 0.126$ in., gives

$$|s_c|_{MAX} = 20\,100 \text{ lbf/in.}^2$$

Example 9: FFT

(a) Derive a Fourier series to represent the voltage waveform shown in Fig. 9.2(a), a square wave with amplitude ± 1V, and period T seconds, by representing one period as an even function.
(b) Repeat (a) using an odd function to represent one period.
(c) Compare the results.
(d) If the period of the square wave is 1 second, plot the sums of each of the two series derived in (a) and (b) above, against time, t, showing that the original square wave is reproduced approximately.



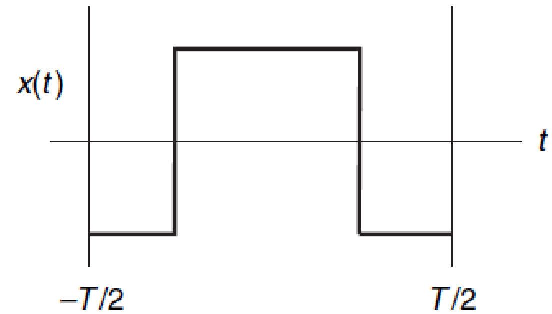
(a)

Example 9: FFT

$$a_0 = 0,$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t \cdot \mathrm{d}t = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\left(\frac{2\pi}{T}\right)t \cdot \mathrm{d}t$$

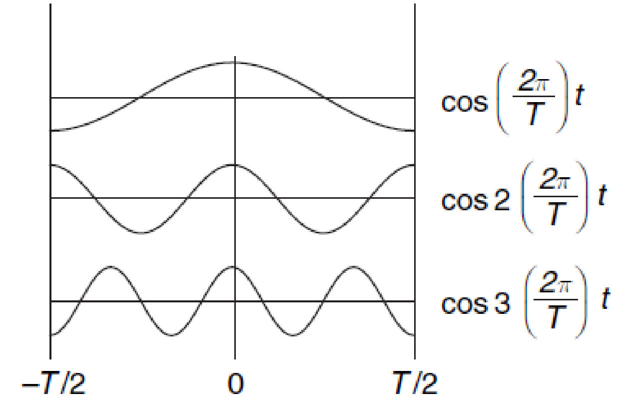
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t \cdot \mathrm{d}t = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\left(\frac{2\pi}{T}\right)t \cdot \mathrm{d}t$$



$$x(t) = a_1\cos\left(\frac{2\pi}{T}\right)t + a_3\,\cos 3\left(\frac{2\pi}{T}\right)t + a_5\,\cos 5\left(\frac{2\pi}{T}\right)t + a_7\,\cos 7\left(\frac{2\pi}{T}\right)t + \cdots$$

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(\frac{2\pi}{T}\right)t\cdot\mathrm{d}t = 4\cdot\frac{2}{T}\int_0^{T/4}\cos\left(\frac{2\pi}{T}\right)t\cdot\mathrm{d}t = 4\cdot\frac{2}{T}\cdot\frac{T}{2\pi}\left[\sin\left(\frac{2\pi}{T}\right)t\right]_0^{T/4} = \frac{4}{\pi}$$

$$a_3 = -\frac{1}{3}\cdot\left(\frac{4}{\pi}\right);\quad a_5 = \frac{1}{5}\cdot\left(\frac{4}{\pi}\right);\quad a_7 = -\frac{1}{7}\cdot\left(\frac{4}{\pi}\right) + \cdots$$



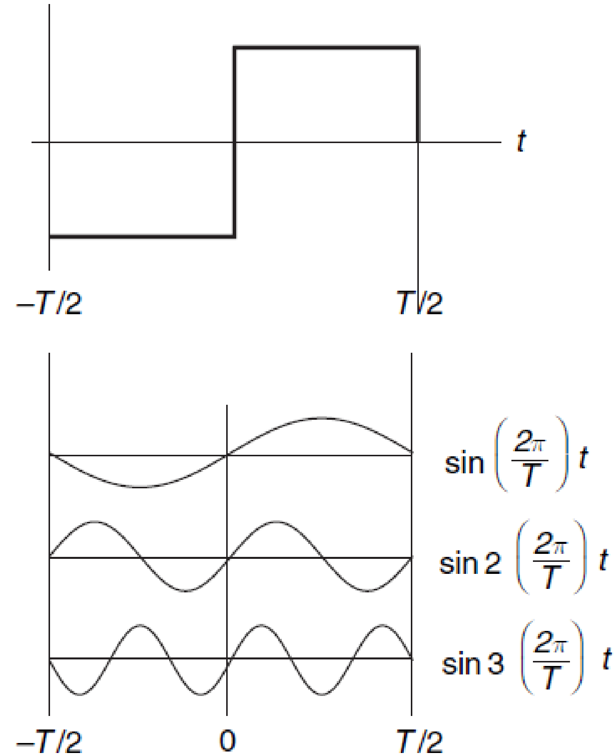
$$x(t) = \frac{4}{\pi}\left[\cos\left(\frac{2\pi}{T}\right)t - \frac{1}{3}\cos\ 3\left(\frac{2\pi}{T}\right)t + \frac{1}{5}\cos\ 5\left(\frac{2\pi}{T}\right)t - \frac{1}{7}\cos\ 7\left(\frac{2\pi}{T}\right)t + \cdots\right]$$

Example 9: FFT

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\left(\frac{2\pi}{T}\right)t \cdot \mathrm{d}t \quad (n = 1, 3, 5, 7, \ldots, \infty)$$

$$b_1 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(\frac{2\pi}{T}\right)t \cdot \mathrm{d}t = 2\cdot\frac{2}{T}\int_{0}^{T/2} \sin\left(\frac{2\pi}{T}\right)\mathrm{d}t$$

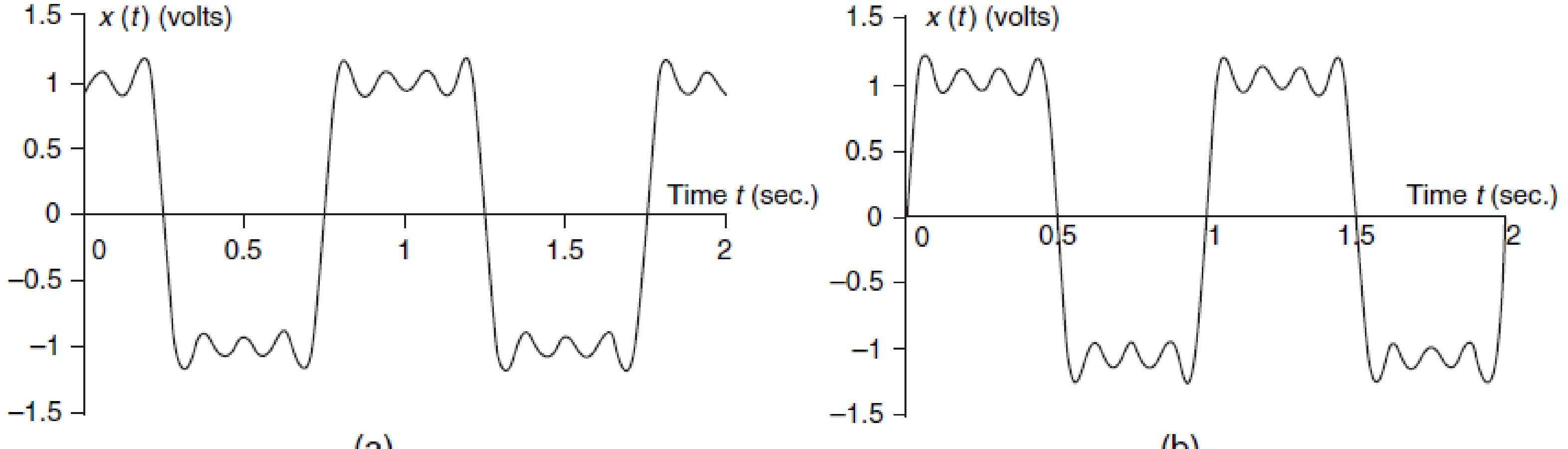
$$= 2\cdot\frac{2}{T}\cdot\frac{T}{2\pi}\left[-\cos\left(\frac{2\pi}{T}\right)t\right]_0^{T/2} = \frac{4}{\pi}.$$



$$b_3 = \frac{1}{3}\cdot\left(\frac{4}{\pi}\right); \quad b_5 = \frac{1}{5}.\left(\frac{4}{\pi}\right); \quad b_7 = \frac{1}{7}\cdot\left(\frac{4}{\pi}\right);$$

$$x(t) = \frac{4}{\pi}\left[\sin\left(\frac{2\pi}{T}\right)t + \frac{1}{3}\sin\,3\left(\frac{2\pi}{T}\right)t + \frac{1}{5}\sin\,5\left(\frac{2\pi}{T}\right)t + \frac{1}{7}\sin\,7\left(\frac{2\pi}{T}\right)t + \cdots\right]$$

Example 9: FFT

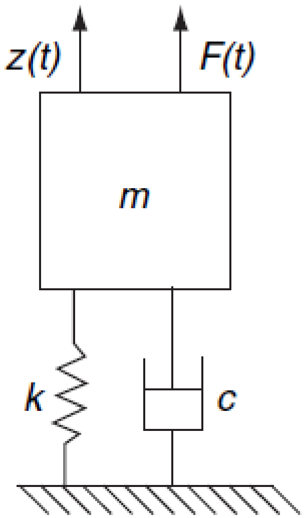


(a) (b)

Example 10: FFT

The vertical motion of a machine tool can be represented schematically by Fig. 9.10. The mass, m, of 200 kg, is carried on elastic supports, so that the natural frequency for vertical motion is 30 Hz, and the viscous damping coefficient, γ, is 0.1 of critical. A mechanism applies a vertical, periodic force, $F(t)$, that can be approximated by a symmetrical square wave of period $T = 0.1$ s, and a magnitude of ± 3000 N.

Plot the vertical displacement time history, $z(t)$, of the machine.



Example 10: FFT

$$F(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right] \quad n = 1, 2, 3, \ldots$$

$$F(t) = a_0 + \sum_{n=1}^{\infty} \left[d_n \cos(n\omega_0 t - \psi_n)\right]$$

$$d_n = \sqrt{a_n^2 + b_n^2} \qquad \psi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$|z|_n = \frac{d_n}{k} \cdot \frac{1}{\sqrt{\left(1 - \Omega_n^2\right)^2 + \left(2\gamma\Omega_n\right)^2}}$$

$$\Omega_n = \frac{n\omega_0}{\omega_u}$$

$$z_n(t) = \frac{d_n}{k} \cdot \frac{1}{\sqrt{\left(1 - \Omega_n^2\right)^2 + \left(2\gamma\Omega_n\right)^2}} \cos(n\omega_0 t - \psi_n - \phi_n) \qquad \phi_n = \tan^{-1} \frac{2\gamma\Omega_n}{1 - \Omega_n^2}$$

Example 10: FFT

$$z(t) = \frac{a_0}{k} + \sum_{n=1}^{\infty} \left[\frac{d_n}{k} \cdot \frac{1}{\sqrt{\left(1-\Omega_n^2\right)^2 + (2\gamma\Omega_n)^2}} \cos(n\omega_0 t - \psi_n - \phi_n) \right], \quad n = 1, 2, 3, \ldots$$

$$\left\langle z^2(t) \right\rangle = \left\langle \left\{ \frac{a_0}{k} + \sum_{n=1}^{\infty} \left[\frac{d_n}{k} A_n \cos(n\omega_0 t - \psi_n - \phi_n) \right] \right\}^2 \right\rangle$$

$$\left\langle z^2(t) \right\rangle = \frac{a_0^2}{k^2} + \frac{1}{2k^2} \sum_{n=1}^{\infty} (d_n A_n)^2$$

The RMS value is $\sqrt{\langle z^2(t) \rangle}$.

Example 10: FFT

$$F(t) = 3000\left(\frac{4}{\pi}\cos\omega_0 t - \frac{4}{3\pi}\cos 3\,\omega_0 t + \frac{4}{5\pi}\cos 5\,\omega_0 t - \frac{4}{7\pi}\cos 7\,\omega_0 t + \cdots\right)$$

$$-\cos\theta = \cos(\theta - \pi),$$

$$F(t) = d_1\cos\omega_0 t + d_3\cos(3\omega_0 t - \pi) + d_5\cos 5\,\omega_0 t + d_7\cos(7\omega_0 t - \pi) + \cdots$$

$$F(t) = a_0 + \sum_{n=1}^{\infty}\left[d_n\cos(n\omega_0 t - \psi_n)\right]$$

$$\omega_0 = 2\pi/T$$

$$a_0 = 0,$$

$$d_n = 0 \quad \text{for even values of } n.$$

$$\begin{aligned} d_1 &= (3000\times 4)/\pi &&= 3819\ \mathrm{N} && \psi_1 = 0\\ d_3 &= (3000\times 4)/(3\pi) &&= 1273\ \mathrm{N} && \psi_3 = \pi\\ d_5 &= (3000\times 4)/(5\pi) &&= 763.9\ \mathrm{N} && \psi_5 = 0\\ d_7 &= (3000\times 4)/(7\pi) &&= 545.7\ \mathrm{N} && \psi_7 = \pi\\ &. \end{aligned}$$

Example 10: FFT

$\omega_u = (2\pi \times 30) = 60\pi$ rad/s = natural frequency of system = 30 Hz;
$m = 200$ kg = mass of machine;
$k = m\omega_u^2 = 200(2\pi \times 30)^2 = 7.106 \times 10^6$ N/m = stiffness of supports;
$\gamma = 0.1$ = viscous damping coefficient.

$$z(t) = \frac{a_0}{k} + \sum_{n=1}^{\infty} \left[\frac{d_n}{k} A_n \cos(n\omega_0 t - \psi_n - \phi_n) \right]$$

$$A_n = \frac{1}{\sqrt{\left(1 - \Omega_n^2\right)^2 + \left(2\gamma\Omega_n\right)^2}}$$

Example 10: FFT

Frequency (Hz)	n	$n\omega_0$ (rad/s)	d_n (N)	ψ_n (rad)	Ω_n	A_n	ϕ_n (rad)
10	1	$20\,\pi$	3819	0	0.3333	1.1218	0.2213
30	3	$60\,\pi$	1273	π	1.0000	5.0000	1.5707
50	5	$100\,\pi$	763.9	0	1.6666	0.5528	3.029
70	7	$140\,\pi$	545.7	π	2.3333	0.2238	3.096
90	9	$180\,\pi$	424.4	0	3.0000	0.1246	3.116
110	11	$220\,\pi$	347.2	π	3.6666	0.0802	3.125
130	13	$260\,\pi$	293.8	0	4.3333	0.0561	3.130