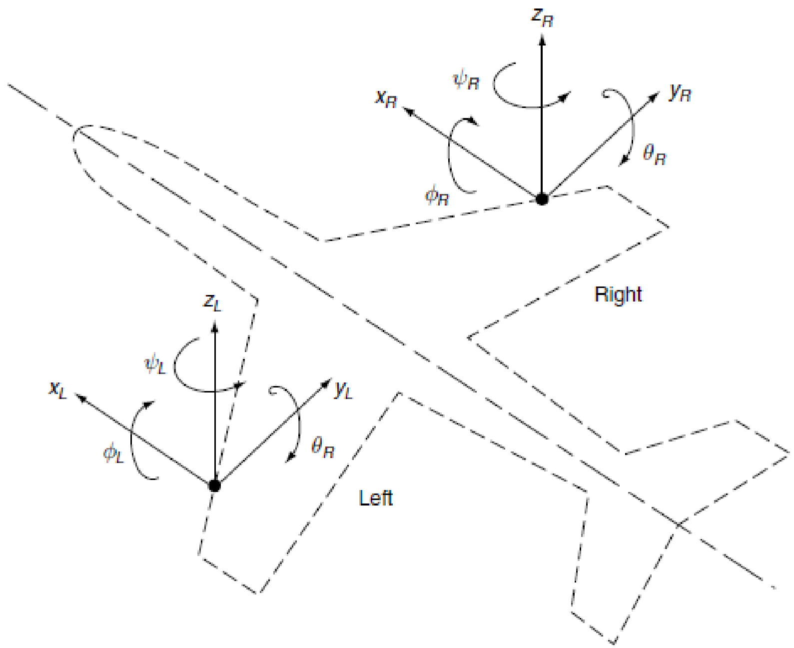
ETS de Ingeniería. Universidad de Sevilla

Introduction to Structural Dynamics for Aircraft Structures



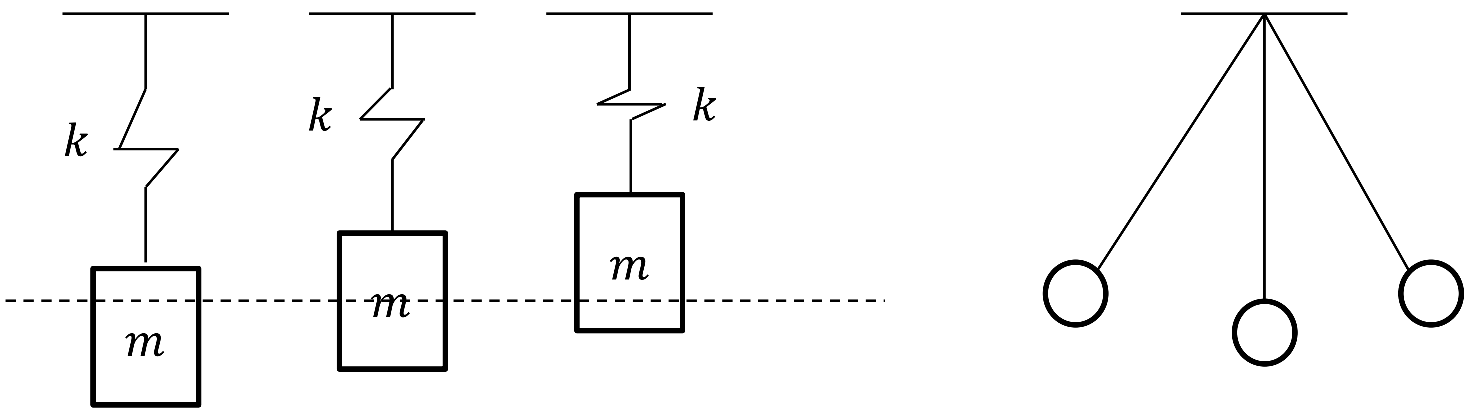
Estructuras Aeronáuticas

Grado en Ingeniería Aeroespacial

Summary

1. Introduction

Vibration → Mechanical oscillations about and equilibrium point
Transformation → Potential Energy (PE) ↔ Kinetic Energy (KE)



PE max
KE = 0

PE = 0
KE max

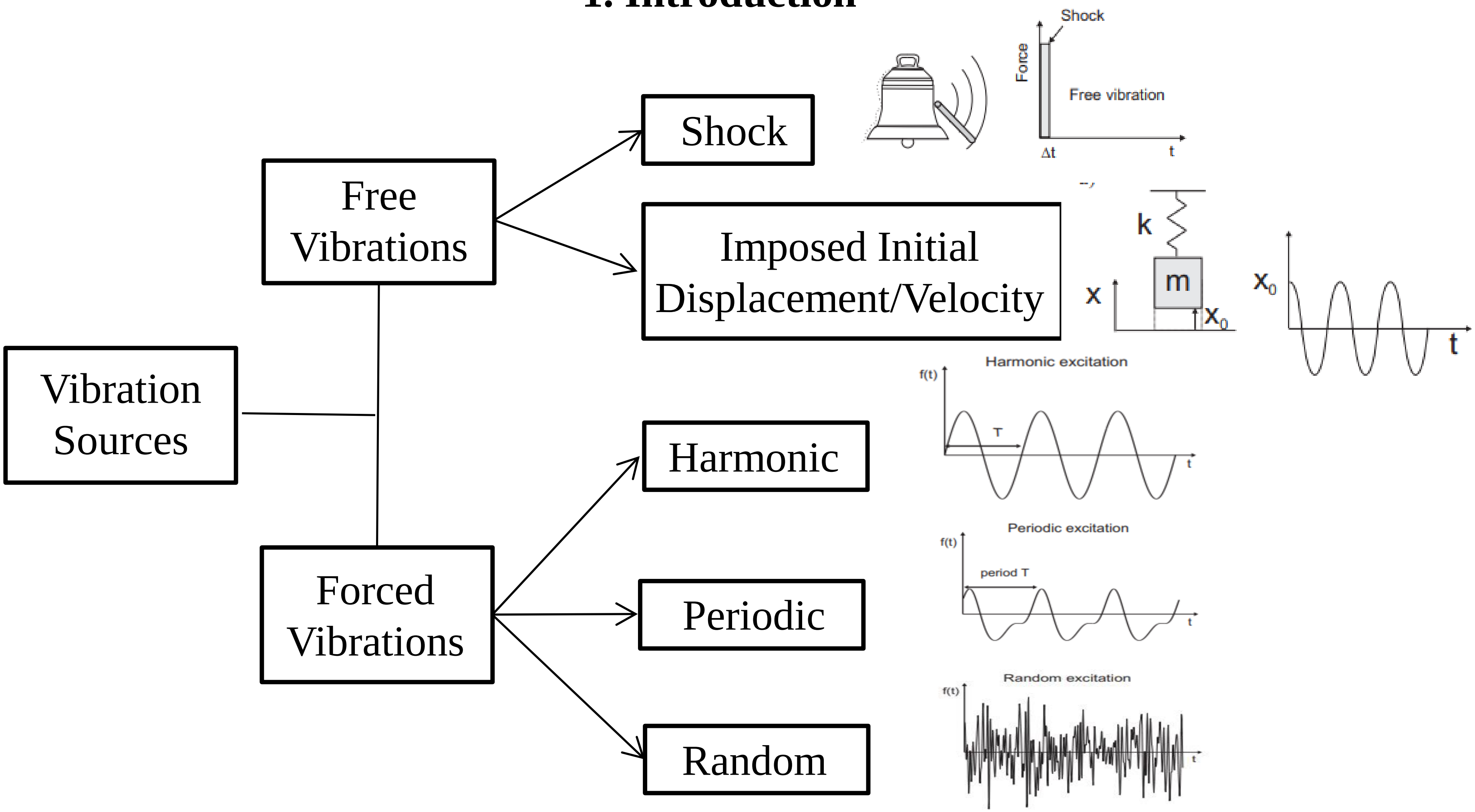
PE max
KE = 0

PE max
KE = 0

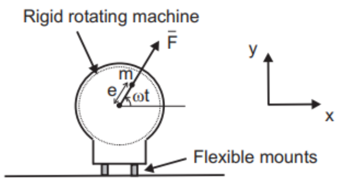
PE = 0
KE max

PE max
KE = 0

1. Introduction



1. Introduction



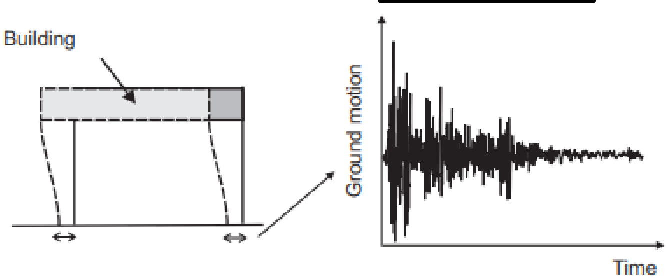
Vibration Sources in Structural Engineering

Internal Sources

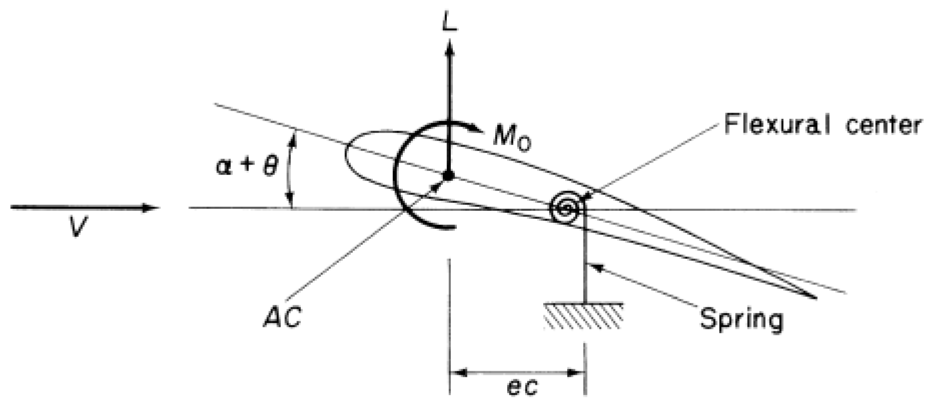
- Ventilation systems
- Elevator and conveyance systems
- Fluid pumping equipment
- Machines and generators
- Aerobics& exercise rooms, human activity

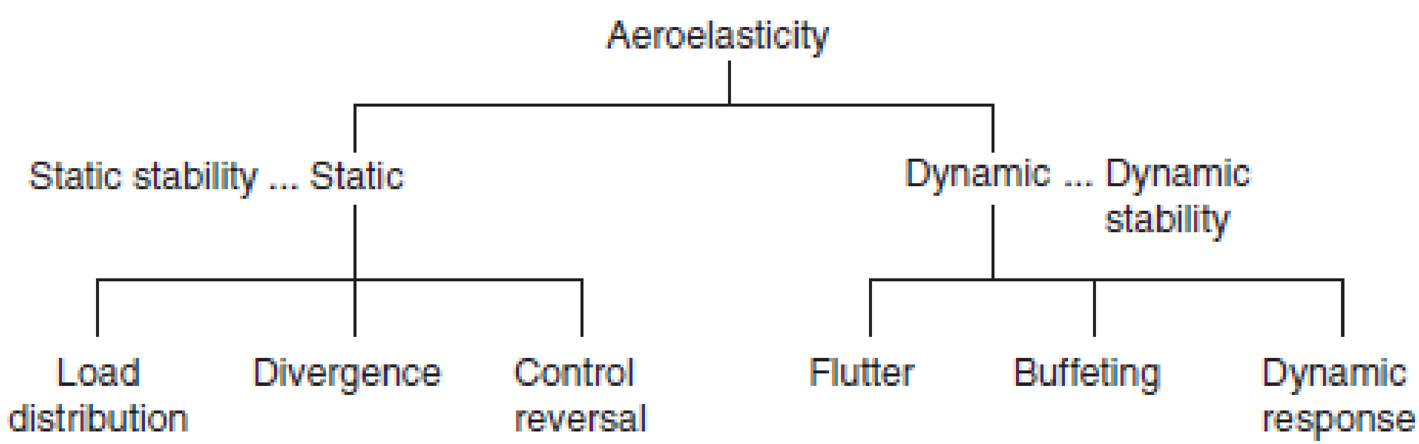
External Sources

- Seismic activity
- Subway, road and rail systems, airplanes
- Construction equipment
- Wind, Waves



1. Introduction





2. Mathematical Tools to Deal with Dynamic Actions

Harmonic Vibration

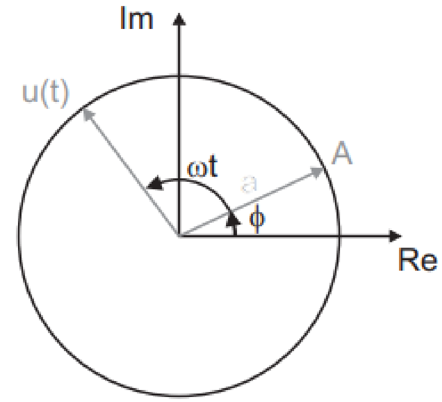
$$\boxed{w = 2\pi f = \frac{2\pi}{T}}$$

Frequency Period

Angular frequency

$$\boxed{u(t) = a\cos(wt + \varphi)}$$

Amplitude Phase angle



$$u(t) = ae^{i(wt+\varphi)} = a\cos(wt + \varphi) + ia\sin(wt + \varphi)$$

$$u(t) = ae^{i\varphi}\, e^{iwt} = Ae^{iwt}$$

$$A = ae^{i\varphi} = a\cos(\varphi) + ia\sin(\varphi)$$

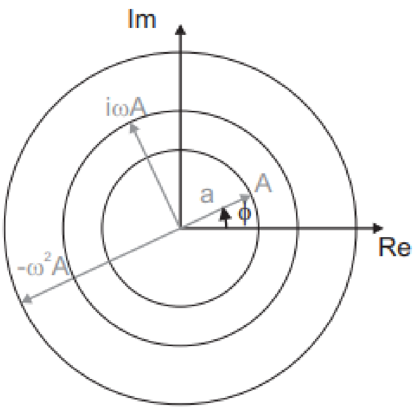
2. Mathematical Tools to Deal with Dynamic Actions

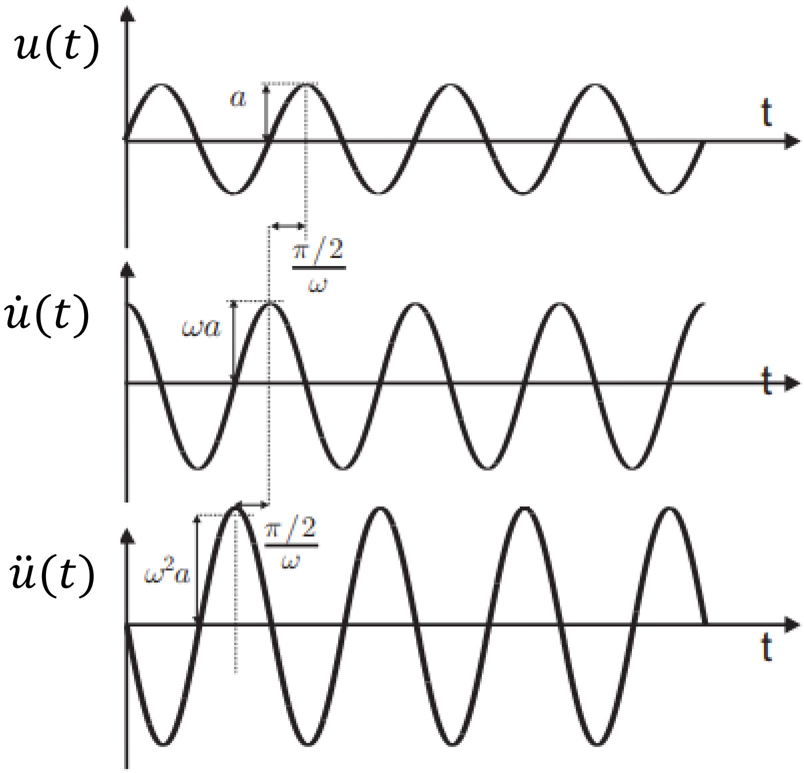
Harmonic Vibration

$$u(t) = Ae^{iwt}$$

$$\dot{u}(t) = iwAe^{iwt} = iwu(t)$$

$$\ddot{u}(t) = -w^2Ae^{iwt} = -w^2u(t)$$





2. Mathematical Tools to Deal with Dynamic Actions

Discrete Fourier Transform (DFT)

Periodic

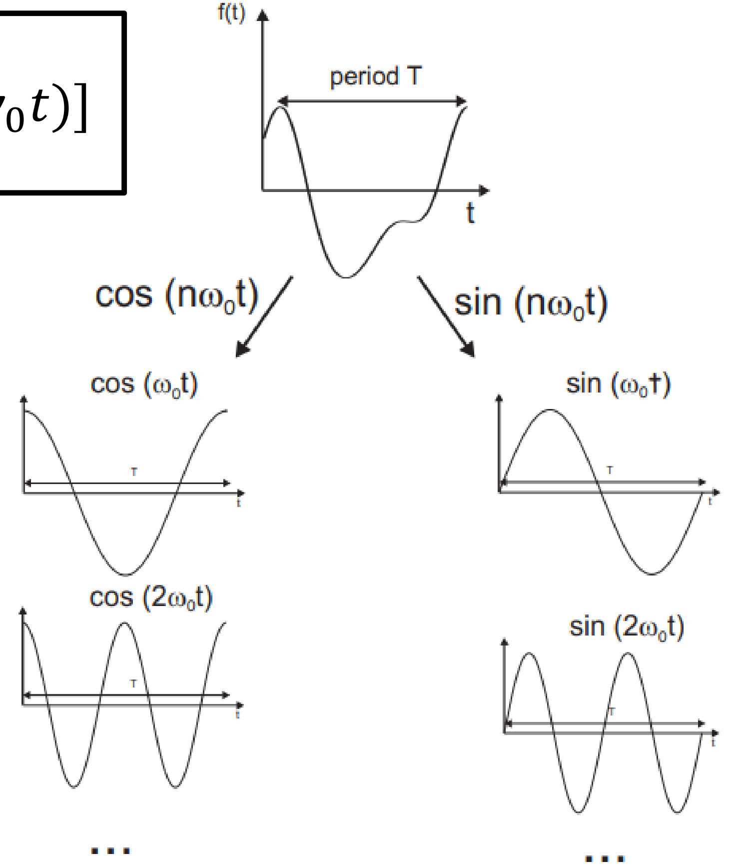
$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n w_0 t) + b_n \sin(n w_0 t)]$$

$$w_0 = \frac{2\pi}{T}$$

$$a_0 = \frac{1}{T}\int_0^T u(t)dt$$

$$a_n = \frac{2}{T}\int_0^T u(t)\cos(n w_0 t)dt$$

$$b_n = \frac{2}{T}\int_0^T u(t)\sin(n w_0 t)dt$$



2. Mathematical Tools to Deal with Dynamic Actions

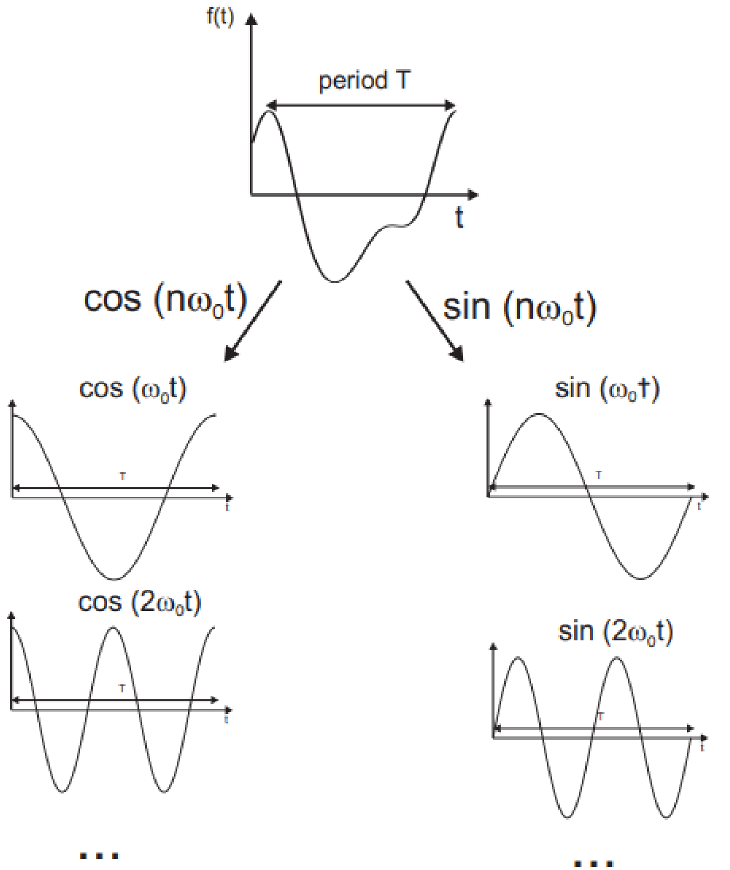
Discrete Fourier Transform (DFT)

$$u(t) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n w_0 t - \varphi_n)$$

$$d_0 = \frac{a_0}{2}$$

$$d_n = \sqrt{{a_n}^2 + {b_n}^2}$$

$$\varphi_n = tan^{-1}(\frac{b_n}{a_n})$$



2. Mathematical Tools to Deal with Dynamic Actions

Discrete Fourier Transform (DFT)→ Complex Formulation

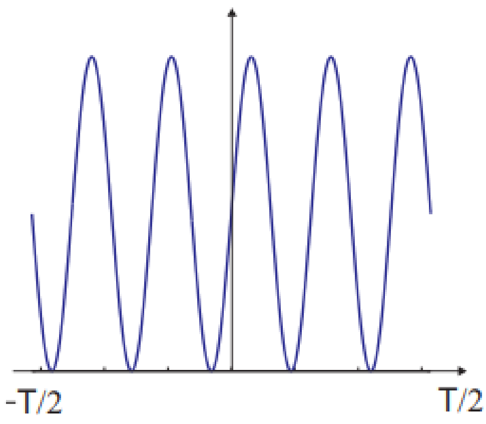
$$\cos(nw_0 t) = \frac{e^{inw_0 t} + e^{-inw_0 t}}{2} \qquad \sin(nw_0 t) = \frac{e^{inw_0 t} - e^{-inw_0 t}}{2i}$$

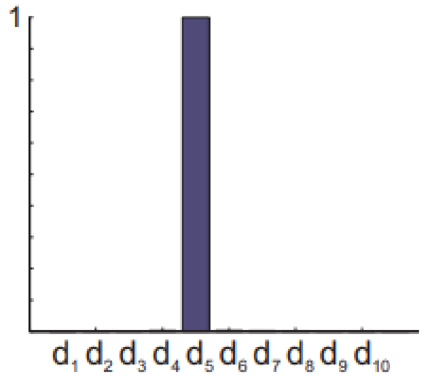
$$u(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{inw_0 t}$$

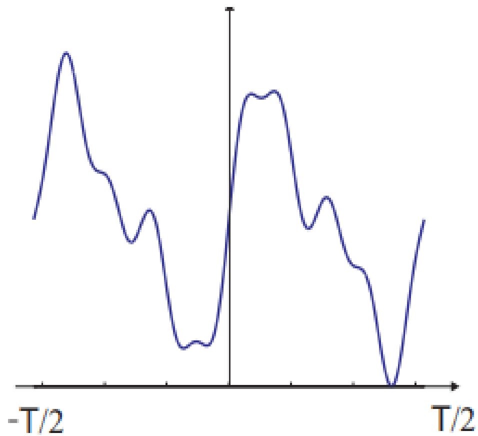
$$c_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)dt \qquad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)e^{inw_0 t}dt$$

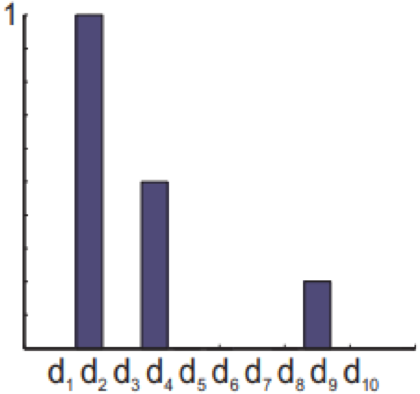
2. Mathematical Tools to Deal with Dynamic Actions

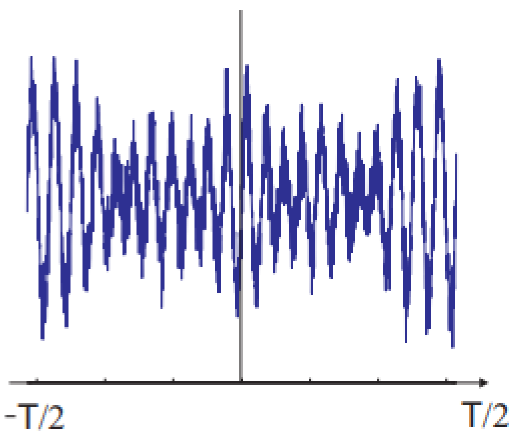
Discrete Fourier Transform (DFT)→ Examples

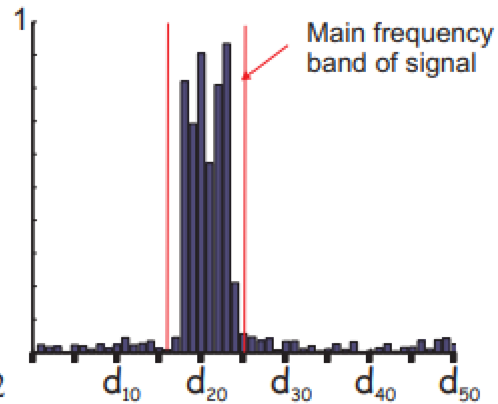












2. Mathematical Tools to Deal with Dynamic Actions

Fast Fourier Transform (FFT)

- The FFT does not refer to a new or different type of Fourier transform. It refers to a very efficient algorithm for computing the DFT.
- The time taken to evaluate a DFT on a computer depends principally on the number of multiplications involved. DFT needs n^2 multiplications. FFT only needs $nlog_2(n)$.
- The central insight which leads to this algorithm is the realization that a DFT of a sequence of n points can be written in terms of two discrete Fourier transforms of length $n/2$.
- Thus if n is a power of two, it is possible to recursively apply this decomposition until we are left with DFT of single points.

Implemented in MATLAB

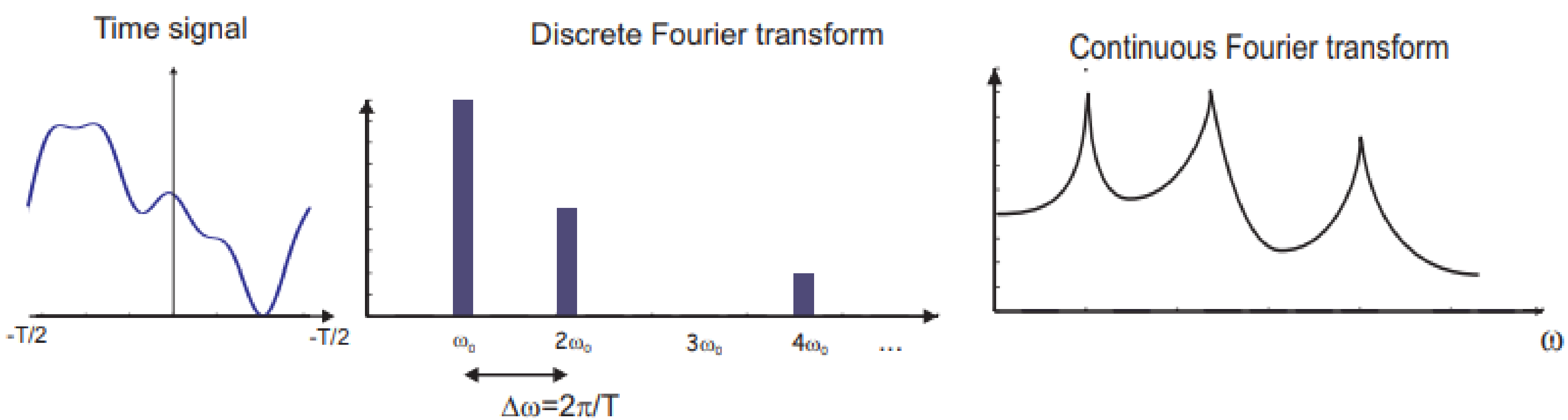
2. Mathematical Tools to Deal with Dynamic Actions

Continuous Fourier Transform (CFT)

$u(t)$ Non periodic → DFT cannot be applied → CFT must be used.

Continuous Variable

$T \to \infty \implies nw_0 \implies \omega \qquad \Delta w = w_0 \implies d\omega$



2. Mathematical Tools to Deal with Dynamic Actions

Continuous Fourier Transform (CFT)

Direct

$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi f t} dt$$

$$u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi f t} df$$

Inverse

$u(t)$	$U(f)$
1	$\delta(f)$
$\delta(t)$	1
$\cos(2\pi f_0 t)$	$\dfrac{\delta(f-f_0)+\delta(f+f_0)}{2}$
$\sin(2\pi f_0 t)$	$\dfrac{\delta(f-f_0)+\delta(f+f_0)}{2i}$
$\sum_{n=-\infty}^{n=\infty} \delta(t-nT)$	$\dfrac{1}{T}\sum_{n=-\infty}^{n=\infty} \delta(f-\dfrac{n}{T})$

2. Mathematical Tools to Deal with Dynamic Actions

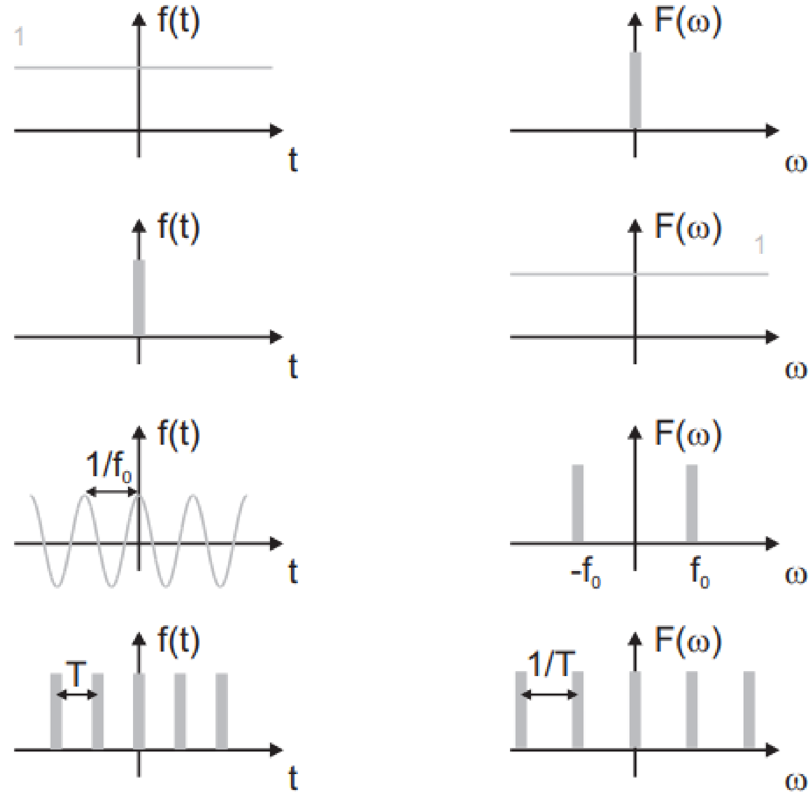
Continuous Fourier Transform (CFT)

Time domain function	Frequency domain function	Property
$a\,f(t) + b\,g(t)$	$a\,F(f) + b\,G(f)$	Linearity
$f(kt)$	$\frac{1}{\lvert k\rvert}F\left(\frac{f}{k}\right)$	Time Scaling
$\frac{1}{k}f\left(\frac{t}{k}\right)$	$F(kf)$	Frequency scaling
$f(t-t_0)$	$e^{-i2\pi f t_0}F(f)$	Time shifting
$f(t)\,e^{i2\pi f_0 t}$	$F(f-f_0)$	Frequency shifting
$f(t)$ real even function	$F(f)$ real even function	
$f(t)$ real odd function	$F(f)$ imag odd function	
$f(t)$ real	$F(-f) = F(f)^*$	

Introduction to Structural Dynamics for Aircraft Structures

2. Mathematical Tools to Deal with Dynamic Actions

Continuous Fourier Transform (CFT)



Estructuras Aeronáuticas
Grado en Ingeniería Aeroespacial

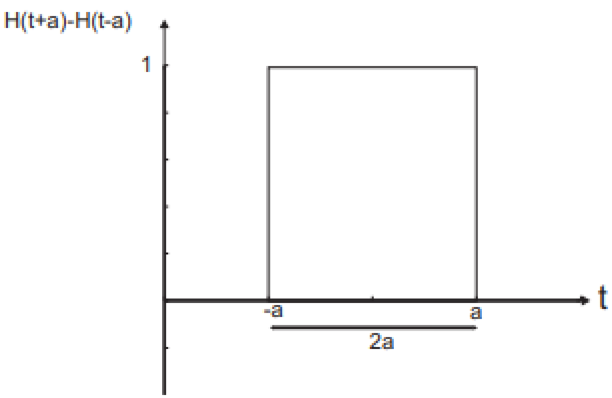
2. Mathematical Tools to Deal with Dynamic Actions

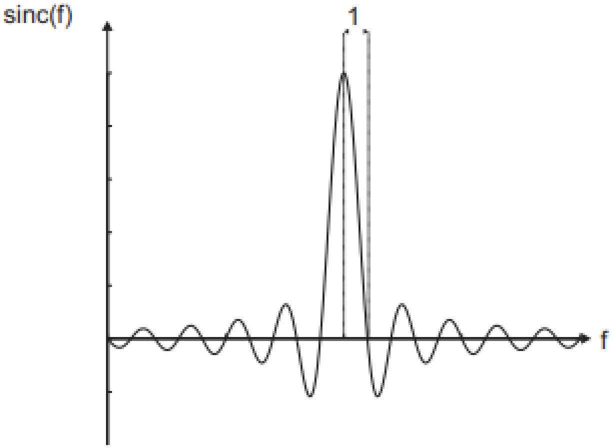
Continuous Fourier Transform (CFT)

Time Domain $\Longrightarrow$ Frequency Domain

$$u(t) = \cos(2\pi f_0 t) \qquad U(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

$$u(t) = \begin{matrix} 1 & -a < t < a \\ 0 & |t| > a \end{matrix} \qquad U(f) = 2asinc(2af) = 2a\frac{\sin(2af)}{\pi 2af}$$





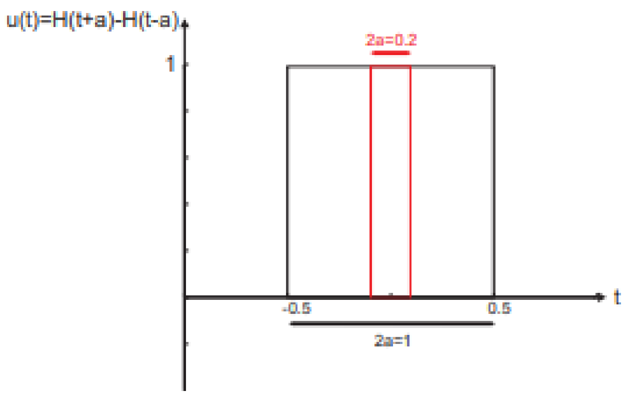
2. Mathematical Tools to Deal with Dynamic Actions

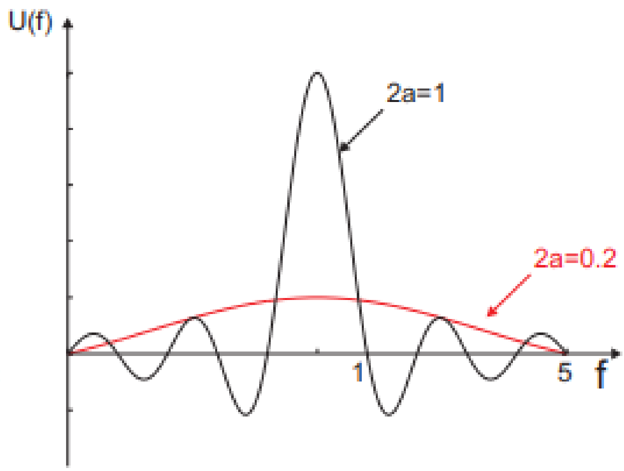
Continuous Fourier Transform (CFT)

Time Domain ⟹ Frequency Domain

$$u(t) = \begin{matrix} 1 & -a < t < a \\ 0 & |t| > a \end{matrix}$$

$$U(f) = 2a\,sinc(2af) = 2a\frac{\sin(2af)}{\pi 2af}$$





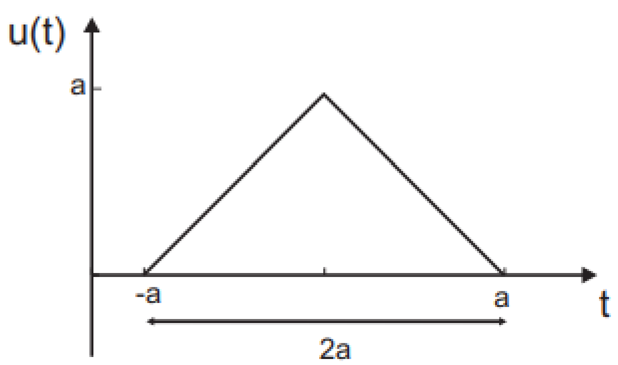
2. Mathematical Tools to Deal with Dynamic Actions

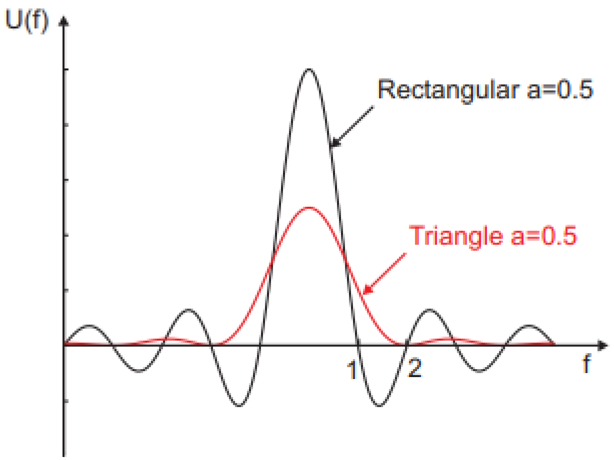
Continuous Fourier Transform (CFT)

Time Domain ⟹ Frequency Domain

$$u(t) = \begin{matrix} a - |t| & -a < t < a \\ 0 & |t| > a \end{matrix}$$

$$U(f) = a^2 sinc^2(af)$$





2. Mathematical Tools to Deal with Dynamic Actions

Convolution Integral

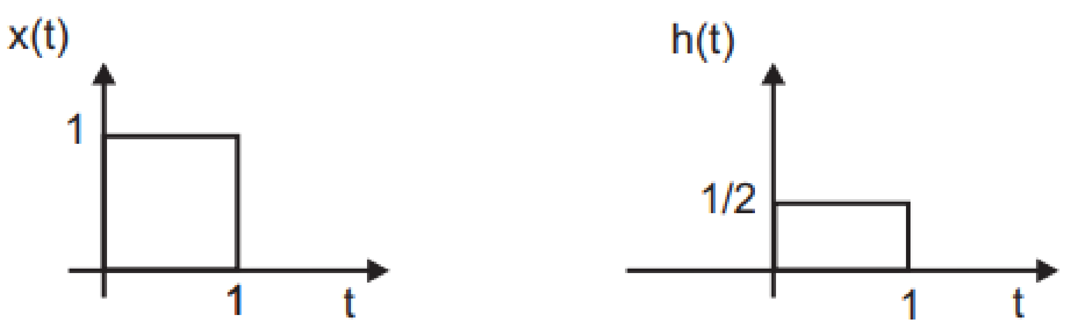
$x(t)$

$h(t)$

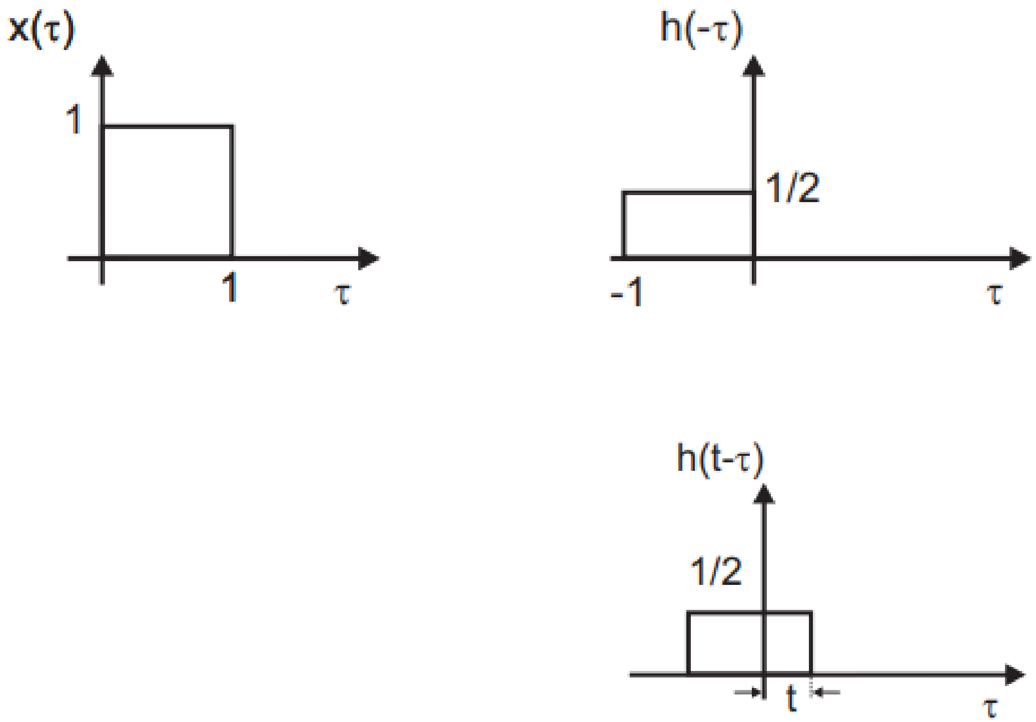
$\Rightarrow$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$y(t) = x(t) * h(t)$$

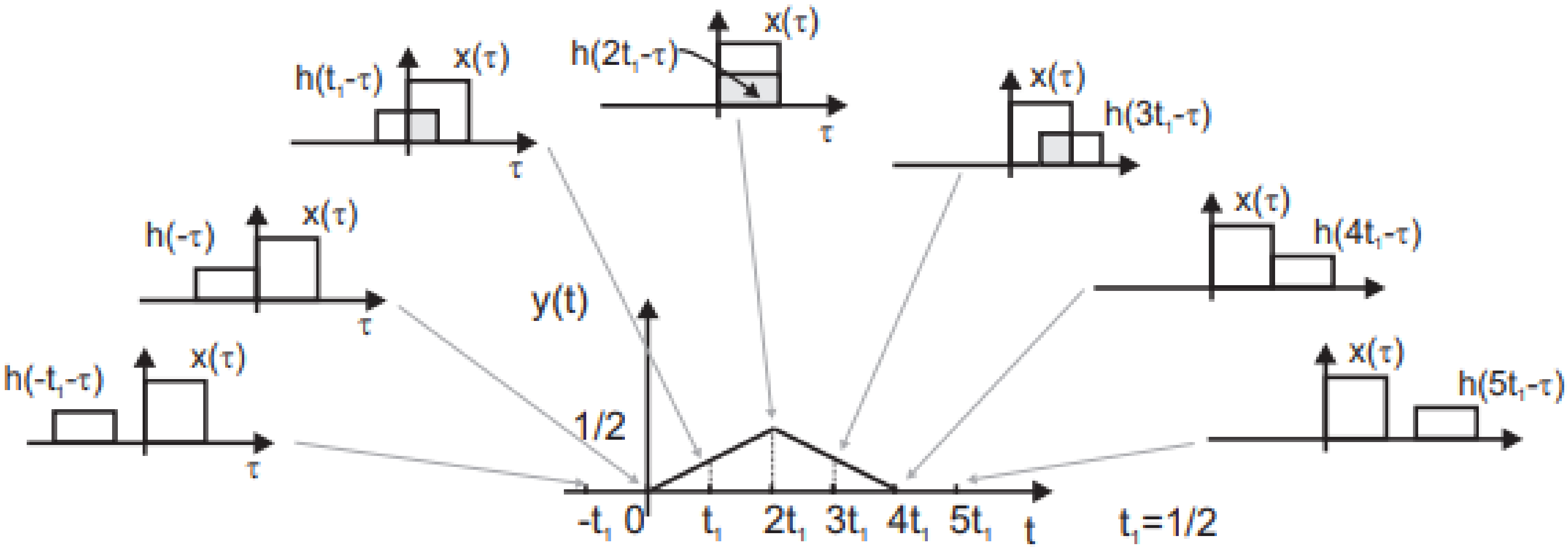


Convolution Integral



2. Mathematical Tools to Deal with Dynamic Actions

Convolution Integral



2. Mathematical Tools to Deal with Dynamic Actions

Convolution Integral

Convolution in the time domain corresponds with a multiplication in the frequency domain:

$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f) \cdot H(f)$$

Convolution in the frequency domain corresponds with a multiplication in the time domain:

$$Y(f) = X(f) * H(f)$$

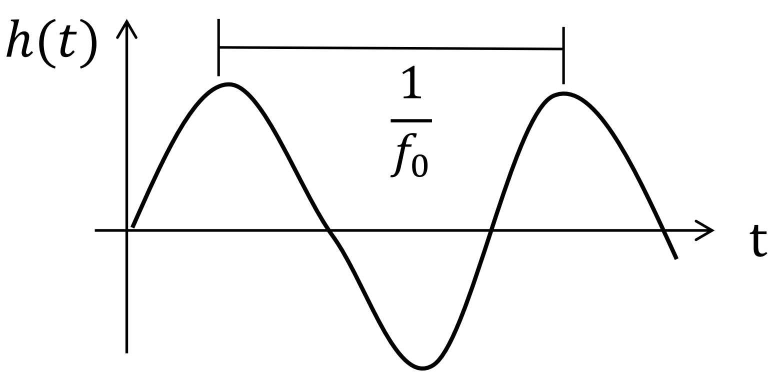
$$y(t) = x(t) \cdot h(t)$$

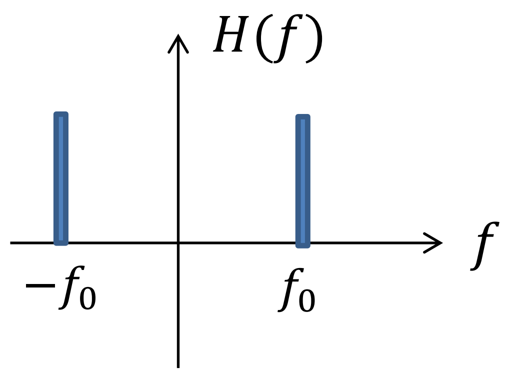
2. Mathematical Tools to Deal with Dynamic Actions

Theorem of Parseval

The energy of a signal computed in the time domain is equal to the energy computed in the frequency domain :

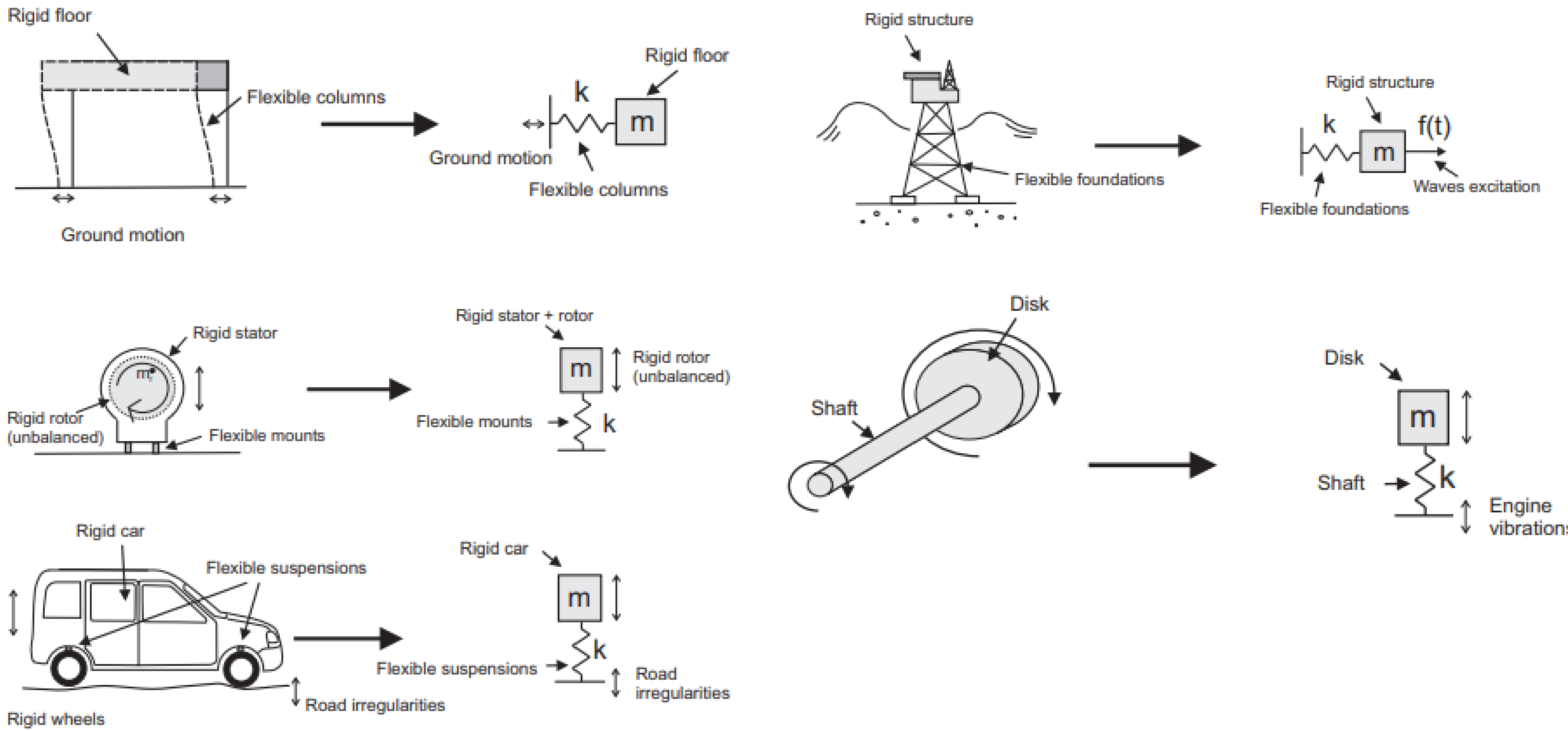
$$\int_{-\infty}^{\infty} h^2(t)dt = \int_{-\infty}^{\infty} |H(f)|^2 df$$





3. Single-Degree of Freedom System (SDOF)

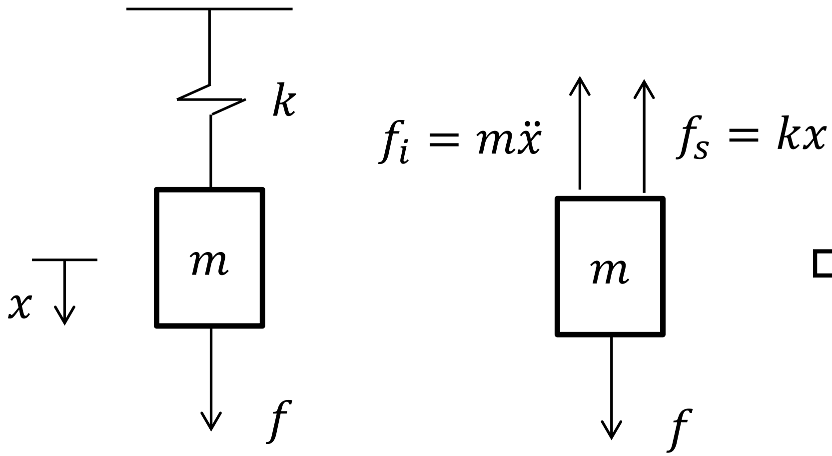
Some examples → Real Structures → SDOF Systems



3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

$$\boxed{m\ddot{x} = \sum F}$$



Time Domain

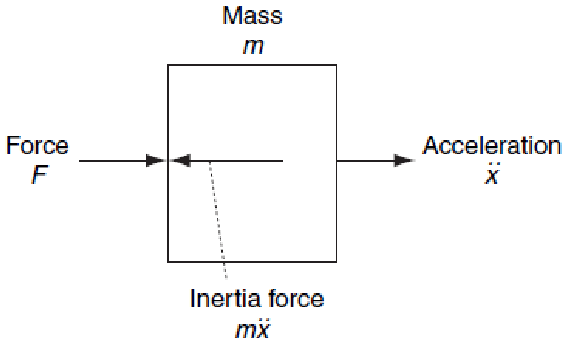
$$\boxed{m\ddot{x} + kx = f}$$

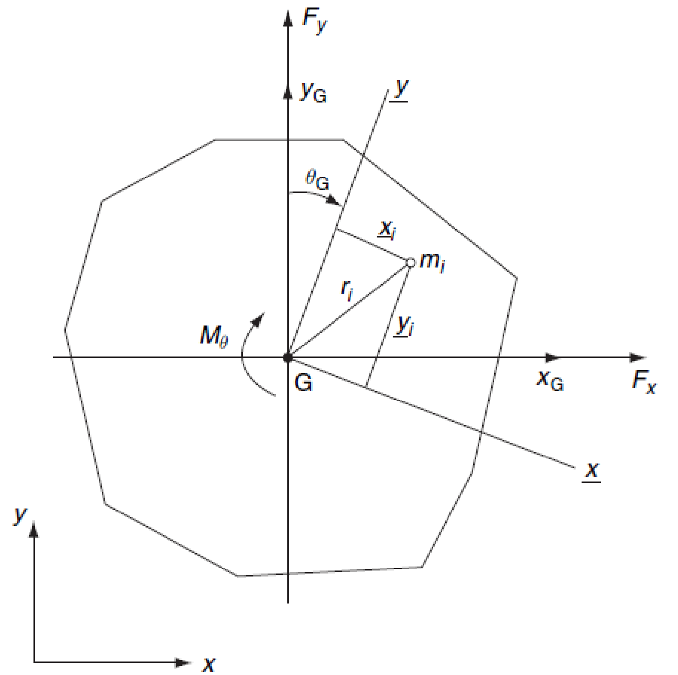
Frequency Domain

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Mass and Inertial





$$F_x + \sum_{i=1}^{n} -m_i\ddot{x}_i = 0; \qquad F_x = m\ddot{x}_G;$$

$$F_y + \sum_{i=1}^{n} -m_i\ddot{y}_i = 0 \qquad F_y = m\ddot{y}_G$$

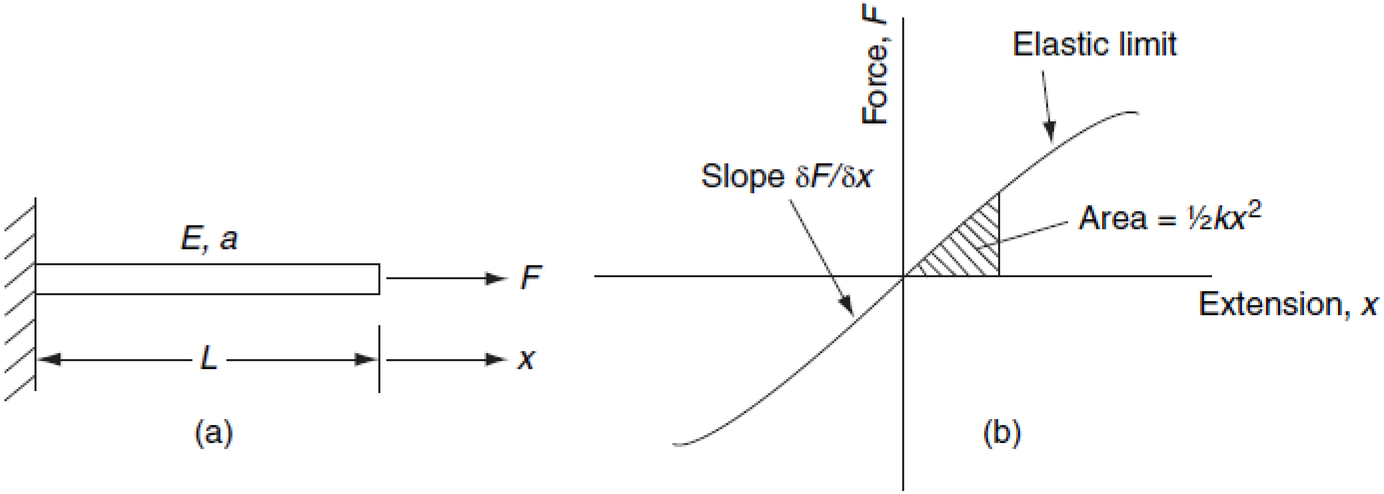
$$M_\theta + \sum_{i=1}^{n} (-m_i r_i^2 \ddot{\theta}_G) \qquad M_\theta = I_G\ddot{\theta}_G$$

$$I_G = \sum_{i=1}^{n} m_i \, r_i^2$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Stiffness

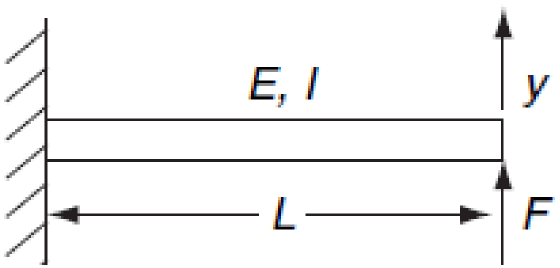


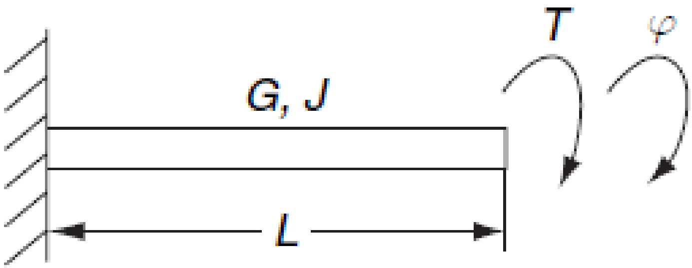
$$k = \frac{\delta F}{\delta x} = \frac{\delta \sigma}{\delta \varepsilon} \cdot \frac{a}{L} = \frac{Ea}{L}$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Stiffness





$$y = \frac{FL^3}{3EI}$$

$$\frac{T}{J} = \frac{G\varphi}{L}$$

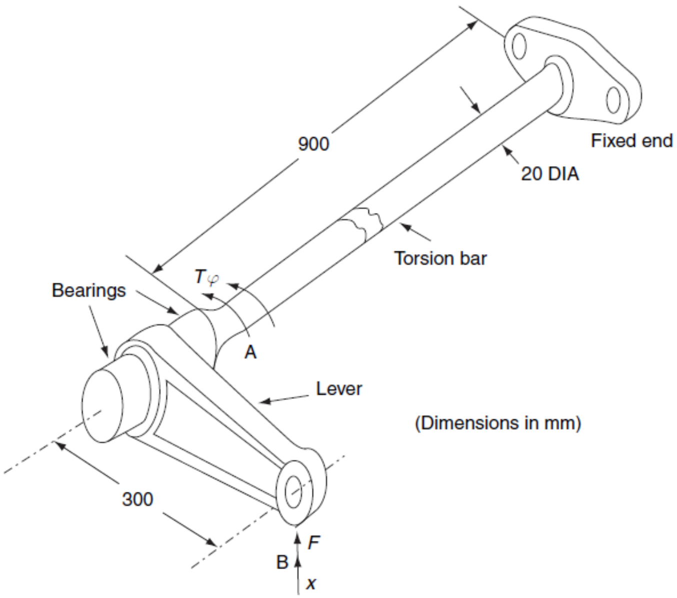
$$k_y = \frac{\delta F}{\delta y} = \frac{3EI}{L^3}$$

$$k_\varphi = \frac{\delta T}{\delta \varphi} = \frac{GJ}{L}$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Example → Equation of Motion



$$k_\varphi = \frac{\delta T}{\delta \varphi} = \frac{GJ}{L_1}$$

$$\delta T = \delta F \cdot L_2;$$

$$\delta \varphi = \delta x / L_2;$$

$$k_x = \frac{\delta F}{\delta x} = \frac{\delta T}{\delta \varphi} \cdot \frac{1}{L_2^2} = \frac{GJ}{L_1 L_2^2} = k_\varphi \cdot \frac{1}{L_2^2}$$

Numerically, $G = 90 \times 10^9$ N/m^2; $J = \pi d^4/32 = \pi(0.02)^4/32 = 15.7 \times 10^{-9}$ m^4; $L_1 = 0.90$ m; $L_2 = 0.30$ m giving $k_x = 17\,400$ N/m or 17.4 kN/m.

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

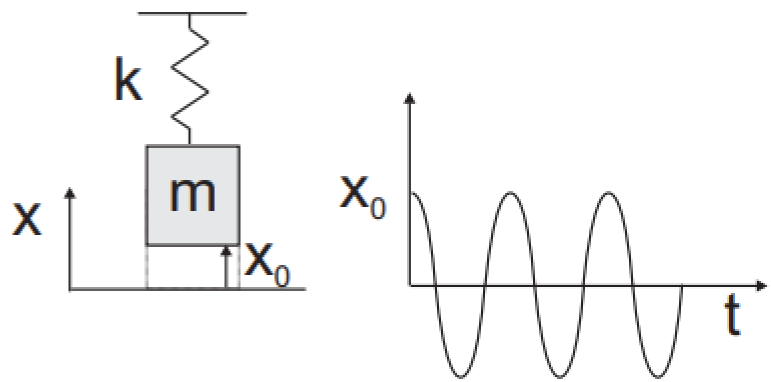
$$\boxed{m\ddot{x} + kx = f} \Rightarrow \boxed{x = x_h + x_p}$$

General Solution

$$m\ddot{x} + kx = 0$$

$$x_h = Ae^{rt}$$

$$x_h(t) = x_0 \cos(w_n t) + \frac{\dot{x}_0}{w_n} \sin(w_n t)$$



Natural Frequency $\quad w_n = \sqrt{k/m}$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Harmonic
Periodic
Random

$\Rightarrow \boxed{m\ddot{x} + kx = f} \Rightarrow \boxed{x = x_h + x_p}$

Particular Solution → CFT

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega}$$

$$F(\omega)e^{-i\omega t} = F(\omega)\cos(\omega t) + iF(\omega)\sin(\omega t)$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Particular Solution

$$F(\omega)e^{-i\omega t} = F(\omega)\cos(\omega t) + iF(\omega)\sin(\omega t)$$

$$\boxed{\begin{aligned} m\ddot{x} + kx &= F\cos(\omega t) \\ x &= a\cos(\omega t + \varphi) \end{aligned}}$$

$$\boxed{\begin{aligned} m\ddot{x} + kx &= F\sin(\omega t) \\ x &= a\sin(\omega t + \varphi) \end{aligned}}$$

$$\boxed{f = Fe^{i\omega t} \quad \Longrightarrow \quad x = Xe^{i\omega t}}$$

$$X = ae^{i\varphi}$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Particular Solution

$$m\ddot{x} + kx = f$$

$$x = Xe^{i\omega t}$$

$$f = Fe^{i\omega t}$$

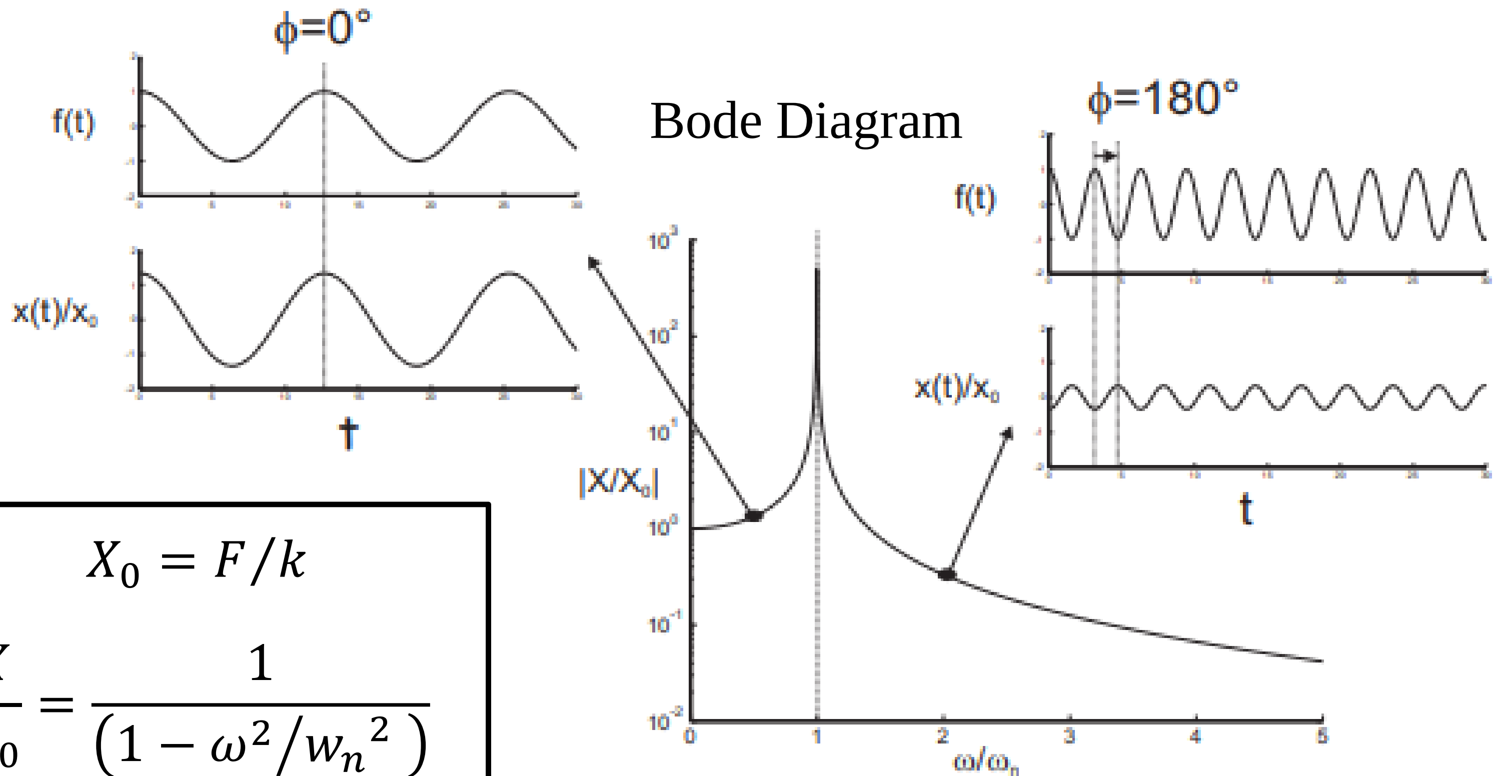
$$(k - \omega^2 m)Xe^{i\omega t} = Fe^{i\omega t}$$

$$X = \frac{1}{(k - \omega^2 m)} F$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Particular Solution



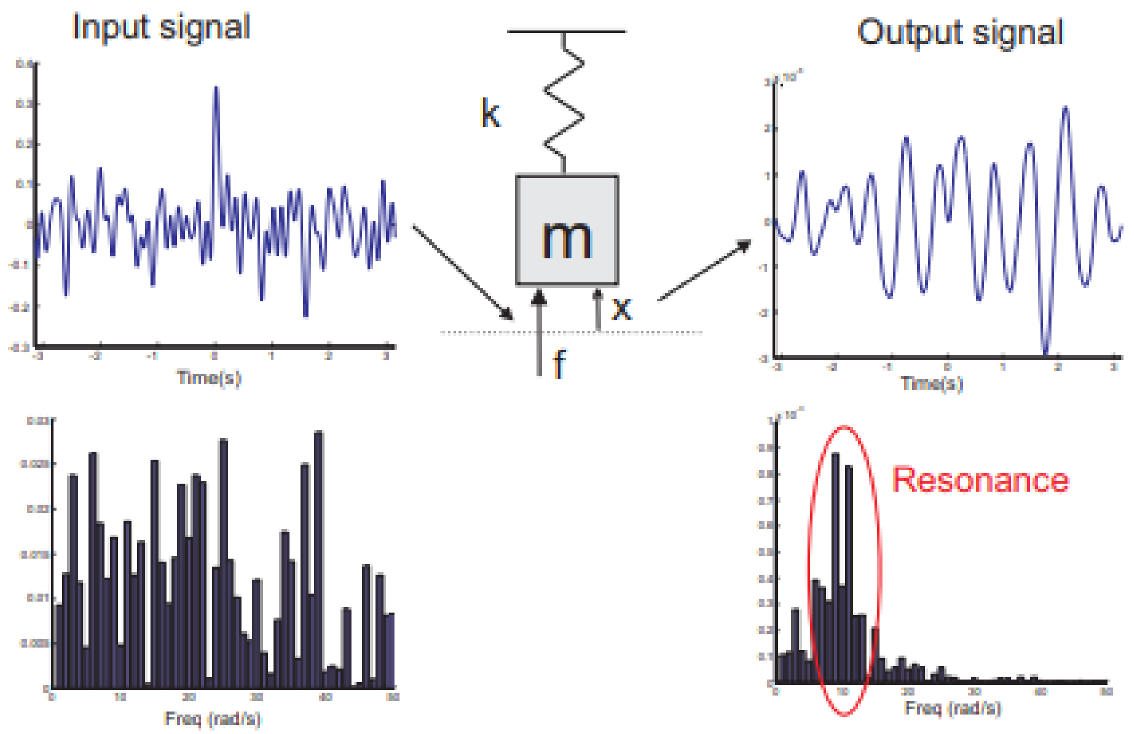
$$X_0 = F/k$$

$$\frac{X}{X_0} = \frac{1}{\left(1 - \omega^2/{w_n}^2\right)}$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Particular Solution



3. Single-Degree of Freedom System (SDOF)

SDOF Systems without damping

Particular Solution

$$\boxed{m\ddot{x} + kx = f}$$

Inverse CFT

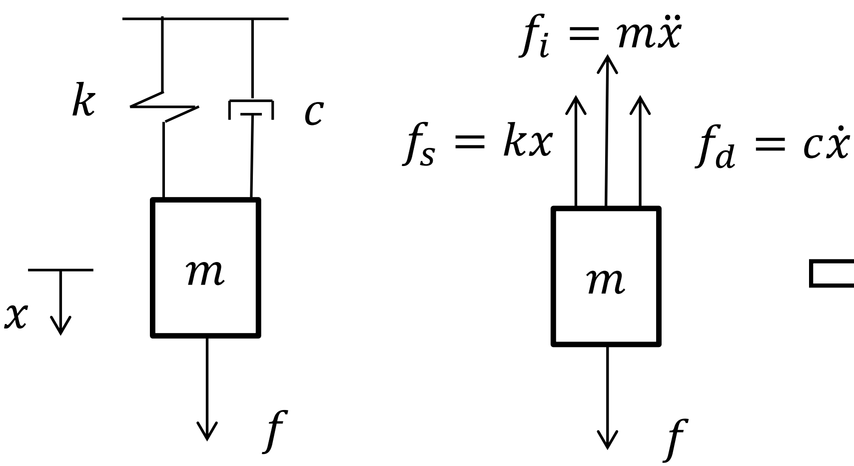
$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{-i\omega t} d\omega}$$

$$\boxed{X(\omega) = \frac{1}{(k - \omega^2 m)} F(\omega)}$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems with damping

$$\boxed{m\ddot{x} = \sum F}$$



Time Domain

$$\boxed{m\ddot{x} + c\dot{x} + kx = f}$$

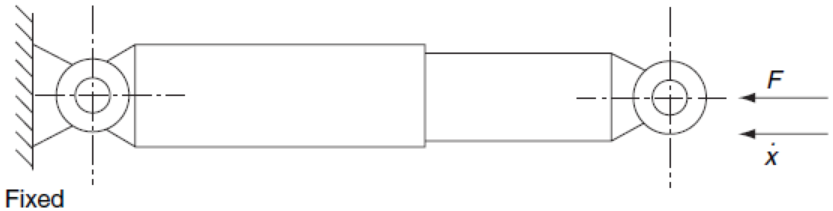
Frequency Domain

3. Single-Degree of Freedom System (SDOF)

SDOF Systems with damping

Damping

$$F = c\dot{x}$$



Example

the constant c is 1500 N/m/s. A test on the unit involves applying a single-peak sinusoidal force of ± 1000 N at each of the frequencies, $f = 1.0$, 2.0 and 5.0 Hz. Calculate the expected single-peak displacement, and total movement, at each of these frequencies.

$$F = 1000 \sin(2\pi f t) \qquad \dot{x} = \frac{F}{c} = \frac{1000 \sin\,(2\pi f t)}{1500}$$

$$x = -\frac{1000 \cos\,(2\pi f t)}{1500(2\pi f)} + x_0$$

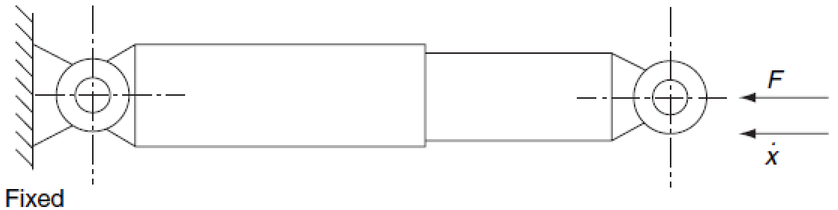
$$|x| = \frac{1000}{1500 \times 2\pi f} = \frac{0.106}{f}\ \mathrm{m}$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems with damping

Damping

$$F = c\dot{x}$$



Example

Frequency f (Hz)	Single-peak displacement $\lvert x\rvert$ (m)	Total movement of damper piston (m)
1	0.106	0.212
2	0.053	0.106
5	0.021	0.042

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Three Methods

- Newton's second law

$$m\ddot{x} + c\dot{x} + kx = f$$

$$I\ddot{\theta} + c_{\theta}\dot{\theta} + k_{\theta}\theta = M$$

- Rayleigh's method
- Principle of Virtual Work
- Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

$$i = 1, 2, 3, \ldots, n.$$

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Principle of Conservation of Energy

(a) Work is done when a force causes a displacement. If both are defined at the same point, and in the same direction, the work done is the product of the force and displacement, measured, for example, in newton-meters (or lbf. -ft). This assumes that the force remains constant. If it varies, the *power*, the instantaneous product of force and velocity, must be integrated with respect to time, to calculate the work done. If a moment acts on an angular displacement, the work done is still in the same units, since the angle is non-dimensional. It is therefore permissible to mix translational and rotational energy in the same expression.

(b) The *kinetic energy*, T, stored in an element of mass, m, is given by $T = \frac{1}{2}m\dot{x}^2$, where $\dot{x}$ is the velocity. By using the idea of a mass moment of inertia, I, the kinetic energy in a rotating body is given by $T = \frac{1}{2}I\dot{\theta}^2$, where $\dot{\theta}$ is the angular velocity of the body.

(c) The *potential energy*, U, stored in a spring, of stiffness k, is given by $U = \frac{1}{2}kx^2$, where x is the compression (or extension) of the spring, not necessarily the displacement at one end. In the case of a rotational spring, the potential energy is given by $U = \frac{1}{2}k_\theta\theta^2$, where k_θ is the angular stiffness, and θ is the angular displacement

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Rayleigh's method

It is applicable only to single-DOF systems, and permits the natural frequency to be found if the kinetic and potential energies in the system can be calculated. The motion at every point in the system (i.e. the mode shape in the case of continuous systems) must be known, or assumed. Since, in vibrating systems, the maximum kinetic energy in the mass elements is transferred into the same amount of potential energy in the spring elements, these can be equated, giving the natural frequency. It should be noted that the maximum kinetic energy does not occur at the same time as the maximum potential energy.

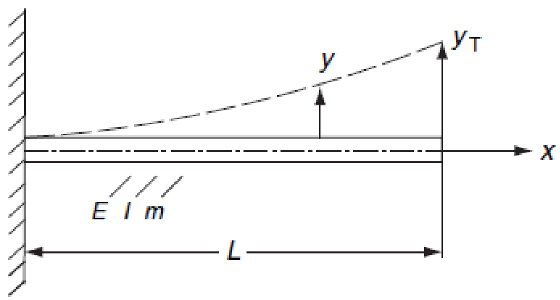
$$\boxed{T_{\max} = U_{\max}.}$$

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Rayleigh's method

Example



$$\frac{y}{y_T} = \left(\frac{x}{L}\right)^2$$

$$y = y_T \frac{x^2}{L^2} \sin \omega t \qquad \dot{y} = \omega_1 y_T \frac{x^2}{L^2} \cos \omega t \qquad T_{max} = \int_0^L \tfrac{1}{2} m \dot{y}^2 = \tfrac{1}{2} \frac{m}{L^4} \omega_1^2 y_T^2 \int_0^L x^4 \cdot dx = \tfrac{1}{10}\, m\, L\, \omega_1^2 y_T^2$$

$$\frac{dy}{dx} = y_T \frac{2x}{L^2} \sin \omega t \qquad \frac{d^2y}{dx^2} = \frac{2y_T}{L^2} \sin \omega t \qquad \left(\frac{d^2y}{dx^2}\right)_{max} = \frac{2y_T}{L^2} \qquad U = \tfrac{1}{2} \int_0^L EI \left(\frac{d^2y}{dx^2}\right)^2 \cdot dx$$

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Rayleigh's method

Example

$$U_{\max} = \frac{2EI \cdot y_T^2}{L^3} \qquad \frac{1}{10} m L \omega_1^2 y_T^2 = \frac{2EI \cdot y_T^2}{L^3}, \qquad \omega_1 = \frac{\sqrt{20}}{L^2}\sqrt{\frac{EI}{m}} = \frac{4.47}{L^2}\sqrt{\frac{EI}{m}}$$

The exact answer $\frac{3.52}{L^2}\sqrt{\frac{EI}{m}}$, so the Rayleigh method is somewhat inaccurate in this case. This was due to a poor choice of function for the assumed mode shape.

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Principle of Virtual Work

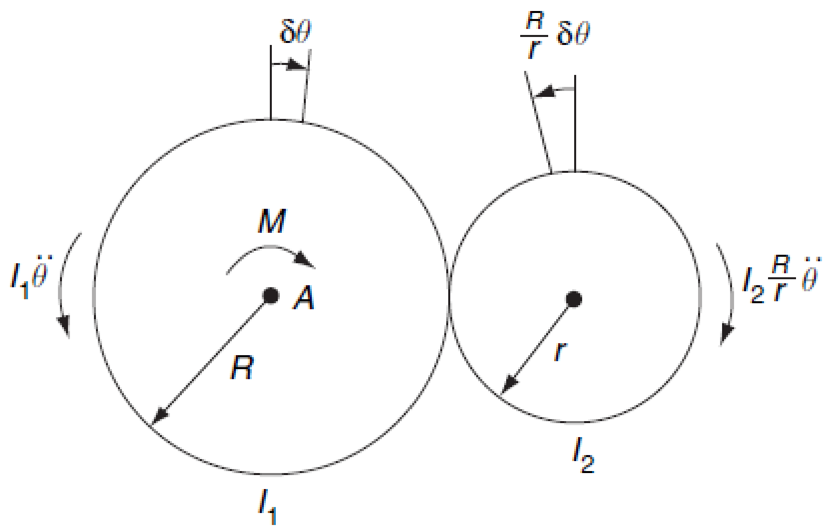
This states that in any system in equilibrium, the total work done by all the forces acting at one instant in time, when a small virtual displacement is applied to one of its freedoms, is equal to zero. The system being 'in equilibrium' does not necessarily mean that it is static, or that all forces are zero; it simply means that all forces are accounted for, and are in balance. Although the same result can sometimes be obtained by diligent application of Newton's second law, and D'Alembert's principle, the virtual work method is a useful time-saver, and less prone to errors, in the case of more complicated systems. The method is illustrated by the following examples.

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Principle of Virtual Work

Example



Left gear: $-I_1\ddot{\theta}$

Right gear: $+I_2\dfrac{R}{r}\ddot{\theta}$

$$M\cdot\delta\theta - I_1\ddot{\theta}\cdot\delta\theta - \frac{R}{r}\delta\theta\cdot I_2\frac{R}{r}\ddot{\theta} = 0$$

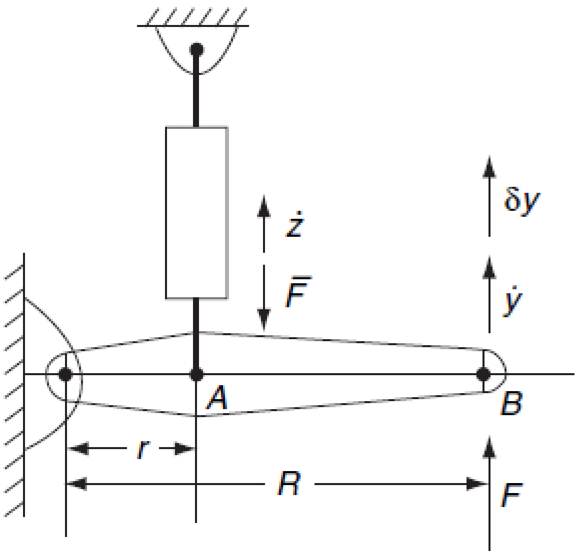
$$M = \left[I_1 + \left(\frac{R}{r}\right)^2 I_2\right]\ddot{\theta}$$

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Principle of Virtual Work

Example



$$\overline{F} = C\dot{z}^2 \operatorname{sgn}(\dot{z})$$

$$\delta y . F + \frac{r}{R}\delta y\left[-C\left(\frac{r}{R}\dot{y}\right)^2\right] = 0$$

$$F = \left(\frac{r}{R}\right)^3 C\dot{y}^2.$$

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Lagrange's equations

$$\boxed{\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}=Q_i\\ &i=1,2,3,\ldots,n.\end{aligned}}$$

For most structures, unless they are rotating, the kinetic energy, T, depends upon the generalized velocities, $\dot{q}_i$, but not upon the generalized displacements q_i, and the term $\partial T/\partial q_i$ can often be omitted.

. Thus, just as we have $T=\frac{1}{2}m\dot{x}^2$ for a single mass, and $U=\frac{1}{2}kx^2$ for a single spring, we can invent the function $D=\frac{1}{2}c\dot{x}^2$ for a single damper c, where c is defined by Eq. (1.34), i.e. $F=c\dot{x}$.

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

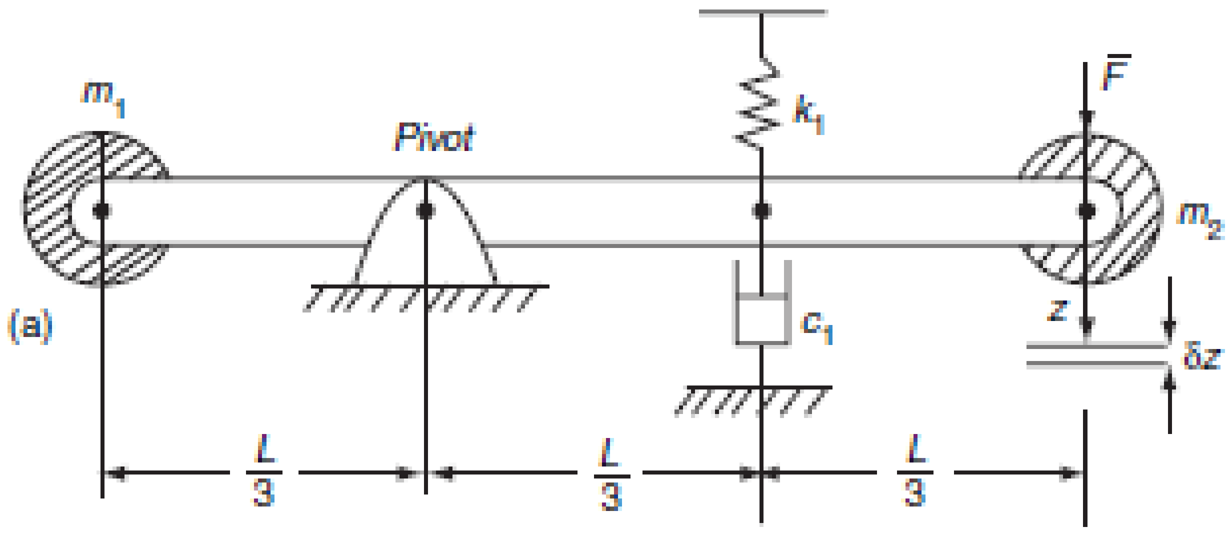
Lagrange's equations

With Damping
$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i, \quad i = 1, 2, 3, \ldots, n.$
Without Damping
$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad L = T - U$

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Example → Equation of Motion



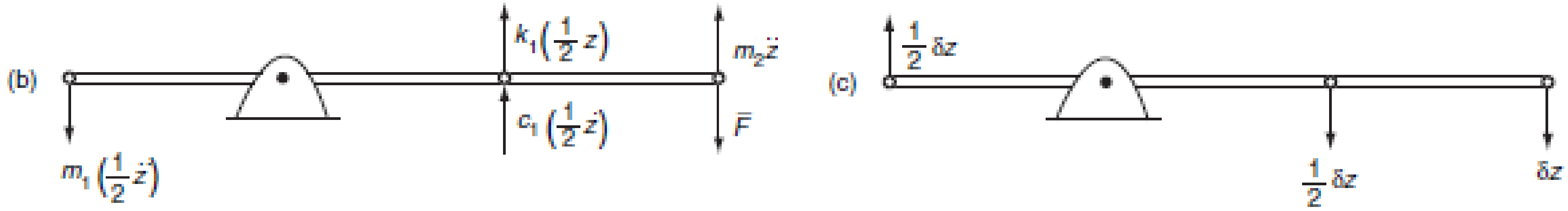
3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Example → Principle of Virtual Work

$$-m_1\left(\frac{1}{2}\ddot{z}\right)\left(\frac{1}{2}\delta z\right)-m_2\ddot{z}\cdot\delta z-k_1\left(\frac{1}{2}z\right)\left(\frac{1}{2}\delta z\right)-c_1\left(\frac{1}{2}\dot{z}\right)\left(\frac{1}{2}\delta z\right)+\overline{F}\cdot\delta z=0$$

$$\left(\frac{1}{4}m_1+m_2\right)\ddot{z}+\left(\frac{1}{4}c_1\right)\dot{z}+\left(\frac{1}{4}k_1\right)z=\overline{F}$$



3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Example → Lagrange's Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right)+\frac{\partial U}{\partial q}+\frac{\partial D}{\partial \dot{q}}=Q$$

$$T=\frac{1}{2}m_1\left(\frac{\frac{1}{3}L}{\frac{2}{3}L}\dot{z}\right)^2+\frac{1}{2}m_2\dot{z}^2=\frac{1}{2}m_1\left(\frac{1}{2}\dot{z}\right)^2+\frac{1}{2}m_2\dot{z}^2$$

$$U=\frac{1}{2}k_1\left(\frac{\frac{1}{3}L}{\frac{2}{3}L}z\right)^2=\frac{1}{2}k_1\left(\frac{1}{2}z\right)^2$$

$$D=\frac{1}{2}c_1\left(\frac{\frac{1}{3}L}{\frac{2}{3}L}\dot{z}\right)^2=\frac{1}{2}c_1\left(\frac{1}{2}\dot{z}\right)^2$$

3. Single-Degree of Freedom System (SDOF)

Setting up the differential equation of motion

Example → Lagrange's Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}}\right) = \left(\frac{1}{4}m_1 + m_2\right)\ddot{z}$$

$$\frac{\partial U}{\partial q} = \frac{\partial U}{\partial z} = \frac{1}{4}k_1 z$$

$$\frac{\partial D}{\partial \dot{q}} = \frac{\partial D}{\partial \dot{z}} = \frac{1}{4}c_1 z$$

$$Q = \overline{F}$$

$$\boxed{\left(\frac{1}{4}m_1 + m_2\right)\ddot{z} + \left(\frac{1}{4}c_1\right)\dot{z} + \left(\frac{1}{4}k_1\right)z = \overline{F}}$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems with damping

$$\boxed{m\ddot{x} + c\dot{x} + kx = f} \Rightarrow \boxed{x = x_h + x_p}$$

General Solution

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x_h = Ae^{rt}$$

Different solutions in terms of ζ

$$r^2 + 2\zeta w_n r + {w_n}^2 = 0$$

$$r = -\zeta \pm i w_n\sqrt{1-\zeta^2} = -\zeta \pm i w_d$$ Damping frequency

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2mw_n}$$ Damping ratio

3. Single-Degree of Freedom System (SDOF)

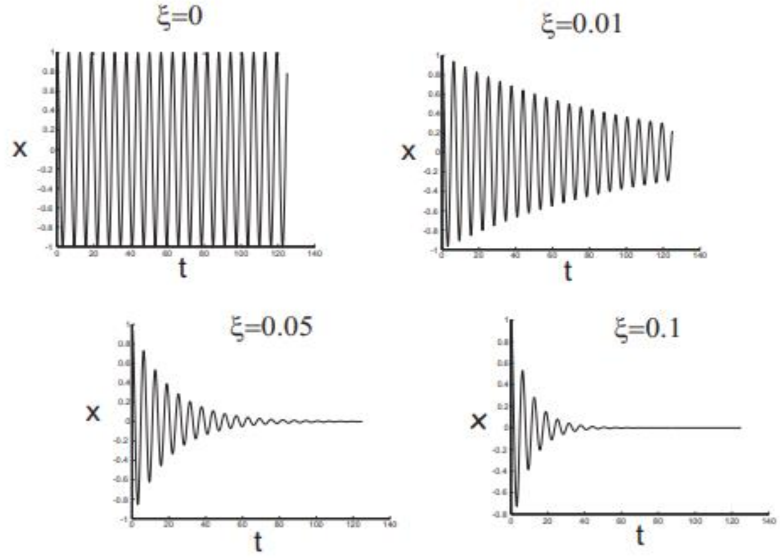
SDOF Systems with damping

General Solution

$\boldsymbol{\zeta < 1}$

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0} \Rightarrow \boxed{x = x_h}$$

$$x_h(t) = e^{-\zeta w_n t}\left(x_0 \cos(w_d t) + \frac{\dot{x}_0 + w_n \zeta x_0}{w_d} \sin(w_d t)\right)$$



3. Single-Degree of Freedom System (SDOF)

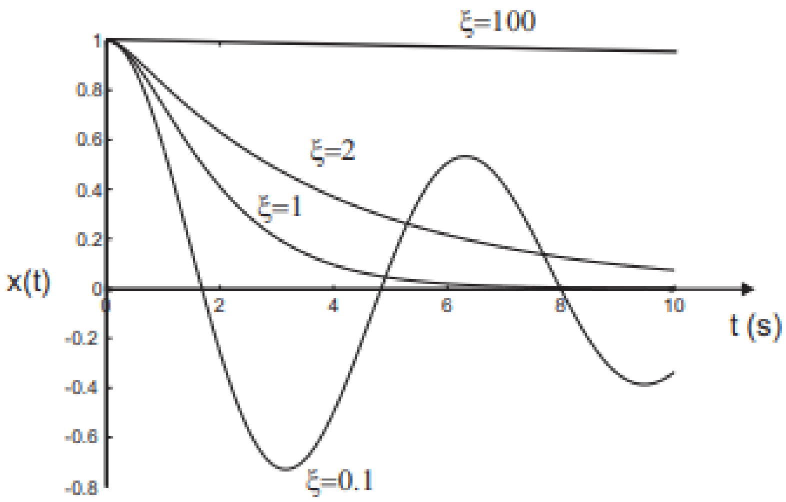
SDOF Systems with damping

General Solution

$\boldsymbol{\zeta > 1}$

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0} \Rightarrow \boxed{x = x_h}$$

$$x_h(t) = e^{-\zeta w_n t}\left(x_0 \cosh(\mu t) + \frac{\dot{x}_0 + w_n \zeta x_0}{\mu} \sinh(\mu t)\right) \qquad \mu = w_n\sqrt{1-\zeta}$$



3. Single-Degree of Freedom System (SDOF)

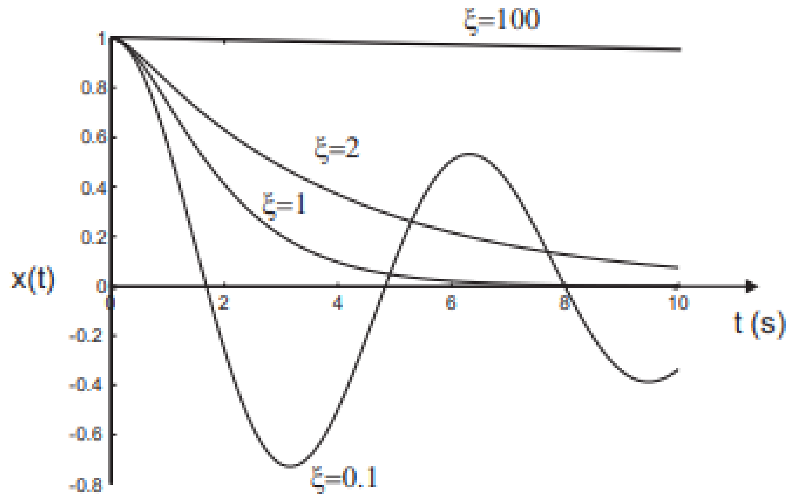
SDOF Systems with damping

General Solution

$\boldsymbol{\zeta = 1}$

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0} \Longrightarrow \boxed{x = x_h}$$

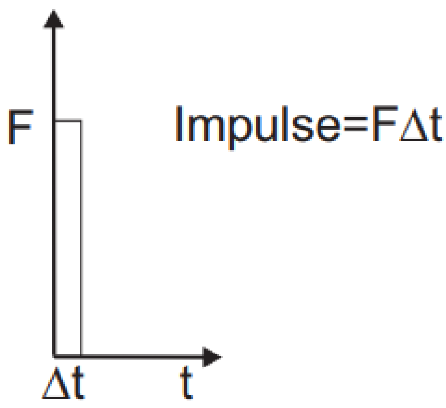
$$x_h(t) = e^{-\zeta w_n t}((\dot{x}_0 + w_n x_0)t + x_0)$$



3. Single-Degree of Freedom System (SDOF)

SDOF Systems with damping

Impulse Response



$$m\dot{x}_0\Big|_{\Delta t} = F\Delta t - \int_0^{\Delta t} kx\,dt - \int_0^{\Delta t} c\dot{x}\,dt$$

$$\dot{x}_0\Big|_{\Delta t} = \frac{F\Delta t}{m}$$

$$x_h(t) = \frac{e^{-\zeta w_n t}}{m w_d}\sin(w_d t)$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems with damping

Particular Solution → CFT

$$m\ddot{x} + c\dot{x} + kx = f$$

$$x = Xe^{i\omega t}$$

$$f = Fe^{i\omega t}$$

$$(k - \omega^2 m + i\omega c)Xe^{i\omega t} = Fe^{i\omega t}$$

$$X = \frac{1}{(k - \omega^2 m + i\omega c)}F$$

3. Single-Degree of Freedom System (SDOF)

SDOF Systems with damping

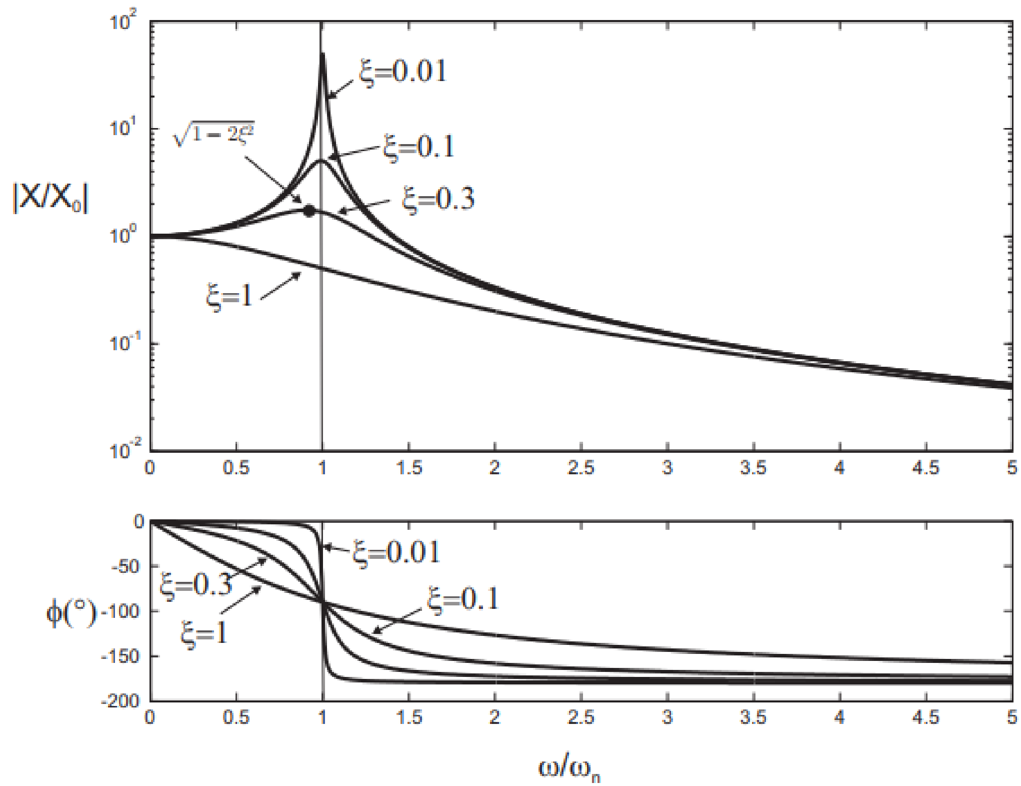
Particular Solution

$$X = \frac{1}{(k - \omega^2 m + i\omega c)} F$$

$$\Downarrow$$

$$\frac{X}{X_0} = \frac{1}{\left(1 - \frac{\omega^2}{{w_n}^2} + 2i\zeta \frac{\omega}{w_n}\right)}$$

Bode Diagram

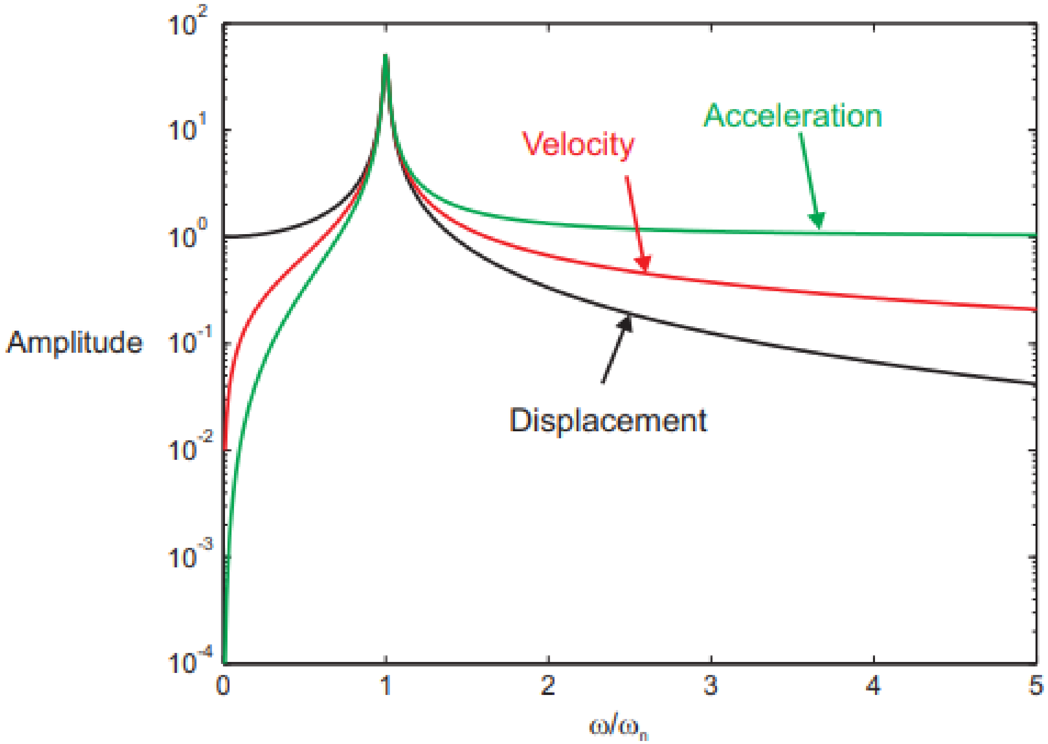


3. Single-Degree of Freedom System (SDOF)

SDOF Systems with damping

Displacement	X
Velocity	$i\omega X$
Acceleration	$-\omega^2 X$

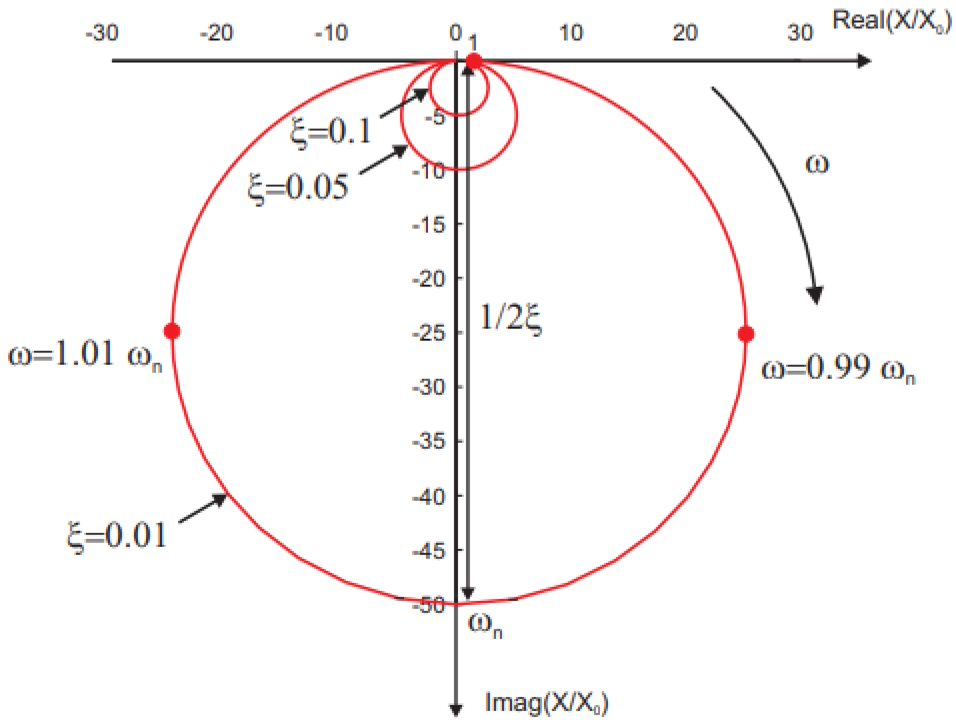
Bode Diagram



3. Single-Degree of Freedom System (SDOF)

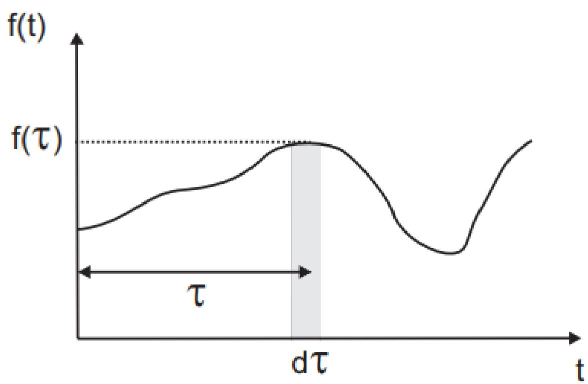
SDOF Systems with damping

Nyquist Diagram



3. Single-Degree of Freedom System (SDOF)

Duhamel's Integral



$$f(\tau)d\tau h(t-\tau) \longleftarrow \text{Impulse Response}$$

$$x(t) = \int_0^t f(\tau)d\tau h(t-\tau)d\tau$$

$$f(t) = 0 \qquad h(t) = 0 \qquad t > 0$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} f(\tau)d\tau h(t-\tau)d\tau = f(t) * h(t)}$$

$$\boxed{f(t) = Fe^{i\omega t}} \longrightarrow x(t) = Xe^{i\omega t} = \int_{-\infty}^{\infty} Fe^{i\omega t} h(t-\tau)d\tau = \int_{-\infty}^{\infty} Fe^{i\omega(t-\tau)} h(\tau)d\tau$$

$$x(t) = Fe^{i\omega t}\int_{-\infty}^{\infty} e^{-i\omega\tau} h(\tau)d\tau = Fe^{i\omega t} H(\omega)$$

3. Single-Degree of Freedom System (SDOF)

Duhamel's Integral

Particular Solution

Transfer Function

$$\boxed{H(\omega) = \frac{X}{F}}$$

Frequency Response Function

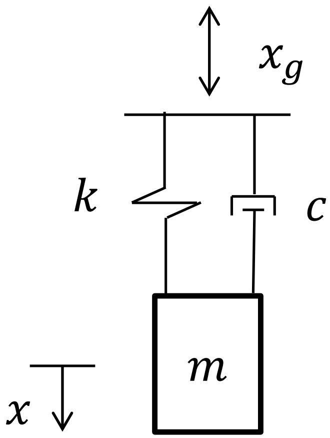
Inverse CFT

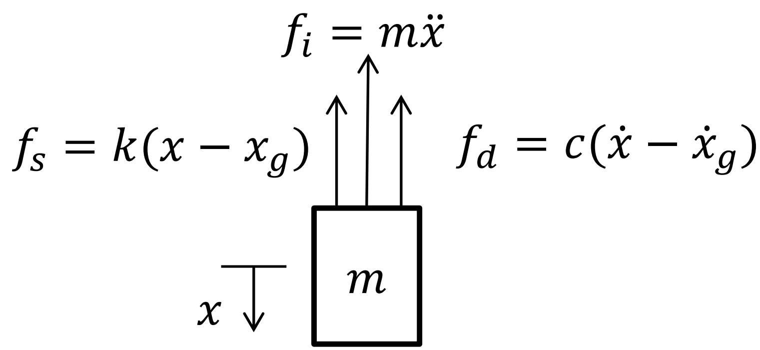
$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{-i\omega t}d\omega}$$

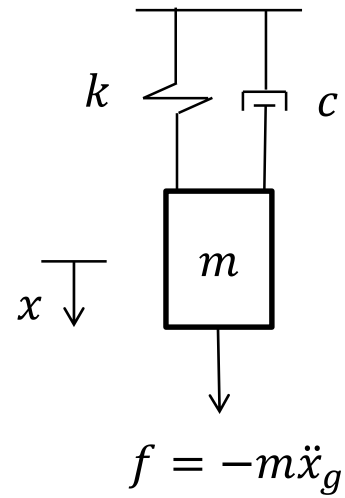
$$\boxed{X(\omega) = H(\omega)F(\omega)}$$

3. Single-Degree of Freedom System (SDOF)

Base Excitation







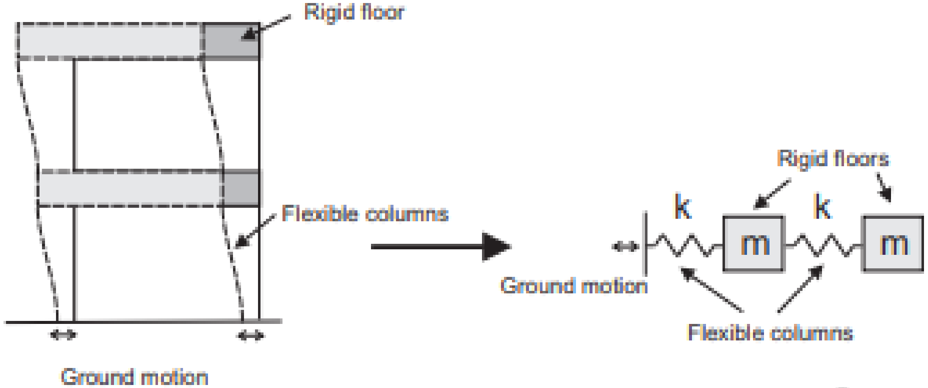
$$m\ddot{x} = -k\left(x - x_g\right) - c(\dot{x} - \dot{x}_g)$$

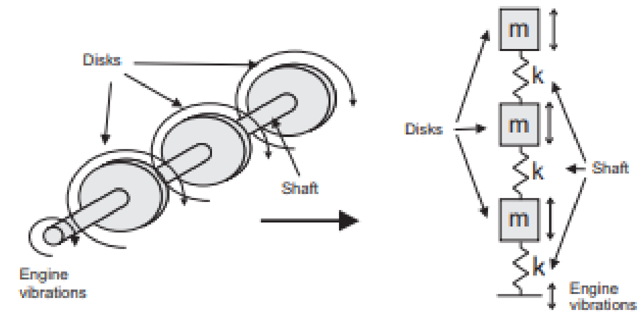
$$x_r = x - x_g$$

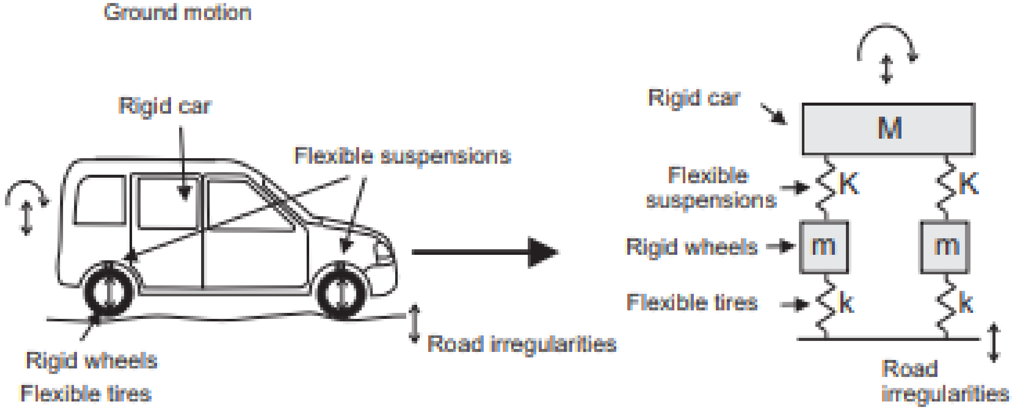
$$\boxed{m\ddot{x}_r + c\dot{x}_r + kx_r = -\mathrm{m}\ddot{x}_g}$$

4. Multiple-Degree of Freedom Systems (MDOF)

Some examples → Real Structures → MDOF Systems

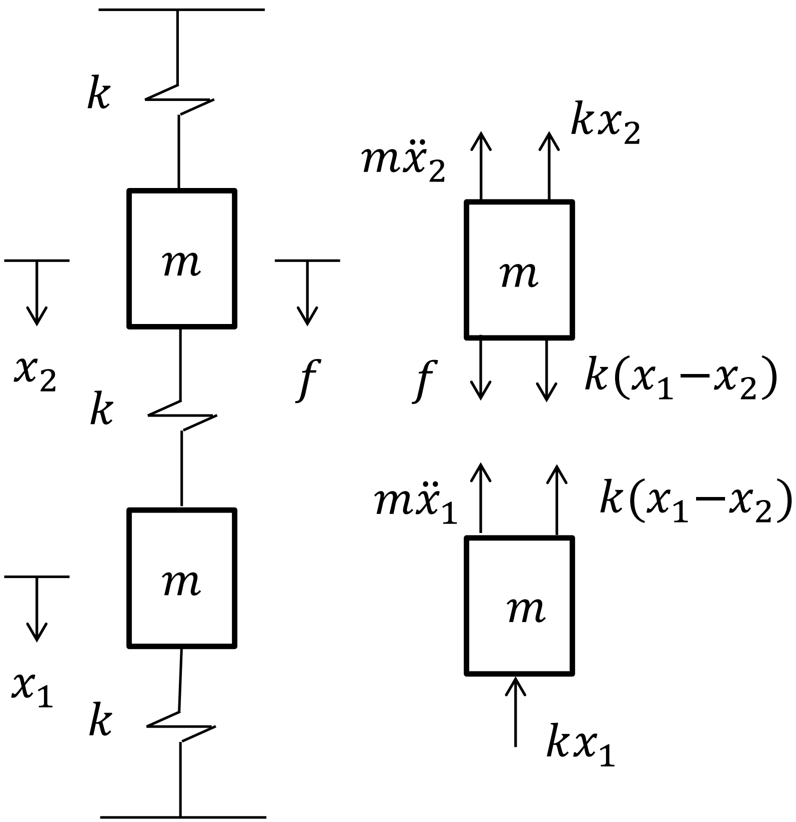






4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems without damping



$$m\ddot{x}_2 = k(x_1 - x_2) - kx_2 + f$$

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$\boxed{[M]\{\ddot{x}\} + [K]\{x\} = \{f\}}$$

4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems without damping

General Solution

$$\boxed{[M]\{\ddot{x}\} + [K]\{x\} = \{0\}}$$

$$\{x\} = \{\psi\}e^{rt}$$

$$([K] + r^2[M])\{\psi\} = \{0\}$$

$$\det([K] + r^2[M]) = 0 \qquad r^2 = -\omega^2$$

Eigenvalue Problem

$$([K] - \omega^2[M])\{\psi\} = \{0\}$$

Natural Frequencies

Mode Shapes

$$\boxed{\{x(t)\} = \sum_{i=1}^{n} (Z_{i1}\cos(\omega_i t) + Z_{i2}\sin(\omega_i t))\psi_i}$$

4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems without damping

Properties of Mode Shapes

Mode Shapes are Orthogonal $[\psi] = [\psi_1 \quad \dots \quad \psi_n]$

$$\boxed{\begin{aligned} [\psi]^T\,[M][\psi] &= diag(\mu_i) \\ [\psi]^T\,[K][\psi] &= diag(\mu_i {\omega_i}^2) \end{aligned}}$$

4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems without damping

Particular Solution $$\{x(t)\} = \sum_{i=1}^{n} z_i(t)\psi_i \qquad x = [\psi]\{z\}$$

$$[\psi]^T[M][\psi]\{\ddot{z}\} + [\psi]^T[K][\psi]\{z\} = [\psi]^T\{f\}$$

$$\boxed{\mu_i\ddot{z}_i + \mu_i{\omega_i}^2 z_i = F_i}$$

$$\boxed{\begin{array}{c} x(t) = \{X\}e^{i\omega t} \\ f(t) = \{F\}e^{i\omega t} \end{array}} \Rightarrow \boxed{([K] - w^2\{M\})\{X\} = \{F\}}$$

Inverting Matrix

Modal Decomposition

4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems without damping

Particular Solution $$X(\omega) = \sum_{i=1}^{n} Z_i(\omega)\psi_i \qquad \{X\} = [\psi]\{Z\}$$

$$([\psi]^T[K][\psi] - \omega^2[\psi]^T[M][\psi])\{Z\} = [\psi]^T\{F\}$$

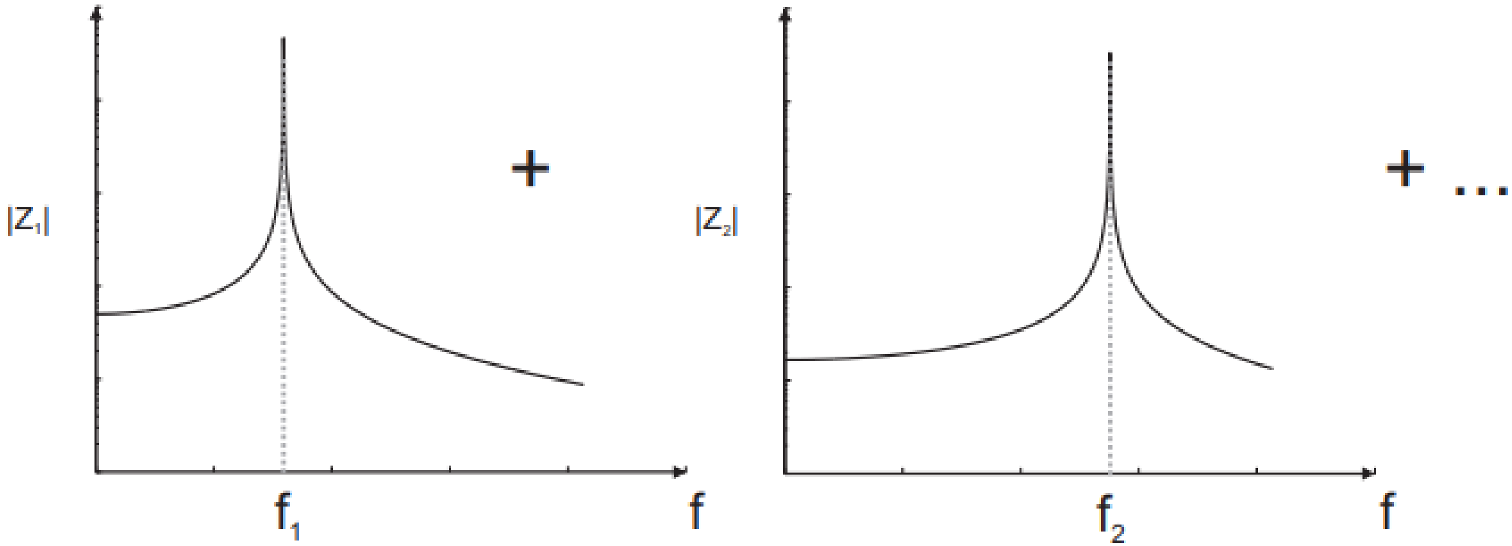
$$Z_j(\omega) = \psi_j{}^T\{F\}\frac{1}{\mu_j(\omega_j{}^2 - \omega)}$$

$$\boxed{\{X(\omega)\} = \sum_{j=1}^{n} Z_j(\omega)\psi_j}$$

4. Multiple-Degree of Freedom Systems (MDOF)

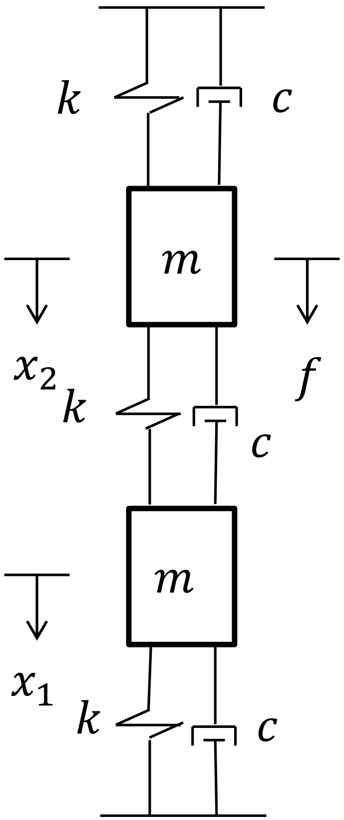
MDOF Systems without damping

Particular Solution



4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems with damping



Equation of motion on each mass

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

$$\boxed{[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f\}}$$

4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems with damping

General Solution

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\}$$

$$\{x\} = \{\psi\}e^{rt}$$

$$([K] + r[C] + r^2[M])\{\psi\} = \{0\}$$

$$\det([K] + r[C] + r^2[M]) = 0$$

Complex eigenvalues and eigenvectors
Not common in structural dynamics

4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems with damping

Particular Solution $\{x(t)\} = \sum_{i=1}^{n} z_i(t)\psi_i \qquad x = [\psi]\{z\}$

$$[\psi]^T[M][\psi]\{\ddot{z}\} + [\psi]^T[C][\psi]\{\dot{z}\} + [\psi]^T[K][\psi]\{z\} = [\psi]^T\{f\}$$

$\downarrow$ Non-diagonal

$[\mathrm{C}] = \alpha[K] + \beta[M]$ Rayleigh

$C_i = 2\mu_i\zeta_i\omega_i$ Modal

$$\boxed{\mu_i\ddot{z}_i + 2\mu_i\zeta_i\omega_i\dot{z}_i + \mu_i{\omega_i}^2 z_i = F_i}$$

4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems with damping

Particular Solution

$$\mu_i \ddot{z}_i + 2\mu_i \zeta_i \omega_i \dot{z}_i + \mu_i {\omega_i}^2 z_i = F_i$$

$$\boxed{\begin{aligned} x(t) &= \{X\} e^{i\omega t} \\ f(t) &= \{F\} e^{i\omega t} \end{aligned}} \Rightarrow \boxed{([K] + i\omega[C] - w^2\{M\})\{X\} = \{F\}}$$

Inverting Matrix

Modal Decomposition

$$X(\omega) = \sum_{i=1}^{n} Z_i(\omega)\psi_i \qquad \{X\} = [\psi]\{Z\}$$

$$([\psi]^T[K][\psi] + i\omega[\psi]^T[C][\psi] - \omega^2[\psi]^T[M][\psi])\{Z\} = [\psi]^T\{F\}$$

4. Multiple-Degree of Freedom Systems (MDOF)

MDOF Systems with damping

Particular Solution

$$\mu_i({\omega_j}^2 - \omega^2 + 2i\zeta_j\omega\omega_j)\, Z_j = F_j$$

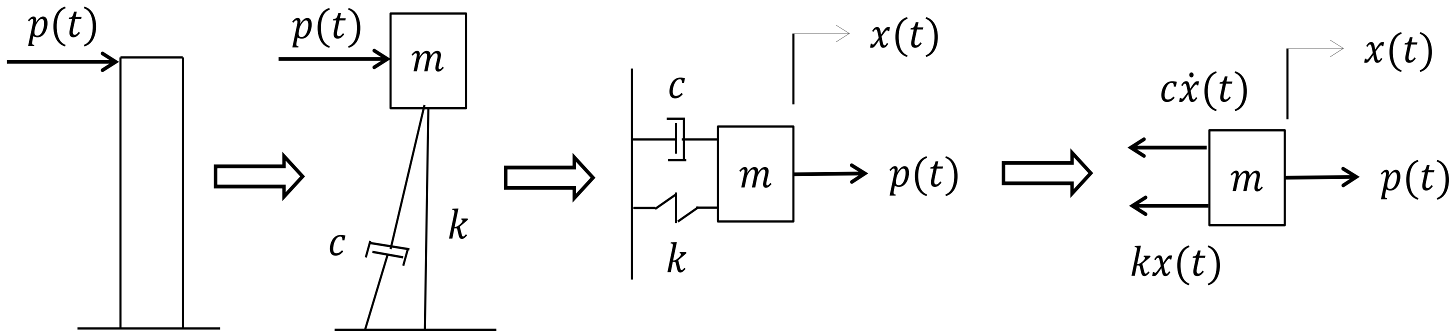
$$Z_j(\omega) = {\psi_j}^T\{F\}\frac{1}{\mu_j({\omega_j}^2 - \omega + 2i\zeta_j\omega\omega_j)}$$

$$\boxed{\{X(\omega)\} = \sum_{j=1}^{n} Z_j(\omega)\psi_j}$$

Introduction to Structural Dynamics for Aircraft Structures

5. Modelling and Analysis in Time Domain

Dynamic Systems in Nodal Coordinates



$$\sum f = ma = m\frac{d^2x(t)}{dt^2}$$

$$\sum f_x = p(t) - c\dot{x}(t) - kx(t) = m\ddot{x}(\mathrm{t})$$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = p(t)$$

Estructuras Aeronáuticas
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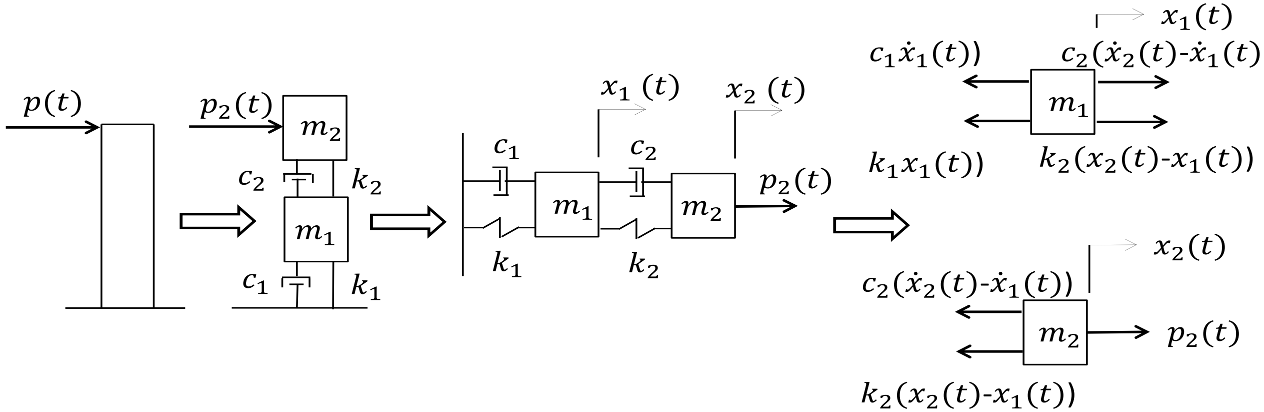
5. Modelling and Analysis in Time Domain

Dynamic Systems in Nodal Coordinates



$$m_1\ddot{x}_1(t) + c_1\dot{x}_1(t) + k_1x_1(t) - \left(c_2(\dot{x}_2(t) - \dot{x}_1(t))\right) - (k_2(x_2(t) - x_1(t))) = 0$$

$$m_2\ddot{x}_2(t) + \left(c_2(\dot{x}_2(t) - \dot{x}_1(t))\right) + (k_2(x_2(t) - x_1(t))) = p(i)$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_2(t) \end{Bmatrix}$$

$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = \{p(t)\}$$

5. Modelling and Analysis in Time Domain

Dynamic Systems in Modal Coordinates

$$[M]\{\ddot{x}(t)\} + [K]\{x(t)\} = \{0\} \Longleftarrow \{x(t)\} = \{\phi\}e^{jwt}$$

$$\left([K] - w^2[M]\right)\{\phi\}e^{j\omega t} = \{0\}$$

$$\boxed{det([K] - w^2[M]) = 0}$$

Natural Frequencies

$$[\Omega] = \begin{bmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & w_{n_f} \end{bmatrix}$$

Vibration Modes

$$[\Phi] = \begin{bmatrix} \phi_{11} & \phi_{21} & \dots & \phi_{n_f 1} \\ \phi_{12} & \phi_{22} & \dots & \phi_{n_f 2} \\ \dots & \dots & \dots & \dots \\ \phi_{1n_g} & \phi_{2n_g} & \dots & \phi_{n_f n_g} \end{bmatrix}$$

5. Modelling and Analysis in Time Domain

Dynamic Systems in Modal Coordinates

Diagonal Matrices

$$[\mathrm{M}_m] = [\Phi]^T[M][\Phi]$$

$$[\mathrm{K}_m] = [\Phi]^T[K][\Phi]$$

Non-diagonal Matrix $[C_m] = [\Phi]^T[C][\Phi]$

Rayleigh Damping Matrix $[C] = \alpha_1[K] + \alpha_2[M]$

Nodal Coordinates ⇨ $\boxed{\{x(t)\} = [\Phi]\{x_m(t)\}}$ ⇦ Modal Coordinates

5. Modelling and Analysis in Time Domain

Dynamic Systems in Modal Coordinates

$$[\Phi]^T[M][\Phi]\{\ddot{x}_m(t)\} + [\Phi]^T[C][\Phi]\{\dot{x}_m(t)\} + [\Phi]^T[K][\Phi]\{x_m(t)\} = [\Phi]^T\{p(t)\}$$

$$[M_m]\{\ddot{x}_m(t)\} + [C_m]\{\dot{x}_m(t)\} + [K_m]\{x_m(t)\} = [\Phi]^T\{p(t)\}$$

$$\{\ddot{x}_m(t)\} + [M_m]^{-1}[C_m]\{\dot{x}_m(t)\} + [M_m]^{-1}[K_m]\{x_m(t)\} = [M_m]^{-1}[\Phi]^T\{p(t)\}$$

$$\{\ddot{x}_m(t)\} + 2[\mathrm{Z}][\Omega]\{\dot{x}_m(t)\} + [\Omega]^2\{x_m(t)\} = [M_m]^{-1}[\Phi]^T\{p(t)\}$$

$[\Omega]^2 = [M_m]^{-1}[K_m]$ Natural Frequencies

$$[\mathrm{Z}] = \begin{bmatrix} \zeta_1 & 0 & \ldots & 0 \\ 0 & \zeta_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & \zeta_{n_f} \end{bmatrix}$$ Damping Ratio

5. Modelling and Analysis in Time Domain

Dynamic Systems in Modal Coordinates

$$[M_m]^{-1}[C_m] = 2[\mathrm{Z}][\Omega]$$

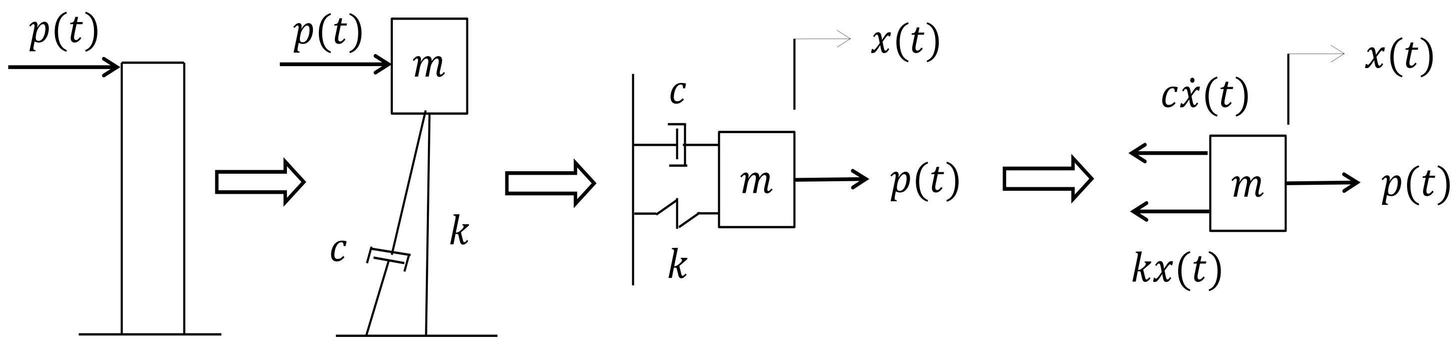
$$[Z] = 0.5[M_m]^{-1}[C_m][\Omega]^{-1} = 0.5[M_m]^{-1}[K_m]^{-1}[C_m]$$

$$[B_m] = [M_m]^{-1}[\Phi]^T\{B_0\}$$ Load Position

$$\boxed{\ddot{x}_{mi}(t) + 2\zeta_i w_i \dot{x}_m(t) + w_i^2 x_m(t) = b_{mi} p(t)}$$

5. Modelling and Analysis in Time Domain

Nodal Coordinates to State Space



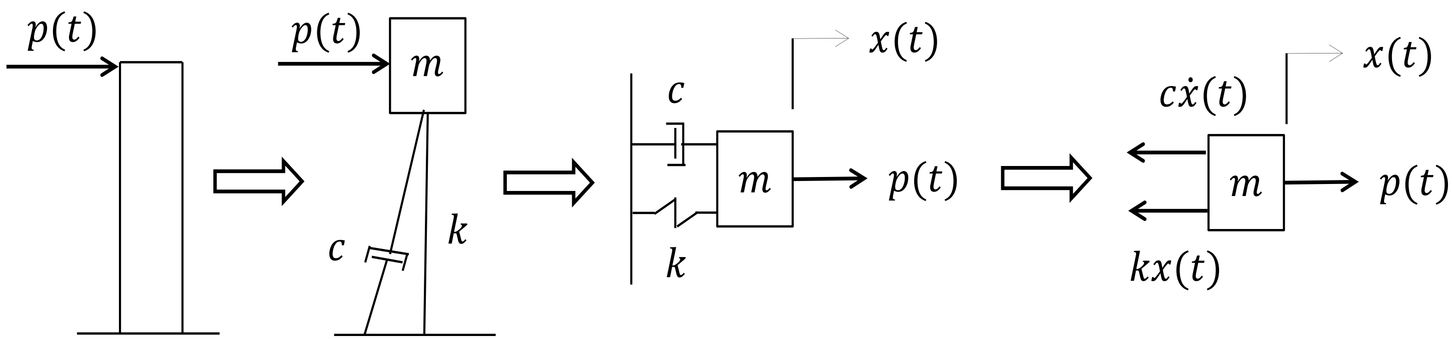
$$\{z(t)\} = \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} \Rightarrow$$

$$\{\dot{z}(t)\} = \begin{Bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{m} \end{Bmatrix} p(t)$$

$$\{y(t)\} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix}$$

5. Modelling and Analysis in Time Domain

Nodal Coordinates to State Space



$$\{\dot{z}(t)\} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{m} \end{Bmatrix} p(t)$$

$$\{y(t)\} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix}$$

$\Rightarrow$

$$\{\dot{z}(t)\} = [A]\{z(t)\} + [B]\{u(t)\}$$

$$\{y(t)\} = [E]\{z(t)\} + [D]\{u(t)\}$$

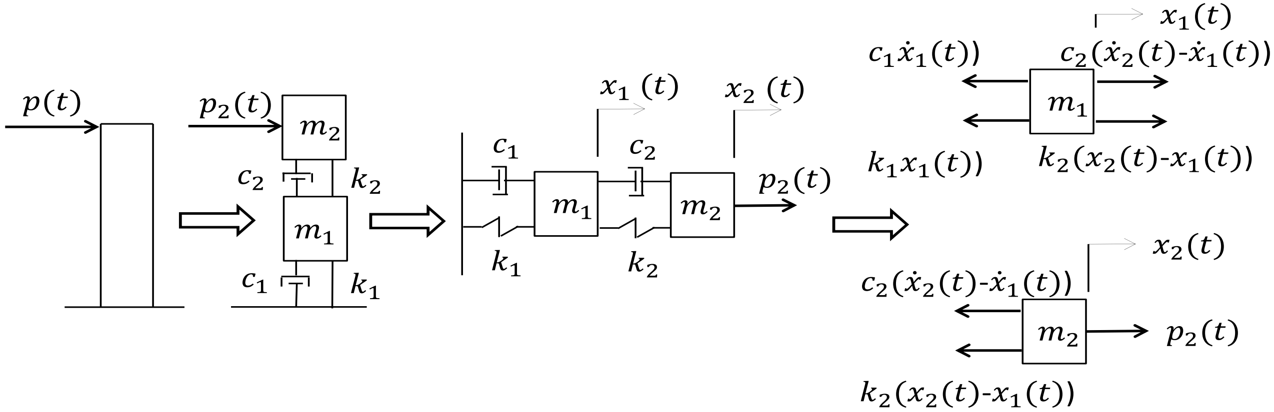
$$[A] = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \quad [B] = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}$$

$$[E] = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad [D] = [0]$$

Introduction to Structural Dynamics for Aircraft Structures

5. Modelling and Analysis in Time Domain

Nodal Coordinates to State Space



$$\{\ddot{x}(t)\} + [M]^{-1}[C]\{\dot{x}(t)\} + [M]^{-1}[K]\{x(t)\} = [M]^{-1}[B_0]\{u(t)\}$$

$$\{y(t)\} = [E_d]\{x(t)\} + [E_v]\{\dot{x}(t)\}$$

$$\Downarrow$$

$$\{\dot{z}(t)\} = [A]\{z(t)\} + [B]\{u(t)\}$$

$$\{y(t)\} = [E]\{z(t)\} + [D]\{u(t)\}$$

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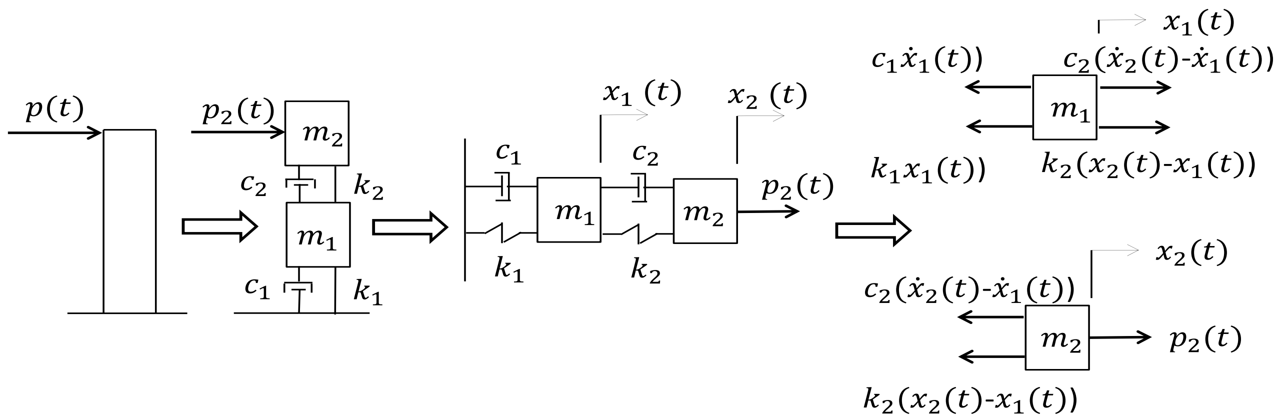
5. Modelling and Analysis in Time Domain

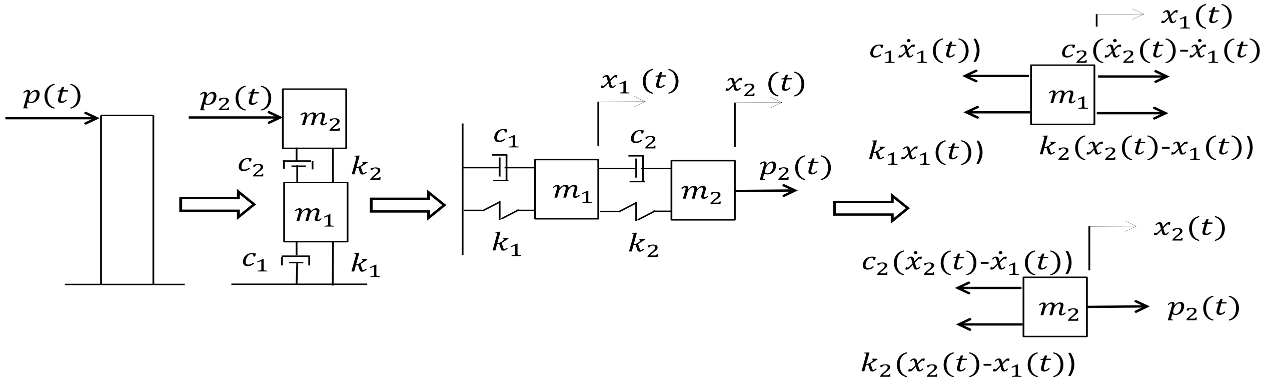
Nodal Coordinates to State Space



$$\{z(t)\} = \begin{Bmatrix} z_1(t) \\ z_2(t) \end{Bmatrix} = \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} \quad \left\{ \begin{aligned} &\dot{z}_1(t) = z_2(t) \\ &\dot{z}_2(t) = -[M]^{-1}[K]z_1(t) - [M]^{-1}[C]z_2(t) + [M]^{-1}[B_0]u(t) \\ &\{y(t)\} = [C_{0d}]z_1(t) + [C_{0v}]z_2(t) \end{aligned} \right.$$

5. Modelling and Analysis in Time Domain

Nodal Coordinates to State Space



$$\{\dot{z}(t)\} = [A]\{z(t)\} + [B]\{u(t)\}$$
$$\{y(t)\} = [E]\{z(t)\} + [D]\{u(t)\}$$

$$\Rightarrow$$

$$[A] = \begin{bmatrix} 0 & I \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix}$$
$$[B] = \begin{bmatrix} 0 \\ -[M]^{-1}[B_0] \end{bmatrix}$$
$$[E] = \begin{bmatrix} E_{0d} & E_{0v} \end{bmatrix}$$
$$[D] = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

5. Modelling and Analysis in Time Domain

Nodal Coordinates to State Space

$$\boxed{\{y(t)\} = [E]\{\ddot{x}(t)\}}$$

Output Accelerations

$$\{\dot{z}(t)\} = \begin{Bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{Bmatrix} \Rightarrow \{y(t)\} = [0 \quad [E_a]]\{\dot{z}(t)\}$$

$$\{y(t)\} = [0 \quad [E_a]][A]\{z(t)\} + [0 \quad [E_a]][B]\{u(t)\}$$

$$\{y(t)\} = [-[E_a][M]^{-1}[K] \quad -[E_a][M]^{-1}[C]]\{z(t)\} + [E_a][M]^{-1}[B_0]\{u(t)\}$$

$$\{y(t)\} = [E]\{z(t)\} + [D]\{u(t)\}$$

$$[E] = -[E_a][[M]^{-1}[K] \quad [M]^{-1}[C]]$$

$$[D] = [E_a][M]^{-1}[B_0]$$

5. Modelling and Analysis in Time Domain

Matlab Implementation. Time Domain Analysis

eig

Valores propios y vectores propios

Sintaxis

```
e = eig(A)
[V,D] = eig(A)
[V,D,W] = eig(A)
```

```
e = eig(A,B)
[V,D] = eig(A,B)
[V,D,W] = eig(A,B)
```

```
[ __ ] = eig(A,balanceOption)
[ __ ] = eig(A,B,algorithm)
```

```
[ __ ] = eig( __ ,eigvalOption)
```

5. Modelling and Analysis in Time Domain

Matlab Implementation. Time Domain Analysis

ss

Create state-space model, convert to state-space model

Syntax

```
sys = ss(A,B,C,D)
sys = ss(A,B,C,D,Ts)
sys = ss(D)
sys = ss(A,B,C,D,ltisys)
sys_ss = ss(sys)
sys_ss = ss(sys,'minimal')
sys_ss = ss(sys,'explicit')
sys_ss = ss(sys, 'measured')
sys_ss = ss(sys, 'noise')
sys_ss = ss(sys, 'augmented')
```

5. Modelling and Analysis in Time Domain

Matlab Implementation. Time Domain Analysis

lsim

Simulate time response of dynamic system to arbitrary inputs

Syntax

```
lsim(sys,u,t)
lsim(sys,u,t,x0)
lsim(sys,u,t,x0,method)
lsim(sys1,...,sysn,u,t)
lsim(sys1,LineSpec1,...,sysN,LineSpecN,u,t)
y = lsim( __ )
[y,t,x] = lsim( __ )
lsim(sys)
```

5. Modelling and Analysis in Time Domain

Matlab Implementation. Time Domain Analysis

ode45

Resolver ecuaciones diferenciales no rígidas: método de orden medio

Sintaxis

```
[t,y] = ode45(odefun,tspan,y0)
[t,y] = ode45(odefun,tspan,y0,options)
[t,y,te,ye,ie] = ode45(odefun,tspan,y0,options)
sol = ode45( __ )
```

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