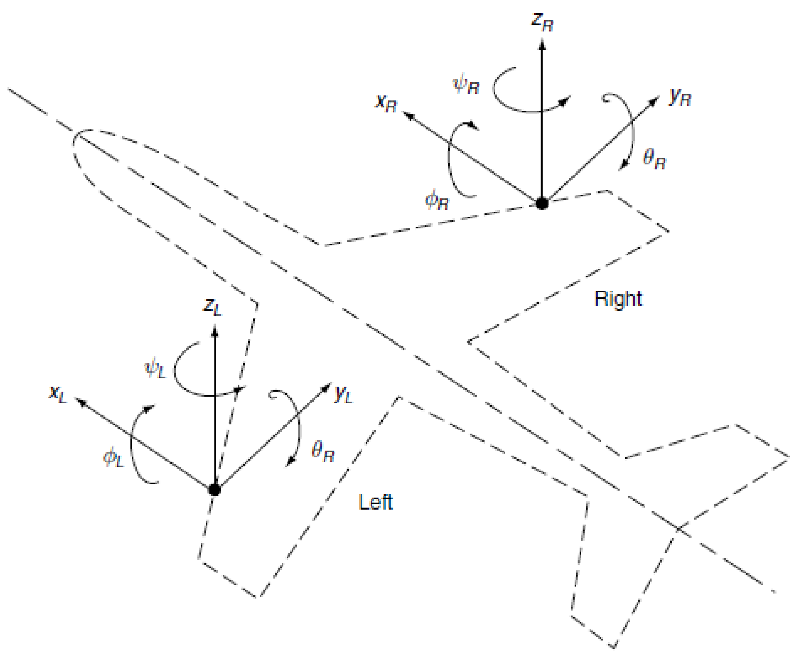
ETS de Ingeniería. Universidad de Sevilla

Examples 2: Structural Dynamics for Aircraft Structures

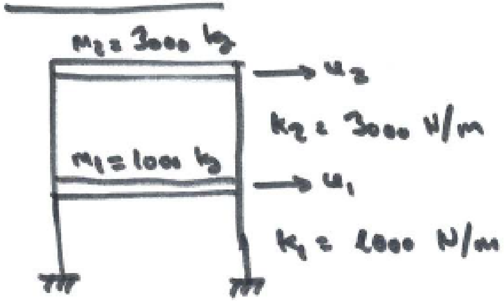


Estructuras Aeronáuticas
Grado en Ingeniería Aeroespacial

Summary

Example 1

EXAMPLE 1



$$[M] = \begin{bmatrix} 1000 & 0 \\ 0 & 3000 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 3000 & -2000 \\ -2000 & 2000 \end{bmatrix}$$

MATLAB $\Rightarrow$ eig(K, M)

$\Downarrow$

$\omega_i^2 \ \{d_i\}$

$\downarrow$

ω_i

1) ω_i and $\{d_i\}$

$$\det\left([K] - \omega^2 [M]\right) = 0$$

$$\det \begin{bmatrix} 3000 - 1000\omega^2 & -2000 \\ -2000 & 2000 - 3000\omega^2 \end{bmatrix} = 1000\left[(3-\omega^2)(2-3\omega^2) - 4\right] = 0$$

$$= 3\omega^4 - 11\omega^2 + 2 = 0$$

$\omega_1^2 = 0'192 \rightarrow \omega_1 = 0'435$ rad/s

$\omega_2^2 = 3'474 \rightarrow \omega_2 = 1'864$ rad/s

Example 1

EXAMPLE 1

$\omega_1 = 0'438$ rad/s

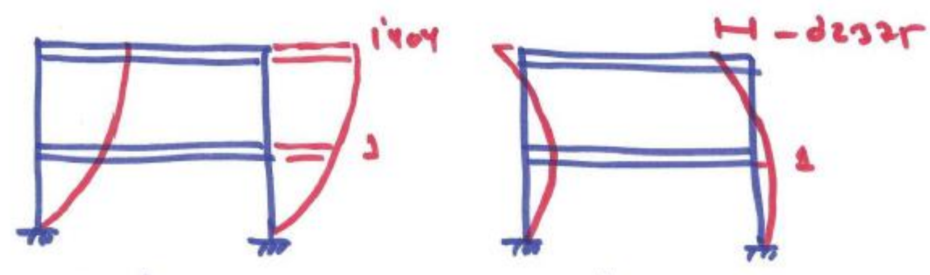
$$\begin{bmatrix} 3000 - 0'192\,1000 & -2000 \\ -2000 & 2000 - 3000\,0'192 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 2808 & -2000 \\ -2000 & 1424 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \phi_1 = \begin{Bmatrix} 1 \\ 1'404 \end{Bmatrix}$$

ASSUMING

$\omega_2 = 1'864$ rad/s

$$\begin{bmatrix} 3000 - 3'478\,1000 & -2000 \\ -2000 & 2000 - 3000\,3'478 \end{bmatrix} \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} -478 & -2000 \\ -2000 & -8425 \end{bmatrix} \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \phi_2 = \begin{Bmatrix} 1 \\ -0'2375 \end{Bmatrix}$$

ASSUMING



$\omega_1 = 0'438$ rad/s $\quad\quad \omega_2 = 1'864$ rad/s

Example 1

EXAMPLE 1

2.) SOLUTION $u(t)$

$$u(t) = \sum_{i=1}^{2} \{\phi_i\} \, \text{sen}(\omega_i t + \theta_i) \Rightarrow u(t) = \begin{Bmatrix} 1 \\ 1{,}404 \end{Bmatrix} \text{sen}(0'438\,t + \theta_1) + \begin{Bmatrix} 1 \\ -0'2375 \end{Bmatrix} \text{sen}(1'864\,t + \theta_2)$$

FIND $\theta_i \rightarrow$ BOUNDARY CONDITIONS $u(0)$, $\dot{u}(0)$

MODAL PROPERTIES

$$[\Phi] = \begin{bmatrix} 1 & 1 \\ 1'404 & -0'2375 \end{bmatrix} \qquad [\Omega^2] = \begin{bmatrix} 0'192 & 0 \\ 0 & 3{,}475 \end{bmatrix}$$

$$[\Phi]^T[M][\Phi] = \begin{bmatrix} 1 & 1 \\ 1'404 & -0'2375 \end{bmatrix}^T \begin{bmatrix} 1000 & 0 \\ 0 & 3000 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1'404 & -0'2375 \end{bmatrix}$$

$$= \begin{bmatrix} 6913'6 & 0 \\ 0 & 1169'2 \end{bmatrix} \quad \text{DECOUPLED}$$

Example 1

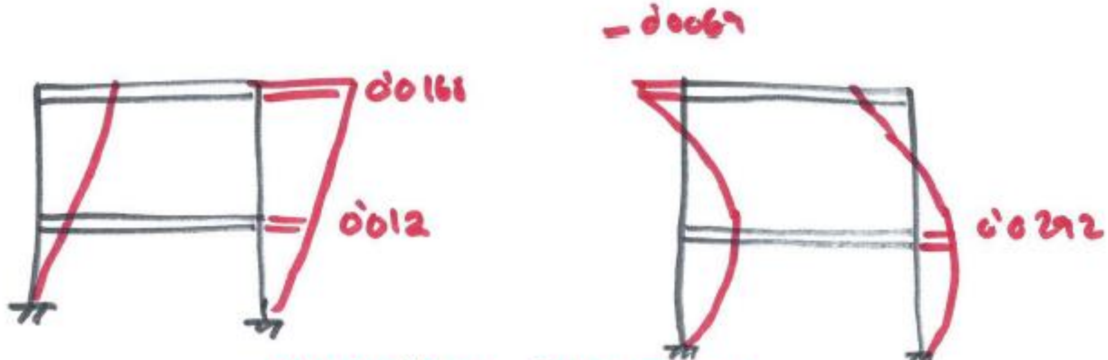
EXAMPLE 1

MASS ORTHONORMALIZED

$$\hat{\Phi}_1 = \frac{1}{\sqrt{6913'6}} \Phi_1 = \begin{Bmatrix} 0'012 \\ 0'0168 \end{Bmatrix} \qquad [\hat{\Phi}]^T [M] [\hat{\Phi}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{\Phi}_2 = \frac{1}{\sqrt{1169'2}} \Phi_2 = \begin{Bmatrix} 0'0292 \\ -0'0069 \end{Bmatrix} \qquad [\hat{\Phi}]^T [K] [\hat{\Phi}] = \begin{bmatrix} 0'192 & 0 \\ 0 & 3,425 \end{bmatrix}$$

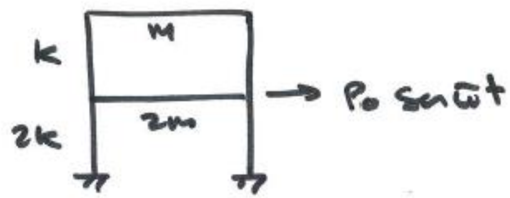
DECOUPLED



NORMALIZED EIGENVECTORS

Example 2

EXAMPLE 2



$$[M] = \begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \qquad [K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

EOM (EQUATION OF MOTION)

$$\begin{bmatrix} 2m & \\ & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_0 \sin \bar{\omega} t \\ 0 \end{Bmatrix}$$

MODAL SUPERPOSITION $\{u\} = [\Phi]\{y\}$

FIND ω_i & $\{d_i\}$

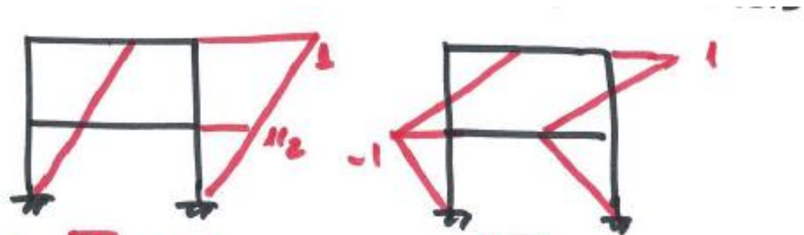
$$\left(-\omega_i^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} m + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} k \right) \{d_i\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\omega_1 = \sqrt{\frac{k}{m}} \rightarrow \{d_1\} = \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$$

$$\omega_2 = \sqrt{\frac{2k}{m}} \rightarrow \{d_2\} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Example 2

EXAMPLE 2



$\omega_1 = \sqrt{\frac{k}{m}}$ rad/s $\qquad \omega_2 = \sqrt{\frac{7k}{m}}$ rad/s

REDUCED SYSTEM

$$[\hat{M}^*] = [\Phi]^T [M][\Phi] = \begin{bmatrix} 1/2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} m \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3/2\, m & 0 \\ 0 & 3m \end{bmatrix}$$

$$[\hat{K}^*] = [\Phi]^T [K][\Phi] = \begin{bmatrix} 1/2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 7k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3/4\, k & 0 \\ 0 & 6k \end{bmatrix}$$

$$[P^*] = [\Phi]^T \{P\} = \begin{bmatrix} 1/2 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} p_0 \sin \bar{\omega} t \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} p_0 \sin \bar{\omega} t \\ -p_0 \sin \bar{\omega} t \end{Bmatrix}$$

SOLUTION IN MODAL COORDINATES

$M_i^* \ddot{y}_i + K_i^* y_i = P_i^* \longmapsto$ SDOF SYSTEM (PHYSICAL COORDINATES)

$$\begin{bmatrix} m\ddot{u} + ku = p_0 \sin \bar{\omega} t \\ u(t) = \frac{\dot{u}(0)}{\omega_n} \sin \omega_n t + u(0) \cos \omega_n t + \frac{p_0}{k} \frac{1}{1-\beta^2} \sin \bar{\omega} t \end{bmatrix}$$

Example 2

EXAMPLE 2

$$m_i^{*}\ddot{y}_i + k_i^{*} y_i = p_i^{*}$$

$$① \rightarrow \frac{3}{2} m \ddot{y}_1 + \frac{3}{4} k y_1 = \frac{1}{2} p_0 \sin \bar{\omega} t \Rightarrow y_1(t) = \frac{2p_0}{3k} \frac{1}{1-\left(\frac{\bar{\omega}}{\omega_1}\right)^2} \sin \bar{\omega} t$$

$$② \rightarrow 3m \ddot{y}_2 + 6k y_2 = -p_0 \sin \bar{\omega} t \Rightarrow y_2(t) = -\frac{p_0}{6k} \left(\frac{1}{1-\left(\frac{\bar{\omega}}{\omega_2}\right)^2} \right) \sin \bar{\omega} t$$

SOLUTION IN PHYSICAL COORDINATES

$$u(t) = \{\phi_1\} y_1(t) + \{\phi_2\} y_2(t)$$

$$u(t) = \frac{p_0}{6k} \begin{Bmatrix} 2 \dfrac{1}{1-\left(\frac{\bar{\omega}}{\omega_1}\right)^2} + \dfrac{1}{1-\left(\frac{\bar{\omega}}{\omega_2}\right)^2} \\ \underbrace{4 \dfrac{1}{1-\left(\frac{\bar{\omega}}{\omega_1}\right)^2}}_{\text{CONTRIBUTION 1st MODE}} - \underbrace{\dfrac{1}{1-\left(\frac{\bar{\omega}}{\omega_2}\right)^2}}_{\text{CONTRIBUTION 2nd MODE}} \end{Bmatrix} \sin \bar{\omega} t$$

Example 3

Stiffness Matrix

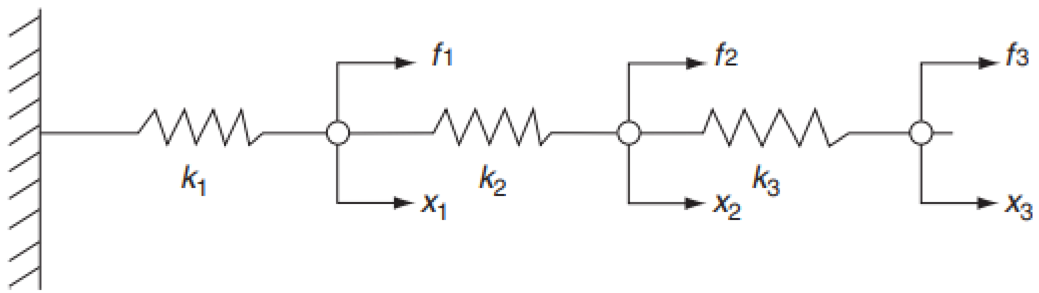
$$\begin{Bmatrix} f_1 \\ f_2 \\ \vdots \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \dots \\ k_{21} & k_{22} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \end{Bmatrix}$$

$$\{f\} = [k]\{x\}$$

Flexibility Matrix

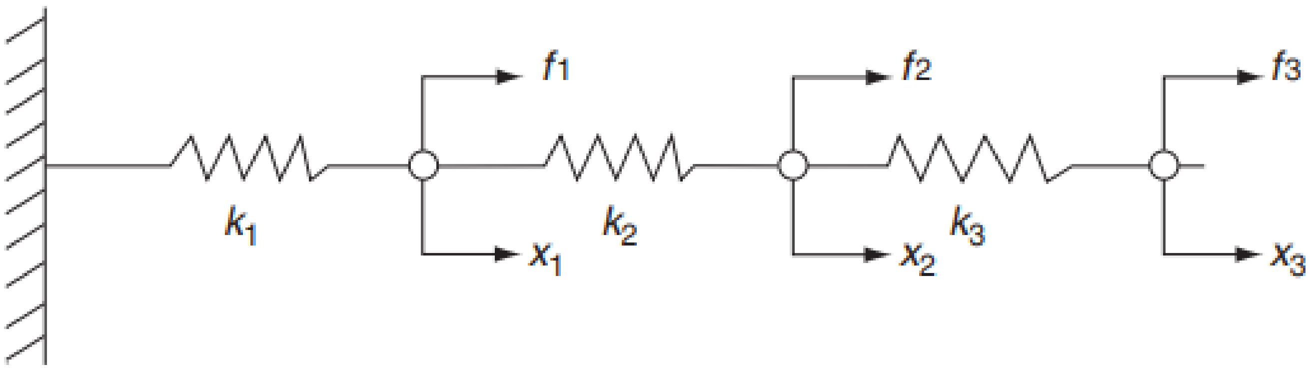
$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots \\ \alpha_{21} & \alpha_{22} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \end{Bmatrix}$$

$$\{x\} = [\alpha]\{f\}$$



Example 3

(a) Derive the stiffness matrix for the chain of springs
(b) Derive the corresponding flexibility matrix.
(c) Show that one is the inverse of the other.



Example 3

Stiffness Matrix

$$\begin{array}{lll}
\text{when}: x_1 = 1 & x_2 = 0 & x_3 = 0 \\
\text{then}: \quad f_1 = k_1 x_1 + k_2 x_2 & f_2 = -k_2 x_1 & f_3 = 0
\end{array}$$

$$\begin{array}{lll}
\text{when}: x_1 = 0 & x_2 = 1 & x_3 = 0 \\
\text{then}: \quad f_1 = -k_2 x_2 & f_2 = k_2 x_2 + k_3 x_2 & f_3 = -k_3 x_2
\end{array}$$

$$\begin{array}{lll}
\text{when}: x_1 = 0 & x_2 = 0 & x_3 = 1 \\
\text{then}: \quad f_1 = 0 & f_2 = -k_3 x_3 & f_3 = k_3 x_3
\end{array}$$

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Example 3

Flexibility Matrix

$f_1 = 1 \qquad f_2 = 0 \qquad f_3 = 0$

$x_1 = 1/k_1 \qquad x_2 = 1/k_1 \qquad x_3 = 1k_1$

$f_1 = 0 \qquad f_2 = 1 \qquad f_3 = 0$

$x_1 = 1/k_1 \qquad x_2 = 1/k_1 + 1/k_2 \qquad x_3 = 1/k_1 + 1/k_2$

$f_1 = 0 \qquad f_2 = 0 \qquad f_3 = 1$

$x_1 = 1/k_1 \qquad x_2 = 1/k_1 + 1/k_2 \qquad x_3 = 1/k_1 + 1/k_2 + 1/k_3$

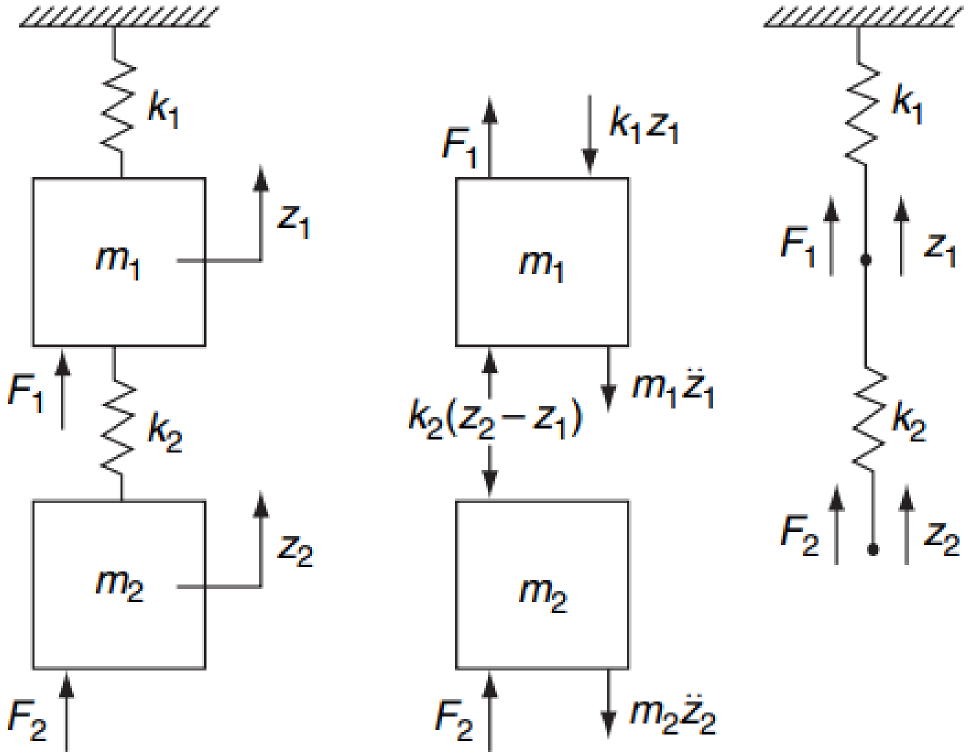
$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2) \\ 1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2 + 1/k_3) \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

Example 3

Stiffness Matrix $\Longleftrightarrow$ *Flexibility Matrix*

$$\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \times \begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_1 & (1/k_1+1/k_2) & (1/k_1+1/k_2) \\ 1/k_1 & (1/k_1+1/k_2) & (1/k_1+1/k_2+1/k_3) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 4



k_1
m_1
z_1
F_1
k_2
m_2
z_2
F_2
F_1
$k_1 z_1$
m_1
$k_2(z_2 - z_1)$
$m_1\ddot{z}_1$
m_2
F_2
$m_2\ddot{z}_2$
k_1
F_1
z_1
k_2
F_2
z_2

Example 4

Equations of Motion from Newton's Second Law and d'Alembert's Principle

m_1: $\Rightarrow$ $$F_1 - m_1\ddot{z}_1 - k_1 z_1 + k_2(z_2 - z_1) = 0$$

m_2: $\Rightarrow$ $$F_2 - m_2\ddot{z}_2 - k_2(z_2 - z_1) = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Example 4

Equations of Motion from Newton's Second Law and d'Alembert's Principle

$$\boxed{m_1:} \quad \Rightarrow \quad F_1 - m_1\ddot{z}_1 - k_1 z_1 + k_2(z_2 - z_1) = 0$$

$$\boxed{m_2:} \quad \Rightarrow \quad F_2 - m_2\ddot{z}_2 - k_2(z_2 - z_1) = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Example 4

Equations of Motion from the Stiffness Matrix

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 - m_1 \ddot{z}_1 \\ F_2 - m_2 \ddot{z}_2 \end{Bmatrix} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix},$$

Example 4

Equations of Motion from Lagrange's Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = Q_i \quad (i = 1, 2)$$

$$T = \tfrac{1}{2} m_1 \dot{z}_1^2 + \tfrac{1}{2} m_2 \dot{z}_2^2$$

$$U = \tfrac{1}{2} k_1 z_1^2 + \tfrac{1}{2} k_2 (z_2 - z_1)^2 = \tfrac{1}{2} k_1 z_1^2 + \tfrac{1}{2} k_2 z_2^2 - k_2 z_1 z_2 + \tfrac{1}{2} k_2 z_1^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}_1}\right) = m_1 \ddot{z}_1 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial q_2}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}_2}\right) = m_2 \ddot{z}_2$$

$$\frac{\partial U}{\partial q_1} = \frac{\partial U}{\partial z_1} = k_1 z_1 - k_2 z_2 + k_2 z_1 \qquad \frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial z_2} = k_2 z_2 - k_2 z_1$$

Example 4

Equations of Motion from Lagrange's Equations

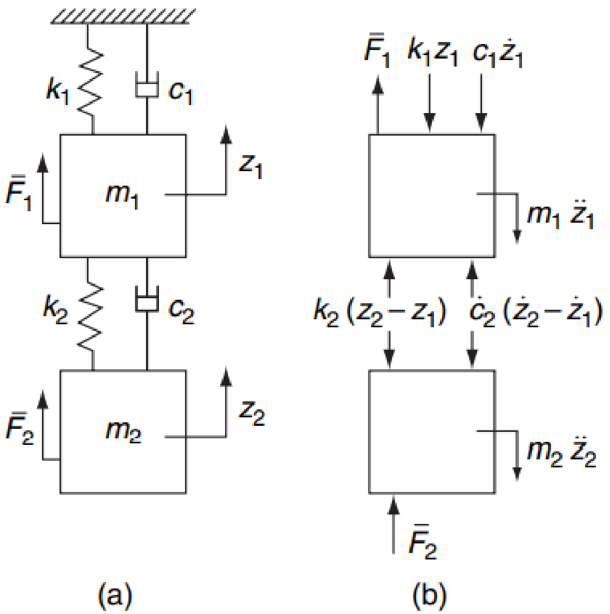
$$m_1\ddot{z}_1 + k_1 z_1 - k_2 z_2 + k_2 z_1 = F_1$$

$$m_2\ddot{z}_2 + k_2 z_2 - k_2 z_1 = F_2$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Example 4

Equations of Motion from Lagrange's Equations



$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}_i}\right) + \frac{\partial D}{\partial \dot{z}_i} + \frac{\partial U}{\partial z_i} = \overline{F}_i \quad (i = 1, 2)$$

$$D = \tfrac{1}{2} c_1 \dot{z}_1^2 + \tfrac{1}{2} c_2 (\dot{z}_2 - \dot{z}_1)^2$$

$$\overline{F}_1 - m_1 \ddot{z}_1 - k_1 z_1 - c_1 \dot{z}_1 + c_2(\dot{z}_2 - \dot{z}_1) + k_2(z_2 - z_1) = 0$$

$$\overline{F}_2 - m_2 \ddot{z}_2 - c_2(\dot{z}_2 - \dot{z}_1) - k_2(z_2 - z_1) = 0$$

Example 4

Equations of Motion from Lagrange's Equations

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} \overline{F}_1 \\ \overline{F}_2 \end{bmatrix}$$

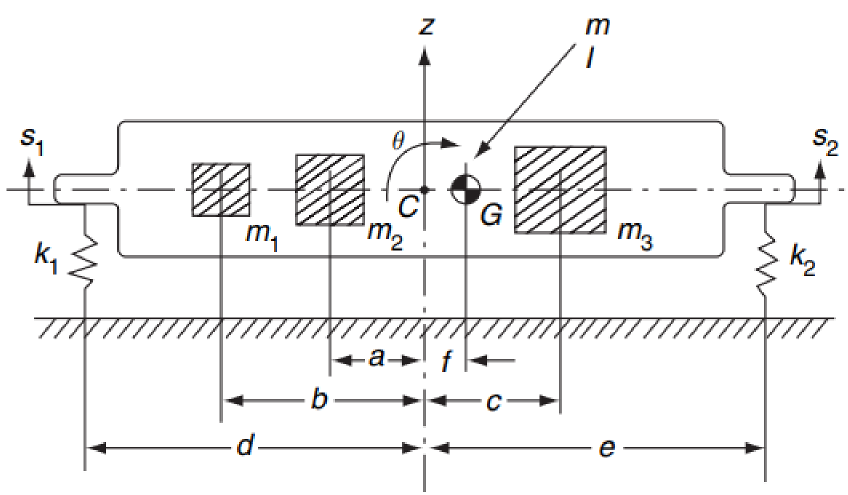
$$\begin{bmatrix} m_{11} & m_{12} & \dots \\ m_{21} & m_{22} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \vdots \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \dots \\ c_{21} & c_{22} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \end{Bmatrix} \begin{bmatrix} k_{11} & k_{12} & \dots \\ k_{21} & k_{22} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \overline{F}_1 \\ \overline{F}_2 \\ \vdots \end{Bmatrix}$$

$$[M]\{\ddot{z}\} + [C]\{\dot{z}\} + [K]\{z\} = \{\overline{F}\}$$

Example 5

The equipment box shown in Fig. 6.2, to be fitted into an aircraft, is vibration isolated by two pairs of springs, of combined stiffness $k_1$and k_2. When empty, the mass of the box is m and its mass center is at G. Its mass moment of inertia about G, when empty, is I. Three heavy items, which can be treated as point masses, m_1, m_2 and m_3, are fixed in the box as shown. The motion of the box is defined by two global coordinates, z and θ, the translation and rotation, respectively, of the reference center, point C.

Use matrix methods to derive the equations of motion of the box and contents in terms of the coordinates z and θ. Assume that the system is undamped.



Example 5

$$[M]\{\ddot{\underline{z}}\} + [K]\{\underline{z}\} = 0 \qquad \{\underline{z}\} = \left\{ \begin{smallmatrix} z \\ \theta \end{smallmatrix} \right\}$$

$$[M] = [X_{\mathrm{m}}]^{\mathrm{T}}[\overline{m}][X_{\mathrm{m}}] \qquad [\overline{m}] = \begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & m_3 \end{bmatrix}$$

$$\begin{Bmatrix} r_m \\ r_I \\ r_{m_1} \\ r_{m_2} \\ r_{m_3} \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \begin{Bmatrix} r_m \\ r_I \\ r_{m_1} \\ r_{m_2} \\ r_{m_3} \end{Bmatrix}_2 = \begin{Bmatrix} -f \\ 1 \\ a \\ b \\ -c \end{Bmatrix} \qquad [X_{\mathrm{m}}] = \begin{bmatrix} 1 & -f \\ 0 & 1 \\ 1 & a \\ 1 & b \\ 1 & -c \end{bmatrix}$$

Example 5

$$[M] = [X_{\mathrm{m}}]^{\mathrm{T}}[\overline{m}][X_{\mathrm{m}}] = \begin{bmatrix} 1 & -f \\ 0 & 1 \\ 1 & a \\ 1 & b \\ 1 & -c \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} 1 & -f \\ 0 & 1 \\ 1 & a \\ 1 & b \\ 1 & -c \end{bmatrix}$$

$$[M] = \begin{bmatrix} (m + m_1 + m_2 + m_3) & (am_1 + bm_2 - cm_3 - fm) \\ (am_1 + bm_2 - cm_3 - fm) & (I + f^2 m + a^2 m_1 + b^2 m_2 + c^2 m_3) \end{bmatrix}$$

Example 5

$$[K] = [X_s]^T [\bar{k}] [X_s]$$

$$\{s\} = [X_s]\{\underline{z}\} \qquad \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix},$$

$$\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \begin{bmatrix} 1 & d \\ 1 & -e \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix}$$

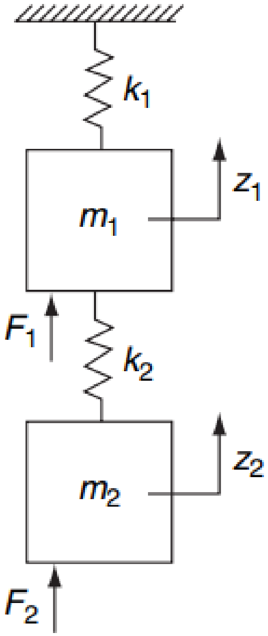
$$[K] = \begin{bmatrix} 1 & d \\ 1 & -e \end{bmatrix}^T \begin{bmatrix} k_1 & \\ & k_2 \end{bmatrix} \begin{bmatrix} 1 & d \\ 1 & -e \end{bmatrix} = \begin{bmatrix} (k_1 + k_2) & (dk_1 - ek_2) \\ (dk_1 - ek_2) & (d^2k_1 + e^2k_2) \end{bmatrix}$$

Example 5

$$[M]\{\ddot{\underline{z}}\} + [K]\{\underline{z}\} = 0$$

$$\begin{bmatrix} (m + m_1 + m_2 + m_3) & (am_1 + bm_2 - cm_3 - fm) \\ (am_1 + bm_2 - cm_3 - fm) & (I + f^2m + a^2m_1 + b^2m_2 + c^2m_3) \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (dk_1 - ek_2) \\ (dk_1 - ek_2) & (d^2k_1 + e^2k_2) \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix} = 0$$

Example 6



(a) Find the eigenvalues and eigenvectors of the undamped system shown in Fig. 6.3 with:

$$m_1 = 1\,\mathrm{kg}; \quad m_2 = 2\,\mathrm{kg}; \quad k_1 = 10\,\mathrm{N/m} \quad k_2 = 10\,\mathrm{N/m}$$

Scale the eigenvectors so that the largest absolute element in each column is set to unity.

(b) Demonstrate that a transformation to modal coordinates using the eigenvectors as modes enables the equations to be written as uncoupled single-DOF systems.

(c) Rescale the eigenvectors so that the mass matrix, in normal mode coordinates, is a unit matrix.

Example 6

Part (a)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = 0$$

$$([K] - \lambda[M])\{\bar{z}\} = 0,$$

$$\left(\begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix} = 0 \qquad \begin{bmatrix} (20 - \lambda) & -10 \\ -10 & (10 - 2\lambda) \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix} = 0$$

Example 6

$$\begin{vmatrix} (20-\lambda) & -10 \\ -10 & (10-2\lambda) \end{vmatrix} = 0,$$

$$(20-\lambda)(10-2\lambda) - 100 = 0,$$

$$\lambda^2 - 25\lambda + 50 = 0.$$

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{2.1922} = 1.480\,\text{rad/s},$$
$$\omega_2 = \sqrt{\lambda_2} = \sqrt{22.807} = 4.775\,\text{rad/s}$$

$$\left(\frac{\bar{z}_1}{\bar{z}_2}\right) = \left(\frac{10}{20-\lambda}\right) \text{ or } \left(\frac{10-2\lambda}{10}\right)$$

$$\lambda = \lambda_1 = 2.192, \quad \left(\frac{\bar{z}_1}{\bar{z}_2}\right)_1 = 0.5615$$

$$\lambda = \lambda_2 = 22.807, \quad \left(\frac{\bar{z}_1}{\bar{z}_2}\right)_2 = -3.5615$$

Example 6

$$\{\phi\}_1 = \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_1 \quad \{\phi\}_2 = \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_2$$

$$\{\phi\}_1 = \begin{Bmatrix} 0.5615 \\ 1 \end{Bmatrix} \quad \{\phi\}_2 = \begin{Bmatrix} 1 \\ -0.2807 \end{Bmatrix}$$

Part (b)

$$\{z\} = [X]\{q\}$$

$$[X] = [\{\phi\}_1\{\phi\}_2] = \begin{bmatrix} 0.5615 & 1 \\ 1 & -0.2807 \end{bmatrix}$$

$$[\underline{M}] = [X]^{\mathrm{T}}[M][X]$$

Example 6

Part (b)

$$[\underline{K}] = [X]^{\mathrm{T}}[K][X]$$

$$[\underline{K}] = \begin{bmatrix} 0.5615 & 1 \\ 1 & -0.2807 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} 0.5615 & 1 \\ 1 & -0.2807 \end{bmatrix} = \begin{bmatrix} 5.075 & 0 \\ 0 & 26.40 \end{bmatrix}$$

$$[\underline{M}]\{\ddot{q}\} + [\underline{K}]\{q\} = \{Q\}$$

$$\begin{bmatrix} 2.315 & 0 \\ 0 & 1.157 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 5.075 & 0 \\ 0 & 26.40 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

$$\underline{m}_{11}\ddot{q}_1 + \underline{k}_{11}q_1 = Q_1 \qquad \underline{m}_{22}\ddot{q}_2 + \underline{k}_{22}q_2 = Q_2$$

Example 6

Part (c)

$$[\underline{M}] = \begin{bmatrix} \underline{m}_{11} & 0 \\ 0 & \underline{m}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

$$[\underline{K}] = \begin{bmatrix} \underline{k}_{11} & 0 \\ 0 & \underline{k}_{22} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

Method(1):

$$\alpha_1 = \frac{1}{\sqrt{2.315}}$$

$$\alpha_i = \frac{1}{\sqrt{\underline{m}_{ii}}}$$

$$\alpha_2 = \frac{1}{\sqrt{1.157}}$$

Example 6

$$\{\phi\}_1^{\text{new}} = \frac{1}{\sqrt{2.315}}\begin{Bmatrix} 0.5615 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.3690 \\ 0.6572 \end{Bmatrix}$$

$$\{\phi\}_2^{\text{new}} = \frac{1}{\sqrt{1.15767}}\begin{Bmatrix} 1 \\ -0.2807 \end{Bmatrix} = \begin{Bmatrix} -0.9294 \\ 0.2609 \end{Bmatrix}$$

$$[X]^{\text{new}} = [\{\phi\}_1\{\phi\}_2] = \begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix}$$

$$[\underline{M}] = [X]^{\mathrm{T}}[M][X] = \begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 6

$$[\underline{K}] = [X]^{\mathrm{T}}[K][X] = \begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix}$$
$$= \begin{bmatrix} 2.192 & 0 \\ 0 & 22.807 \end{bmatrix}$$

Method (2):

$$\underline{m}_{ii} = \{\underline{\phi}\}_i^{\mathrm{T}}[M]\{\underline{\phi}\}_i$$

$$\{\underline{\phi}\}_1 = \begin{Bmatrix} 0.5615 \\ 1 \end{Bmatrix} \quad \text{and} \quad \{\underline{\phi}\}_2 = \begin{Bmatrix} -3.561 \\ 1 \end{Bmatrix}$$

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \alpha_i = \left(\frac{1}{\{\underline{\phi}\}_i^{\mathrm{T}}[M]\{\underline{\phi}\}_i} \right)^{\frac{1}{2}}$$

Example 6

Method (2):

$$\alpha_1 = 0.6572 \qquad \alpha_2 = 0.2609.$$

$$\{\phi\}_1 = \alpha_1\{\underline{\phi}\}_1 = 0.6572\begin{Bmatrix} 0.5615 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.3690 \\ 0.6572 \end{Bmatrix}$$

$$\{\phi\}_2 = \alpha_2\{\underline{\phi}\}_2 = 0.260958\begin{Bmatrix} -3.561 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.9294 \\ 0.2609 \end{Bmatrix}$$

Example 7

The simple structure shown in Fig. 6.5 consists of a beam, considered massless, with constant EI, free to bend vertically, with two concentrated masses, m_1 and m_2, located as shown. Numerical values are

$$L = 4\,\mathrm{m}; \quad EI = 2 \times 10^6\,\mathrm{N\,m^2}; \quad m_1 = 10\,\mathrm{kg}; \quad m_2 = 8\,\mathrm{kg}$$

Tests on a similar structure have suggested that the viscous damping coefficient for both normal modes should be 0.02 critical.

(a) Derive the flexibility matrix for the system in terms of the global coordinates z_1 and z_2, and the external forces F_1 and F_2.

(b) Use the flexibility matrix, with mass data, to find the normal modes of the system. Sketch the mode shapes and express them in orthonormal form. Check the results by showing that the orthonormal eigenvectors transform the original mass matrix in global coordinates to a unit matrix in normal mode coordinates.

(c) Write the equations of motion of the system in normal mode coordinates.

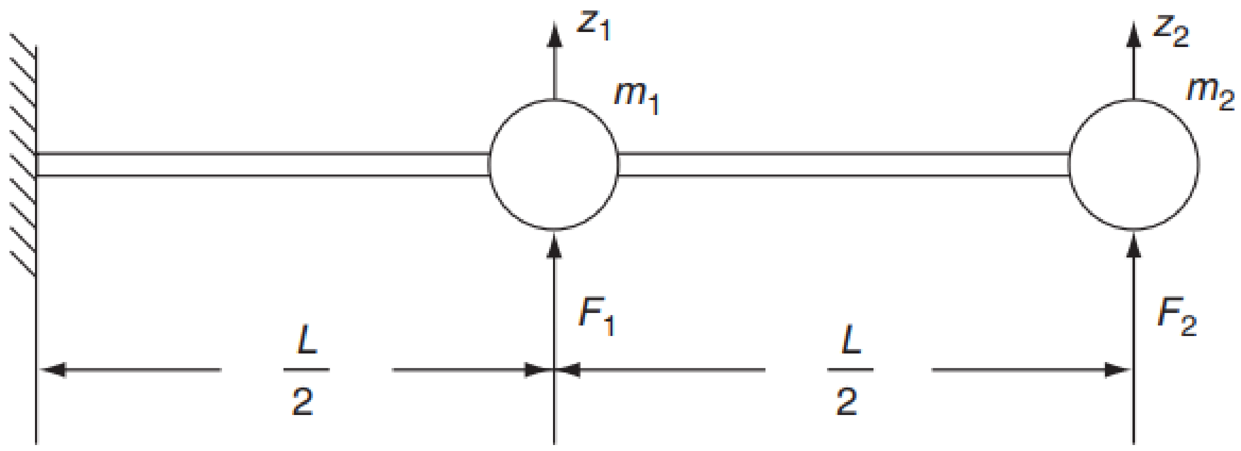
(d) Use the normal mode summation method to calculate the time history of z_1 when F_1 is a step force of 1000 N and F_2 is zero, i.e.:

$$F_1 = 1000H(t) \quad F_2 = 0$$

where $H(t)$ is the Heaviside unit step function.

(e) Plot the displacement history of z_1.

Example 7



Example 7

Calculation of Flexibility Influence Coefficients

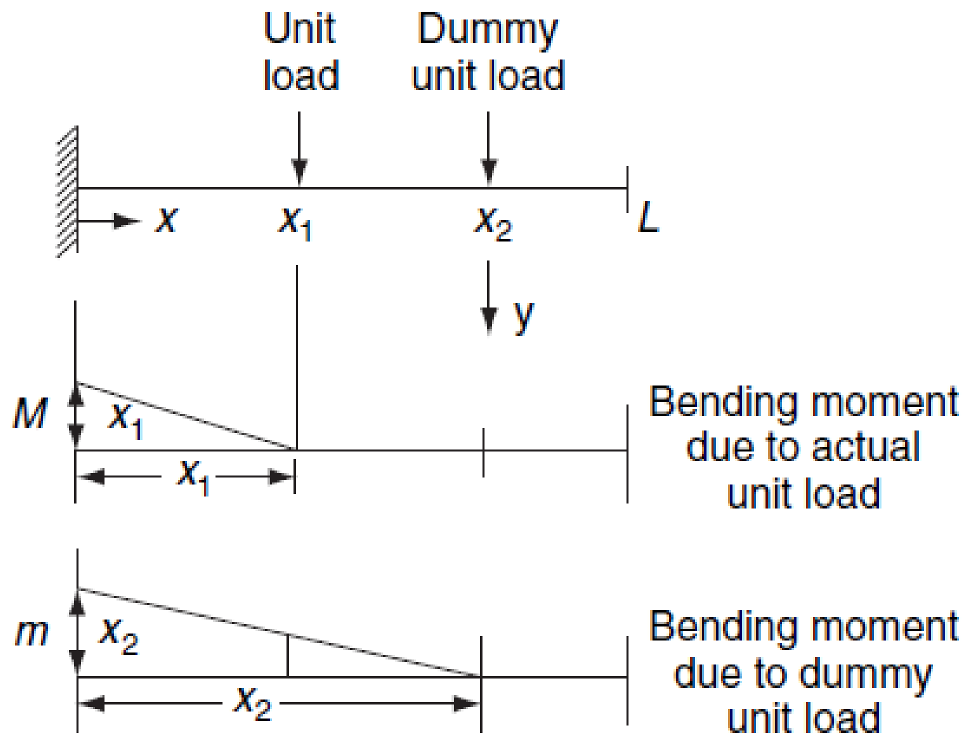
$$U_{\mathrm{B}} = \int \frac{M^2}{2EI} \mathrm{d}x \qquad M = EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \qquad U_{\mathrm{B}} = \frac{1}{2} \int EI \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \right)^2 \mathrm{d}x$$

Castigliano's first theorem

$$y_P = \frac{\partial U_{\mathrm{B}}}{\partial P} = \int \frac{M}{EI} \cdot \frac{\partial M}{\partial P} \cdot \mathrm{d}x \qquad \boxed{y = \int \frac{Mm}{EI} \cdot \mathrm{d}x} \qquad m = \frac{\partial M}{\partial P}$$

Example 7

Calculation of Flexibility Influence Coefficients



Example 7

Calculation of Flexibility Influence Coefficients

$$\begin{array}{ll} M = (x_1 - x) & 0 < x < x_1 \\ M = 0 & x_1 < x < L \end{array}$$

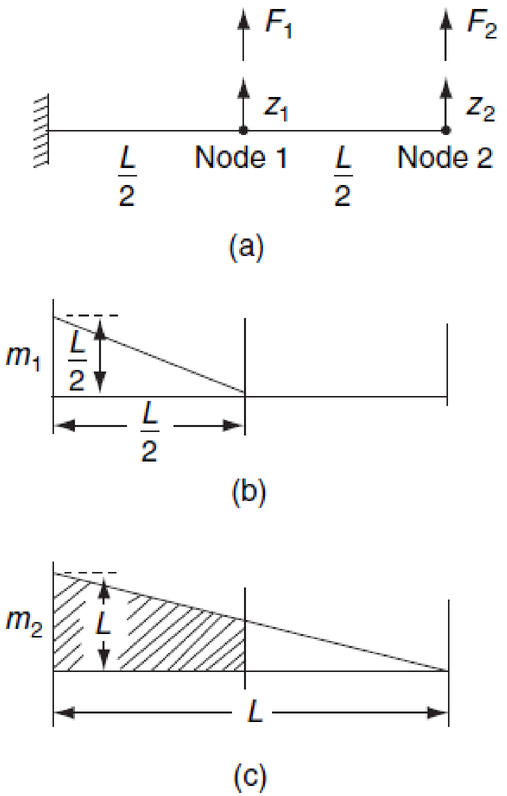
$$\begin{array}{ll} m = (x_2 - x) & 0 < x < x_2 \\ m = 0 & x_2 < x < L \end{array}$$

$$y = \int_0^{x_1} \frac{Mm}{EI} \cdot \mathrm{d}x = \int_0^{x_1} \frac{(x_1 - x)}{EI}(x_2 - x) \cdot \mathrm{d}x$$

$$\boxed{\phi = \int \frac{Tt}{GJ} \mathrm{d}x}$$

Example 7

Calculation of Flexibility Influence Coefficients



$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

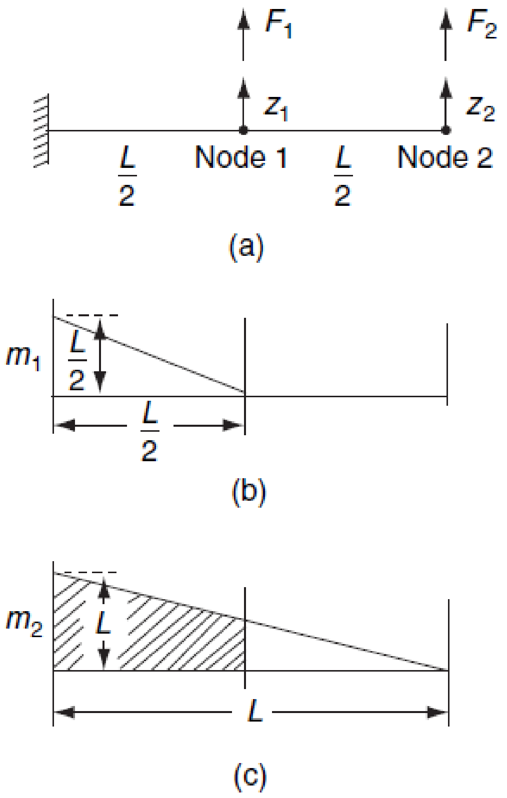
$$[\alpha] = [K]^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

$$\alpha_{11} = \int_0^{\frac{L}{2}} \frac{\left(\frac{L}{2} - x\right)}{EI} \left(\frac{L}{2} - x\right) \cdot \mathrm{d}x$$

$$\alpha_{11} = \frac{1}{EI} \times \left[\frac{1}{2}\left(\frac{L}{2}\right)^2\right] \times \left[\frac{2}{3}\left(\frac{L}{2}\right)\right] = \frac{L^3}{24EI}$$

Example 7

Calculation of Flexibility Influence Coefficients



$$\alpha_{12} = \frac{1}{EI}\left\{\left[\frac{1}{2}\left(\frac{L}{2}\right)^2 \times \frac{L}{3}\right] + \left[\left(\frac{L}{2}\right)^2 \times \frac{L}{4}\right]\right\} = \frac{5L^3}{48EI}$$

$$\alpha_{21} = \alpha_{12} = \frac{5L^3}{48EI}$$

$$\alpha_{22} = \frac{L^3}{3EI}$$

$$[\alpha] = [K]^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \frac{L^3}{EI}\begin{bmatrix} \frac{1}{24} & \frac{5}{48} \\ \frac{5}{48} & \frac{1}{3} \end{bmatrix}$$

Example 7

Part (a):

$$[\alpha] = [K]^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \frac{L^3}{EI} \begin{bmatrix} \frac{1}{24} & \frac{5}{48} \\ \frac{5}{48} & \frac{1}{3} \end{bmatrix}$$

Part (b):

$$[M]\{\ddot{z}\} + [C]\{\dot{z}\} + [K]\{z\} = \{F\}$$

$$[M]\{\ddot{z}\} + [K]\{z\} = \{0\} \qquad \{z\} = \{\bar{z}\}e^{i\omega t},$$

$$\left([K] - \omega^2 [M]\right)\{\bar{z}\} = 0$$

$$\left([K]^{-1}[M] - \frac{1}{\omega^2}[I]\right)\{\bar{z}\} = 0$$

Example 7

$$[K]^{-1} = \frac{L^3}{EI}\begin{bmatrix} \frac{1}{24} & \frac{5}{48} \\ \frac{5}{48} & \frac{1}{3} \end{bmatrix} \qquad [M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\left(\frac{L^3}{48EI}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 8 \end{bmatrix} - \frac{1}{\omega^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix} = 0.$$

$$\left(\begin{bmatrix} 20 & 40 \\ 50 & 128 \end{bmatrix} - \Lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix} = 0$$

$$\Lambda = \frac{48EI}{L^3\omega^2} = \frac{1.5 \times 10^6}{\omega^2}$$

$$EI = 2 \times 10^6, \text{ and } L = 4,$$

Example 7

$$\begin{vmatrix} (20-\Lambda) & 40 \\ 50 & (128-\Lambda) \end{vmatrix} = (20-\Lambda)(128-\Lambda) - 2000 = 0$$

$$\Lambda^2 - 148\Lambda + 560 = 0$$

$$\Lambda = \frac{148 \pm \sqrt{148^2 - 4(560)}}{2} = 74 \pm 70.114$$

$$\Lambda_1 = 144.11, \text{ and } \Lambda_2 = 3.885,$$

$$\omega_1 = 102.02 \text{ rad/s and } \omega_2 = 621.30 \text{ rad/s.}$$

Example 7

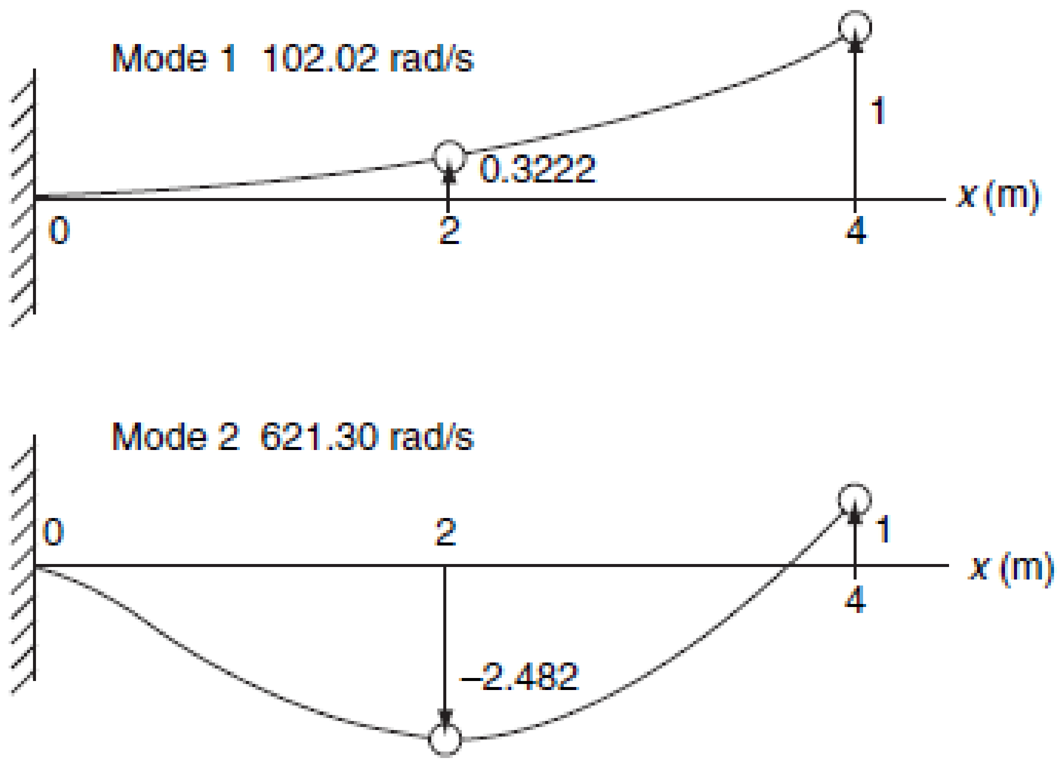
$$\left(\frac{\bar{z}_1}{\bar{z}_2}\right)_1 = \frac{-40}{20-\Lambda_1} = \frac{-40}{20-144.11} = 0.3222$$

$$\left(\frac{\bar{z}_1}{\bar{z}_2}\right)_2 = \frac{-40}{20-\Lambda_2} = \frac{-40}{20-3.885} = -2.482$$

$$\{\underline{\phi}\}_1 = \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.3222 \\ 1 \end{Bmatrix}$$

$$\{\underline{\phi}\}_2 = \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_2 = \begin{Bmatrix} -2.482 \\ 1 \end{Bmatrix}$$

Example 7



Example 7

$$\alpha_i = \left(\frac{1}{\{\underline{\phi}\}_i^{\mathrm{T}} [M] \{\underline{\phi}\}_i} \right)^{\frac{1}{2}}$$

$$\alpha_1 = \left(\begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_1^{\mathrm{T}} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_1 \right)^{-\frac{1}{2}} = \begin{Bmatrix} 0.3222 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 10 & 0 \\ 0 & 8 \end{bmatrix} \begin{Bmatrix} 0.3222 \\ 1 \end{Bmatrix} = 0.3326$$

$$\alpha_2 = \left(\begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_2^{\mathrm{T}} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_2 \right)^{-\frac{1}{2}} = \begin{Bmatrix} -2.482 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 10 & 0 \\ 0 & 8 \end{bmatrix} \begin{Bmatrix} -2.482 \\ 1 \end{Bmatrix} = 0.1198$$

Example 7

$$\{\phi\}_1 = \alpha_1\{\underline{\phi}\}_1 = 0.3326\begin{Bmatrix} 0.3222 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.1071 \\ 0.3326 \end{Bmatrix}$$

$$\{\phi\}_2 = \alpha_2\{\underline{\phi}\}_2 = 0.1198\begin{Bmatrix} -2.4822 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.2975 \\ 0.1198 \end{Bmatrix}$$

$$[X] = [\{\phi\}_1\{\phi\}_2] = \begin{bmatrix} 0.1071 & -0.2975 \\ 0.3326 & 0.1198 \end{bmatrix}$$

$$\{z\} = [X]\{q\} \qquad \{Q\} = [X]^{\mathrm{T}}\{F\}$$

$$[\underline{M}] = [X]^{\mathrm{T}}[M][X]$$

Example 7

$$[\underline{M}] = \begin{bmatrix} 0.1071 & -0.2975 \\ 0.3326 & 0.1198 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 10 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 0.1071 & -0.2975 \\ 0.3326 & 0.1198 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

Part (c):

$$[M]\{\ddot{z}\} + [K]\{z\} = \{F\}$$

$$[\underline{M}]\{\ddot{q}\} + [\underline{K}]\{q\} = \{Q\}$$

$$[\underline{M}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [\underline{K}] = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

Example 7

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

$$\underline{c}_{ii} = 2\gamma_i \omega_i \underline{m}_{ii}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2\gamma_1\omega_1 & 0 \\ 0 & 2\gamma_2\omega_2 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

$\omega_1 = 102.02$, $\omega_2 = 621.30$ and $\gamma_1 = \gamma_2 = 0.02$:

$$2\gamma_1\omega_1 = 4.080, \quad 2\gamma_2\omega_2 = 24.85, \quad \omega_1^2 = 10408, \quad \omega_2^2 = 386020$$

Example 7

$$\{z\} = [X]\{q\} \quad \text{or:} \quad \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 0.1071 & -0.2975 \\ 0.3326 & 0.1198 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$\{Q\} = [X]^{\mathrm{T}}\{F\}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} 0.1071 & -0.2975 \\ 0.3326 & 0.1198 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} 0.1071 & 0.3326 \\ -0.2975 & 0.1198 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Example 7

Part (d):

$$\ddot{q}_1 + 2\gamma_1\omega_1\dot{q}_1 + \omega_1^2 q_1 = Q_1$$

$$\ddot{q}_2 + 2\gamma_2\omega_2\dot{q}_2 + \omega_2^2 q_2 = Q_2$$

$$Q_1 = 0.1071F_1 + 0.3326F_2$$

$$Q_2 = -0.2975F_1 + 0.1198F_2$$

$$F_1 = 1000H(t), \quad F_2 = 0$$

$$Q_1 = (0.1071 \times 1000)H(t)$$

$$Q_2 = (-0.2975 \times 1000)H(t)$$

Example 7

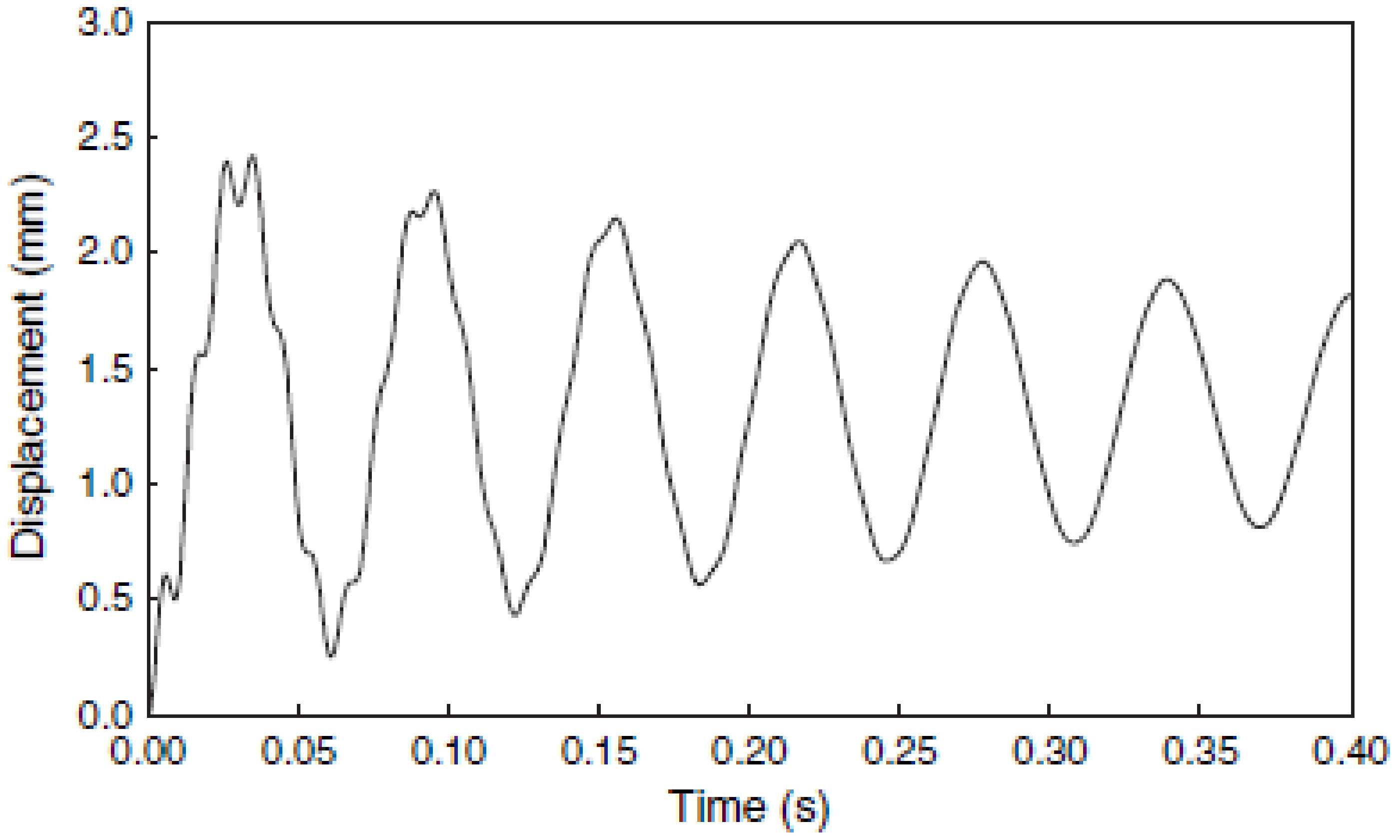
$$z = \frac{a}{m\omega_n^2}\left[1 - e^{-\gamma\omega_n t}\left(\cos\,\omega_d t + \frac{\gamma}{\sqrt{(1-\gamma^2)}}\sin\,\omega_d t\right)\right]$$

$$q_1 = \frac{0.1071 \times 1000}{\omega_1^2}\left[1 - e^{-\gamma_1\omega_1 t}\left(\cos\,\omega_{1d} t + \frac{\gamma_1}{\sqrt{(1-\gamma_1^2)}}\sin\,\omega_{1d} t\right)\right]$$

$$q_2 = \frac{-0.2975 \times 1000}{\omega_2^2}\left[1 - e^{-\gamma_2\omega_2 t}\left(\cos\,\omega_{2d} t + \frac{\gamma_2}{\sqrt{(1-\gamma_2^2)}}\sin\,\omega_{2d} t\right)\right]$$

$$z_1 = 0.1071q_1 - 0.2975q_2$$

Example 7


3.0
2.5
2.0
1.5
1.0
0.5
0.0
Displacement (mm)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
Time (s)