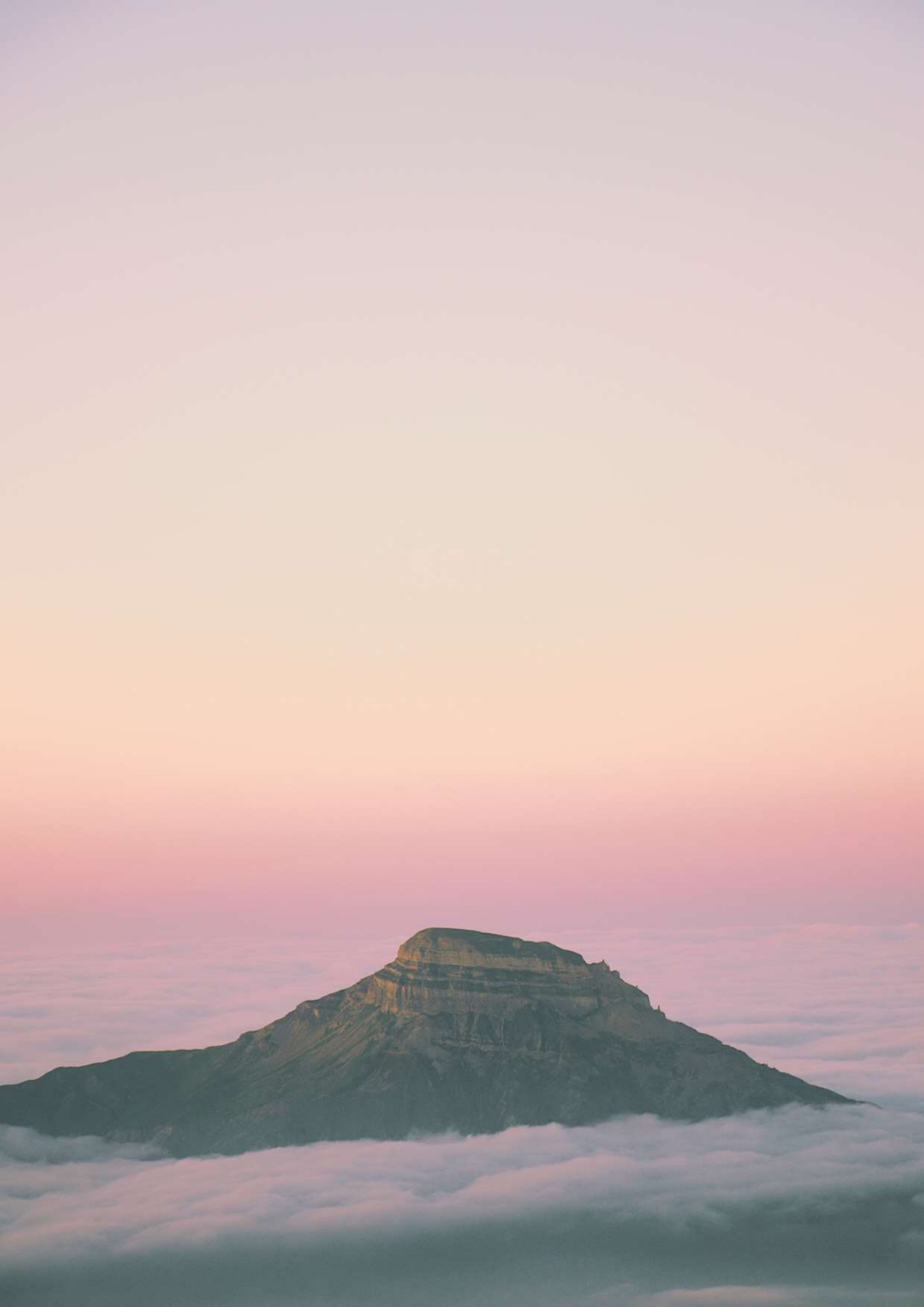

$\Gamma_1 \vdash \sim$

$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to_i$ | $\frac{}{\Gamma \vdash A}\ h(ss: A \in \Gamma)$ | $\frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B} \to_e(A)$ | $\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \bot_c$

$\sim\sim A = \sim(A \to \bot) = (A \to \bot) \to \bot$ (on groupe : $(\sim \ldots) = (\ldots \to \bot)$)

Règles dérivées :

$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}$ aff | $\frac{\Gamma \vdash A \quad \Gamma \vdash \sim A}{\Gamma \vdash B}$ absurde | $\frac{\Gamma \vdash A \to B}{\Gamma \vdash \sim B \to \sim A}$ contraposée

$\frac{\Gamma \vdash A}{\Gamma \vdash \sim\sim A} dn_i$ | $\frac{\Gamma \vdash \sim\sim A}{\Gamma \vdash A} dn_e$

Exercice 1. *Prouver les séquents suivants (en utilisant les deux formats de preuve) :*

- $P \to Q \vdash \sim Q \to \sim P$
- $\vdash P \to \sim\sim P \to Q$
- $\sim(P \to Q) \vdash \sim Q$

1. Supposons $P \to Q$
2. { Supposons $\sim Q$
3. Supposons P
4. Q [$\to_e$, 1, 3]
5. $\bot$ [$\sim_e$, 2, 4]
6. } $\sim Q \to \sim P$ [$\to_i$, 1, 2-5]

1. Supposons $P \to Q$
2. { Supposons $\sim Q$ // obj. $\sim P$
3. { Supposons P // obj. $\bot$
4. Q [$\to_e$, 1, 3]
5. $\bot$ [$\sim_e$, 2, 4]
6. } $\sim P$ [$\to_i$, 3-5]
7. } $\sim Q \to \sim P$ [$\to_i$, 2-6]

$\frac{\frac{}{\Gamma, \sim Q, P \vdash \sim Q}\ h \quad \frac{}{\Gamma, \sim Q, P \vdash P}\ h}{\Gamma, \sim Q, P \vdash \sim P(P)}$

$\frac{\Gamma, \sim Q, P \vdash \bot}{P \to Q \vdash \sim Q \to \sim P} \to_i \quad \sim P(Q)$

$\vdash P \to \neg P \to Q$

{

1. Suppose P // obj $\neg P \to Q$
2. Suppose $\neg P$ // obj Q
3. $\bot$ [mp, 1, 2]
4. Q [$\bot_c$, 3]

}

5. $P \to \neg P \to Q$ [$\to_i$, 1–4]

$\neg (P \to Q) \vdash \neg Q$

1. Suppose $\neg (P \to Q)$

{

2. Suppose Q // obj $\bot$

{

3. Suppose P // Q

}

4. $P \to Q$ [$\to_i$, 3]
5. $\bot$ [$\neg_e$, 1, 4]

}

6. $\neg Q$ [$\to_i$, 2–5]

$$\dfrac{\dfrac{}{\Gamma, Q \vdash \neg(P \to Q)}\,h \qquad \dfrac{\dfrac{}{\Gamma, Q, P \vdash Q}\,h}{\Gamma, Q \vdash P \to Q}\,\to_i}{\dfrac{\neg(P \to Q) \quad P \to Q}{\Gamma, Q \vdash \bot}\,\neg_e}$$

$$\dfrac{\neg(P \to Q) \quad \Gamma, Q \vdash \bot}{\neg(P \to Q) \vdash \neg Q}\,\to_i$$

$P \to Q, P \to \neg Q \vdash P \to S$

1. Suppose $P \to Q$
2. Suppose $P \to \neg Q$

{

3. Suppose P // obj S
4. Q [$\neg_e$, 1, 3]
5. $\neg Q$ [$\neg_e$, 2, 3]
6. S [absurde, 4, 5]

}

7. $P \to S$ [$\to_i$, 3–6]

} or $\bot$ [$\neg_e$, 4, 5]; S [$\bot_c$, 6]

~P -> P ⊢ ~~P

1. Suppose ~P -> P
   {
2. Suppose ~P (IOH) ⊥
3. P [~P, 1, 2]
4. ⊥ [~P, 2, 3]
   }
   ~~P [->, 1, 2-4]

$P \to Q, R \to \sim Q, P \to R \vdash P \to S$

1. Suppose P -> Q
2. Suppose R -> ~Q
3. Suppose P -> R
   {
4. Suppose P (IP) {
5. R [~P, 3, 4]
6. ~Q [mp, 2, 5]
7. Q [~P, 1, 4]
8. S [absurdity, 6, 7]
   }
9. P -> S [->, 4-8]

~~P ⊢ ~P -> ~Q

1. Suppose ~~P
   {
2. Suppose ~P (IOH) ~Q
3. Suppose Q (IOH) ⊥
4. ⊥ [~P, 1, 2]
   }
5. ~P -> ~Q [->, 2-4]

$P \to \neg\neg P \vdash \neg\neg P$

1. Oh suppose $P \to \neg\neg P$
   {
2. Oh suppose P // obj $\bot$
3. $\neg\neg P$ [$\to_e$, 1, 2]
4. $\bot$ [$\neg_e$, 2, 3]
   }
5. $\neg\neg P$

$\neg P \to \neg Q \vdash \neg\neg Q \to \neg\neg P$

1. Oh suppose $\neg P \to \neg Q$
   {
2. Oh suppose $\neg\neg Q$
3. Oh suppose $\neg P$ // obj $\bot$
4. $\neg Q$ [$\neg_e$, 1, 3]
5. $\bot$ [$\neg_e$, 2, 4]
   }
6. $\neg\neg Q \to \neg\neg P$ [$\to_i^1$, 2-5]

$P \to Q \vdash \neg\neg P \to \neg\neg Q$

1. Oh suppose $P \to Q$
   {
2. Oh suppose $\neg\neg P$ // obj $\neg\neg Q$
3. Oh suppose $\neg Q$ // $\bot$
4. $\neg Q \to \neg P$ [contrapose, 1]
5. $\neg P$ [$\neg_e$, 3, 4]
6. $\bot$ [$\neg_e$, 2, 5]
   }
   $\neg\neg P \to \neg\neg Q$ [$\to_i^1$, 2-6]

Correction

$\neg(\neg P \to Q) \vdash \neg P$

1. Suppose $\neg(\neg P \to Q)$
   {
2. Suppose P // $\bot$
   {
3. Suppose $\neg P$ // Q
4. $\bot$ [$\neg$, 2, 3]
5. Q [$\bot_e$, 4]
   }
6. $\neg P \to Q$ [$\to_i$, 3–5]
7. $\bot$ [$\neg$, 1, 6]
   }
8. $\neg P$ [$\to_i$, 2–7]