Notes on Calabi-Yau Theorem

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Introduction

In 1955 Calabi posed a question: Does a compact $K\ddot{a}hler$ manifold with vanishing first Chern class admit a Ricci flat metric? This conjecture was later generalized: Let M be a compact $K\ddot{a}hler$ manifold and T represents the first Chern Class of M in $H^2(M)$. If there exists a $K\ddot{a}hler$ metric in each cohomology class such that its Ricci tensor is precisely T? In 1970s Yau [1] given a positive answer to this conjecture by reducing this problem to solving a non-linear elliptic differential equation on M. Later in 1980s with the advent of more advanced tools, Cao [2] came to a proof using Ricci Flow method by deforming the initial metric smoothly to the required limit metric. It should be remarked here in both Yau's and Cao's paper the extistence of the $K\ddot{a}hler - Einstein$ Metric was solved when the Chern Class is negative. Yau proved the existence by generalizing the right hand side of the complex Monge-Ampere equations, which we will see in the next section.

The proof of the Calabi-Yau theorem has some interesting application in Algebraic geometry and $K\ddot{a}hler$-Geometry. One can refer to the talks given by Yau[5] and Jean Pierre[3]

This note aims at providing some works on Calabi-Yau Theorem and some relating results. In the first part of this note I will review the proof of Yau in details, and in the second part I will briefly sketch the proof ascribing to Cao. In the third part I would like to settle down to some toy examples of Riemann surfaces (i.e. Hyper-elliptic Curves, Fermat Curves). Meanwhile, out of curiousity I pose 4 questions on this topics, though some of them might have been carefully studied. At the end of this note a remark on the survey of complex Monge-Ampere equation[6] will be given.

Continuity Method

Formulation

By Global $\partial\bar{\partial}$-lemma and the result of Chern we know that $[\rho_\omega] = 2\pi c_1(M)$. That is the say for every *Kähler* metric there exists a real-valued function f such that:

$$T_{i\bar{j}} = R_{i\bar{j}} + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} \tag{1}$$

where $R_{i\bar{j}}$ is components of Ricci tensor. Follow the local computation in our class we will have:

$$R_{i\bar{j}} = -\frac{\partial^2 \log\det(g)}{\partial z^i \partial \bar{z}^j} \tag{2}$$

The problem of the existence of the required metric $h_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}$ reduced to solving the following equation:

$$\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\log(\det(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j})) - \log\det(g_{i\bar{j}}) + f\right) = 0 \tag{3}$$

Maximum principle then shows it is equivalent to:

$$\det(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j})\det(g_{i\bar{j}})^{-1} = \exp(F) \tag{4}$$

for some F.

Yau proved that when F satisfies $\int_M \exp(F)\mathbf{d}V_g = Vol(M)$ and F is smooth, (6) admits a unique smooth solution up to a constant. Actually more is true:

The Monge-Ampere equation with more general right hand side Let M be a *Kähler* manifold and $F(x,t)$ be a smooth function on $M \times \mathbb{R}$ where $\frac{\partial F}{\partial t} \geq 0$. Suppose that for some smooth function ψ on M such that $\int_M \exp(F(x,\psi(x)))\mathbf{d}V_g = Vol(M)$ then the equation:

$$\det(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j})\det(g_{i\bar{j}})^{-1} = \exp(F(x,\varphi)) \tag{5}$$

admits a unique smooth solution up to constant. And $\omega + i\partial\bar{\partial}\varphi$ defines a *Kähler* form. For simplicity, we omit a more general case of the right side here. One can refer to the last section of [1] for more detailed. This theorem bring us to the existence problem of *Kähler*-Einstein metric with $c_1(M)$ negative. We choose a *Kähler* form in $-c_1(M)$, but the Ricci form lies in $c_1(M)$. Then there exists a smooth real fuction such that $\omega + \rho_\omega = i\partial\bar{\partial}f$. Let $F(z,\psi) = \psi + f$. It is easy to check that $\exp(-f+f) = 1$ the condition in (5) holds. Now (5) has a smooth solution φ and $\omega_1 = \omega + i\partial\bar{\partial}\varphi$ defines a *Kähler* form. Taking logarithm and derivative, we will have

$$\rho_{\omega_1} = -i\partial\bar{\partial}(f+\varphi) + \rho_\omega = -\omega - i\partial\bar{\partial}f = -\omega_1 \tag{6}$$

And henceforth ω_1 is *Kähler*-Einstein.

Uniqueness of the solution

Let $\omega_i = \omega_0 + \partial\bar{\partial}\varphi_i$ be two solutions then one has (without loss of generality we can assume one of them is the initial metric)

$$\det(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}) \det(g_{i\bar{j}})^{-1} = 1 \tag{7}$$

Carefully choose a coordinate chart such that $g_{i\bar{j}} = \delta_{ij}$ and $\frac{\partial^2}{\partial z^i \partial \bar{z}^j} = \delta_{ij} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}$ at p . Then we obtain $\frac{1}{2} \triangle_g \varphi = \varphi_{i\bar{i}} = \varphi_{i\bar{i}} + n - n \geq n \prod_{i=1}^n (1 + u_{i\bar{i}})^{\frac{1}{n}} - n = 0$. Maximum principle then shows that $\varphi = 0$, which proves the uniqueness.

Existence of the solution

Main Theorem

Let ω be the $K\ddot{a}hler$ form of M, then the equation is to say $(\omega + i\partial\bar{\partial}\varphi)^m = \exp(F)\omega^m$ have a solution. Integrating in M, we obtain $\int \exp(F) \mathrm{d}V_g = Vol_g(M)$ (Since ω_1, ω_2 is homologous). We shall prove that if $F \in C^k, k \geq 3$, and F satisfies the equality, then we can find a solution φ where $\varphi \in C^{k-1,\alpha}, \forall \alpha, 0 < \alpha \leq 1$ of the differential equation.

Def:

$$S = \{t \in [0,1] | \text{the equation } \frac{(\omega + \partial\bar{\partial}\varphi)^m}{\omega^m} = Vol(M) (\int \exp(tF))^{-1} \exp(tF) \text{ has a solution in } C^{k+1,\alpha}\} \tag{8}$$

It easy to check that $t = 0$ one has a solution $\varphi = 0$. We only need to show that S is both open and closed , and henceforth we obtain the required $K\ddot{a}hler$ metric. We will see in the estimate of the lower bound of $1 + \varphi_{i\bar{i}}$ that $g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}$ (Since $\prod_{i=1}^m (1 + \bar{\varphi}_{i\bar{i}}) = \exp(tF) > 0$) is actually positive definite.

Schauder Estimate [4]:

Let Ω is an open domain in $\mathbb{R}^n$, and let $u \in C^{2,\alpha}(\Omega)$ be a bounded solution in Ω of the equation:

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f \tag{9}$$

where $f \in C^\alpha(\Omega)$ and there are positive constants satisfy:

$$\begin{aligned} &a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2 \\ &|a^{ij}|^{(0)}_{0,\alpha,\Omega}, |b^i|^{(1)}_{0,\alpha,\Omega}, |c|^{(2)}_{0,\alpha,\Omega} \leq \Lambda \end{aligned} \tag{10}$$

Then

$$|u|^*_{2,\alpha,\Omega} \leq C(n, \alpha, \lambda, \Lambda)(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha,\Omega}) \tag{11}$$

Where

$$[f]^{(\sigma)}_{k,0;\Omega} = [f]^{(\sigma)}_{k;\Omega} = \sup_{\substack{x\in\Omega \\ |\beta|=k}} d_x^{k+\sigma}|D^\beta f(x)| \tag{12}$$
$$[f]^{(\sigma)}_{k,0;\Omega} = \sup_{\substack{x,y\in\Omega \\ |\beta|=k}} d_{x,y}^{k+\sigma+\alpha}\frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha}, 0 < \alpha \le 1$$
$$|f|^{(\sigma)}_{k;\Omega} = \sum_{j=0}^{k}[f]^{(\sigma)}_{j;\Omega}$$
$$|f|^{(\sigma)}_{k,\alpha;\Omega} = [f]^{(\sigma)}_{k;\Omega} + [f]^{(\sigma)}_{k,\alpha;\Omega}$$
$$|\cdot|^* = |\cdot|^{(0)}$$
$$[\cdot]^* = [\cdot]^{(0)}$$
$$d_x = dist(x, \partial\Omega)$$
$$d_{x,y} = dist(x,y)$$

If the coefficients have better regularity, differetiating the equation yields:

$$a^{ij}D_{ij}D_k u + b^i D_i D_k u + cD_k u = D_k f - (D_k a^{ij})D_{ij}u - D_k b^i D_i u - (D_k c)u \tag{13}$$

Then

$$|D_k u|^*_{2,\alpha,\Omega} \le C(n,\alpha,\lambda,\Lambda)(|D_k u|_{0,\Omega} + |\tilde{f}|^{(2)}_{0,\alpha,\Omega}) \tag{14}$$

Repeat this process and apply the fact that the compact manifold M can be cover by finite charts whose closures are contains in some new chart (d_x has a uniformly positive lower bound (in Riemannian distance function)). So one has the Schauder estimate on compact manifold:

$$a^{ij}\xi_i\xi_j \ge \lambda|\xi|^2 \tag{15}$$
$$|a^{ij}|_{k,\alpha}, |b^i|_{k,\alpha}, |c|_{k,\alpha} \le \Lambda$$
$$u \in C^{k+2,\alpha}$$

then

$$|u|_{k+2,\alpha} \le C(n,\alpha,\lambda,\Lambda)(|u|_{k,\alpha} + |f|_{k,\alpha}) \tag{16}$$

S is opened:

We define two hyper-plane $A := \{\varphi \in C^{k+1,\alpha}, \int \varphi\omega_1^n = 0\}$, $B := \{g \in C^{k-1,\alpha}, \int g\omega_1^n = Vol(M)\}$ A, B are both closed space in Banach space. We define a functional

$$G : A \to B|\varphi \mapsto G(\varphi) = \det(h_{i\bar{j}})\det(g_{i\bar{j}})^{-1}\triangle_h\varphi \tag{17}$$

It remained to show that G is a continuous open map from A to B. Then we compute the differential of G at φ_0 (let $h_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}$), $\mathrm{d}G_{\varphi_0}(\varphi) = \det(h)\det(g)^{-1}\triangle_h\varphi$. According to Hodge theory we know that $\mathrm{d}G_{\varphi_0}(\varphi) = f, f \in C^{k-1,\alpha}$ has a weak solution in H^1 iff $\int f\mathrm{d}Vol_g = 0$. That is to say every element in $f \in T_{G(\varphi_0)}B$ has a weak solution. And Schauder theory tells us that the weak solution lies in $C^{k+1,\alpha}$. In another word $\mathrm{d}G$ is a surjective bounded linear functional from $T_{\varphi_0}A$ to $T_{G(\varphi_0)}B$. Injective is trivial because in compact manifold $\triangle\varphi = 0, \int\varphi = 0$ implies $\varphi = 0$. And henceforth $\mathrm{d}G_{\varphi_0}$ has a bounded inverse.

Moreover

$$\mathrm{d}G : A \to \mathscr{L}(A, \tilde{B})|\phi \mapsto \mathrm{d}G_\phi \tag{18}$$

is continuous, where

$$\tilde{B} := \{g \in C^{k-1,\alpha}, \int g\omega_1^n = 0\} \tag{19}$$

Construct $\Phi(\varphi) = \varphi + \mathrm{d}G_{\varphi_0}^{-1}(f - G(\varphi))$ and apply contracting mapping principle (same word for the Inverse function theorem in finite dimensional case) we deduce that G is a local homeomorphism. Q.E.D.

S is closed

Suppose now $\{t_q\}$ is a sequence contianed in S and that $t_q \to t_0$. We want to show that the corresponding $C^{k+1,\alpha}$ solution φ_q converges to some φ_0. Normalizing, we can assume $\int_M \varphi_q = 0$. To see this we differentiating the equation: (Let $h^q_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2 \varphi_q}{\partial z^i \partial \bar{z}^j}$)

$$\det(h^q_{i\bar{j}}) = \mathrm{Vol}(M)\Big(\int_M \exp(t_q F)\Big)^{-1} \exp(t_q F)\det(g_{i\bar{j}}) \tag{20}$$

Differentiating with respect to z^p , we have:

$$\det(h^q_{i\bar{j}})\sum_{k,l}(h^q)^{k\bar{l}}\frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\frac{\partial \varphi_q}{\partial z^p}) = \mathrm{Vol}(M)\Big(\int_M \exp(t_q F)\Big)^{-1}\frac{\partial}{\partial z^p}\Big(\exp(t_q F)\det(g_{i\bar{j}})\Big) \tag{21}$$

$$\det(h^q_{i\bar{j}})\sum_{k,l}(h^q)^{k\bar{l}}\frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\frac{\partial \varphi_q}{\partial \bar{z}^p}) = \mathrm{Vol}(M)\Big(\int_M \exp(t_q F)\Big)^{-1}\frac{\partial}{\partial \bar{z}^p}\Big(\exp(t_q F)\det(g_{i\bar{j}})\Big)$$

$\sum_{k,l}(h^q)^{k\bar{l}}\frac{\partial g_{k\bar{l}}}{\partial z^p}$ vanishes because in this coordinate $\frac{\partial g_{k\bar{l}}}{\partial z^p} = 0$. The estimate of second terms will show that the PDEs are uniformly elliptic (λ, Λ can be chosen to be independent on q); meanwhile the estimate of third terms shows that the coefficients are Holder continuous with exponent $\alpha, 0 < \alpha \leq 1$ since $\{\sup|(\varphi_q)_{i\bar{j}k}|\}$ has a uniform bound independent of q. One can find the $C^{2,\alpha}$ estimate of $\frac{\partial \varphi}{\partial z^p}, \frac{\partial \varphi}{\partial \bar{z}^p}$. Iterating this process, $C^{k+1,\alpha}$-estimate can be found. And consequently φ_q is a bounded sequence in $C^{k+1,\alpha}$ (which will be explained in the estimate up to third term), Azela-Ascoli theorem then implies that φ_q has a subsequence converges uniformly to φ_0 in C^{k+1}. φ_0 is the required solution when $t = t_0$. Given that $k \geq 2$, $\varphi_0 \in C^{2,\alpha}$. Apply Schauder estimate again, $\varphi_0 \in C^{k+1,\alpha}$.

Estimate up to second Term

For the sake of simplicity we will use Einstein summation convention unless mentioned. Since the computation can be carried out point-wise, we can always choose a coordinate near p such that (No convention):

$$g_{i\bar{j}}(p) = \delta_{ij}, \frac{\partial g_{i\bar{j}}}{\partial z^k}(p) = \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k}(p) = 0 \tag{22}$$

$$\frac{\partial^2 \varphi}{\partial z^k \partial \bar{z}^l}(p) = \delta_{kl}\varphi_{k\bar{k}}(p)$$

Let $R_{i\bar{j}}$ (resp. $\tilde{R}_{i\bar{j}}$) be the Ricci tensor with respect to $g_{i\bar{j}}$ (resp. $h_{i\bar{j}}$). Let $\triangle$ be the laplacian of $g_{i\bar{j}}$. $\triangle f = g^{i\bar{j}} f_{;i\bar{j}} = g^{i\bar{j}} f_{i\bar{j}}$. And we denote the normalize Laplacian of $h_{i\bar{j}}$ $\triangle' f = h^{i\bar{j}} f_{i\bar{j}}$. It should be noticed here $\triangle'$ is the Laplacian of $h_{i\bar{j}}$ in the usual sense. Let $\omega, \tilde{\omega}$ be the *Kähler* form of g, h repectively. $\tilde{\omega} = \omega + \partial\bar{\partial}\varphi$. Thanks for our choice of the coordinate, $\partial\omega = \bar{\partial}\omega = 0$. To see this:

$$\triangle' f = ni\frac{\partial\bar{\partial} f \wedge \tilde{\omega}^{n-1}}{\tilde{\omega}^n} + \langle \mathrm{d}f, * - i(\partial - \bar{\partial})\tilde{\omega}^{n-1}\rangle = h^{i\bar{j}} f_{i\bar{j}} \tag{23}$$

Differentiating the equation we have:

$$g^{k\bar{l}}(\tilde{R}_{k\bar{l}} - R_{k\bar{l}}) = -\triangle F \tag{24}$$

$$\begin{aligned} &\triangle'(\triangle\varphi) - \triangle F = \\ &h^{k\bar{l}}\frac{\partial^2}{\partial z^k\partial\bar{z}^l}(g^{i\bar{j}}\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}) + g^{k\bar{l}}(\tilde{R}_{k\bar{l}} - R_{k\bar{l}}) = \\ &h^{k\bar{l}}\frac{\partial^2}{\partial z^k\partial\bar{z}^l}(g^{i\bar{j}}\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}) + g^{k\bar{l}}g^{i\bar{j}}\frac{\partial^2 g_{i\bar{j}}}{\partial z^k\partial\bar{z}^l} - g^{k\bar{l}}h^{i\bar{j}}\frac{\partial^2 h_{i\bar{j}}}{\partial z^k\partial\bar{z}^l} - g^{k\bar{l}}\frac{\partial h^{i\bar{j}}}{\partial z^k}\frac{\partial h_{i\bar{j}}}{\partial\bar{z}^l} = \\ &h^{k\bar{l}}\frac{\partial^2 g^{i\bar{j}}}{\partial z^k\partial\bar{z}^l}\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j} + g^{k\bar{l}}g^{i\bar{j}}\frac{\partial^2 g_{i\bar{j}}}{\partial z^k\partial\bar{z}^l} - g^{k\bar{l}}h^{i\bar{j}}\frac{\partial^2 g_{i\bar{j}}}{\partial z^k\partial\bar{z}^l} + g^{k\bar{l}}h^{s\bar{j}}h^{i\bar{t}}\frac{\partial^3\varphi}{\partial z^s\partial\bar{z}^t\partial z^k}\frac{\partial^3\varphi}{\partial z^i\partial\bar{z}^j\partial\bar{z}^l} \end{aligned} \tag{25}$$

Now we will not use Einstein summation convention:

$$\begin{aligned} &\sum_{i,k} -(1+\varphi_{k\bar{k}})^{-1}\frac{\partial^2 g_{i\bar{i}}}{\partial z^k\partial\bar{z}^k}\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^i} + \frac{\partial^2 g_{i\bar{i}}}{\partial z^k\partial\bar{z}^k} - \frac{1}{1+\varphi_{i\bar{i}}}\frac{\partial^2 g_{i\bar{i}}}{\partial z^k\partial\bar{z}^k} + \sum_j (1+\varphi_{i\bar{i}})^{-1}(1+\varphi_{j\bar{j}})^{-1}\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}} \\ &= \sum_{i,k}(1+\varphi_{k\bar{k}})^{-1}R_{k\bar{k}i\bar{i}}\varphi_{i\bar{i}} - R_{k\bar{k}i\bar{i}} + \frac{1}{1+\varphi_{i\bar{i}}}R_{k\bar{k}i\bar{i}} + \sum_j (1+\varphi_{i\bar{i}})^{-1}(1+\varphi_{j\bar{j}})^{-1}\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}} \\ &= \left(\sum_{i,j,k}(1+\varphi_{i\bar{i}})^{-1}(1+\varphi_{j\bar{j}})^{-1}\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}}\right) - \sum_{i,k} R_{i\bar{i}k\bar{k}}\frac{\varphi_{i\bar{i}}(\varphi_{k\bar{k}} - \varphi_{i\bar{i}})}{(1+\varphi_{k\bar{k}})(1+\varphi_{i\bar{i}})} \\ &= \left(\sum_{i,j,k}\frac{\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})}\right) - \sum_{i,k} R_{i\bar{i}k\bar{k}}\frac{(\varphi_{k\bar{k}} - \varphi_{i\bar{i}})^2}{2(1+\varphi_{k\bar{k}})(1+\varphi_{i\bar{i}})} \end{aligned} \tag{26}$$

Summerizing we see that

$$\triangle'(\triangle\varphi) \geq \triangle F + \left(\sum_{i,j,k}\frac{\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})}\right) + \left(\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\left[\sum_{i,l}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} - m^2\right] \tag{27}$$

On the other hand

$$\triangle'\varphi = \sum_i \frac{\varphi_{i\bar{i}}}{1+\varphi_{i\bar{i}}} \tag{28}$$

It should be remarked here that h is positive definite $\varphi_{i\bar{i}} + 1 > 0$ then $m + \triangle\varphi > 0$.

Our estimate of the second derivatives can be sketched out as follow

1. estimate of $\inf\varphi, \sup\varphi \to m + \triangle\varphi$
2. the estimate of Green function on $M \to \sup\varphi$
3. the estimate of $m + \triangle\varphi \to \sup|\nabla\varphi|$
4. estimate of $m + \triangle\varphi \to 1 + \varphi_{i\bar{i}}$

Estimate of $m + \triangle\varphi$:

we claim that:

$$0 < m + \triangle\varphi \leq C_1 \exp\left(C(\varphi - \inf_M \varphi)\right) \tag{29}$$

For $C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}} > 1, C_1$ is a constant depend only on:

$$\sup_M(-\triangle F), \sup_M |\inf_{i\neq l} R_{i\bar{i}l\bar{l}}|, C, \sup_M F \tag{30}$$

We begin with the following equation:

$$\begin{aligned}\triangle'\Big(\exp(-C\varphi)(m+\triangle\varphi)\Big) &= C^2 h^{i\bar{j}}\varphi_i\varphi_{\bar{j}}\exp(-C\varphi)(m+\triangle\varphi) \\ &- C\exp(-C\varphi)\Big(h^{i\bar{j}}\big(\varphi_i(\triangle\varphi)_{\bar{j}} + (\triangle\varphi)_i\varphi_{\bar{j}}\big)\Big) \\ &- \exp(-C\varphi)\Big(C\triangle'\varphi(m+\triangle\varphi) - \triangle'(\triangle\varphi)\Big)\end{aligned} \tag{31}$$

Since we have

$$Ch^{i\bar{j}}\big(\varphi_i(\triangle\varphi)_{\bar{j}} + (\triangle\varphi)_i\varphi_{\bar{j}}\big) \le C^2(m+\triangle\varphi)h^{i\bar{j}}\varphi_i\varphi_{\bar{j}} + (m+\triangle\varphi)^{-1}h^{i\bar{j}}(\triangle\varphi)_i(\triangle\varphi)_{\bar{j}} \tag{32}$$

This implies:

$$\begin{aligned}&\triangle'\Big(\exp(-C\varphi)(m+\triangle\varphi)\Big) \\ &\ge -\exp(-C\varphi)(m+\triangle\varphi)^{-1}h^{i\bar{j}}(\triangle\varphi)_i(\triangle\varphi)_{\bar{j}} - \exp(-C\varphi)\Big(C\triangle'\varphi(m+\triangle\varphi) - \triangle'(\triangle\varphi)\Big) \\ &\ge -\exp(-C\varphi)(m+\triangle\varphi)^{-1}h^{i\bar{j}}(\triangle\varphi)_i(\triangle\varphi)_{\bar{j}} - C\exp(-C\varphi)(\triangle'\varphi)(m+\triangle\varphi) \\ &+ \exp(-C\varphi)\Bigg(\triangle F + \Bigg(\sum_{i,j,k}\frac{\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})}\Bigg) + \Big(\inf_{i\neq l}R_{i\bar{i}l\bar{l}}\Big)\Bigg[\sum_{i,l}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} - m^2\Bigg]\Bigg)\end{aligned} \tag{33}$$

Since φ is a real function, $\overline{\varphi_{i\bar{j}k}} = \varphi_{\bar{i}j\bar{k}}$. One can show that:

$$\begin{aligned}(m+\triangle\varphi)^{-1}\sum_{k,i,j}\frac{\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}}}{1+\varphi_{k\bar{k}}} - \sum_{i,j,k}\frac{\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} &\le m\sum_{i,k}\frac{|\varphi_{i\bar{i}k}|^2}{1+\varphi_{k\bar{k}}} - \sum_{i,j,k}\frac{\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \\ &= (m+\triangle\varphi)^{-1}\sum_{i,k,l}\frac{|\varphi_{i\bar{i}k}|^2(1+\varphi_{l\bar{l}})^{\frac{1}{2}}}{(1+\varphi_{k\bar{k}})(1+\varphi_{l\bar{l}})^{\frac{1}{2}}} - \sum_{i,j,k}\frac{\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \\ &\le \sum_{i,j,k}\frac{|\varphi_{k\bar{k}j}|^2}{(1+\varphi_{j\bar{j}})(1+\varphi_{i\bar{i}})} - \sum_{i,j,k}\frac{\varphi_{k\bar{i}j}\varphi_{\bar{k}i\bar{j}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \le 0\end{aligned} \tag{34}$$

Then:

$$\begin{aligned}&\triangle'\Big(\exp(-C\varphi)(m+\triangle\varphi)\Big) \ge \\ &\exp(-C\varphi)\Bigg(\triangle F + \Big(\inf_{i\neq l}R_{i\bar{i}l\bar{l}}\Big)\Bigg[\sum_{i,l}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} - m^2\Bigg]\Bigg) \\ &- C\exp(-C\varphi)(\triangle'\varphi)(m+\triangle\varphi)\end{aligned} \tag{35}$$

Now let us notice the following inequality:

$$\sum_i\frac{1}{1+\varphi_{i\bar{i}}} \ge \Bigg(\frac{\sum_i(1+\varphi_{i\bar{i}})}{\prod_i(1+\varphi_{i\bar{i}})}\Bigg)^{\frac{1}{m-1}} \tag{36}$$

And original PDE yields: $\prod_i(1+\varphi_{i\bar{i}}) = \exp(F)$. One can deduce that

$$\sum_i\frac{1}{1+\varphi_i} \ge (m+\triangle\varphi)^{\frac{1}{m-1}}\exp(-\frac{F}{m-1}) \tag{37}$$

And hence:

$$\begin{aligned}
&\triangle'\Big(\exp(-C\varphi)(m+\triangle\varphi)\Big) \\
&\geq \exp(-C\varphi)\Big(\triangle F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\Big) \\
&\quad - C\exp(-C\varphi)m(m+\triangle\varphi) \\
&+ \exp(-C\varphi)\Big(C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\Big)(m+\triangle\varphi)^{\frac{m}{m-1}}\exp(-\frac{F}{m-1})
\end{aligned} \tag{38}$$

we choose p reaching the maximum value of $\exp(-C\varphi)(m+\triangle\varphi)$. At this point:

$$\begin{aligned}
(m+\triangle\varphi)^{1+\frac{1}{m-1}} &\leq C\exp(\frac{F}{m-1})m(m+\triangle\varphi) \\
&+ \exp(\frac{F}{m-1})(m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - \triangle F)
\end{aligned} \tag{39}$$

It is obvious that $y^{1+\frac{1}{m-1}} \leq ay+b$ implies $y \leq (2a)^{m-1} + (2b)^{\frac{m-1}{m}}$. If $F_q := t_q F$, φ_q be the corresponding solutions, there exists a constant C_1 depends only on F, g, M such that $(m+\triangle\varphi)(p_q) \leq C_1$. That is to say:

$$0 < m + \triangle\varphi_q \leq C_1 \exp\Big(C(\varphi_q - \inf\varphi_q)\Big) \tag{40}$$

We will use this result to give the estimate of $\sup\varphi$.

Estimate of $\sup\varphi$

To archieve this goal we introduce green function $G(p,q)$ on M. Fix p, $G(p,q)$ is define on M but p, and $\triangle_q G(p,q) = 0$ where $G(p,q)$ is defined. Now choose constant K such that $G(p,q)+K \geq 0$. Then green formula shows that

$$\begin{aligned}
\varphi_(p) &= -\int_M G(p,q)\triangle\varphi \mathrm{d}q \\
&= -\int_M \Big(G(p,q)+K\Big)\triangle\varphi \mathrm{d}q \\
&\leq m\int_M \Big(G(p,q)+K\Big)\mathrm{d}q
\end{aligned} \tag{41}$$

Therefore:

$$\sup_M \varphi \leq m \sup_{p\in M}\int_M \Big(G(p,q)+K\Big)\mathrm{d}q \tag{42}$$

Utilize this result and the normalizing condition of φ, $\int_M \varphi = 0$. One obtains L^1 estimate of φ:

$$\begin{aligned}
\int_M |\varphi| &\leq \int_M \sup_M \varphi - \int_M \varphi + \int_M \sup_M \varphi \\
&= 2\sup_M \varphi \mathrm{Vol}(M) \\
&= 2m\mathrm{Vol}(M)\sup_{p\in M}\int_M \Big(G(p,q)+K\Big)\mathrm{d}q
\end{aligned} \tag{43}$$

Estimate of $\sup(|\nabla\varphi|)$

Rewrite $0 < m + \triangle\varphi \leq C_1\exp\Big(C(\varphi - \inf_M \varphi)\Big)$ as

$$\triangle\varphi = f \tag{44}$$

where $-m \le f \le C_1 \exp(C(\sup\varphi - \inf\varphi))$ Then by Schauder estimate there is a constant C_2 that depends only on $M, C_1 \exp(C\sup\varphi)$ such that:

$$\sup_M |\nabla\varphi| \le C_2 \left(\exp(-C\inf\varphi) + \int_M |\varphi| \right) \tag{45}$$

Since we have an estimate of $\sup\varphi$, $\int_M |\varphi|$ there exists a constant depending on M, C such that:

$$\sup_M |\nabla\varphi| \le C_3(\exp(-C\inf\varphi) + 1) \tag{46}$$

Estimate of $\inf\varphi$

Let N be any positive number , then we have the following inequality:

$$\begin{aligned} \triangle'(\exp(-N\varphi)(m+\triangle\varphi)) \ge \exp(-N\varphi)\left(\triangle F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right) \\ - N\exp(-N\varphi)m(m+\triangle\varphi) \\ + \exp(-N\varphi)\left(N + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp(\frac{-F}{m-1})(m+\triangle\varphi)^{\frac{m}{m-1}} \end{aligned} \tag{47}$$

Choosing $\frac{1}{2}N + \inf_{i\neq l} R_{i\bar{i}l\bar{l}} \ge 0$. Apply inverse young inequality:

$$\frac{1}{2}\exp(\frac{-F}{m-1})(m+\triangle\varphi)^{\frac{m}{m-1}} \ge 2m(m+\triangle\varphi) - C_4\exp(F) \tag{48}$$

C_2 is a positive constant only depends on m. Likewise one can choose $C_5 = \sup_q \sup_M C_4 \exp(F_q)$

$$\begin{aligned} &\exp(F_q)\triangle'\Big(\exp(-N\varphi_q)(m+\triangle\varphi_q)\Big) \\ &\ge \exp(F_q - N\varphi_q)\left(\triangle F_q - m^2\inf_{i\neq l} R_{i\bar{i}l\bar{l}} - NC_5\right) \\ &\qquad + N\exp(F_q - N\varphi_q)m^2 \\ &\qquad + Nm\exp(\inf F_q)\exp(-N\varphi_q)\triangle\varphi_q \\ &\ge \exp(F_q - N\varphi_q)\left(\triangle F_q - m^2\inf_{i\neq l} R_{i\bar{i}l\bar{l}} - NC_5\right) \\ &\qquad + N\exp(F_q - N\varphi_q)m^2 \\ &+ m\exp(\inf F_q)\Big(-\triangle\exp(-N\varphi_q) + N^2\exp(-N\varphi_q)|\nabla\varphi_q|^2\Big) \end{aligned} \tag{49}$$

Integrating on M and apply Green formula:

$$\begin{aligned} \int_M |\nabla\exp(-\frac{1}{2}N\varphi_q)|^2 = \frac{1}{4}N^2\int_M \exp(-N\varphi_q)|\nabla\varphi_q|^2 \le \\ \frac{1}{4}C_6 m^{-1}\exp(\inf F_q)\int_M \exp(-N\varphi_q) \end{aligned} \tag{50}$$

C_4 is a constant depends only on M, N, F, g. We claim that for each N satisfying:

$$\frac{1}{2}N + \inf_{i\neq l} R_{i\bar{i}l\bar{l}} \ge 0 \tag{51}$$

and every φ satisfying (41)(48) yields an estimate of $\int_M \exp(-N\varphi)$ (depending on N, F, M, g). Suppose now there exists a sequence such that $\lim_{i\to\infty}\int_M \exp(-N\varphi_i) = \infty$. Then we define:

$$\exp(-N\tilde{\varphi}_i) = \exp(-N\varphi_i)\Big(\int_M \exp(-N\varphi_i)\Big)^{-1} \tag{52}$$

It follows from (48) that the sequence $\int_M |\nabla \exp(-\frac{1}{2}N\tilde{\varphi}_i|^2$ is uniformly bounded. Sobolev inequality and Compactly embedding theorem implies there exists a subsequence that converges in L^2 to some function f. Without loss of generality we assume this subsequence is $\{\exp(-\frac{1}{2}N\tilde{\varphi}_i)\}$. It is easy to see that $\|f\|_{L^2} = 1$.

On the other hand, we know that, for any $\lambda > 0$, Chebeshev inequality shows that:

$$\mathrm{Vol}(x|\lambda \le |\varphi(x)|) \le \int_M |\varphi| \tag{53}$$

Since $\lim_{i\to\infty}\int_M \exp(-\frac{1}{2}N\varphi_i) = \infty$, we conclude that for i is large enough;

$$\begin{aligned} &\mathrm{Vol}(x|\lambda \le \exp(-\frac{1}{2}N\tilde{\varphi}_i)) \\ &\le \mathrm{Vol}(x|0 < \frac{2\log\lambda}{N} + \frac{1}{N}\log\int_M \exp(-N\varphi_i) \le |\varphi_i|) \\ &\le \Big(\frac{2\log\lambda}{N} + \frac{1}{N}\log\int_M \exp(-N\varphi_i)\Big)^{-1}\int_M |\varphi_i| \end{aligned} \tag{54}$$

By φ_i is L^1 uniformly bounded one can conclude that:

$$\lim_{i\to\infty}\mathrm{Vol}(x|\lambda \le \exp(-\frac{1}{2}N\tilde{\varphi}_i)) = 0 \tag{55}$$

That is to say $\forall\lambda > 0$, $\mathrm{Vol}(x|\lambda \le f) = 0$. Define $\alpha_f(\lambda) = \mathrm{Vol}(x|\lambda \le f)$. We observe that $f \ge 0, a.e.$ $\int_M f^2 = -\int_0^{+\infty}\lambda^2 d\alpha_f(\lambda) = 0$, which contradicts to $\int_M f^2 = 1$. Now we come to a powerful conclusion that $\int_M \exp(-N\varphi_q) \le C_5$, C_5 is a constant depending on M, N, F, g.

We now renormalize φ such that $\sup_M \varphi_q \le -1$. Let p be the point where φ_q reaches its minimum. Choose a geodesic ball B with radius $-\frac{1}{2}(\inf\varphi_q)C_3^{-1}\exp(C\inf\varphi_q - 1)$. In this Ball φ_q os no greater than $-\frac{1}{2}\inf_M \varphi_q$. Since we can assume $-\inf_M \varphi$ such that the radius is less than injective radius. And we choos N so large that $\frac{1}{2}N + \inf_{i\neq l} R_{i\bar{i}l\bar{l}} \ge 0$ and $N \ge 4mC$

$$\exp(-\frac{N}{2}\inf_M \varphi_q)\mathrm{Vol}(B) \le \int_B \exp(-N\varphi_q) \le C_5(M, N, F, g) \tag{56}$$

And there exists a constant C_6 depending only on $(M, g), C$ such that:

$$\mathrm{Vol}(B) \ge C_6(-\inf\varphi_q)^m \exp(mC\inf\varphi_q) \tag{57}$$

Summarizing:

$$C_6(-\inf\varphi_q)^m \exp(-\frac{N}{4}\inf\varphi_q) \le C_5 \tag{58}$$

Simplifying and using the fact that $-\inf\varphi_q \ge 1$ we will have:

$$-\inf\varphi_q \le C_7(M, N, F, g, C) \tag{59}$$

Combine with the estimate of $\sup_M \varphi_q$ when $\int_M \varphi_q = 0$ we have the estimate of $\sup_M |\varphi_q|$.

Estimate of $1 + \varphi_{i\bar{i}}$

Finally the original equation $\prod_i^m(1 + (\varphi_q)_{i\bar{i}}) = \exp(F_q)$ shows that

$$1 + (\varphi_q)_{i\bar{i}} \geq \exp(F_q)(m + \triangle\varphi_q)^{m-1} \tag{60}$$

In conclusion we have prove the following proposition:

Proposition 1 Let M be a compact *Kähler* manifold. Let φ_q be a sequence of real-value function in $C^4(M)$ such that $\int_M \varphi_q = 0$. Suppose $\det(g_{i\bar{j}} + \frac{\partial^2\varphi_q}{\partial z^i \partial \bar{z}^j})\Big(\det(g_{i\bar{j}})\Big)^{-1} = \exp(F_q), F_q = t_q F$. Then there exists positive constant that depend only on M, g, F such that $\sup_q \sup_M |\varphi_q| \leq C_1$, $\sup_q \sup_M |\nabla\varphi_q| \leq C_2$, $0 < C_3 \leq 1 + (\varphi_q)_{i\bar{i}} \leq C_4$ for all i, q.

Estimate up to Third Term

In this section we estimate the third derivative of $(\varphi_q)_{i\bar{j}k}$. We consider the function (";" denote the covariant derivative with respect to g) :

$$S = h^{i\bar{r}} h^{\bar{j}s} h^{k\bar{t}} \varphi_{;i\bar{j}k}\varphi_{;\bar{r}s\bar{t}} \tag{61}$$

We introduce the following notation: $A \simeq B$ if $|A - B| \leq K_1\sqrt{S} + K_2$ where K_1, K_2 are constants that can be estimated. We also say that $A \cong B$ if $|A - B| \leq K_3 S + K_4\sqrt{S} + K_5$ where K_3, K_4, K_5 are constant that can be estimated.

By Ricci identity (Since $\varphi_{;\alpha\beta}$ is $(0,2)$-tensor):

$$\varphi_{;i\bar{j}k\bar{\beta}\alpha} = \varphi_{;i\bar{j}\bar{\beta}k\alpha} + \Big(\varphi_{;i\bar{p}} R^{\bar{p}}_{\bar{\beta}\bar{j}\alpha} - \varphi_{p\bar{\beta}} R^{p}_{i\alpha\bar{j}}\Big)_{;\alpha} \tag{62}$$

By the estimate of $m + \triangle\varphi$ we know that $\varphi_{;i\bar{j}k\bar{\beta}\alpha} \simeq \varphi_{;i\bar{j}\bar{\beta}k\alpha}$. By the same reasoning we have $\varphi_{;i\bar{j}k\bar{\beta}\alpha} \simeq \varphi_{;ik\bar{j}\bar{\beta}\alpha}, \varphi_{;i\bar{j}k\bar{\beta}\alpha} \simeq \varphi_{;i\bar{j}k\alpha\bar{\beta}}$. Moreover $\varphi_{;i\bar{j}k\bar{\beta}\alpha} \simeq \varphi_{;i\bar{\beta}\alpha\bar{j}k}$.

Then $\nabla g = 0$ shows

$$\begin{aligned} F_{;k\bar{l}} &= \Big(\log\det(h_{i\bar{j}}) - \log\det(g_{i\bar{j}})\Big)_{;k\bar{l}} \\ &= h^{i\bar{j}}\varphi_{;i\bar{j}k\bar{l}} - h^{t\bar{j}} h^{i\bar{n}} \varphi_{;t\bar{n}\bar{l}}\varphi_{;i\bar{j}k} \end{aligned} \tag{63}$$

Differentiating again we have:

$$h^{i\bar{j}}\varphi_{;i\bar{j}k\bar{l}s} = h^{i\bar{t}} h^{n\bar{j}} \varphi_{;i\bar{j}k\bar{l}}\varphi_{;n\bar{t}s} + F_{;k\bar{l}s} + \Big(h^{t\bar{j}} h^{i\bar{n}} \varphi_{;t\bar{n}\bar{l}}\varphi_{;i\bar{j}k}\Big)_{;s} \tag{64}$$

Using these result, we are able to compute the Laplacian of S:

$$
\begin{aligned}
\triangle' S = h^{p\bar q} \Big(& \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right)_{;p\bar q} \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t} + \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right)_{;p} [\varphi_{;i\bar j k\bar q} \varphi_{;\bar r s\bar t} + \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t\bar q}] \\
& + \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right)_{;\bar q} [\varphi_{;i\bar j kp} \varphi_{;\bar r s\bar t} + \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t p}] \\
& + \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right) [\varphi_{;i\bar j k\bar q} \varphi_{;\bar r s\bar t p} + \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t p\bar q} + \varphi_{;i\bar j kp\bar q} \varphi_{;\bar r s\bar t} + \varphi_{;i\bar j kp} \varphi_{;\bar r s\bar t\bar q}] \\
\cong h^{p\bar q} \Big(& \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right)_{;p\bar q} \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t} + \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right)_{;p} [\varphi_{;i\bar j k\bar q} \varphi_{;\bar r s\bar t} + \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t\bar q}] \\
& + \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right)_{;\bar q} [\varphi_{;i\bar j kp} \varphi_{;\bar r s\bar t} + \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t p}] + \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right) [\varphi_{;i\bar j k\bar q} \varphi_{;\bar r s\bar t p} + \varphi_{;i\bar j kp} \varphi_{;\bar r s\bar t\bar q}] \Big) \\
& + h^{p\bar q} h^{i\bar r} h^{\bar j s} h^{k\bar t} [\varphi_{;i\bar j k} (F_{;s\bar r\bar t} + h^{p\bar n} h^{w\bar q} \varphi_{;w\bar n\bar t} \varphi_{;p\bar q s\bar r} + (h^{p\bar l} h^{w\bar\gamma} \varphi_{;p\bar\gamma\bar r} \varphi_{;w\bar l s})_{;\bar t}) \\
& + \varphi_{;\bar r s\bar t} \left(F_{i\bar j k} + h^{p\bar n} h^{w\bar q} \varphi_{;w\bar n k} \varphi_{;p\bar q i\bar j} + (h^{p\bar l} h^{w\bar\gamma} \varphi_{;p\bar\gamma\bar j} \varphi_{;w\bar l i})_{;k}\right)] \\
\cong h^{p\bar q} \Big(& \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right)_{;p\bar q} \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t} + \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right)_{;p} [\varphi_{;i\bar j k\bar q} \varphi_{;\bar r s\bar t} + \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t\bar q}] \\
& + \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right)_{;\bar q} [\varphi_{;i\bar j kp} \varphi_{;\bar r s\bar t} + \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t p}] + \left(h^{i\bar r} h^{\bar j s} h^{k\bar t}\right) [\varphi_{;i\bar j k\bar q} \varphi_{;\bar r s\bar t p} + \varphi_{;i\bar j kp} \varphi_{;\bar r s\bar t\bar q}] \Big) \\
& h^{i\bar r} h^{\bar j s} h^{k\bar t} [\varphi_{;i\bar j k} (h^{p\bar n} h^{w\bar q} \varphi_{;w\bar n\bar t} \varphi_{;p\bar q s\bar r} + (h^{p\bar l} h^{w\bar\gamma} \varphi_{;p\bar\gamma\bar r} \varphi_{;w\bar l s})_{;\bar t}) \\
& + \varphi_{;\bar r s\bar t} \left(h^{p\bar n} h^{w\bar q} \varphi_{;w\bar n k} \varphi_{;p\bar q i\bar j} + (h^{p\bar l} h^{w\bar\gamma} \varphi_{;p\bar\gamma\bar j} \varphi_{;w\bar l i})_{;k}\right)]
\end{aligned} \tag{65}
$$

The last line follows from the Cauchy Schwartz inequality and the fact that $\varphi_{\bar r s\bar t p\bar q} \simeq \varphi_{p\bar q\bar r s\bar t}$. We can obtain:

$$
\begin{aligned}
\triangle' S \cong\ & h^{i\bar r} h^{\bar j s} h^{k\bar t} h^{p\bar n} h^{w\bar q} \left(\varphi_{;i\bar j k} \varphi_{;w\bar n\bar t} \varphi_{;p\bar q s\bar r} + \varphi_{;\bar r s\bar t} \varphi_{;w\bar n k} \varphi_{;p\bar q i\bar j}\right) \\
& + h^{i\bar r} h^{\bar j s} h^{k\bar t} h^{p\bar l} h^{w\bar\gamma} \left(\varphi_{;i\bar j k} \varphi_{;p\bar\gamma\bar r\bar t} \varphi_{;w\bar l s} + \varphi_{;i\bar j k} \varphi_{;p\bar\gamma\bar r} \varphi_{;w\bar l s\bar t} + \varphi_{;\bar r s\bar t} \varphi_{;p\bar\gamma\bar j k} \varphi_{;w\bar l i} + \varphi_{;\bar r s\bar t} \varphi_{;p\bar\gamma\bar j} \varphi_{;w\bar l ik}\right) \\
& - h^{i\bar r} h^{\bar j s} h^{k\bar t} h^{w\bar\gamma} h^{p\bar\beta} h^{\alpha\bar l} \left(\varphi_{;i\bar j k} \varphi_{;\alpha\bar\beta\bar t} \varphi_{;p\bar\gamma\bar r} \varphi_{;w\bar l s} + \varphi_{\bar r s\bar t} \varphi_{;\alpha\bar\beta k} \varphi_{;p\bar\gamma\bar j} \varphi_{;w\bar l i}\right) \\
& - h^{i\bar r} h^{\bar j s} h^{k\bar t} h^{p\bar l} h^{w\bar\beta} h^{\alpha\bar\gamma} \left(\varphi_{;i\bar j k} \varphi_{;\alpha\bar\beta\bar t} \varphi_{;p\bar\gamma\bar r} \varphi_{;w\bar l s} + \varphi_{\bar r s\bar t} \varphi_{;\alpha\bar\beta k} \varphi_{;p\bar\gamma\bar j} \varphi_{;w\bar l i}\right) \\
& + h^{p\bar q} h^{i\bar r} h^{\bar j s} h^{\bar k t} [\varphi_{;i\bar j k\bar q} \varphi_{;\bar r s\bar t p} + \varphi_{;i\bar j kp} \varphi_{;\bar r s\bar t\bar q}] \\
& - h^{p\bar q} \left(h^{i\bar\beta} h^{\alpha\bar r} h^{\bar j s} h^{k\bar t} + h^{i\bar r} h^{\bar j\alpha} h^{\bar\beta s} h^{k\bar t} + h^{i\bar r} h^{\bar j s} h^{k\bar\beta} h^{\alpha\bar t}\right) \\
& \times \left(\varphi_{;\alpha\bar\beta p} [\varphi_{;i\bar j k\bar q} \varphi_{;\bar r s\bar t} + \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t\bar q}] + \varphi_{;\alpha\bar\beta\bar q} [\varphi_{;i\bar j kp} \varphi_{;\bar r s\bar t} + \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t p}]\right) \\
& - h^{p\bar q} \left(h^{i\bar\beta} h^{\alpha\bar r} h^{\bar j s} h^{k\bar t} + h^{i\bar r} h^{\bar\beta s} h^{\bar j\alpha} h^{k\bar t} + h^{i\bar r} h^{\bar j s} h^{\alpha\bar t} h^{k\bar\beta}\right) \varphi_{;\alpha\bar\beta p\bar q} \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t} \\
& + h^{p\bar q} \Big(h^{i\bar l} h^{\gamma\bar\beta} h^{\alpha\bar r} h^{\bar j s} h^{k\bar t} + h^{i\bar\beta} h^{\alpha\bar l} h^{\gamma\bar r} h^{\bar j s} h^{k\bar t} + h^{i\bar\beta} h^{\alpha\bar r} h^{\bar j\gamma} h^{\bar l s} h^{k\bar t} + h^{i\bar\beta} h^{\alpha\bar r} h^{\bar j s} h^{k\bar l} h^{\gamma\bar t} \\
& + h^{i\bar l} h^{\gamma\bar r} h^{\bar\beta s} h^{\bar j\alpha} h^{k\bar t} + h^{i\bar r} h^{\bar\beta\gamma} h^{\bar l s} h^{\bar j\alpha} h^{k\bar t} + h^{i\bar r} h^{\bar\beta s} h^{\bar j\gamma} h^{\bar l\alpha} h^{k\bar t} + h^{i\bar r} h^{\bar\beta s} h^{\bar j\alpha} h^{k\bar l} h^{\gamma\bar t} \\
& + h^{i\bar l} h^{\gamma\bar r} h^{\bar j s} h^{\alpha\bar t} h^{k\bar\beta} + h^{i\bar r} h^{\bar j\gamma} h^{\bar l s} h^{\alpha\bar t} h^{k\bar\beta} + h^{i\bar r} h^{\bar j s} h^{\alpha\bar l} h^{\gamma\bar t} h^{k\bar\beta} + h^{i\bar r} h^{\bar j s} h^{\alpha\bar t} h^{k\bar l} h^{\gamma\bar\beta}\Big) \varphi_{;\gamma\bar l\bar q} \varphi_{;\alpha\bar\beta p} \varphi_{;i\bar j k} \varphi_{;\bar r s\bar t}
\end{aligned} \tag{66}
$$

Use the normal coordinate, and we obtain:

$$
\begin{aligned}
\triangle' S \cong & \sum (1+\varphi_{;i\bar{i}})^{-1}(1+\varphi_{;j\bar{j}})^{-1}(1+\varphi_{;k\bar{k}})^{-1}(1+\varphi_{;p\bar{p}})^{-1} \\
& \times \left(\varphi_{;i\bar{j}k\bar{p}}\varphi_{;\bar{i}j\bar{k}p} + \varphi_{;i\bar{j}kp}\varphi_{;\bar{i}j\bar{k}\bar{p}}\right) + \\
& \sum (1+\varphi_{;i\bar{i}})^{-1}(1+\varphi_{;j\bar{j}})^{-1}(1+\varphi_{;k\bar{k}})^{-1}(1+\varphi_{;p\bar{p}})^{-1}(1+\varphi_{;w\bar{w}})^{-1} \\
& \times \Big(\varphi_{;i\bar{j}k}\varphi_{;w\bar{p}\bar{k}}\varphi_{;p\bar{w}j\bar{i}} + \varphi_{;\bar{i}j\bar{k}}\varphi_{;w\bar{p}k}\varphi_{;p\bar{w}i\bar{j}} + \varphi_{;i\bar{j}k}\varphi_{;p\bar{w}\bar{i}\bar{k}}\varphi_{;w\bar{p}j} + \varphi_{;i\bar{j}k}\varphi_{;p\bar{w}\bar{i}}\varphi_{;w\bar{p}j\bar{k}} + \varphi_{;\bar{i}j\bar{k}}\varphi_{;p\bar{w}\bar{j}k}\varphi_{;w\bar{p}i} + \varphi_{;\bar{i}j\bar{k}}\varphi_{;p\bar{w}\bar{j}}\varphi_{;w\bar{p}ik} \\
& - \varphi_{;w\bar{i}p}[\varphi_{;i\bar{j}k\bar{p}}\varphi_{;\bar{w}j\bar{k}} + \varphi_{;i\bar{j}k}\varphi_{;\bar{w}j\bar{k}\bar{p}}] - \varphi_{;w\bar{i}\bar{p}}[\varphi_{;i\bar{j}kp}\varphi_{;\bar{w}j\bar{k}} + \varphi_{;i\bar{j}k}\varphi_{;\bar{w}j\bar{k}p}] \\
& - \varphi_{;j\bar{w}p}[\varphi_{;i\bar{j}k\bar{p}}\varphi_{;\bar{i}w\bar{k}} + \varphi_{;i\bar{j}k}\varphi_{;\bar{i}w\bar{k}\bar{p}}] - \varphi_{;j\bar{w}\bar{p}}[\varphi_{;i\bar{j}kp}\varphi_{;\bar{i}w\bar{k}} + \varphi_{;i\bar{j}k}\varphi_{;\bar{i}w\bar{k}p}] \\
& - \varphi_{;w\bar{k}p}[\varphi_{;i\bar{j}k\bar{p}}\varphi_{;\bar{i}j\bar{w}} + \varphi_{;i\bar{j}k}\varphi_{;\bar{i}j\bar{w}\bar{p}}] - \varphi_{;w\bar{k}\bar{p}}[\varphi_{;i\bar{j}kp}\varphi_{;\bar{i}j\bar{w}} + \varphi_{;i\bar{j}k}\varphi_{;\bar{i}j\bar{w}p}] \\
& - \varphi_{;w\bar{i}p\bar{p}}\varphi_{;i\bar{j}k}\varphi_{;\bar{w}j\bar{k}} - \varphi_{;j\bar{w}p\bar{p}}\varphi_{;i\bar{j}k}\varphi_{;\bar{i}w\bar{k}} - \varphi_{;w\bar{k}p\bar{p}}\varphi_{;i\bar{j}k}\varphi_{;\bar{i}j\bar{w}}\Big) + \\
& \sum (1+\varphi_{;i\bar{i}})^{-1}(1+\varphi_{;j\bar{j}})^{-1}(1+\varphi_{;k\bar{k}})^{-1}(1+\varphi_{;p\bar{p}})^{-1}(1+\varphi_{;w\bar{w}})^{-1}(1+\varphi_{;l\bar{l}})^{-1} \\
& \times \Big(-\varphi_{;i\bar{j}k}[\varphi_{;l\bar{p}\bar{k}}\varphi_{;p\bar{w}\bar{i}}\varphi_{;w\bar{l}j} + \varphi_{;l\bar{w}\bar{k}}\varphi_{;p\bar{l}\bar{i}}\varphi_{;w\bar{p}j}] - \varphi_{;\bar{i}j\bar{k}}[\varphi_{;l\bar{p}k}\varphi_{;p\bar{w}\bar{j}}\varphi_{;w\bar{l}i} + \varphi_{;l\bar{w}k}\varphi_{;p\bar{l}\bar{j}}\varphi_{;w\bar{p}i}] \\
& + \varphi_{;l\bar{i}\bar{p}}\varphi_{;w\bar{l}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{w}j\bar{k}} + \varphi_{;w\bar{l}\bar{p}}\varphi_{;l\bar{i}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{l}w\bar{k}} + \varphi_{;j\bar{l}\bar{p}}\varphi_{;w\bar{i}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{w}l\bar{k}} + \varphi_{;l\bar{k}\bar{p}}\varphi_{;w\bar{i}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{w}j\bar{l}} \\
& + \varphi_{;l\bar{i}\bar{p}}\varphi_{;j\bar{w}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{l}w\bar{k}} + \varphi_{;w\bar{l}\bar{p}}\varphi_{;j\bar{w}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{i}l\bar{k}} + \varphi_{;j\bar{l}\bar{p}}\varphi_{;l\bar{w}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{i}w\bar{k}} + \varphi_{;l\bar{k}\bar{p}}\varphi_{;j\bar{w}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{i}w\bar{l}} \\
& + \varphi_{;l\bar{i}\bar{p}}\varphi_{;w\bar{k}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{l}j\bar{w}} + \varphi_{;j\bar{l}\bar{p}}\varphi_{;w\bar{k}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{i}l\bar{w}} + \varphi_{;l\bar{w}\bar{p}}\varphi_{;w\bar{k}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{i}j\bar{l}} + \varphi_{;w\bar{k}\bar{p}}\varphi_{;l\bar{w}p}\varphi_{;i\bar{j}k}\varphi_{;\bar{i}j\bar{l}}\Big)
\end{aligned}
\tag{67}
$$

Combining (63) we deduce that

$$
\begin{aligned}
\triangle' S \cong & \sum (1+\varphi_{i\bar{i}})^{-1}(1+\varphi_{j\bar{j}})^{-1}(1+\varphi_{k\bar{k}})^{-1}(1+\varphi_{\alpha\bar{\alpha}})^{-1} \\
& \times \left(\left|\varphi_{;\bar{i}j\bar{k}\alpha} - \sum_p \varphi_{;\bar{i}p\bar{k}}\varphi_{;\bar{p}j\alpha}(1+\varphi_{p\bar{p}})^{-1}\right|^2 + \left|\varphi_{;i\bar{j}k\alpha} - \sum_p (\varphi_{;\bar{p}i\alpha}\varphi_{;p\bar{j}k} + \varphi_{;\bar{p}ik}\varphi_{;p\bar{j}\alpha})(1+\varphi_{p\bar{p}})^{-1}\right|^2\right) \geq 0
\end{aligned}
\tag{68}
$$

We know that $\triangle'\triangle\varphi_q \geq \sum(1+(\varphi_q)_{k\bar{k}})^{-1}(1+(\varphi_q)_{i\bar{i}})^{-1}|(\varphi_q)_{k\bar{i}\bar{j}}|^2 - K_6$ where K_6 is constant that only depends on M, g, F. Since the RHS of (68) is positive, letting K_7 be a large positive constant, we find:

$$\triangle'(S + K_7\triangle\varphi_q) \geq K_8 S - K_9 \tag{69}$$

At the point where $S + K_7\triangle\varphi_q$ reaches its maximum we will have K_8 is positive constant:

$$K_8(S + K_7\triangle\varphi_q) \leq K_9 + K_8 K_7 \triangle\varphi_q \tag{70}$$

So we will have $S \leq K_{10}(1 + \sup\triangle\varphi - \inf\triangle\varphi)$. Let $h^{i\bar{j}}\xi_i\bar{\xi}_j \geq M|\xi|^2$ We have

$$S \geq M^3 \sum_{ijk} |\varphi_{;i\bar{j}k}|^2 \tag{71}$$

Then we will have estimate of $\varphi_{;i\bar{j}k}$. Combine with the estimate of second term we have an estimate of $\varphi_{i\bar{j}k}$.

Ricci Flow Method

Motivation

Let $T_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$ represents the first Chern class. We shall consider the complex version of Hamilton's equation of the following type:

$$\frac{\partial h_{i\bar{j}}}{\partial t} = -\tilde{R}_{i\bar{j}} + T_{i\bar{j}}, h_{i\bar{j}} = g_{i\bar{j}}, \text{ at } t = 0 \tag{72}$$

$\tilde{R}_{i\bar{j}}$ denotes the Ricci tensor of metric $h_{i\bar{j}}$. If we can prove the solution exists for all long time and $h_{i\bar{j}}$ converges uniformly in t as t goes to infinity and that $\frac{\partial h_{i\bar{j}}}{\partial t}$ converges uniformly to a constant, then $h_{i\bar{j}}(\infty)$ is want we want. By global $\partial\bar{\partial}$- lemma we will have:

$$T_{i\bar{j}} = R_{i\bar{j}} + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} \tag{73}$$

And we let:

$$h_{i\bar{j}}(t) = g_{i\bar{j}} + \frac{\partial^2 u(t)}{\partial z^i \partial \bar{z}^j} \tag{74}$$

Then we will have:

$$\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\log\det(g_{k\bar{l}} + \frac{\partial^2 u(t)}{\partial z^k \partial \bar{z}^l}) - \log\det(g_{k\bar{l}})\right) + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} \tag{75}$$

Applying Maximal principle on compact manifolds:

$$\frac{\partial u}{\partial t} = \log\det(g_{k\bar{l}} + \frac{\partial^2 u(t)}{\partial z^k \partial \bar{z}^l}) - \log\det(g_{k\bar{l}}) + f + \varphi(t) \tag{76}$$

It is easy to see $\varphi(t)$ must obey the compatibility condition:

$$\int_M \exp(\frac{\partial u}{\partial t} - f)\mathrm{d}V_g = \exp(\varphi(t))\mathrm{Vol}(M) \tag{77}$$

According to the standard theory of nonlinear parabolic equation the solution exists for a short time. Let $\triangle_t$ be the laplacian correspondence to $h_{i\bar{j}}(t)$ And let $L_t = \triangle_t - \frac{\partial}{\partial t}$.

Long time existence

In this section we assume u to be the solution of the initial value problem:

$$\frac{\partial u}{\partial t} = \log\det(g_{k\bar{l}} + \frac{\partial^2 u(t)}{\partial z^k \partial \bar{z}^l}) - \log\det(g_{k\bar{l}}) + f \tag{78}$$
$$u(x,0) = 0$$

Differentiating with respect to t we have:

$$\frac{\partial^2 u}{\partial t^2} = \triangle_t\left(\frac{\partial u}{\partial t}\right) \tag{79}$$

It then follows from maximum principle:

$$\max_M\left|\frac{\partial u}{\partial t}\right| \leq \max_M |f| \tag{80}$$

We now normalize u

$$v := u - \frac{1}{\mathrm{Vol}(M)}\int_M u\mathrm{d}V_g \tag{81}$$

RMK: As we have seen in the continuity method that we required $\int_M \varphi = 0$

prop: Let $[0, T)$ be the maximal time interval. The C^∞ norm of v are uniformly bounded for all $t \in (0, T)$ and consequentely $T = \infty$. Moreover there exists a time sequence $t_n \to \infty$ such that $v(x, t_n)$ converges in C^∞ topology to a smooth function $v_\infty(x)$ on M as $n \to \infty$.

sketch of proof:

Differentiating on z^k ($\bar{z}^k$ yields similar result) we have:

$$L_t\left(\frac{\partial u}{\partial z^k}\right) = g^{i\bar{j}}\frac{\partial g_{i\bar{j}}}{\partial z^k} - h^{i\bar{j}}\frac{\partial_{i\bar{j}}}{\partial z^k} - \frac{\partial f}{\partial z^k} \tag{82}$$

Using Prior Estimate we will have:

1. $0 < m + \triangle u \le C_1 \exp\left(C_0\left(u - \inf_{M\times[0,T)} u\right)\right)$
2. $\sup_{M\times[0,T)} |v| < C_2$
3. Using (2) we have $0 < m + \triangle v = m + \triangle u \le C_3$
4. Now use Schauder estimate: $\sup_{M\times[0,T)} |\nabla v| \le C_4\left(\sup_{M\times[0,T)} |\triangle v| + \sup_{M\times[0,T)} |v|\right) \le C_5$
5. Using compatiblity condition we will have
$$\exp(-2\max_M |f|)\mathrm{d}V \le \mathrm{d}V_t = \exp\left(\frac{\partial u}{\partial t} - f\right)\mathrm{d}V \le \exp(2\max_M |f|)\mathrm{d}V \tag{83}$$
7. This shows in normal coordinate: $\prod_i^m(1 + u_{i\bar{i}}) \ge C_6$ which shows that $1 + u_{i\bar{i}} \ge \frac{C_6}{C_3^{m-1}}$.
8. As it was carried out in continiuty method on can compute $S = h^{i\bar{r}}h^{\bar{j}s}h^{k\bar{t}}v_{i\bar{j}k}v_{\bar{r}s\bar{t}}$. We can show $L_t(S + C_7\triangle v) \ge C_8 S - C_9$ ultimately S is bounded.

Using Schauder estimate and Arzela-Ascoli theorem we proof the theorem.

The uniform convergence

Now Assume u is the solution for the initial value problem:

$$\frac{\partial u}{\partial t} = \log\det(g_{k\bar{l}} + \frac{\partial^2 u(t)}{\partial z^k \partial \bar{z}^l}) - \log\det(g_{k\bar{l}}) + f \tag{84}$$
$$u(x, 0) = 0$$

Theorem 1: Let M be a compact manifold of dimension n and let $g_{ij}(t)$ $o \le t < \infty$ be a family of Riemanian metrics on M with the following properties:

1. $C_1 g_{ij}(0) \le g_{ij}(t) \le C_2 g_{ij}(0)$
2. $\left|\frac{\partial g_{ij}}{\partial t}\right| \le C_3 g_{ij}(0)$
3. $R_{ij}(t) \ge -K g_{ij}(0)$

where C_i, K is constant independent of t. Let $\triangle_t$ be the Laplacian of $g_{ij}(t)$. if $\varphi(x, t)$ is a solution of the equation:

$$\left(\triangle_t - \frac{\partial}{\partial t}\right)\varphi(x, t) = 0 \tag{85}$$

on $M \times [0, \infty)$, then for ant $\alpha > 1$, we have

$$\begin{aligned} &\sup_M \varphi(x, t_1) \\ &\leq \inf_M \varphi(x, t_2) \left(\frac{t_2}{t-1} \right)^{\frac{n}{2}} \exp\left(\frac{C_2^2 d^2}{4(t_2 - t_1)} + \left(\frac{n\alpha K}{2(\alpha-1)} + C_2 C_3 (n+A) \right) \right) \end{aligned} \tag{86}$$

where d is the diameter of m measured by $g_{ij}(0)$, $A = \sup \|\nabla^2 \log \varphi\|$ and $0 < t_1 < t_2 < \infty$.

Proposition As $t \to \infty, v(x,t)$ converges to the function $v_\infty(x)$ in C^∞ topology and that $\frac{\partial u}{\partial t}$ converges to a constant in C^∞ topology.

sketch of proof:

Step 1:

we show $h_{i\bar{j}}$ satisfies the aforementioned condition:

Let $F = \frac{\partial u}{\partial t}$. it is easy to see:

$$\begin{aligned} \left(\triangle_t - \frac{\partial}{\partial t} \right) F = 0 \\ F(x, 0) = f(x) \end{aligned} \tag{87}$$

It follows from maximal principle for parabolic equation that for $0 < t_1 < t_2$

$$\begin{aligned} \sup_M F(x, t_2) < \sup_M F(x, t_1) < \sup_M f \\ \inf_M F(x, t_2) > \inf_M F(x, t_1) > \inf_M f \end{aligned} \tag{88}$$

Using the result in (83) one can show that the conditions in *Theorem 1* are satisfied for $h_{i\bar{j}}$.

Step 2:

Define

$$\omega(t) = \sup_M F(x,t) - \inf_M F(x,t) \tag{89}$$

and shows that $\omega(t) \leq C_4 \exp(-at)$ for some constant C_4, a. Define

$$\begin{aligned} \varphi(x,t) = \frac{\partial u}{\partial t} - \frac{1}{\mathrm{Vol}(M)} \int_M \frac{\partial u}{\partial t} \mathrm{d}V_t \\ E(t) = \frac{1}{2} \int_M \varphi^2 \mathrm{d}V_t \end{aligned} \tag{90}$$

One has the estimate:

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -\frac{1}{2} \int_M |\nabla_t \varphi|^2 \mathrm{d}V_t \leq -\frac{\lambda_1(t)}{2} \int_M \varphi^2 \mathrm{d}V_t = -\lambda_1(t) E \tag{91}$$

$\lambda_1(t)$ is the principle eigenvalue of the operator $\triangle_t$. Since $h(t)$ is equivalent to g (independent of t) We know from spectral geometry that $K_2 \lambda_1(0) \leq \lambda_1(t) \leq K_1 \lambda_1(0)$. Henceforth we will have:

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} \leq C_4 E \\ E \leq C_5 \exp(-C_4 t) \end{aligned} \tag{92}$$

Moreover $\mathrm{d}V_t$ is uniformly equivalent to $\mathrm{d}V$ we also have

$$\int_M \varphi^2 \mathrm{d}V \le C_6 \exp(-C_4 t) \tag{93}$$

Step 3:

For any $0 < s < s'$ we have:

$$\int_M |v(x,s) - v(x,s')| \mathrm{d}V \le C_7 \int_s^\infty \exp(-2C_4 t) + C_8 \int_s^\infty \exp(-at) \tag{94}$$

This shows that as $t \to \infty$ are Cauchy in L^1 norm so $v(x,t)$ converges to $V_\infty(x)$in L^1. It is easy to show that V_∞ is actually v_∞ and $v(x,t)$ converges to v_∞ in C^∞ topology. Consequently $\frac{\partial u}{\partial t}$ converges to $\frac{\partial u}{\partial t}(x,\infty) = \log\det(g_{k\bar{l}} + \frac{\partial^2 v_\infty(t)}{\partial z^k \partial \bar{z}^l}) - \log\det(g_{k\bar{l}}) + f$. But the estimate of $\omega(t)$ shows $\frac{\partial u}{\partial t}(x,\infty) \equiv C$.

Main Theorem

We can find a smooth function u on $M \times [0,\infty)$ which solves the initial problem:

$$\frac{\partial u}{\partial t} = \log\det(g_{k\bar{l}} + \frac{\partial^2 u(t)}{\partial z^k \partial \bar{z}^l}) - \log\det(g_{k\bar{l}}) + f \tag{95}$$
$$u(x,0) = 0$$

We have show that $h_{i\bar{j}}(t) = g_{i\bar{j}} + \frac{\partial^2 v}{\partial z^i \partial \bar{z}^j}$ converges to smooth function $h_{i\bar{j}}(\infty)$ and

$$\frac{\partial}{\partial t}\frac{\partial^2 v}{\partial z^i \partial \bar{z}^j} = \frac{\partial}{\partial t}\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\frac{\partial u}{\partial t} \tag{96}$$

hence

$$\frac{\partial h_{i\bar{j}}}{\partial t}(\infty) = 0 \tag{97}$$

Differentiating (90)

$$\frac{\partial h_{i\bar{j}}}{\partial t} = -\tilde{R}_{i\bar{j}} + T_{i\bar{j}} \tag{98}$$

We conclude that $\tilde{R}_{i\bar{j}} = T_{i\bar{j}}$, which prove the theorem.

It should be remarked that the heat deformation method applies to the existence problem of *Kähler*-Einstein metrics on compact *Kähler* manifold M with negative Chern class. In this case the evolution equation for the metrics has the form:

$$\frac{\partial h_{i\bar{j}}}{\partial t} = -\tilde{R}_{i\bar{j}} - h_{i\bar{j}} \tag{99}$$

Using same method presented above one can show that the limit metric $h_{i\bar{j}}(\infty)$ is the required *Kähler* -Einstein metric.

Some Examples

In this section we are to discuss the first Chern class and Ricci forms of some curves and their compactification.

1. Hyper-elliptic curves $\{(x,y)\in\mathbb{C}^2|y^2=\prod_{k=1}^{2g+1}(x-a_k), a_k \text{ are distinct}\}$ and its compactification in $\mathbb{CP}^2$ (adding one point in the ∞). Using Riemann-Hurwitz formula we can show that the compatified curve is an orientable compact surface of genus g
2. Fermat Curves $\{(x,y)\in\mathbb{C}^2|x^d+y^d=1\}$. It compatification is given by adding d points $[1,\xi_d^j,0], 1\le j<d$ in $\mathbb{CP}^2$ where ξ_d is the d-th root of unity.The compatified curve is a orientable compact surface of genus $\frac{(d-1)(d-2)}{2}$

Compact Riemann surfaces, with a Hermitian metric are of course *Kähler*, for every 2-form on M is closed. In the perceeding examples we take the metric on M to be the pull back metric of $M\hookrightarrow\mathbb{CP}^2$. In local computation the *Kähler* form has the form of $\sqrt{-1}e^f\mathbf{d}z\wedge\mathbf{d}\bar{z}$, where f is some real function. *Kähler* form on $\mathbb{CP}^2$ can choose to $\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(1+|z|^2)=\frac{\sqrt{-1}}{2}\left(\frac{\mathbf{d}z^\beta\wedge\mathbf{d}\bar{z}^\beta}{1+|z|^2}-\frac{z^\alpha\bar{z}^\beta\mathbf{d}z^\beta\wedge\mathbf{d}\bar{z}^\alpha}{(1+|z|^2)^2}\right)$. Now we consider a coordinate chart $U\to\mathbb{CP}^2|x\mapsto[g,h,1]$or $x\mapsto[1,g,h]$ then the pullback of the *Kähler* form is(let $g'=\frac{\partial g}{\partial z}$(resp. h)) $\omega=\frac{\sqrt{-1}}{2}\frac{(|g'|^2+|h|^2+|gh'-g'h|^2)\mathbf{d}x\wedge\mathbf{d}\bar{x}}{(1+|g|^2+|h|^2)^2}$

Compactified Hyper-elliptic curves

Its image in $\mathbb{CP}^2$ can be covered by $\{[x,y,1]\},\{[1,y,z]\}$. There are three kinds of charts in Hyper-elliptic curves: 1. corrdinate balls which do not contain a_i,∞: $\psi_p^{-1}(x)=[p+x,\sqrt{\prod_{k=1}^{2g+1}(p+x-a_k)},1]$; 2. coordinate balls at a_i: $\psi_i^{-1}(x)=[a_i+x^2,x\sqrt{\prod_{k=1,k\neq i}^{2g+1}(x^2+a_i-a_k)},1]$; 3. at ∞: $\psi_\infty^{-1}(x)=[1,\frac{1}{x^{2g-1}}\sqrt{\prod_{k=1}^{2g+1}(1-a_kx^2)},x^2]$.

Compactified Fermat Curves

Its Image in $\mathbb{CP}^2$ can be covered by $\{[x,y,1]\},\{[1,y,z]\}$. There are also three kinds of charts in this curve: $p\neq\xi_d^j$, then $\psi_p^{-1}(x)\mapsto[x+p,\xi_d^j(1-(x+p)^d)^{1/d},1]$; $p=\xi_d^j$ then $\psi_j^{-1}(x)\mapsto[x^d+\xi_d^j,x(-\prod(z^d+\xi_d^j-\xi_d^k))^{1/d}]$; $p=\infty_j, 1\le j\le d$ $\psi_{\infty_j}^{-1}(x)=[1,\xi_d^j(1-x^d)^{1/d},x]$.

The Chern connection is $\mathbf{d}+\partial f$ and the curvature turns out to be $\bar{\partial}\partial f$. Henceforth the first Chern class is $-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f$. Moreover we can see that the Ricci form $\rho=-\sqrt{-1}\partial\bar{\partial}\log\det e^f=2\pi c_1(M)$. In this case $2\pi c_1(M)$ coincides with the Ricci form. Now by global $\partial\bar{\partial}$-lemma any representative of the first Chern class has the form $-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(f+\phi)$, where ϕ is a globally define real function. It is easy to see that $\tilde{\omega}=e^{f+\phi}\mathbf{d}z\wedge\mathbf{d}\bar{z}$ is the *Kähler* form whose Ricci form is exactly the very representative. In the case of Riemann surfaces F (as we have defined in the Contuity Method) has less restrction on the regularity(i.e. $F\in C^2(M)$ instead of $C^{3,\alpha}(M)$; or more generally $F\in W^{2,p}(M)$).

Remark Let $G=(g_{i\bar{j}})$ then the Ricci curvature of G and $tr(F_G)$ coincides. For

$$\begin{aligned}Ricci&=\bar{\partial}\partial\log\det(G)=\bar{\partial}\partial\log\det(\bar{G})\\&=\bar{\partial}tr(\bar{G}^{-1}\partial\bar{G})=-tr(\bar{G}^{-1}(\bar{\partial}G)\wedge\bar{G}^{-1}\partial\bar{G})+tr(\bar{G}^{-1}\bar{\partial}\partial\bar{G})\\&=tr(\mathbf{d}(\bar{G}^{-1}\partial\bar{G})+\bar{G}^{-1}\partial\bar{G}\wedge\bar{G}^{-1}\partial\bar{G})=tr(F_G)\end{aligned}\tag{100}$$

Also we observe that those curves are non-compact, yet the above computation shows that every representative of Riemann surface are the Ricci form of a $K\ddot{a}hler$ metric. Although the geometry on Riemann surfaces is rather simple, yet the above discussion leads me into much deepper questions.

Question 1 Can we generlize the Calabi-Yau Theorem to some non-compact cases?

Question 2 Under what circumstances can a $K\ddot{a}hler$ Manifold(M, g, J) embed into a compact $K\ddot{a}hler$ Manifold$(\tilde{M}, g, J)$? Actually it is easy to see that if M can be embedded in to a larger compact one, 1 automatically holds.

Question 3 If 2 holds what additional condition should be added such that $\tilde{M} - M$ is a finite union of subvarieties whose complex codimension are at least 1.? Since we have already seen that we we compactify Hyper-elliptic curves and Fermat curves by adding some points of infinity.

Question 4 What if the representative of the first Chern class is not that smooth can we still find out a $K\ddot{a}hler$ metric such that its Ricci form is precisely $2\pi c_1(M)$, for example if the complex Monge-Ampere equation admits a solution when $F \in C^{2,\alpha}$. More generally what if we define ∂F in the weak sense (i.e. define $\eta \in \Gamma(\bigwedge^p T^*M)$ (not necessarily smooth). $(\partial\eta, \lambda) = \int_M \partial\eta \wedge *\lambda = (\eta, \partial^*\lambda)$ for all smooth section λ in $\bigwedge^{p+1} T^*M$. In this case what will the existence and regularity problem of the correspoding Monge-Ampere equation be like.It should be remarked that in both Yau's and Cao's approaches the estimate of $\triangle' S$ is heavily depends on $F \in C^3$.

Some Survey on Complex Monge-Ampere Equation

This part follows a survey on Complex Monge-Ampere equations[6].

Dirichlert boundary problem Let M be a compact $K\ddot{a}hler$ manifold with smooth boundary Let $F(z, \varphi)$ be a smooth, strictly positive function of the variables z and φ, and let φ_b be a smooth function on ∂M. Consider the Dirichlet problem:

$$\frac{(\omega + i\partial\bar{\partial}\varphi)^n}{\omega^n} = F(z, \varphi), \varphi = \varphi_b \text{ on } \partial M \tag{101}$$

If $F(z, \varphi)_\varphi \geq 0$ and the problem admits a smooth subsolution, that is, a smooth function$\underline{\varphi}$ satisfying:

$$\frac{(\omega + i\partial\bar{\partial}\underline{\varphi})^n}{\omega^n} > F(z, \underline{\varphi}), \underline{\varphi} = \varphi_b \text{ on } \partial M \tag{102}$$

Then the Dirichlet problem admits a unique smooth solution . This theorem offers a partial solution to Question 1 when $F(z, \varphi) = F(z)$ be the corresonding potential function.

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