MODULAR INTERPRETATION OF CONFIGURATION SPACE $X(2,4)$

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Abstract. In this talk, I will describe the configuration space of $X(2,4)$ and $X\{2,4\}$ using Weierstrass elliptic function. Next I will use hyper-geometry function and periodic integral to show those configuration s-paces have intimate connections with tesselation of complex plane and representation of $\pi_1(\mathbf{C}-\{0,1\})$.

Some useful notation:

(1) $D_z = \frac{\mathrm{d}}{\mathrm{d}z}$
(2) $\theta_z = z\frac{\mathrm{d}}{\mathrm{d}z}$
(3) $(M)_k = M.(M+1)\ldots(M+k-1)$
(4) linear system $E(a,b,c;z)$: Euler-Gauss hyper-geometric differential equation $((\theta_z+a)(\theta_z+b)+(\theta_z+c)(\theta_z+1)\frac{1}{z})f(z)=0$ or $(z(1-z)D_z^2+(c-(a+b+1)z)D_z-ab)f(z)=0$
(5) hyper-geometric function: $F(a,b,c;z) := \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k$
(6) $\mathbf{\Gamma}(2) := \{g|g\in SL(2,Z), g\equiv I(mod\ 2)\}$

1. Preliminary

Theorem 1.1 (The local existence problem 1). *Let $B(0,r)$ be an open neighborhood of the origin. Let $(A(z)) = (a_{ij}(z))_{n\times n}$ be a matrix function, where $a_{ij}(z)$ is analytic in $B(0,r)$. Then the differential equation $D_zF(z)+A(z)F(z)$ assumes n linearly independent holomorphic solutions in $B(0,r)$.*

proof: observe that $\sum_{n=0}^{\infty}a_nz^n$ converges in $B(0,r)$ iff $\forall\rho\in(0,1)$ $\exists M_\rho$ such that $|a_n|\le M_\rho(\rho r)^{-n}$. Let $A(z)=\sum_{n=0}^{\infty}A_nz^n, A_n\in M(n,\mathbf{C})$. If such $F(z)=\sum_{n=0}^{\infty}F_nz^n, F_n\in\mathbf{C}^n$ exists, then it must yield to the following relation:

$$(n+1)F_{n+1}+\sum_{k=0}^{n}A_{n-k}F_k=0 \tag{1.1}$$

To prove this theorem it is suffice to show that such M_ρ exists for $||F_n|| := \sqrt{|x_1|^2+\cdots+|x_n|^2}$. Since $a_{ij}(z))$ is analytic in $B(0,r)$, $||A_n|| := \sqrt{tr(A_nA_n^*)} < M_\rho(\rho r)^{-n-1}$. We claim that $||F_k|| < \frac{(M_\rho)_k}{k!}(\rho r)^{-k}||F_0||$. The inequality holds when $n=0$, and by induction $||F_{n+1}|| < \frac{1}{n+1}\sum_{k=0}^{n}M_\rho(\rho r)^{-n-1}\frac{\Gamma(M_\rho+k)}{\Gamma(M_\rho)k!}||F_0||$

$=\frac{(M_\rho)_{n+1}}{(n+1)!}(\rho r)^{-n-1}||F_0||$. Moreover we observe that $(1-z)^{M_\rho} = \sum_{n=0}^{\infty} \frac{(M_\rho)_n}{n!} z^n$ converges in $B(0,1)$. So we obtain an inequality: $\frac{(M_\rho)_n}{n!} < N_\sigma \sigma^{-n}$, and henceforth $||F_n|| < N_{\sqrt{\rho}}||F_0||(\rho r)^{-n}$. Let $\{F(z)^{(k)}\}$ be the solution of the differential equation, and $\{F(0)^{(k)}\}$ is linearly independent. We conclude, by calculating their Wronskian, $\{F(z)^{(k)}\}$ are linearly independent for each $z \in B(0,1)$.

Theorem 1.2 (The local existence problem 2)**.** *Let $B(0,r)$ be an open neighborhood of the origin. Let $(B(z)) = (B_{ij}(z))_{n\times n}$ be a matrix function, where $b_{ij}(z)$ is analytic in $B(0,r)$. If B_0 is semisimple and does not have eigenvalues in negative integer, the differential equation $\theta_z F(z) + B(z)F(z)$ assumes n linearly independent solutions of the form $z^{s_i}G_i(z)$, where $det(s_iI + B_0) = 0$ and $G_i(z)$ is holomorphic in a neighborhood of the origin.*

Proof: One can use Frobenius method to show that the formal power series $F(z) = z^s \sum_{n=0}^{\infty} F_n z^n$ is the solution to $\theta_z F(z) + B(z)F(z)$ iff $(sI + B_0)F_0 = 0$ and $((s + n + 1)I + B_0)F_{n+1} + \sum_{k=0}^{n} B_{n+1-k}F_k = 0$. Since $c_1||A|| \leq sup_{v,||v||=1}||Av|| \leq c_2||A||$, we obtain $||((s_i + k + 1)I + B_0)^{-1}|| < K(k+1)^{-1}$, K is a constant. It is easy to show $||F_n|| < \frac{(M_\rho)_n K^n}{n!}(\rho r)^{-n}||F_0||$. The rest of the proof is trivial.

Theorem 1.3 (Analytic continuation along curves in a homotopy equivalent class)**.** *Let $F(z)$ is a local solution of a differential equation on an path-connected open subset Ω. If $\gamma_1(0) = \gamma_2(0) = z_0, \gamma_1(1) = \gamma_2(1) = z_1$, and if γ_1 is homotopy to γ_2 in Ω relative to z_0, z_1, the analytic continuation of $F(z)$ along γ_1, γ_2 give the same value at z_1.*

proof: Denote the homotopy between γ_1, γ_2 as $h : [0,1] \times [0,1] \to \Omega$. According to the local existence theorem, each point in Ω has an open neighborhood that admits a unique solution. This gives rise to an open covering of Ω. Since h is continuous, we can define an open covering on $[0,1] \times [0,1]$. Thanks to Lebsgue number lemma, one can divide this square equally into n^2 smaller ones$\{Q_{ij}\}$, each of which is contained in an element of the covering. By tube lemma, Q_{k1}, Q_{kn} can be chosen to lie in the preimage of the neighborhood of z_0, z_1 respectively. By induction F assumes a same value on the rightest boundary of $\cup_0^k Q_{ln}$.

Remark 1.4. This proposition implies that the analytic continuation induces a finite dimensional representation of $\pi_1(\Omega)$.

Theorem 1.5 (Schwartz reflection principal)**.**

proof: One can refer to any textbook on complex analysis.

2. Configuration spaces

Definition 2.1. *Define $X(k,n), k \leq n$ the quotient space of $GL(k)\backslash M^*(k,n)/H(n)$, where $M^*(k,n)$ is the subset of $M(k,n)$ with no degenerated k-minors. $H(n)$ consists of diagonal matrixs. Define $X\{k,n\}$ to be $S_n\backslash X(k,n)$.*

Remark 2.2. We can view $X(2,n)$ as the equivalent class of n distinct points on $\mathbf{CP^1}$ under mobius transformation.

In this talk I focus on the case when $n=4, k=2$, and to honest this is as far as I can go for the geometry of $X(n,k)$ is rather complicated.

Proposition 2.3. $X(2,4) \backsimeq \mathbf{C} - \{0,1\}$

proof: One can obtain a canonical element in the equivalent class.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \lambda \end{bmatrix} \tag{2.1}$$

Proposition 2.4. *S_4 acting on $X(2,4)$.*

proof: Let $N = \{(12)(34),(13)(24),(14)(23)\}$ $X(2,4)$ is invariant under N action. $(12) : \lambda \mapsto \frac{1}{\lambda}$, $(13) : \lambda \mapsto 1-\lambda$. $(123) : \lambda \mapsto 1 - \frac{1}{\lambda}$.

Lemma 2.5. *Let G be the subgroup of automorphism of rational function field and G is generated by $\frac{1}{\lambda}$, $1-\lambda$, then $\mathbf{C}(x)^G = \mathbf{C}(j(\lambda))$, where $j(\lambda) = \frac{4(\lambda^2-\lambda+1)^3}{27\lambda^2(1-\lambda)^2}$*

Hints: find out the subgroup of S_3.

Proposition 2.6. $X\{2,4\} \backsimeq \mathbf{C}$

proof: $j(\lambda) : \mathbf{C} - \{0,1\} \to \mathbf{C}$ is surjective (fundamental theorem of calculus). Injective of $j(\lambda)$ from $X\{2,4\}$ can be shown by studying the fix points of the group action.

Proposition 2.7. *Basic facts about Weierstrass elliptic function Let L be a lattice of rank 2 spanned by ω_1, ω_2 $\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L - \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$*

(1) $g_2 = 60 \sum_{\omega \in L-\{0\}} \frac{1}{\omega^4}$, $g_3 = 140 \sum_{\omega \in L-\{0\}} \frac{1}{\omega^6}$. $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.
(2) $\mathbf{C}/L \simeq \{z_1 : z_2 : z_3 | z_2^2 z_3 = 4z_1^3 - g_2 z_1 z_3^2 - g_3 z_3^3\}$, $z \mapsto \wp(z) : \wp'(z) : 1$
(3) Since $\wp'(z)$ is an odd elliptic function, then $\wp'(z)$ has three zeros at $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. And this implies $e_1 = \wp(\frac{\omega_1}{2})$, $e_2 = \wp(\frac{\omega_2}{2})$, $e_3 = \wp(\frac{\omega_1+\omega_2}{2})$ are the only three zeros of the cubic $4z^3 - g_2 z - g_3 = 0$.
(4) modular function: define $\tau = \frac{\omega_2}{\omega_1}$ and $\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}$.
(5) $\mathbf{SL(2,Z)}$ acting on λ. $\lambda(\tau+2) = \lambda(\frac{1}{2\tau+1}) = \lambda(\tau)$ and $\lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}$, $\lambda(\frac{-1}{\tau}) = 1 - \lambda(\tau)$. Henceforth this action can factor through $\mathbf{SL(2,Z)/\Gamma(2)}$.

Proposition 2.8. $\mathbf{H}/\mathbf{\Gamma}(2) \backsimeq \mathbf{C} - \{0,1\}$

proof: set $\omega_1 = 1$ then

$$e_3 - e_2 = \pi^2 \sum_{n=-\infty}^{\infty} \frac{1}{\cos^2 \pi(n-\frac{1}{2})\tau} - \frac{1}{\sin^2 \pi(n-\frac{1}{2})\tau} \tag{2.2}$$

$$e_1 - e_2 = \pi^2 \sum_{n=-\infty}^{\infty} \frac{1}{\cos^2 \pi n\tau} - \frac{1}{\sin^2 \pi(n-\frac{1}{2})\tau} \tag{2.3}$$

Use argument principle.

Proposition 2.9. $\mathbf{H/SL(2,Z) \simeq C}$

proof: trivial.

A natural question is if one can generalize this result, say, interpret $\mathrm{C}, \mathrm{C}-\{0,1\}$ as the quotient spaces of the subset of $\mathrm{CP}^1, \mathrm{C}, \mathrm{H}$ (or unit disc)? The answer to this question is "yes". Via hyper-geometric integrations we can construct such maps.

3. Hyper-geometric function and loaded path

One can check the solutions of $E(a,b,c;z)$ around $0, 1, \infty$ are the following:

Let $\xi = \frac{1}{z}$, $\zeta = 1 - z$.
And by computation one find:

(1) At z=0. B_0 :

$$\begin{bmatrix} 0 & -1 \\ 0 & c-1 \end{bmatrix} \tag{3.1}$$

So we obtain two solutions: $F(a,b,c;z)$,$z^{1-c}F(a+1-c, b+1-c, 2-c; z)$

(2) At z=1. B_0

$$\begin{bmatrix} 0 & -1 \\ 0 & a+b-c \end{bmatrix} \tag{3.2}$$

So we obtain two solution: $F(a,b,a+b+c-1;1-z)$, $(1-z)^{c-a-b}F(c-b, c-a, c-a-b+1; 1-z)$

(3) At $z = \infty$. B_0

$$\begin{bmatrix} 0 & -1 \\ ab & -a-b \end{bmatrix} \tag{3.3}$$

two solution would be $z^{-a}F(a, 1+a-c, a-b+1; \frac{1}{z})z^{-b}F(1+b-c, b, b-a+1; \frac{1}{z})$

Definition 3.1. *Define Schwartz map Pev :* $\mathbf{H} \longrightarrow \mathbf{CP^1}$ $z \mapsto f_1(z) : f_2(z)$, *where* f_1, f_2 *are two given linearly independent solutions of* $E(a,b,c;z)$.

We take a deeper look at the image of upper half plane under the Schwartz map. It should be noted here that given the parameters a, b, c Schwartz is unique up to Mobius transformations. If a, b, c are real it is easy to observe that there are two linearly independent real solutions of the aforementioned differential equation. We conclude that the Schwartz map maps

$(\infty,0),(0,1),(1,\infty)$ diffeomorphically into arcs on Riemann Sphere. And the image of $(\infty,0)\cup(0,1)\cup(1,\infty)$ is the boundary of a triangle. And via s-tudying the three singular points of the solutions of $E(a,b,c;z)$, we conclude that the angles at the singular points. They are:

(1) $\pi|1-c|$ at $f(0)$
(2) $\pi|c-a-b|$ at $f(1)$
(3) $\pi|a-b|$ at $f(\infty)$

If $|1-c|,|c-a-b|,|a-b|<1$ then by trivial observation and argument principle, the Schwartz maps are bi-holomorphic maps between upper half plane and Schwartz triangles. By reflection principle we can extend the domain of Schwartz maps to $\mathrm{H}\cup\bar{\mathrm{H}}\cup I$, I is one of the following: $(\infty,0),(0,1),(1,\infty)$. Here we do not want the triangles overlapping again and again, so it is convenient to assume the angles of the triangles to be the quotient of pi. $|1-c|=\frac{1}{p},|c-a-b|=\frac{1}{q},|a-b|=\frac{1}{r}$ where $p,q,r\in\{2,3,\dots\}\cup\{\infty\}$. There are three cases:

(1) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$

$$\begin{array}{ccc} p & q & r \\ 2 & 2 & n \\ 2 & 3 & 3 \\ 2 & 3 & 4 \\ 2 & 3 & 5 \end{array} \tag{3.4}$$

(2) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$

$$\begin{array}{ccc} p & q & r \\ 2 & 2 & \infty \\ 2 & 3 & 6 \\ 2 & 4 & 4 \\ 3 & 3 & 3 \end{array} \tag{3.5}$$

(3) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ there are infinitely many possibilities.

It is easy to see that the reflected images of the cases above are simple connected, and by uniformization theorem of Riemann Surfaces, we know that the reflected images are the following: $\mathrm{CP}^1,\mathrm{C},\mathrm{D}$. We state the following proposition without proof:

Proposition 3.2. *The Schwartz maps induce the following isomorphism;*

(1) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$

$$\mathrm{C}-\{0,1\}\backsimeq(\mathrm{CP}^1-\{\text{vertices of the triangle}\})/\Gamma(p,q,r) \tag{3.6}$$

(2) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$

$$\mathrm{C}-\{0,1\}\backsimeq(\mathrm{C}^1-\{\text{vertices of the triangle}\})/\Gamma(p,q,r) \tag{3.7}$$

(3) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$

$$\mathrm{C}-\{0,1\} \backsimeq (\mathrm{D}^1-\{\text{vertices of the triangle}\})/\Gamma(p,q,r) \tag{3.8}$$

$\Gamma(p,q,r)$ are so called triangle groups. They are given by the projective representations of the fundamental group of $\mathrm{C}-\{0,1\}$. Suppose $x\in \mathrm{H}, \gamma_0, \gamma_1$ are the simple loops based at x and go around $1,0$ counter-clockwise. Analytic continuation along the two loops give rise to the generators of the triangle groups. To see this more clearly we turn to the Schwartz reflection principle. Continuation along γ_0 is precisely reflect first the upper half plane with respect to $(\infty,0)$, and then its image with respect to $(0,1)$. Details of this proposition can be found in [2, 4]

The monodromy groups $\Gamma(2,3,\infty), \Gamma(\infty,\infty,\infty)$ are conjugate in $PGL(2,\mathrm{C})$ to $PSL(2,\mathrm{Z}), \Gamma(2)$ respectively [2]. Using these fact we can revisit the configuration spaces of $X(2,4), X\{2,4\}$.

Proposition 3.3. *The Schwartz map of hypergeometric equation $E(\frac{1}{2},\frac{1}{2},1)$ gives an isomorphism $X(2,4)\backsimeq \mathrm{C}-\{0,1\}\backsimeq \mathrm{H}/\Gamma(2)$ which is the inverse of the λ function; $E(\frac{1}{12},\frac{5}{12},)$ gives an isomorphism: $X\{2,4\}\backsimeq \mathrm{H}/PSL(2)$ which is the inverse map of the $J-$invariant function.*

Now we investigate further into the solutions of hypergeometric equations via integrate along some loop. This will eventually lead us to the configuration spaces of $X(2,n)$.

Proposition 3.4. *$\mathscr{R}c>\mathscr{R}a>0$ then*
$F(a,b,c;z)=\frac{\Gamma(c)}{\Gamma(a-1)\Gamma(c-a)}\int_0^1 t^a(1-t)^{c-a-1}(1-tz)^{-b}\mathrm{d}t$

proof: by direct computation.

It is better to get rid of the restriction of the real part of a,c. One can generalize this formula by integrating the integrand along Pochhammer loop around $0,1$ γ instead of interval $(0,1)$.

Let γ_1 be the path consists of $\overrightarrow{(\frac{1}{2},1-\epsilon)}$, counter-clockwise circle $C(1,\epsilon)$ and $\overrightarrow{(1-\epsilon,\frac{1}{2})}$. γ_2 and γ_1 are symmetric about $\mathscr{R}\mathscr{E}z=\frac{1}{2}$. Define γ to be $\gamma_1*\gamma_2^{-1}*\gamma_1^{-1}*\gamma_2$. One may observe that $[\gamma_1]$, $[\gamma_2]$ generate $\pi_1(\mathbf{C}-\{0,1\},\frac{1}{2})$, and that $[\gamma]$ is a commutator of this group.

Proposition 3.5. *let γ be the Pochhammer loop around any two point of $\{0,1,\frac{1}{z},\infty\}$, then $\int_\gamma t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\mathrm{d}t$ is a solution of $E(a,b,c;z)$.*

proof: Define Euler kernel $K(t,z,\lambda)=(1-tz)^\lambda$,$P(D_z)=(a+D_z)(b+D_z)-(c+D_z)(1+D_z)\frac{1}{z}$. $Q(t,\theta_t)=\sum a_j(t)D_t^j$ and $Q^*=\sum(-1)^jD_t^ja_j(t)$, $G(t)=t^{a-1}(1-t)^{c-a-1}$. Since the continuation of G along γ gives the same value. So we can integrate by part.

(3.9)
$$\int G(t)P(D_z)K(t,z,b) = \int G(t)Q(t,\theta_t)K(t,z,b{+}1) = \int Q^*(t,\theta_t)G(t)K(t,z,b{+}1) = 0$$

In order to study the the continuation of the preceding integrations, we introduce the hypergeometric integration.

$$I(x) := \int_C \Pi_{j=1}^n (x_{1j}t_1 + x_{2j}t_2)^{\beta_j} \mathrm{d}\log(\frac{(x_{1p}t_1 + x_{2p}t_2)}{(x_{1q}t_1 + x_{2q}t_2)}) \tag{3.10}$$

Where $\beta_1 + \ldots + \beta_n = 0$ and C is a undetermined path. We hope this integration is defined on $X(2,n)$, in another words, $I(x)$ is independent of the choice of the representatives of $\{x_{1j} : x_{2j}$. Computation shows that $I(gx) = I(x), g \in GL(2), I(xh) = I(x)\Pi h_j^{\beta_j}, h \in H_n$. However, we can amend this lapses by defining $J(x) := I(x)(D_x^\beta)^{-\frac{1}{n-2}}$, where $D_x^\beta = \Pi_{i\neq j}\det(D_x(ij))^{\beta_i}$,

$$D_x(ij) = \begin{bmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{1j} \end{bmatrix} \tag{3.11}$$

Now we study the canonical form of $X(2,4)$, let z_0 lies in the upper half plane and γ_0, γ_1 are counterclock -wise loops around $0, 1$

$$x = \begin{bmatrix} 1 & 0 & 1 & z \\ 0 & 1 & 1 & 1 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 1 & z_0 \\ \frac{z-z_0}{z_0(z-1)}) & 1 & 1 & 1 \end{bmatrix} \tag{3.12}$$

From this prospective, we turn the continuation problem of the integrand with respect to z to the deformation of the integrated path on the complex plane. To make this clearly, we introduce ”loaded path” which is the formal abelian group tensor with the integrand on the path. We briefly describe the results obtained from this observation [2].

Proposition 3.6. *Under some suitable basis of loaded path, the matrices of the representation are given by:*

$$\gamma_0 \mapsto \begin{bmatrix} c_1c_2 & 0 \\ c_2(1-c_1) & 1 \end{bmatrix} \tag{3.13}$$

$$\gamma_1 \mapsto \begin{bmatrix} c_1c_2 & 1-c_3 \\ 0 & c_2c_3 \end{bmatrix} \tag{3.14}$$

where $c_i = \exp(2\pi i a_i)$, $a_1 = a-1, a_2 = c-a-1, a_3 = -b$

Remark 3.7. For example if $a_i \in \frac{1}{2} + \mathbf{Z}$, they become

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{3.15}$$

which generate $\Gamma(2)$ up to ± 1

References

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