MORSE THEORY

(ref "Morse Theory by J.Milnor")

Morse Lemma Let p be a nondegenerated critical point of f on M, then there exists a chart such that $f = c - (x^1)^2 - \ldots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2$, where λ is called the index of f at p. We obtain a fact that non-degenerated cirtical points are isolated.

Homotopy type in Term of Critical Values

Def* $M^a = f^{-1}(-\infty, a]$

Thm 1 Let f be a smooth real value function on M, and suppose $f^{-1}[a,b]$ is compact and contains no critical point. Then M^a is differeomorphic to M^b. Moreover M^a is a deformation retract of M^b.

Sketch of proof: choose a Riemann metric on M and then Let $f^{-1}[a,b] \subset\subset U, df \neq 0$ in U. Choose $\rho = 1/\langle \nabla f, \nabla f \rangle$ in $f^{-1}[a,b]$ and *support* lies in U. Let $X = \rho \nabla f$ there exists a one parameter group ϕ_t associate to X. Then we have $\frac{df \circ \phi_t}{dt} = \langle X, \nabla f \rangle$=1. Now consider ϕ_{b-a}. It carries M^a diffeomorphically to M^b. And $r_t(p) = p$ if $f(p) \leq a$; $r_t(p) = \phi_{t(f(p)-a)}(p)$ else.

Thm 2 Let f be a smooth fucntion on M, and let p be a nondegenerated critical point with index λ. Setting $f(p) = c$, and suppose that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and contains no cirtical point other than p, for some $\epsilon > 0$. Then, for all sufficiently small ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with e^λ attached.

sketch of proof: First choose a nghd of p mentioned in Morse Lemma and ϵ as required and $e^\lambda = \{(x^1)^2 + \ldots + (x^\lambda)^2 < \epsilon, x^{\lambda+1} = \ldots = x^n = 0\}, B(p, \sqrt{2\epsilon}) \subset U$ in this Theorem. We claim that there exists a function F such that: 1, $F^{-1}[-\infty, c+\epsilon] = M^{c+\epsilon}$; 2,The cirtical point of F are the same as those of f; 3,The region $F^{c-\epsilon}$ is a deformation retract of $M^{c+\epsilon}$. Construction of $F = f - \mu((x^1)^2 + \ldots + (x^\lambda)^2 + 2(x^{\lambda+1})^2 + \cdots + 2(x^n)^2)$. Where $\mu(0) > \epsilon, \mu(r) = 0$ (for $r \geq 2\epsilon$) , $-1 < \mu' \leq 0$. Denote $F^{-1}(-\infty, c-\epsilon]$ as $M^{c-\epsilon} \cup H, H = \ldots$. Moreover we can prove that $M^{c-\epsilon} \cup e^\lambda$ is a defoemation retract of $M^{c-\epsilon} \cup H$.

RMK: M^c is homotopic to $M^{c+\epsilon}$.

Thm 3 If f is a diifferentiable function on a manifold M with no critical points, and if M^a is compact, then M have a homotopy type of a CW-complex, with one cell of dimension λ for each critical point of index λ.

The Morse Inequalities

Def $R_\lambda(X,Y)$ = λ th Betti number of (X,Y)=rank over F of $H_\lambda(X,Y;F)$.

Lemma 4: R_λ is subadditive since we have the exact sequence:
$\to H_\lambda(Y,Z) \to H_\lambda(X,Z) \to H_\lambda(X,Y) \to$

pf: Since dim is a additive function, we have
$dim(ker) - R_\lambda(Y,Z) + R_\lambda(X,Z) - R_\lambda(X,Y) + dim(im) = 0$.

def 5 Euler characteristic $\chi(X,Y) = \sum(-1)^\lambda R_\lambda(X,Y)$

Lemma 6: χ is additive.

pf
$\chi(X,Y) + \chi(Y,Z) = \sum_\lambda(-1)^\lambda[R_\lambda(X,Z) + dim(ker(\partial_\lambda)) + dim(im(\partial_{\lambda+1}))] = \chi(X,Z)$
.

Thm 7 Let M be a compact manifold and f be a differentiable function on M with isolated, nondegenerated, critical points, and let $a_1 < \ldots < a_k$ be such that M^{a_i} contains exactly i cirtical points (W.L.O.G. we can choose f satisfying this condition by perturbation). Then
$dimH_*(M^{a_i}, M^{a_{i-1}}) = dimH_*(M^{a_{-1}} \cup e^{\lambda_i}, M^{a_{i-1}}) = dimH_*(e^{\lambda_i}, \partial e^{\lambda_i}) = \delta_{\lambda *}$.
Let C_λ denote the number of critical point of index λ. Then we conclude the weak morse inequalities $R_\lambda(M) \leq \sum_{i=1}^k R_\lambda(M^{a_i}, M^{a_{i-1}}) = C_\lambda$. And
$\sum_\lambda(-1)^\lambda R_\lambda(M) = \chi(M) = \sum_{i=1}^k \chi(M^{a_i}, M^{a_{i-1}}) = \sum_\lambda(-1)^\lambda \sum_{i=1}^k R_\lambda(M^{a_i}, M^{a_{i-1}}) = \sum_\lambda(-1)^\lambda C_\lambda$
.

Thm 8 (the more subtle one) $S_\lambda(X,Y)$ is sub additive, where $S_\lambda(X,Y)=\sum_{i=0}^{\lambda}R_i(X,Y)(-1)^{\lambda-i}$. We obseve that $rank(\partial)=\sum_{i=0}^{\lambda}[R_i(X,Y)-R_i(X,Z)+R_i(Y,Z)](-1)^{\lambda-i}$ so we obtain $S_\lambda(Y,Z)-S_\lambda(X,Z)+S_\lambda(X,Y)\geq 0$. Applying this function on $\emptyset\subset M^{a_1}\subset\ldots\subset M^{a_k}$. And hence $S_\lambda(M)\leq\sum_{i=1}^{k}S_\lambda(M^{a_i},M^{a_{i-1}})=\sum_{k=1}^{\lambda}C_k(-1)^{\lambda-k}$. Henceforth we know that $\sum_{k=1}^{\lambda}C_k(-1)^{\lambda-k}\geq\sum_{k=1}^{\lambda}R_k(M)(-1)^{\lambda-k}$.

Cor 9 Suppose that $C_{\lambda+1}=0$ then we know that $R_{\lambda+1}=0$, by the inequality above we obtain $\sum_{k=1}^{\lambda}C_k(-1)^{\lambda-k}=\sum_{k=1}^{\lambda}R_k(M)(-1)^{\lambda-k}$. Also suppose that $C_{\lambda-1}=0$ so we can deduce that $R_\lambda=C_\lambda$.

Cor 10 Let $M_t(f)=\sum_{p\text{ critical}}t^{\lambda_p}=\sum_{i=0}^{n}C_it^i$ and $P_t(M,K)=\sum_i t^i dim(H_i(M,K))$. We know that $M_{-1}-P_{-1}=0$ and hence $M_t-P_t=(1+t)R(t)$ we use power series of $\frac{1}{1+t}$ and the subtle morse inequality we know that the coefficients of $R(t)$ are non-negative.

Thm 11 Let $\tilde{P}_t(M,K)=\sum_i t^i dim(H^i(M,K))$. $M_t-\tilde{P}_t=(1+t)\tilde{R}_t$, where $\tilde{R}_t$ is a polynomial with non-negative coefficients.

pf: using the axioms for cohomology alike, we deduce that $H^*(M^{a_i},M^{a_{i-1}})\backsimeq H^*(M^{a_{i-1}}\cup e^{\lambda_i},M^{a_{i-1}})\backsimeq H^*(e^{\lambda_i},\partial e^{\lambda_i})\backsimeq 0$ if $*\neq\lambda_i$; K if $*=\lambda_i$. Define $\tilde{\chi},\tilde{R}_\lambda$ alike, by the same reasoning we obtain $M_t-\tilde{P}_t=(1+t)\tilde{R}_t$.