MODULAR INTERPRETATION OF CONFIGURATION SPACE $X(2,4)$

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Abstract. In this talk, I will describe the configuration space of $X(2,4)$ and $X\{2,4\}$ using Weierstrass elliptic function. Next I will use hypergeometry function and periodic integral to show those configuration spaces have intimate connections with tesselation of complex plane and representation of $\pi_1(\mathbf{C}-\{0,1\})$.

Some useful notation:

(1) $D_z = \frac{\mathrm{d}}{\mathrm{d}z}$
(2) $\theta_z = z\frac{\mathrm{d}}{\mathrm{d}z}$
(3) $(M)_k = M.(M+1)\dots(M+k-1)$
(4) linear system $E(a,b,c;z)$: Euler-Gauss hyper-geometric differential equation $((\theta_z+a)(\theta_z+b)+(\theta_z+c)(\theta_z+1)\frac{1}{z})f(z)=0$ or $(z(1-z)D_z^2+(c-(a+b+1)z)D_z-ab)f(z)=0$
(5) hyper-geometric function: $F(a,b,c;z) := \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k$
(6) $\mathbf{\Gamma}(2) := \{g|g\in SL(2,Z), g\equiv I(mod\ 2)\}$

1. Preliminary

Theorem 1.1 (The local existence problem 1)**.** *Let $B(0,r)$ be an open neighborhood of the origin. Let $(A(z)) = (a_{ij}(z))_{n\times n}$ be a matrix function, where $a_{ij}(z)$ is analytic in $B(0,r)$. Then the differential equation $D_zF(z)+A(z)F(z)$ assumes n linearly independent holomorphic solutions in $B(0,r)$.*

proof: observe that $\sum_{n=0}^{\infty}a_nz^n$ converges in $B(0,r)$ iff $\forall\rho\in(0,1)\ \exists M_\rho$ such that $|a_n|\le M_\rho(\rho r)^{-n}$. Let $A(z)=\sum_{n=0}^{\infty}A_nz^n, A_n\in M(n,\mathbf{C})$. If such $F(z)=\sum_{n=0}^{\infty}F_nz^n, F_n\in\mathbf{C}^n$ exists, then it must yield to the following relation:

$$(n+1)F_{n+1}+\sum_{k=0}^{n}A_{n-k}F_k=0 \tag{1.1}$$

To prove this theorem it is suffice to show that such M_ρ exists for $||F_n|| := \sqrt{|x_1|^2+\dots+|x_n|^2}$. Since $a_{ij}(z))$ is analytic in $B(0,r)$, $||A_n|| := \sqrt{tr(A_nA_n^*)} < M_\rho(\rho r)^{-n-1}$. We claim that $||F_k|| < \frac{(M_\rho)_k}{k!}(\rho r)^{-k}||F_0||$. The inequality holds when $n=0$, and by induction $||F_{n+1}|| < \frac{1}{n+1}\sum_{k=0}^{n}M_\rho(\rho r)^{-n-1}\frac{\Gamma(M_\rho+k)}{\Gamma(M_\rho)k!}||F_0||$

$=\frac{(M_\rho)_{n+1}}{(n+1)!}(\rho r)^{-n-1}||F_0||$. Moreover we observe that $(1-z)^{M_\rho}=\sum_{n=0}^{\infty}\frac{(M_\rho)_n}{n!}z^n$ converges in $B(0,1)$. So we obtain an inequality: $\frac{(M_\rho)_n}{n!}<N_\sigma\sigma^{-n}$, and henceforth $||F_n||<N_{\sqrt{\rho}}||F_0||(\rho r)^{-n}$. Let $\{F(z)^{(k)}\}$ be the solution of the differential equation, and $\{F(0)^{(k)}\}$ is linearly independent. We conclude, by calculating their Wronskian, $\{F(z)^{(k)}\}$ are linearly independent for each $z\in B(0,1)$.

Theorem 1.2 (The local existence problem 2)**.** *Let $B(0,r)$ be an open neighborhood of the origin. Let $(B(z))=(B_{ij}(z))_{n\times n}$ be a matrix function, where $b_{ij}(z)$ is analytic in $B(0,r)$. If B_0 is semisimple and does not have eigenvalues in negative integer, the differential equation $\theta_z F(z)+B(z)F(z)$ assumes n linearly independent solutions of the form $z^{s_i}G_i(z)$, where $det(s_iI+B_0)=0$ and $G_i(z)$ is holomorphic in a neighborhood of the origin.*

Proof: One can use Frobenius method to show that the formal power series $F(z)=z^s\sum_{n=0}^{\infty}F_nz^n$ is the solution to $\theta_zF(z)+B(z)F(z)$ iff $(sI+B_0)F_0=0$ and $((s+n+1)I+B_0)F_{n+1}+\sum_{k=0}^{n}B_{n+1-k}F_k=0$. Since $c_1||A||\le sup_{v,||v||=1}||Av||\le c_2||A||$, we obtain $||((s_i+k+1)I+B_0)^{-1}||<K(k+1)^{-1}$, K is a constant. It is easy to show $||F_n||<\frac{(M_\rho)_nK^n}{n!}(\rho r)^{-n}||F_0||$. The rest of the proof is trivial.

Theorem 1.3 (Analytic continuation along curves in a homotopy equivalent class)**.** *Let $F(z)$ is a local solution of a differential equation on an path-connected open subset Ω. If $\gamma_1(0)=\gamma_2(0)=z_0,\gamma_1(1)=\gamma_2(1)=z_1$, and if γ_1 is homotopy to γ_2 in Ω relative to z_0,z_1, the analytic continuation of $F(z)$ along γ_1,γ_2 give the same value at z_1.*

proof: Denote the homotopy between γ_1,γ_2 as $h:[0,1]\times[0,1]\to\Omega$. According to the local existence theorem, each point in Ω has an open neighborhood that admits a unique solution. This gives rise to an open covering of Ω. Since h is continuous, we can define an open covering on $[0,1]\times[0,1]$. Thanks to Lebsgue number lemma, one can divide this square equally into n^2 smaller ones$\{Q_{ij}\}$, each of which is contained in an element of the covering. By tube lemma, Q_{k1}, Q_{kn} can be chosen to lie in the preimage of the neighborhood of z_0,z_1 respectively. By induction F assumes a same value on the rightest boundary of $\cup_0^kQ_{ln}$.

Remark 1.4. This proposition implies that the analytic continuation induces a finite dimensional representation of $\pi_1(\Omega)$.

Theorem 1.5 (Schwartz reflection principal)**.**

proof: One can refer to any textbook on complex analysis.

2. Configuration spaces

Definition 2.1. *Define $X(k,n),k\le n$ the quotient space of $GL(k)\backslash M^*(k,n)/H(n)$, where $M^*(k,n)$ is the subset of $M(k,n)$ with no degenerated k-minors. $H(n)$ consists of diagonal matrixs. Define $X\{k,n\}$ to be $S_n\backslash X(k,n)$.*

Remark 2.2. We can view $X(2,n)$ as the equivalent class of n distinct points on $\mathbf{CP^1}$ under mobius transformation.

In this talk I focus on the case when $n=4, k=2$, and to honest this is as far as I can go for the geometry of $X(n,k)$ is rather complicated.

Proposition 2.3. $X(2,4) \backsimeq \mathbf{C} - \{0,1\}$

proof: One can obtain a canonical element in the equivalent class.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \lambda \end{bmatrix} \tag{2.1}$$

Proposition 2.4. *S_4 acting on $X(2,4)$.*

proof: Let $N = \{(12)(34), (13)(24), (14)(23)\}$ $X(2,4)$ is invariant under N action. $(12) : \lambda \mapsto \frac{1}{\lambda}$, $(13) : \lambda \mapsto 1-\lambda$. $(123) : \lambda \mapsto 1 - \frac{1}{\lambda}$.

Lemma 2.5. *Let G be the subgroup of automorphism of rational function field and G is generated by $\frac{1}{\lambda}$, $1-\lambda$, then $\mathbf{C}(x)^G = \mathbf{C}(j(\lambda))$, where $j(\lambda) = \frac{4(\lambda^2-\lambda+1)^3}{27\lambda^2(1-\lambda)^2}$*

Hints: find out the subgroup of S_3.

Proposition 2.6. $X\{2,4\} \backsimeq \mathbf{C}$

proof: $j(\lambda) : \mathbf{C} - \{0,1\} \to \mathbf{C}$ is surjective (fundamental theorem of calculus). Injective of $j(\lambda)$ from $X\{2,4\}$ can be shown by studying the fix points of the group action.

Proposition 2.7. *Basic facts about Weierstrass elliptic function Let L be a lattice of rank 2 spanned by ω_1, ω_2 $\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L-\{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$*

(1) $g_2 = 60 \sum_{\omega \in L-\{0\}} \frac{1}{\omega^4}$, $g_3 = 140 \sum_{\omega \in L-\{0\}} \frac{1}{\omega^6}$. $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.
(2) $\mathbf{C}/L \simeq \{z_1 : z_2 : z_3 | z_2^2 z_3 = 4z_1^3 - g_2 z_1 z_3^2 - g_3 z_3^3\}$, $z \mapsto \wp(z) : \wp'(z) : 1$
(3) Since $\wp'(z)$ is an odd elliptic function, then $\wp'(z)$ has three zeros at $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. And this implies $e_1 = \wp(\frac{\omega_1}{2})$, $e_2 = \wp(\frac{\omega_2}{2})$, $e_3 = \wp(\frac{\omega_1+\omega_2}{2})$ are the only three zeros of the cubic $4z^3 - g_2 z - g_3 = 0$.
(4) modular function: define $\tau = \frac{\omega_2}{\omega_1}$ and $\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}$.
(5) $\mathbf{SL(2,Z)}$ acting on λ. $\lambda(\tau+2) = \lambda(\frac{1}{2\tau+1}) = \lambda(\tau)$ and $\lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}$, $\lambda(\frac{-1}{\tau}) = 1 - \lambda(\tau)$. Henceforth this action can factor through $\mathbf{SL(2,Z)/\Gamma(2)}$.

Proposition 2.8. $\mathbf{H}/\mathbf{\Gamma}(2) \backsimeq \mathbf{C} - \{0,1\}$

proof: set $\omega_1 = 1$ then

$$e_3 - e_2 = \pi^2 \sum_{n=-\infty}^{\infty} \frac{1}{\cos^2 \pi(n-\frac{1}{2})\tau} - \frac{1}{\sin^2 \pi(n-\frac{1}{2})\tau} \tag{2.2}$$

$$e_1 - e_2 = \pi^2 \sum_{n=-\infty}^{\infty} \frac{1}{\cos^2 \pi n\tau} - \frac{1}{\sin^2 \pi(n-\frac{1}{2})\tau} \tag{2.3}$$

Use argument principle.

Proposition 2.9. $\mathbf{H/SL(2,Z)} \simeq \mathbf{C}$

proof: trivial.

A natural question is if one can generalize this result, say, interpret $\mathrm{C}, \mathrm{C}-\{0,1\}$ as the quotient spaces of the subset of $\mathrm{CP}^1, \mathrm{C}, \mathrm{H}$ (or unit disc)? The answer to this question is "yes". Via hyper-geometric functions we can construct such maps.

3. Hyper-geometric function and loaded path

One can check the solutions of $E(a,b,c;z)$ around $0, 1, \infty$ are the following:

Let $\xi = \frac{1}{z}$, $\zeta = 1 - z$.
And by computation one find:

(1) At z=0. B_0 :

$$\begin{bmatrix} 0 & -1 \\ 0 & c-1 \end{bmatrix} \tag{3.1}$$

So we obtain two solutions: $F(a,b,c;z)$,$z^{1-c}F(a+1-c, b+1-c, 2-c;z)$

(2) At z=1. B_0

$$\begin{bmatrix} 0 & -1 \\ 0 & a+b-c \end{bmatrix} \tag{3.2}$$

So we obtain two solution: $F(a,b,a+b+c-1;1-z)$, $(1-z)^{c-a-b}F(c-b, c-a, c-a-b+1; 1-z)$

(3) At $z = \infty$. B_0

$$\begin{bmatrix} 0 & -1 \\ ab & -a-b \end{bmatrix} \tag{3.3}$$

two solution would be $z^{-a}F(a, 1+a-c, a-b+1; \frac{1}{z})z^{-b}F(1+b-c, b, b-a+1; \frac{1}{z})$

Definition 3.1. *Define Schwartz map Pev :* $\mathbf{H} \longrightarrow \mathbf{CP^1}$ $z \mapsto f_1(z) : f_2(z)$, *where* f_1, f_2 *are two given linearly independent solutions of* $E(a,b,c;z)$.

We take a deeper look at the image of upper half plane under the Schwartz map. It should be noted here that given the parameters a, b, c Schwartz is unique up to Mobius transformations. If a, b, c are real it is easy to observe that there are two linearly independent real solutions of the aforementioned differential equation. We conclude that the Schwartz map maps

$(\infty,0),(0,1),(1,\infty)$ diffeomorphically into arcs on Riemann Sphere. And the image of $(\infty,0)\cup(0,1)\cup(1,\infty)$ is the boundary of a triangle. And via studying the three singular points of the solutions of $E(a,b,c;z)$, we conclude that the angles at the singular points. They are:

(1) $\pi|1-c|$ at $f(0)$
(2) $\pi|c-a-b|$ at $f(1)$
(3) $\pi|a-b|$ at $f(\infty)$

If $|1-c|,|c-a-b|,|a-b|<1$ then by trivial observation and argument principle, the Schwartz maps are bi-holomorphic maps between upper half plane and Schwartz triangles.

Proposition 3.2. *$\mathscr{R}\mathscr{E}c > \mathscr{R}\mathscr{E}a > 0$ then*
$F(a,b,c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^a(1-t)^{c-a-1}(1-tz)^{-b}\mathrm{d}t$

proof: by direct computation.

I would like to get rid of the restriction of the real part of a,c. One can generalize this formula by integrating the integrand along Pochhammer loop around $0,1$ γ instead of interval $(0,1)$.

Let γ_1 be the path consists of $\overrightarrow{(\frac{1}{2},1-\epsilon)}$, counter-clockwise circle $C(1,\epsilon)$ and $\overrightarrow{(1-\epsilon,\frac{1}{2})}$. γ_2 and γ_1 are symmetric about $\mathscr{R}\mathscr{E}z=\frac{1}{2}$. Define γ to be $\gamma_1*\gamma_2^{-1}*\gamma_1^{-1}*\gamma_2$. One may observe that $[\gamma_1]$, $[\gamma_2]$ generate $\pi_1(\mathbf{C}-\{0,1\},\frac{1}{2})$, and that $[\gamma]$ is a commutator of this group.

Proposition 3.3. *let γ be the Pochhammer loop around any two point of $\{0,1,\frac{1}{z},\infty\}$, then $\int_\gamma t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\mathrm{d}t$ is a solution of $E(a,b,c;z)$.*

proof: Define Euler kernel $K(t,z,\lambda)=(1-tz)^\lambda P(D_z)=(a+D_z)(b+D_z)-(c+D_z)(1+D_z)\frac{1}{z}$. $Q(t,\theta_t)=\sum a_j(t)D_t^j$ and $Q^*=\sum(-1)^jD_t^ja_j(t)$, $G(t)=t^{a-1}(1-t)^{c-a-1}$. Since the continuation of G along γ gives the same value. So we can integrate by part.

(3.4)
$$\int G(t)P(D_z)K(t,z,b)=\int G(t)Q(t,\theta_t)K(t,z,b{+}1)=\int Q^*(t,\theta_t)G(t)K(t,z,b{+}1)=0$$

References

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