The topology of the gauge group

Fiber bundles associated to principal bundles

Let P be a principals G-bundle over X; the action of G on itself be the adjoint action (i.e. $g.q = g^{-1}qg$). Define $Ad(P) := P \times_G G$, it is a fiber bundle over X. We define the "Gauge group" of principal bundle P as the smooth section from the based space to the associate bundle.

$$\mathscr{G}(P) := \Gamma(Ad(P))$$

$\Gamma(Ad(P))$ forms a group under pointwise action multiplication of the section at each fiber. It can infact be identified with the group of G-equivariant smooth maps from P to G (adjoint action). To see this

$$[(p, g_p)] \leftrightarrow f : p \mapsto g_p$$

it is G equivariant.

Moreover it can be shown that Gauge group can be identified with $Aut(P)$.

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$$\begin{tikzcd}
P \arrow[rr, "F"] \arrow[dr, "\pi"] & & P \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

$F : P \mapsto P | p \mapsto pf(p)$ F is G equivariant (in the sense of right multiplication). On the other hand in local coordinate $F(x, g) = (x, F(x)g)$, $F(x)$ is the deasired section. The following section is to describe the topology of Classifying space $B\mathscr{G}(P)$ when M is Riemann surfaces and G is the unitary group.

Classifying space

Recall the definition of classifying space of a group G is a topological space BG equipped with a principal G bundle EG where EG is weakly contractible. Then We have the following properties:

1. If BG exists then BG can be chose to be a CW-complex. (Otherwise one can construct a CW complex that is weak equivalent to BG)
2. if X is a CW-complex then $[X, BG] \leftrightarrow P_G(X)$ the set of homotopy map from X to BG is isomorphic to the set of $P_G(X)$ the set of isomorphic equivalent class of principal G bundles over X. In fancy words BG represents the functor $P_G : CW \to Set$.

Proposition 7.1 *Let X be an arbitrary space, P a principal G-bundle over X. Suppose that B is a CW-complex and that $f, g : B \longrightarrow X$ are homotopic maps. Then the pullbacks f^*P, g^*P are isomorphic as principal G-bundles over B.*

Let $F : B \times I \longrightarrow X$ be a homotopy from f to g. By considering the pullback F^*P, we reduce at once to proving the following lemma:

Lemma 7.2 *Let $Q \longrightarrow B \times I$ be a principal G-bundle, Q_0 its restriction to $B \times 0$. Then Q is isomorphic to $Q_0 \times I$. In particular Q_0 is isomorphic to Q_1.*

Proof: By Proposition 2.1, it is enough to construct a morphism $Q \longrightarrow Q_0 \times I$. By Proposition 6.1, this is equivalent to constructing a section s of $Q \times_G (Q_0 \times I) \longrightarrow B \times I$. But by Proposition 6.1 again, we have a section s_0 on B_0. By a general property of Serre fibrations, any section defined over B_0 extends to a section over $B \times I$. This completes the proof of the Lemma, and of Proposition 7.1.

In other words, $B \mapsto \mathcal{P}_G(B)$ is a homotopy functor from CW-complexes to sets. Thus a homotopy equivalence induces a bijection on $\mathcal{P}_G(-)$. In particular:

Theorem 7.4 *Suppose $P \longrightarrow B$ is a principal G-bundle with P weakly contractible. Then for all CW-complexes X, the map $\phi : [X, B] \longrightarrow \mathcal{P}_G X$ given by $f \mapsto f^*P$ is bijective.*

We then call B a *classifying space* for G, and P a *universal G-bundle*. We will see below that the converse of Theorem 7.4 holds also.

Proof: Suppose P is weakly contractible. We first show ϕ is onto. Let $Q \longrightarrow B$ be a principal G-bundle. Then the Serre fibration $Q \times_G P \longrightarrow B$ has weakly contractible fibre and therefore admits a section. By Proposition 6.1, this section determines a G-equivariant map $\tilde{f} : Q \longrightarrow P$. Let $f : B \longrightarrow X$ denote the induced map on orbit spaces. Then $Q \cong f^*P$, as desired.

Now suppose given maps $f_0, f_1 : B \longrightarrow X$ and an isomorphism $\psi : f_0^*P \xrightarrow{\cong} f_1^*P$. Let Q denote the principal G-bundle $(f_0^*P) \times I$ over $B \times I$, and consider the local product $\rho : Q \times_G P \longrightarrow B \times I$. The equivariant maps $Q_0 = f_0^*P \longrightarrow P$ and $Q_1 = f_0^*P \xrightarrow{\psi} f_1^*P \longrightarrow P$ define a section of ρ over $B \times 0 \cup B \times 1$. Since the fibre is weakly contractible, this section extends over all of $B \times I$, and so determines a G-map $Q \longrightarrow P$. Passing to orbit spaces yields a homotopy $B \times I \longrightarrow X$ from f_0 to f_1. This shows ϕ is injective.

Theorem 5.1 *In the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] \arrow[ur, dotted, "h"] & B \end{tikzcd}$$

suppose that p is Serre fibration and i is a subcomplex inclusion. Then if either i or p is a weak equivalence, the filler h exists.

Proof: Suppose first that p is a weak equivalence. We will construct h inductively over $X^n \cup A$. The case $n = 0$ is easy, since X^0 is discrete. At the inductive step, we reduce to the special case

$$\begin{tikzcd} S^{n-1} \arrow[r, "f_\alpha"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ D^n \arrow[r, "g_\alpha"'] \arrow[ur, dotted, "h_\alpha"] & B \end{tikzcd}$$

Here $g_\alpha = g \circ \phi_\alpha$ and $f_\alpha = h^{n-1} \circ \psi_\alpha$, where ϕ_α, ψ_α are respectively the characteristic map and attaching map for a typical n-cell e^n_α.

Now any map $D^n \longrightarrow B$ is homotopic rel S^{n-1} to a map that is constant on $D^n(1/2)$, the disc of radius $1/2$. In view of Proposition 2.5, we may therefore assume that $g_\alpha(D^n(1/2)) \equiv b_0$ for some $b_0 \in B$. Let W denote the annulus consisting of $\{x \in D^n : 1/2 \leq\mid x \mid\leq 1\}$. Then W is homeomorphism to $S^{n-1} \times I$. Since p is a Serre fibration, there is a lift h'_α defined on W. Now observe that h'_α maps the sphere of radius $1/2$ into the fibre $p^{-1}b_0$. Since p is a weak equivalence, the long exact homotopy sequence shows that this fibre is weakly contractible. Hence h'_α extends to a map $h_\alpha : D^n \longrightarrow E$, and by construction it lifts g_α. This completes the proof in the case p is a weak equivalence.

Now suppose i is a weak equivalence. Then by Theorem 2.6, A is a deformation retract of X. Let r denote the retraction. Since $ir \sim 1_X$ rel A, $gir \sim g$ rel A. But gir clearly admits a lift in the diagram—namely, fr—and hence g lifts by Proposition 2.5.

3. Moreover BG is unique up to homotopy in CW because of Yoneda's Lemma: $\mathrm{Nat}(Hom(-, BG), Hom(-, \tilde{B}G)) = Hom(BG, \tilde{B}G)$ And henceforth this two spaces are homotopic.

We use $\mathrm{Map}_P(M, BG)$ to denote the map that induces the principal bundle P over M. In the view of the aforementioned properties We know that $\mathrm{Map}_P(M, BG)$ is a connected component in $\mathrm{Map}(M, BG)$.

prop let BG be the classifying space for G the in homotopy theory $B\mathscr{G}(P) = \mathrm{Map}_P(M, BG)$.

proof: Let $\mathrm{Map}_\mathrm{G}(P, EG)$ be the G equivariant map. And $\mathscr{G}(P)$ acts naturally by $\phi.h = h \circ \phi$ (Automorphism).

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$$
\begin{tikzcd}
P \arrow[r, "h"] \arrow[d] \arrow[dr, dotted, "\tilde{h}"] & EG \arrow[d] \\
M = P/G \arrow[r, dotted, "\bar{h}"] & BG = EG/G
\end{tikzcd}
$$

$\tilde{h}$ is the compositioin of h and the projection. Since $\tilde{h}$ is G-invarian, it induces $\bar{h} : M \to BG$ And $P = \bar{h}^* EG$. It is easy to see that $h \circ \phi, \phi \in Aut(P)$ induces the same map as h, which implies $\mathrm{Map}_P(M, BG) = \mathrm{Map}_{\mathrm{G}}(P, EG)/\mathscr{G}(P)$. If BG is paracompact and locally contractible, which is easily arranged, the quotient map will be a locally trivial fibring, as follows easily from the homotopy properties of firings. The total space $\mathrm{Map}_{\mathrm{G}}(P, E)$ is contractible such that this is a universal bundle for $\mathscr{G}(P)$, and $B\mathscr{G}(P) = \mathrm{Map}_{\mathrm{P}}(M, BG)$ as was asserted.

Case $U(1)$: The unitray group $U(1)$

$BU(1) = \mathbb{CP}^\infty, EU(1) = V_1^U(\mathbb{C}^\infty)$ the later is one-dimensional subspace equibed with unitray frame of the subspace. The following serre fibration $S^1 \to V_1^U(\mathbb{C}^\infty) \to \mathbb{CP}^\infty$ induces long exact sequence of homotopy groups:

$$\to \pi_{k+1}(\mathbb{CP}^\infty) \to \pi_k(S^1) \to \pi_k(V_1^U(\mathbb{C}^\infty)) \to \pi_k(\mathbb{CP}^\infty) \to$$

We deduce that $\mathbb{CP}^\infty$ is a Eilenberg-Maclane space of type $K(\mathbb{Z}, 2)$. To compute the homotopy type of $\mathscr{G}(U(1))$, we need the following theorem.

Thorem (Thom). Let $Y = K(\pi, n)$ then $\mathrm{Map}(X, Y) = \Pi_q K(H^q(X, \pi); n - q)$, where X is a CW complex with finite cells.

proof: We first look into the cases when $X = S^q$. We fix a base point in each S^k and in Y. And suppose $K(n, \pi)$ is path connected (It is obvious that one can easily achieve this when $n > 1$, and when $n = 0$ and π disceret, $K(n, \pi) = \pi$. We will obmit this case in the following discussion) then $\mathrm{Map}(X, Y)$ is homotopy to $\mathrm{Map}_*(X, Y) \times Y$. To see this we adopt (unreduced) suspension model for $S^q = \Sigma S^{q-1}$. There is a deformation of $\mathrm{Map}(X, Y)$ to its subset $Map_1 = \{f \in \mathrm{Map}(X, Y) | f(x, t) = f(x, 0), \forall t \le \frac{1}{2}\}$.

$$F : \text{Map} \times I \to \text{Map}_1$$
$$F(f,s)(x,t) = f(x,(1+s)(t-s)), t \geq s$$
$$F(f,s)(x,t) = f(x,0), t \leq s$$

Now assign each point p of Y a continous curve γ_p from the based point to p. Obsever that:

$$G : \text{Map}_1 \times I \to \text{Map}_*$$
$$G(f,s)(x,t) = f(x,t), t \geq s/2$$
$$G(f,s)(x,t) = \gamma_{f(x,0)}(1-2t), t \leq s/2$$

Map_* is adjoint to suspension which implies

$$\text{Map}_*(S^p, \text{Map}_*(S^q, Y)) = \text{Map}_*(S^p \wedge S^q, Y) = \text{Map}_*(S^{p+q}, Y)$$

Reduce to homotopy equivalence, it turns out that $\text{Map}_*(S^q, Y)$ happens to be the Eilenberg-Maclane space $K(\pi, n-q)$. Combining with the preceeding results, this prove Thom theorem when X is a sphere.

Suppose now that $X = X_1 \sqcup X_2$ is disjoint union of two CW-complexes. Since $\text{Map}(X,Y) = \text{Map}(X_1,Y) \times \text{Map}(X_2,Y)$ and $K(H^q(X,\pi), n-q) = K(H^q(X_1,\pi), n-q) \times K(H^q(X_2,\pi), n-q)$. The theorem reduces to X is a connected CW-complex...(I do not know how to prove this but I found a proof in MSE)

We claim that $\text{Map}(X,Y)$ (up to homotopy equivalence) and $\Pi_q K(H^q(X,\pi), n-q)$ both represent a functor from CW to *Abel Group*, and Yoneda's Lemma says that the spaces are homotopy equivalence. First:

$$[Z, \text{Map}(X, K(\pi,n))] \simeq [Z \times X, K(\pi,n)] \simeq H^n(Z \times X, \pi)$$
$$[Z, \Pi_q K(H^q(X,\pi); n-q)] = \bigoplus_q [Z, K(H^q(X,\pi); n-q)] \simeq \bigoplus_q H^{n-q}(Z, H^q(X,\pi))$$

$[Z, K(\pi,n)] = H^n(Z,\pi)$ comes from the axiomatic description of cohomology theory in CW complex. (Dimension; Exactness; Excision; Additivity; Equivalence)

For Riemann surface M of genus g this yields $\text{Map}_P(M, BG) = S^1 \times \ldots \times S^1 \times \mathbb{CP}^\infty$. Kunneth formula show that the Poincare series $P_t(\mathscr{G}(P) = (1+t)^{2g}(1+t^2+t^4+\ldots) = (1+t)^{2g}/(1-t^2)$. When P is a $U(1)$ bundle over S^4 the Thom theorem show that $\text{Map}(S^4, BG) = K(\mathbb{Z},2) = \mathbb{CP}^\infty$. (It implies that any $U(1)$ bundle over S^4 is trivial).

Case $U(n), n > 1$:

Now it is no longer true that BG is an Eilenberg-Maclane space. Since the fibration induces the following long exact sequence:

$$\to \pi_{k+1}(G_n\mathbb{C}^\infty) \to \pi_k(U(n)) \to \pi_k(V_n^U(\mathbb{C}^\infty)) \to \pi_k(G_n\mathbb{C}^\infty) \to$$

When $k > 0$ $\pi_{k+1}(G_n\mathbb{C}^\infty) = \pi_k(U(n))$. The latter one is very hard to compute. However, over the rationals Q, BG is simply a product of Eilenberg-maclane spaces: $BU(n) \simeq_Q K(\mathbb{Z},2) \times \ldots \times K(\mathbb{Z},2n)$

proof of this claim: We already know that $[BU(n), K(\mathbb{Z}, 2i)] = H^{2i}(BU(n), \mathbb{Z})$. Hence every class in the $2i$-th cohomology group induces a homotopy class of maps from $BU(n)$ to $K(\mathbb{Z}, 2i)$. We define $\omega : (BU(1))^n \to BU(n)$ be the map indued by the embedding of $U(1)^n \to U(n)$. Since this subgroup is closed under permutation, application of the classifiying sapce functor to conjugation by permutation of the factor of $BU(1)$, and it induces the identity map on $BU(n)$. We know that $H^*(BU(1)^n) = \mathbb{Z}[x_1 \ldots x_n]$ x_i is of degree two and let σ_i be the fundemanal symmetric polynomials. ω^* is the monomophism from $H^*(BU(n), \mathbb{Z})$ to $\mathbb{Z}[\sigma_1 \ldots \sigma_n]$. Passing to $\mathbb{Q}$ we get an isomorphism. Chern class c_i is defined to be the unique element in $H^{2i}(BU(n), \mathbb{Z})$ such that $\omega^*(c_i) = \sigma_i$. Since $H^*(K(\mathbb{Z}, 2\mathbb{k}), \mathbb{Q}) = \mathbb{Q}[x]$ where $|x| = 2k$ and $H^*(K(\mathbb{Z}, 2k-1), \mathbb{Q}) = \wedge_{\mathbb{Q}}[x]$, $|x| = 2k-1$.

The preceeding discussion yields $P_t(K(\mathbb{Z}, 2k)) = 1/(1-t^{2k})$, $P_t(K(\mathbb{Z}, 2k-1)) = 1 + t^{2k-1}$. Hence

$$P_t(\mathrm{Map}_P(M, K(\mathbb{Z}, 2k))) = (1+t^{2k-1})^{2g}/((1-t^{2k})(1-t^{2k-2})), k \geq 2$$
$$P_t(\mathrm{Map}_P(M, K(\mathbb{Z}, 2))) = (1+t)^{2g}/(1-t^2)$$

Collecting the results we obtain:

$$P_t \mathrm{Map}_P(M, BU(n)) = \prod_{k=1}^{n}(1+t^{2k-1})^{2g}/(\prod_{k=1}^{n-1}(1-t^{2k})^2(1-t^{2n}))$$

(It remain to show that if $X \simeq_{\mathbb{Q}} Y$ the for finite CW complex M, $\mathrm{Map}(M, X) \simeq_{\mathbb{Q}} \mathrm{Map}(M, Y)$).

Rational homotopy equivalence:

Def: A simply connected space X is rational if the following, equivalent conditions are satisfied:

1. $\pi_* X$ is a $\mathbb{Q}$-vector space.
2. $\tilde{H}_*(X, \mathbb{Z})$ is a $\mathbb{Q}$-vector space.
3. $\tilde{H}_*(\Omega X, Z)$ is a $\mathbb{Q}$-vector space.

Def: Let X be a simple connected space. A continuous map $l : X \to Y$ is a rationalization of X if Y is a simpley connected and rational and $\pi_* l \otimes \mathbb{Q} : \pi_* X \otimes \mathbb{Q} \to \pi_* Y \otimes \mathbb{Q} = \pi_* Y$ is an isomorphism.

THEOREM 1.5. *Let X be a simply connected space. There exists a relative CW-complex (X_0, X) with no zero-cells and no one-cells such that the inclusion $j : X \to X_0$ is a rationalization. Furthermore, if Y is a simply connected, rational space, then any continuous map $f : X \to Y$ can be extended over X_0, i.e., there is a continuous map $g : X_0 \to Y$, which is unique up to homotopy, such that*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & X_0 \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

DEFINITION 1.7. A continuous map $\varphi : X \to Y$ between simply connected spaces is a *rational homotopy equivalence* if the following, equivalent conditions are satisfied.

(1) $\pi_*(\varphi) \otimes \mathbb{Q}$ is an isomorphism.
(2) $\mathrm{H}_*(\varphi; \mathbb{Q})$ is an isomorphism.
(3) $\mathrm{H}^*(\varphi; \mathbb{Q})$ is an isomorphism.
(4) $\varphi_0 : X_0 \to Y_0$ is a weak homotopy equivalence.

Actually more is ture:

prop The space $\mathrm{Map}_P(M, BU(n))$ under consideration, is free of torsion.

We have a cofibration:

$$\bigvee_{2g} S^1 \to M \to S^2$$

which, by the exactness of the mapping functor, gives rise to the fibring

$$
\begin{tikzcd}
\mathrm{Map}_*(S^2, BU(n)) \arrow[rr] & & \mathrm{Map}_*(M, BU(n)) \arrow[dd] \\
& & \\
& & \mathrm{Map}_*(\vee S^1, BU(n
\end{tikzcd}
$$

By the adjoint property of Ω , this diagram can be rewritten as (since the loop space $\Omega BU(n) = U(n)$)

$$
\begin{tikzcd}
\Omega U(n) \arrow[rr] & & \mathrm{Map}_*(M, BU(n)) \arrow[dd] \\
& & \\
& & \prod_{i=1}^{2g} U(n)
\end{tikzcd}
$$

By the same principle mentioned in the incomplete prook of Thom theorem We have:

$$\begin{tikzcd}
\mathrm{Map}_*(M, BU(n)) \arrow[r] & \mathrm{Map}(M, BU(n)) \arrow[d] \\
& BU(n)
\end{tikzcd}$$

Now recall that $BU(n), U(n), \Omega U(n)$ are all torsion-free. Any non-trivial homology-twisting or a non-trivial coefficient system, would be detectable over $\mathbb{Q}$ and produce a Poincare series for the middle term that would be smaller than the product of the Poincare series of the factors. On the other hand by Thom's theorem: $P_t \mathrm{Map}_P(M, BU(n)) = \prod_{k=1}^{n} (1 + t^{2k-1})^{2g} / (\prod_{k=1}^{n-1} (1 - t^{2k})^2 (1 - t^{2n}))$ (computation is needed here)

Prop for any subrgoup of $\mathscr{G}'$ of $\mathscr{G}$ of finite index, $B\mathscr{G}'$ is torsion-free and has the same Poincare series as $B\mathscr{G}$.

In the second fibration, the fundamental group of the base, namely $\Gamma = \pi_1(U(n))^{2g} = H_1(M, \mathbb{Z})$. ($S^1 \times \ldots \times S^{2n-1}$), acts trivially on the cohomology of the fibre $\Omega U(n)$ (Beacuse of the torsion free property???). This implies that the cohomology is unaltered on lifting to a finites covering corresponding to a subgroup Γ' of finite index in Γ. Moreover from the last two diagram we see that:

$$\pi_1(B\mathscr{G}) = \pi_1(U(n))^{2g} = \Gamma$$
$$0 = \pi_1(BU(n)) \to \pi_1(\mathrm{Map}(M, BU(n))) \to \pi_1(\mathrm{Map}_*(M, BU(n))) \to \pi_0(BU(n))) = 0$$
$$0 = \pi_2(U(n)) = \pi_1(\Omega U(n)) \to \pi_1(\mathrm{Map}_*(M, BU(n))) \to \pi_1(U(n) \times \ldots) \to \pi_1(U(n)) \to \ldots$$

(Since two maps homotopy in $\mathrm{Map}_*(S^2, BU(n))$ are homotopy in $\mathrm{Map}_*(M, BU(n))$ the last map in the third line is injective)

But $\pi_1(B\mathscr{G}) = \pi_0(\mathscr{G})$ is the group component of $\mathscr{G}$. Hence a subgroup Γ' of Γ of finite index corresponds to a subgroup $\mathscr{G}'$ of $\mathscr{G}$ of finite index. $\mathscr{G}'$ is an admissable subrgroup of $\mathscr{G}$. Hence $B\mathscr{G}' = E\mathscr{G}/\mathscr{G}'$

$$
\begin{tikzcd}
\Omega U(n) \arrow[r] \arrow[d] & \mathrm{Map}_*(M, BU(n)) \arrow[d] \\
\Omega U(n) \times I \arrow[r] \arrow[ur, dotted] & U(n)\times\ldots
\end{tikzcd}
$$

Last time we have shown that the Poincare series for $\mathscr{G}(U(n))$ via rational homotopy theory:

$$P_t \mathrm{Map}_P(M, BU(n)) = \prod_{k=1}^{n}(1+t^{2k-1})^{2g} / (\prod_{k=1}^{n-1}(1-t^{2k})^2(1-t^{2n})) \tag{1}$$

And we obtain three fibration:

$$\begin{tikzcd}
\mathrm{Map}_*(S^2, BU(n)) \arrow[r] & \mathrm{Map}_*(M, BU(n)) \arrow[d] \\
& \mathrm{Map}_*(\vee S^1, BU(n
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega U(n) \arrow[r] & \mathrm{Map}_*(M, BU(n)) \arrow[d] \\
& \prod_{i=1}^{2g} U(n)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}_*(M, BU(n)) \arrow[r] & \mathrm{Map}(M, BU(n)) \arrow[d] \\
 & \\
 & BU(n)
\end{tikzcd}$$

Using the aforedmentioned result we can show that $B\mathscr{G} = \mathrm{Map}_{\mathrm{P}}(M, BU(n))$ is torsion free (in $\mathbb{Z}$). This week we try to provide a description for the generator for the cohomology ring of $B\mathscr{G}$. (RMK: This will eventually enable us to describe the corresponding generators for the cohomology of the moduli space of the stable bundles. It also provides an indenpendent proof of the cohomological triviality of the fibration without using Thom's theorem)

Prop for any subrgoup of $\mathscr{G}'$ of $\mathscr{G}$ of finite index, $B\mathscr{G}'$ is torsion-free and has the same Poincare series as $B\mathscr{G}$.

In the second fibration, the fundamental group of the base, namely $\Gamma = \pi_1(U(n))^{2g} = H_1(M, \mathbb{Z})$. ($S^1 \times \ldots \times S^{2n-1}$), acts trivially on the cohomology of the fibre $\Omega U(n)$ (Beacuse of the torsion free property???). This implies that the cohomology is unaltered on lifting to a finites covering corresponding to a subgroup Γ' of finite index in Γ. Moreover from the last two diagram we see that:

$$\begin{aligned}
&\pi_1(B\mathscr{G}) = \pi_1(U(n))^{2g} = \Gamma \\
&0 = \pi_1(BU(n)) \to \pi_1(\mathrm{Map}(M, BU(n))) \to \pi_1(\mathrm{Map}_*(M, BU(n))) \to \pi_0(BU(n))) = 0 \\
&0 = \pi_2(U(n)) = \pi_1(\Omega U(n)) \to \pi_1(\mathrm{Map}_*(M, BU(n))) \to \pi_1(U(n) \times \ldots) \to \pi_1(U(n)) \to \ldots
\end{aligned} \tag{2}$$

(Since two maps homotopy in $\mathrm{Map}_*(S^2, BU(n))$ are homotopy in $\mathrm{Map}_*(M, BU(n))$ the last map in the third line is injective)

But $\pi_1(B\mathscr{G}) = \pi_0(\mathscr{G})$ is the group component of $\mathscr{G}$. Hence a subgroup Γ' of Γ of finite index corresponds to a subgroup $\mathscr{G}'$ of $\mathscr{G}$ of finite index. $\mathscr{G}'$ is an admissable subrgroup of $\mathscr{G}$. Hence $B\mathscr{G}' = E\mathscr{G}/\mathscr{G}'$

$$\begin{tikzcd}
\Omega U(n) \arrow[r] \arrow[d] & \mathrm{Map}_*(M, BU(n)) \arrow[d] \\
\Omega U(n) \times I \arrow[r] \arrow[ur, dotted] & U(n) \times \ldots
\end{tikzcd}$$

We begin by considering the natural envaluatioin map:

$$e : \mathrm{Map}(M, BU(n)) \times M \to BU(n) \tag{3}$$

Pulling back the universal vector bundle over $BU(n)$ we then get a vector bundle V over $B\mathscr{G} \times M$. Since M hanse torsion the Kunneth formula gives, for integral cohomology:

$$H^{2r}(B\mathscr{G} \times M) \cong H^{2r}(B\mathscr{G}) \oplus H^{2r-1}(B\mathscr{G}) \otimes H^1(M) \oplus H^{2r-2}(B\mathscr{G}) \otimes H^2(M) \tag{4}$$

Taking the Chern class $c_r(V)$ and decomposing it, we get from this Kunneth decomposition elements.

$$\begin{aligned}
a_r &\in H^{2r}(B\mathscr{G}), \\
b_r^j &\in H^{2r-1}(B\mathscr{G}), j = 1, \ldots, 2g \\
f_r &\in H^{2r-2}(B\mathscr{G}),
\end{aligned} \tag{5}$$

relative to a basis of $H^1(M)$. But to integral generators we need K-theory instead of analogous to (5)

Define $K(X)$ to be the Grothendieck group defined as follow: ($[E]$ denote the isomophism class of vector bundle E then the isomorphism class of vector bundle over X form a semiring) $F(X)$ is the free abelian group generated by $\{[E]\}$

$$K(X) := F(X)/ < [m \otimes n] - m - n > \tag{6}$$

We now recall the definition of reduce K-Theory :

$$\begin{aligned}
&C \to C^2 \to C^+ \\
&X \mapsto (X, \emptyset), (X, Y) \mapsto X/Y, \text{with Y/Y as the based point}
\end{aligned} \tag{7}$$

define $\tilde{K}(X), (X, x_0) \in C^+$ to be the kernel of $K(X) \to K(x_0)$, $K(X, Y) := \tilde{K}(X/Y)$, $K^{-n}(X) = \tilde{K}(S^n \wedge (X^+))$

Prop Periodicity theorem: For any space X and $n \leq 0$, the map $[X] \otimes [Y] \mapsto [\pi^*(E_X) \otimes \pi^*(E_Y)]$, $K^{-2}(point) \otimes K^{-n}(X) \to K^{-n-2}(X)$ induces an isomorphism $\beta : K^{-n}(X) \to K^{-n-2}(X)$.

By kunneth formula again we have $K^0(B\mathscr{G} \times M) \cong K^1(B\mathscr{G}) \otimes K^1(M) \oplus K^0(B\mathscr{G}) \otimes K^0(M)$.

Now we have

$$K^0(M) = \mathbb{Z} \oplus \mathbb{Z} \tag{8}$$

with two generators, the first one is the trivial line bundle and the second by the reduced line-bundls of Chern class 1 ([H]-1). There are two way to obtain a bundls on $B\mathscr{G}$:

one is pulling back the universal bundle using evaluation map on $B\mathscr{G} \times M$ and decompose it in $K^0(B\mathscr{G} \times M)$(i.e. projection on the third component); another way is to ultilize $f_!$ where $f : B\mathscr{G} \times M \to B\mathscr{G}$ is the projecting map.

Finally take the Chern class of W we get an infinite sequence of elements:

$$e_r \in H^{2r}(B\mathscr{G}) \tag{9}$$

Prop: The element a_r, b_r^j, e_r constructed above are multiplicative generators of the integral cohomology ring of $B\mathscr{G}$. The elements a_r, b_r^j, f_r are multiplicative generators of the rational cohomology ring.

First, The cofibrations imply

$$\begin{tikzcd}
\mathrm{Map}_*(M, BU(n)) \arrow[r] & \mathrm{Map}(M, BU(n)) \arrow[d] \\
 & BU(n)
\end{tikzcd}$$

a_r give the Chern classes of $BU(n)$ and generate its cohomology (Since $H^*(BU(n))$ can be generated by its Chern class)

$$\begin{tikzcd}
\Omega U(n) \arrow[r] & \mathrm{Map}_*(M, BU(n)) \arrow[d] \\
 & \prod_{i=1}^{2g} U(n)
\end{tikzcd}$$

The classes b_r^j for fixed j are easily seen to give the generators for the cohomology of $\Omega U(n)$. Now we have the stabilization map:

$$i : \Omega U(n) \to \Omega U \tag{10}$$

U is the stable unitary group. The periodicity(Bott) theorem gives a homotopy equivalence.

$$\Omega U \cong Z \times BU \tag{11}$$

So that $H(\Omega U)$ is a polynomial ring on all the universal Chern classes $c_1, c_2, \ldots$ pulling back by i we get classes in $H(\Omega U(n))$ These coincide with the classes e_r introducd above in the view of the relation between K-theory and the periodicity theorem. To prove this, it remains to show that i^* is surjective in cohomology, or in steps that the inculsions

$$j : \Omega U(n) \to \Omega U(n+1) \tag{12}$$

have this property for all n. This can be deduced from the explicit decription of $H^*(\Omega U(n))$ given by Bott (The Spaces of Loops on a Lie Group)

Prop Assume that the Chern class k of P and the rank n are coprime. Then the inclusion of the constant central $U(1)$ in $\mathscr{G}$ induces a surjection

$$H^2(B\mathscr{G}, \mathbb{Z}) \to H^2(BU(1), \mathbb{Z}). \tag{13}$$

Using the cohomological triviality of the fibrations. It will be enough to check the subjectivity when M is the 2-sphere S^2. In the case $\mathscr{G}$ is connected and

$$H^2(B\mathscr{G}) \cong H^1(\mathscr{G}, Z) \tag{14}$$

so we are reduced to checking surjectivity of

$$H^1(\mathscr{G}, Z) \to H^2(BU(1), Z) \tag{15}$$

or equivalently that

$$\pi_1(U(1)) \to \pi_1(\mathscr{G}) \tag{16}$$

gives a direct summand of $\pi_1(\mathscr{G})$. Now $\pi_1(\mathscr{G}) \cong \pi_2(b\mathscr{G})$ and since $M = S^2$, this can be calculated from the fibration, which gives the short exact sequence:

$$0 \to \pi_3(U(n)) \to \pi_1(\mathscr{G}) \to \pi_1(U(n)) \to 0 \tag{17}$$

Thus $\pi_1(\mathscr{G})$ is free abelian on two generators. Note that the projection in the last column is given by evaluation at the point of $M = S^2$.

A more convenient descrption of the above short exact sequence is given in terms of $K - theory$. Let E be the vector bundle difined by P and write $\mathscr{G}(E)$ for $\mathscr{G}(E)$. then to every map

$$f : S^1 \to \mathscr{G}(E) \tag{18}$$

we form the bundle E_f over $M \times S^2$ by using f as clutching data and consider the element

$$[E_f] - [E_1] \in K(M \times S^2, M \times point) \tag{19}$$

The assignment $(f) \mapsto [E_f] - [E_1]$ gives an isomorphism

$$\pi_1(\mathscr{G}) \to K(M \times S^2, M \times point) \tag{20}$$

https://mathoverflow.net/questions/110654/about-mf-atiyah-and-r-botts-1983-paper

Now utilize the above description we shall now prove the following lemma, ture for all pairs (n, k) with $0 < k < n$.

Lemma let E be the direct sum of k copies of H and $n - k$ tirvial factors and let $U(k) \times U(n-k)$ be the corresponding constant automorphisms of E. Then the induce map

$$\pi_1(U(k) \times U(n-k)) \to \pi_1(\mathscr{G}(E)) \tag{21}$$

is an isomorphism

Corollary If $(n, k) = 1$ the homomorphism

$$\pi_1(U(1)) \to \pi_1(\mathscr{G}(E)) \tag{22}$$

coming from the constant central automorphisms, is a direct summand.

It should be remarked here our use of K theory in this proof becomes very natural if we consider briefly the situation for manifolds M of arbitary dimension.