In the attempt to analyse in detail the features of the described ultraproduct, which I intend to use for philosophical purposes, I encountered difficulties that I list below in the form of proofs of contradiction so that it becomes clear that something went wrong even though what that in particular is, is still unclear to me.

I use "$\mathbb{N}$", "$\mathbb{R}$", "$\{1, 2, ...\}$", "$\aleph_0$" as symbols of the metalanguage and always write respectively "$\omega^{\mathcal{M}}$", "$1^{\mathcal{M}}$", ... as their interpretation in a model $\mathcal{M}$. I don't write "$\cdot^{\mathcal{M}}$" when the symbols are preceded by "$\mathcal{M} \models$". I also write "$\Rightarrow$", "$\Leftrightarrow$" as the metalinguistic logical consequence.
Let $\{s_i \in \mathcal{L}_\epsilon : i \in \mathbb{N}\}$ s.t., for a theory T of $\mathcal{L}_\epsilon$:
(i): $\forall_{i \leq \mathbb{N}}(T \not\models s_i \wedge T \not\models \neg s_i)$
(ii): $\forall_{i,j \in \mathbb{N}}(i \neq j \Rightarrow T + s_i \models \neg s_j)$.
From (i) I derive that $\forall_{i \in \mathbb{N}} Con(T) \rightarrow Con(T + s_i)$, then choose a model of $T + s_i$ and call it $\mathfrak{U}_i$ s.t.
(iii): for a function f, $\mathfrak{U}_i \models f : \{\emptyset\} \rightarrow \omega \wedge (f(\emptyset) = i \leftrightarrow s_i)$.
Then, for a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, define $\mathfrak{U} := \prod_{i \in \mathbb{N}} \mathfrak{U}_i/\mathcal{U}$, if not specified $\mathcal{U}$ is a free variable, if bounded then write "$\mathfrak{U}_{\mathcal{U}}$".
For a proposition in the metalanguage φ and the ultraproduct $\mathfrak{U}$ define $B_{\mathfrak{U}}(\varphi) := \{i \in \mathbb{N} : \mathfrak{U}_i \models \varphi\}$, note that this definition is independent from the ultrafilter $\mathcal{U}$.

Lemma (0). *For $\varphi \in \mathcal{L}_\epsilon$, $(\mathfrak{U} \models \varphi) \Rightarrow (|B_{\mathfrak{U}}(\varphi)| \geq \aleph_0)$*

Proof. Let $\varphi \in \mathcal{L}_\epsilon$ and assume $\mathfrak{U} \models \varphi$, by Łos Theorem derive that $\models \varphi \Rightarrow \{i \in \mathbb{N} : \mathfrak{U}_i \models \varphi\} \subset \mathcal{U}$ and since $\mathcal{U}$ is non-principal derive $A \in \mathcal{U} \rightarrow |A| \geq \aleph_0$, hence $|\{i \in \mathbb{N} : \mathfrak{U}_i \models \varphi\}| \geq \aleph_0$.

Lemma (0'). *For $\varphi \in \mathcal{L}_\epsilon$, $|B_{\mathfrak{U}}(\varphi)| \geq \aleph_0 \Rightarrow \exists_{\mathcal{U}}(\mathfrak{U}_{\mathcal{U}} \models \varphi)$*

Proof. Let $\varphi \in \mathcal{L}_\epsilon$ and assume $|B_{\mathfrak{U}}(\varphi)| \geq \aleph_0$ then one can generate a non-principal ultrafilter on $\mathbb{N}$ from $B_{\mathfrak{U}}(\varphi)$, its existence proves the claim.

Theorem (I). $\mathfrak{U} \models \exists_{i \in \omega}(f(\emptyset) = i)$

Proof. Note $B_{\mathfrak{U}}(\exists_{i \in \omega}(f(\emptyset) = i)) = \{j \in \mathbb{N} : \mathfrak{U}_j \models \exists_{i \in \omega}(f(\emptyset) = i)\} = \mathbb{N}$, the last step follows from (iii) and construction of $\mathfrak{U}_i$. Since $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then $\mathbb{N} \in \mathcal{U}$ hence, by Łos Theorem, the claim follows.

Theorem (II). $\forall_{i \in \mathbb{N}}(\mathfrak{U} \models f(\emptyset) \neq i)$

Proof. Let $i \in \mathbb{N}$, then note $|B_{\mathfrak{U}}(f(\emptyset) = i)| = 1 < \aleph_0$ by (ii), therefore, by 0, $\mathfrak{U} \not\models f(\emptyset) = i$ and by completeness of models derive the claim.

Lemma (II^+). *For $g : \mathbb{N} \rightarrow \omega^{\mathfrak{U}}$, $\forall_{i \in \mathbb{N}}(\mathfrak{U} \models f(\emptyset) \neq g(i))$*

Proof. Let $g : \mathbb{N} \rightarrow \omega^{\mathfrak{U}}$ and let $i \in \mathbb{N}$ then note $|B_{\mathfrak{U}}(f(\emptyset) = g(i))| = 1 < \aleph_0$ by (ii), therefore, by 0, $\mathfrak{U} \not\models f(\emptyset) = g(i)$ and by completeness of models derive the claim.

Theorem ($\perp$.1). $\perp$

Proof. From (I) define $\alpha^{\mathfrak{U}}$ s.t. $\mathfrak{U} \models f(\emptyset) = \alpha$. Let $g : \mathbb{N} \rightarrow \omega^{\mathfrak{U}}$, from ($\text{II}^+$) derive $\alpha \in g(\mathbb{N}) \Rightarrow \perp$. Since the mapping of g is free, set $g : 1 \mapsto \alpha^{\mathfrak{U}}$, from $\alpha \in g(\mathbb{N})$ and $\alpha \in g(\mathbb{N}) \Rightarrow \perp$, derive the claim.

Theorem ($\perp$.2). $\perp$

Proof. Let i be an ordinal and claim $\mathfrak{U} \models f(\emptyset) \neq i$. If $i \in \mathbb{N}$, from (II) derive $\mathfrak{U} \models f(\emptyset) \neq i$, assert also that $i \notin \mathbb{N} \Rightarrow \mathfrak{U} \models f(\emptyset) \neq i$. Assume $i \notin \mathbb{N}$, then from (iii) and construction of $\mathfrak{U}_i$ derive $\{j \in \mathbb{N} : \mathfrak{U}_j \models f(\emptyset) = i\} = \{j \in \mathbb{N} : \mathfrak{U}_j \models f(\emptyset) = j\}$ and since $i \notin \mathbb{N}$, $\{j \in \mathbb{N} : \mathfrak{U}_j \models f(\emptyset) = i\} = \emptyset$, $|\emptyset| < \aleph_0$ hence, by 0, $\mathfrak{U} \not\models f(\emptyset) = i$ and by completeness of models derive $\mathfrak{U} \models f(\emptyset) \neq i$.
Conclude that for any ordinal i, $\mathfrak{U} \models f(\emptyset) \neq i$ and from (I) derive $\perp$.