Some Exercises in Applied Linear Algebra

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1 Fundamentals Review

Exercise 1. Prove the Pythagorean Theorem in the case $n = 2$ via explicit computation.

Proof: Recall that we can write Euclidean length as $\|x\|^2$, with Euclidean space an inner product space. For $n = 2$, we state the Pythagorean Theorem

$$\|\sum_{i=1}^{2} x_i\|^2 = \sum_{i=1}^{2} \|x_i\|^2,$$

and expand

$$\|x_1+x_2\|^2 = \langle x_1+x_2, x_1+x_2\rangle = \langle x_1, x_1\rangle+\langle x_1, x_2\rangle+\langle x_2, x_1\rangle+\langle x_2, x_2\rangle = \|x_1\|^2+\|x_2\|^2,$$

by the properties of additivity and symmetry, which is the sum $\sum_{i=1}^{2} \|x_i\|^2$.

The sum $\sum_{i=1}^{2} \|x_i\|^2$ is $\|x_1\|^2 + \|x_2\|^2$ as the total Euclidean length, which we can proceed to write as $\|x_1 + x_2\|^2$, or the sum $\| \sum_{i=1}^{2} x_i\|^2$. We have now proven

$$\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

for the case $n = 2$ via explicit computation. ∎

Exercise 2.

1. Prove that all eigenvals of a Hermitian matrix are real.
2. Prove that if x, y are distinct eigvectors corresponding to distinct eigenvals of a Hermitian matrix, $x \perp y$.

Proof: We begin our proof by applying the fundamental theorem of algebra to the characteristic polynomial $p_A(t) = \det(tI - A)$ of our matrix A.

Let $p_A(t)$ be a degree n polynomial in the variable t with complex coefficients. Then $p_A(t)$ has exactly n complex roots counted with multiplicity and A has n complex eigenvalues with corresponding eigenvectors. In particular, there must be at least one complex eignevalue λ_1 and eigenvector x_1.

For any eigenpair (λ, x) with $x \neq 0$,

$$\lambda\langle x_1, x_2\rangle = \langle \lambda x_1, x_2\rangle = \langle Ax_1, x_2\rangle.$$

Since A is Hermitian,

$$\langle Ax_1, x_2\rangle = \langle Ax_1, x_2\rangle = \langle x_1, Ax_2\rangle = \overline{\lambda_1}\langle x_1, x_2\rangle,$$

so $\lambda_1 = \overline{\lambda_1}$.

We now proceed to prove b.). Consider $Y^{n-1} = \text{span}(x)^{\perp}$ as the orthogonal complement of x. As an invariant subspace under A, we check

$$\langle Ay, x\rangle = \langle y, Ax\rangle = \langle y, \lambda_1 x\rangle = 0.$$

The operator $A \restriction Y^{n-1}$ suggests that A must have at least one eigenvalue λ_2 corresponding to $y \in Y^{n-1} \perp x$. Therefore $x \perp y$ with distinct eigenvalues λ_1, λ_2.

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Exercise 3. The matrix $A = I + uv*$ is a rank one pertubation of the identity. If A is invertible, it's inverse has form $A = I + uv*$ for some scalar α. Give an expression for α. Additionally, for what u and v is A singular? If it's singular, what is null(A)?

Proof: See that

$$\begin{aligned}
AA^{-1} &= (I + uv^*)(I + \alpha uv^*) \\
&= I + uv^* + \alpha uv^* + uv^*\alpha uv^* \\
&= I + uv^* + \alpha uv^* + \alpha u(v^*u)v^* \\
&= I + (1 + \alpha + \alpha uv^*)uv^*,
\end{aligned}$$

so we solve $\alpha = -\frac{1}{1+uv^*}$ as our expression for α.

Suppose $uv^* = -1$. We compute

$$\begin{aligned}
Au &= (I + uv^*)u \\
&= u + u(v^*u) \\
&= u - u \\
&= 0,
\end{aligned}$$

so a nonzero vector u exists s.t. $Au = 0$ and A is singular. The null space consists of the m-vectors u. ∎

Exercise 4. Prove that if W is an arbitrary nonsingular matrix, the function $\|\cdot\|_W$ is a norm.

Proof: Let $\|\cdot\|_W$ be a norm defined by

$$\|x\|_W = \|Wx\|.$$

We will prove $\|\cdot\|_W$ is a vector norm for a nonsingular matrix W in proving the norm conditions.

1. Since $\|\cdot\|_W$ is a norm, $\|Wx\| \geq 0$ with $\|x\|_W = 0$ implying $\|Wx\| = 0$. Recall that $\|Wx\| = 0 \iff Wv = 0$. As W is nonsingular, its determinant is nonzero, thus $Wv = 0$ must have trivial solution $v = 0$. Therefore it is true that $\|x\|_W = 0 \iff x = 0$ and we have proven the property of positive definiteness.

2. We now prove the triangle inequality given vectors x, y in the normed vector space. See that

$$\|x + y\|_W = \|W(x + y)\| = \|Wx + Wy\|,$$

and by $\|\cdot\|_W$ as the underlying norm, we have

$$\|Wx + Wy\| \leq \|Wx\| + \|Wy\|.$$

See that $\|Wx\| + \|Wy\| = \|x\|_W + \|y\|_W$, therefore

$$\|x + y\|_W \leq \|x\|_W + \|y\|_W$$

satisfies the triangle inequality.

3. Let α be a scalar element as we prove absolute homogeneity. See that

$$\|\alpha x\|_W = \|W(\alpha x)\| = \|\alpha(Wx)\|$$

and since $\|\cdot\|$ is a norm, $\|\alpha(Wx)\| = |\alpha|\|Wx\|$. Since $|\alpha|\|Wx\| = |\alpha|\|x\|_W$, we have $\|\alpha x\|_W = |\alpha|\|x\|_W$.

Having proven 1) positive-definiteness, 2) the triangle inequality, and 3) absolute homogeneity of $\|\cdot\|_W$ defined by $\|x\|_W = \|Wx\|$, we have shown that $\|\cdot\|_W$ is a valid vector norm given an arbitrary nonsingular W.

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Exercise 5. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show$\rho(A) \leq \|A\|$.

Proof: Let $\|\cdot\|$ be any norm on $\mathbb{C}^m$ as well as on the induced matrix norm $\mathbb{C}^{m\times m}$. See that

$$\|Ax\| = \|\lambda x\| = \lambda\|x\|,$$

and recall that $\|Ax\| \leq \|A\|\|x\|$ for any vector x, so $|\lambda|\|x\| \leq \|A\|\|x\|$. Dividing, we have $|\lambda| \leq \|A\|$ true for any induced matrix norm $\|A\|$. Define $\rho(A) = \max_i |\lambda_i|$. Then $\rho(A) \leq \|A\|$, completing our proof. ∎

Exercise 6. Let λ be an eigenvalue of an orthogonal matrix. Show that $|\lambda| = 1$.

Proof: Let λ be an eigenvalue of an orthogonal matrix Q, By definition of an orthogonal matrix, $QQ^T = I$. Taking det $(QQ^T) = \det(I)$, recall det $Q = \det Q^T$, so det $(Q)^2 = \det(I) = 1$ (as the determinant of the identity matrix). Thus the eigenvalues of Q must be ± 1 and $|\lambda| = 1$. ∎

Exercise 7. A permutation matrix is orthogonal and hence from part its eigenvalues satisfy $|\lambda| = 1$. Give an example of a 4 by 4 permutation matrix whose eigenvalues are $\lambda = \{\pm 1, \pm i\}$.

Proof: Suppose we have eigenvalues $\{\pm 1, \pm i\}$. Then we have the characteristic polynomial $p(\lambda) = (\lambda^2 - 1)(\lambda^2 + 1)$ with factors corresponding to the respective matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

We build a 4x4 block permutation matrix with these same eigenvalues

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and characteristic polynomial $p(\lambda) = \lambda^4 - 1$.

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Exercise 8. Verify the induced matrix norm is indeed a norm.

Proof: Let $\|\cdot\|$ be a norm on a vector space V. For an $m \times n$ matrix A we define

$$\|A\| = \sup_{x \in V x \neq 0} \frac{\|Ax\|}{\|x\|},$$

and prove it satisfies the norm conditions.

1. First note that $\|A\| \geq 0$ since $\|Ax\| \geq 0$. Now consider that if $A = 0$, then $\|Ax\| = 0$ and $\|A\| = 0$. Conversely, if $\|A\| = 0$, then $\|Ax\| = 0$ for every x with A as the zero operator. Therefore $\|A\| = 0 \iff A = 0$ and we have proven positive definiteness.

2. For any scalar α, see that

$$\|\alpha A\| = \sup_{x \in V x \neq 0} \frac{\|\alpha A x\|}{\|x\|} = \sup_{x \in V x \neq 0} \frac{|\alpha| \|Ax\|}{\|x\|} = |\alpha| \sup_{x \in V x \neq 0} \frac{\|Ax\|}{\|x\|} = |\alpha| \|A\|.$$

3. For matrices A, B see that

$$\|A + B\| = \sup_{x \in V x \neq 0} \frac{\|(A+B)x\|}{\|x\|} = \sup_{x \in V x \neq 0} \frac{\|Ax + Bx\|}{\|x\|}$$

and

$$\sup_{x \in V x \neq 0} \frac{\|Ax + Bx\|}{\|x\|} \leq \sup_{x \in V x \neq 0} \frac{\|Ax\| + \|Bx\|}{\|x\|}.$$

Having proven 1) positive-definiteness, 2) absolute homogeneity, and 3) the triangle inequality, we have confirmed that the induced matrix norm is indeed a norm.

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Exercise 9. Prove the infinity matrix norm is the max row sum.

Proof: Let $\|\cdot\|_\infty$ denote the vector infinity norm $\|x\|_\infty$. For an $m \times n$ matrix $A = (a_i j)$ we define

$$\|A\|_\infty = \sup_{x \in V x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}.$$

Considering $\|Ax\|_\infty$, see that

$$\|Ax\|_\infty = \sum_j |(Ax)_j| = \sum_j |\sum_i a_{ij} x_i| \leq \sum_j \sum_i |a_{ij}||x_i| = \sum_i \sum_j |a_{ij}||x_i|$$

and

$$\sum_i \sum_j |a_{ij}||x_i| \leq \left(\max_i \sum_j |a_{ij}| \right) \cdot \sum_i |x_i| = \left(\max_i \sum_j |a_{ij}| \right) \cdot \|x\|_\infty.$$

Thus

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \max_i \sum_j |a_{ij}|$$

for all $x \neq 0$. The $\max_i \sum_j |a_{ij}|$ is the $\max_i \sum_j |a_{kj}|$ for some index k, so consider a "signed vector" e_k of form $[1, -1]$. Then $\|x\|_\infty = 1$ and $\|Ae_k\| = \sum_j |a_{ik}|$. See that

$$\sup_{x \in V x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \max_i \sum_j |a_{ij}| = \sum_j |a_{ik}| = \frac{\|Ae_k\|_\infty}{\|e_k\|_\infty} \leq \sup_{x \in V x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \|A\|_\infty,$$

and we have proven the infinity norm as our max row sum. ∎