Progress Report

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Background and progress

Stochastic analysis is analysis involving stochastic processes such as Brownian motions. Stochastic analysis is not only a fundamental field of mathematics, but it also has wide-ranging applications in physics, biology and engineering. However, its most important application lies in mathematical finance, where stock processes are often modelled by differential equations involving Brownian motions.

In classical stochastic analysis, we only deal with ideal stochastic processes which are either martingale or Markovian. While this ideal assumption leads to beautiful mathematics, it is often too ideal for applications, and there is an increasing demand to develop stochastic analysis for non-martingale and non-Markov processes. For instance, mathematical finance community has recently found tremendous interest in rough volatility models, where the volatility process is modelled by a very rough fractional Brownian motion, a stochastic processe which is neither martingale nor Markovian.

The *fractional Brownian motion* B with Hurst parameter H is a centered Gaussian process with

$$\mathbb{E}[|B_t - B_s|^2] = |t-s|^{2H}.$$

It gives a class of most masmatically tructable stochastic processes that are non-martingale and non-Markovian, and since the popularisation by Mandelbrot it has shown up in many applications, including the rough volatility model discussed before. Therefore, it is important to develop a mathematical technique to analyse fractional Brownian motions.

In my previous work with my supervisor Nicolas Perkowski, we have advanced our understanding of fractional Brownian motions. In the work, we extended the stochastic sewing lemma first proposed by Khoa Lê. By so doing, we showed: (i) the construction of stochastic integrals along fractional Brownian motions with low regularity, (ii) the new representation of local time via discretisation and (iii) regularity improvement of diffusion coefficients of fractional stochastic differential equations.

Planned work

1. *Excursions of non-Markov processes*. An excursion X is a continuous path with $X_0 = X_T = 0$ but with $X_t \neq 0$ for $t \in (0,T)$. Since an excursion takes either non-negative or non-positive value, it can be used to model some price process in long or short position. So far, excursions are studied only for Markov processes such as diffusions. However, in view of increasing use of fractional Brownian motions in mathematical finance, it is very important to develop a theory of excursions for non-Markov processes such as fractional Brownian motion.

   For this purpose, we need Ito's point of view in excursions. He observed that the key to excursion theory is the excursion point process, which is a point process counting the number of excursions. If the underlying process is Markovian, then the excursion point process becomes a Poisson point process, and a lot of analysis can be conducted. On the other hand, if the underlying process is non-Markovian, the excursion point process is no longer Poissson and this leads to mathematical difficulty. Our goal is that, by taking advantage of ergodicity and asymptotic independence, we prove some asymptotic properties of its excursion point process. Especially, in the limit of time going to infinity, we hope that a natural candidate of excursion measures appears.

2. *$1/H$-variation of fractional Brownian motion along Lebesgue partitions*. For a fractional Brownian motion B with Hurst parameter H, the following convergence is known:

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left| B_{\frac{k}{2^n}T} - B_{\frac{k+1}{2^n}T} \right|^{1/H} = \mathbb{E}[|B_1|^{1/H}]T \quad \text{a.s.}$$

   A similar result holds for deterministic partitions if their diameters dacay sufficiently fast. We would like to understand the $1/H$-variation along *Lebesgue partitions*. They are the random partitions defined as follows. For each $n \in \mathbb{N}$ we set $T_0^n := 0$ and inductively

$$T_k^n := \inf\{t > T_{k-1}^n : |B_t - B_{T_{k-1}^n}| = 2^{-n}\}.$$

Then, the nth Lebesgue partition π^n is given by $\{[T_{k-1}^n, T_k^n] : T_k^n \leq T\}$. Our goal is to prove the existence of the limit of $1/H$-variations along the Lebesgue partitions $(\pi^n)_{n=1}^{\infty}$, namely,

$$\lim_{n \to \infty} 2^{-nH} \#\{k : T_k^n \leq T\}.$$

Numerical simluation shows that the limit along the Lebesgue partitions differs from that along deterministic partitions, which is a highly non-Markovian characteristic.

3. *Regularization by noise.* An ill-posed differential equation can sometimes restore well-posedness in the presence of noise. This phenomena is called regularization by noise, and is now a very active field of research in stochastic analysis. We have two projects in this topic: regularization by noise of $L_q L_p$-drift in the whole sub-critical regime and regularization by noise of diffusion coefficient.

Collateral Activities

Teaching

- Teaching, Stochastic II Exercise, Summer term of 2022

Seminars and lectures

- IRTG seminar (2021-22), Rough path seminar (2021-22, Berlin), Group seminar (2021-22, run by Nicolas Perkowski), Oxford stochastic analysis seminar (2022 winter), ICL stochastic analysis seminar (2022 winter).

Presentations in conferences and workshops

Past Information can be found at
`https://docs.google.com/spreadsheets/d/1Hq3xC7zkISbYOwTR03Kgfj2qLucA5ffEd2mheP6sg2M/edit#gid=0`.

Planned

- Kyushu probability seminar, December 2022.

External Collaborations

- Oleg Butkovsky, WIAS, Berlin, Germany (ongoing)
- Rama Cont, University of Oxford, Oxford, UK (ongoing)
- Purba Das, University of Michigan, Michigan, US (ongoing)
- Khoa Lê, University of Leeds, Leeds, UK (ongoing)
- Rafał Łochowski, Warsaw school of economics, Warsaw, Poland (ongoing)
- Avi Mayorcas, TU Berlin, Berlin, Germany (ongoing)
- Willem van Zuijlen, WIAS, Berlin, Germany (completed)

Publications

- T. Matsuda, "Integrated density of states of the Anderson Hamiltonian with two-dimensional white noise", Stochastic Processes and their Applications,Volume 153, 2022, Pages 91-127
- T. Matsuda and N. Perkowski, "An extension of the stochastic sewing lemma and its applications to fractional stochastic calculus", 2022, preprint.
- T. Matsuda and W. van Zuijlen, "Anderson Hamiltonian with singular potential", appear soon as preprint.