Fractional Stochastic Calculus via Stochastic Sewing

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Abstract

The thesis explores stochastic calculus for fractional Brownian motion. Our approach builds upon a novel technique called stochastic sewing, originally introduced by Lê [*Electron. J. Probab.* 25:1-55, 2020]. The stochastic sewing has been effectively used to obtain sharp estimates on stochastic Riemann sums.

The main result of the thesis is an extension of Lê's stochastic sewing, which we refer to as the shifted stochastic sewing. This extension takes advantage of asymptotic decorrelation in stochastic Riemann sums and can be seen as a combination of Lê's stochastic sewing and the asymptotic independence formulated by Picard [*Ann. Probab.* 36(6): 2235-2279, 2008].

As applications of the shifted stochastic sewing, we address two important problems in fractional stochastic calculus. Firstly, we characterize the local time of the fractional Brownian motion via level crossings, extending the classical work of Lévy to the fractional setting. Secondly, we establish the pathwise uniqueness of Young and rough differential equations driven by fractional Brownian motion. This result optimizes the regularity of the noise coefficient, which is consistent with the Brownian setting.

Additionally, we demonstrate strong regularization by fractional noise for differential equations with integrable drifts. This result can be viewed as a fractional analogue of the celebrated work by Krylov and Röckner [*Probab. Theory Relat. Fields* 131: 154–196, 2005].

Introduction

During the 19th century, Robert Brown made a significant observation regarding the irregular movement of particles within a medium. This motion, now widely recognized as Brownian motion, or the Wiener process in honor of the mathematician who laid its mathematical groundwork, has had a profound impact on various fields, including modern mathematics. Brownian motion, denoted by W, is a centered Gaussian process characterized by the following property (in one dimension):

$$\mathbb{E}[(W_t - W_s)^2] = t - s, \quad s < t.$$

The process exhibits independent increments and possesses martingale and Markovian properties, which paved the way for the development of a comprehensive theory on Brownian motion. In the 1940s, Itô initiated the field of stochastic calculus, which involves the calculus with respect to Brownian motion. This field has evolved into one of the most fruitful areas in mathematics, as demonstrated in the monograph [RY99].

However, in practical applications, Brownian motion is often considered too ideal. To address this, the *fractional Brownian motion* B^H, indexed by $H \in (0, 1)$, was introduced. It is a centered Gaussian process characterized by the following property:

$$\mathbb{E}[(B_t^H - B_s^H)^2] = (t - s)^{2H}, \quad s < t.$$

The parameter H represents the roughness of the process, as depicted in Figure 1. When $H = 1/2$, the process reduces to the standard Brownian motion. In other cases, the process exhibits correlated increments and is neither a martingale nor Markovian. Kolmogorov [Kol40] first introduced this process, and it was later popularized by Mandelbrot [MV68; Man82]. Naturally, the field of *fractional stochastic calculus* emerged to handle calculus involving fractional Brownian motion.

Since fractional Brownian motion is neither a martingale nor Markovian, many of the arguments used in Itô's stochastic calculus cannot be directly applied to fractional stochastic calculus. Consequently, researchers have developed two main tools in fractional stochastic calculus. The first tool involves pathwise arguments, such as Young's integration theory

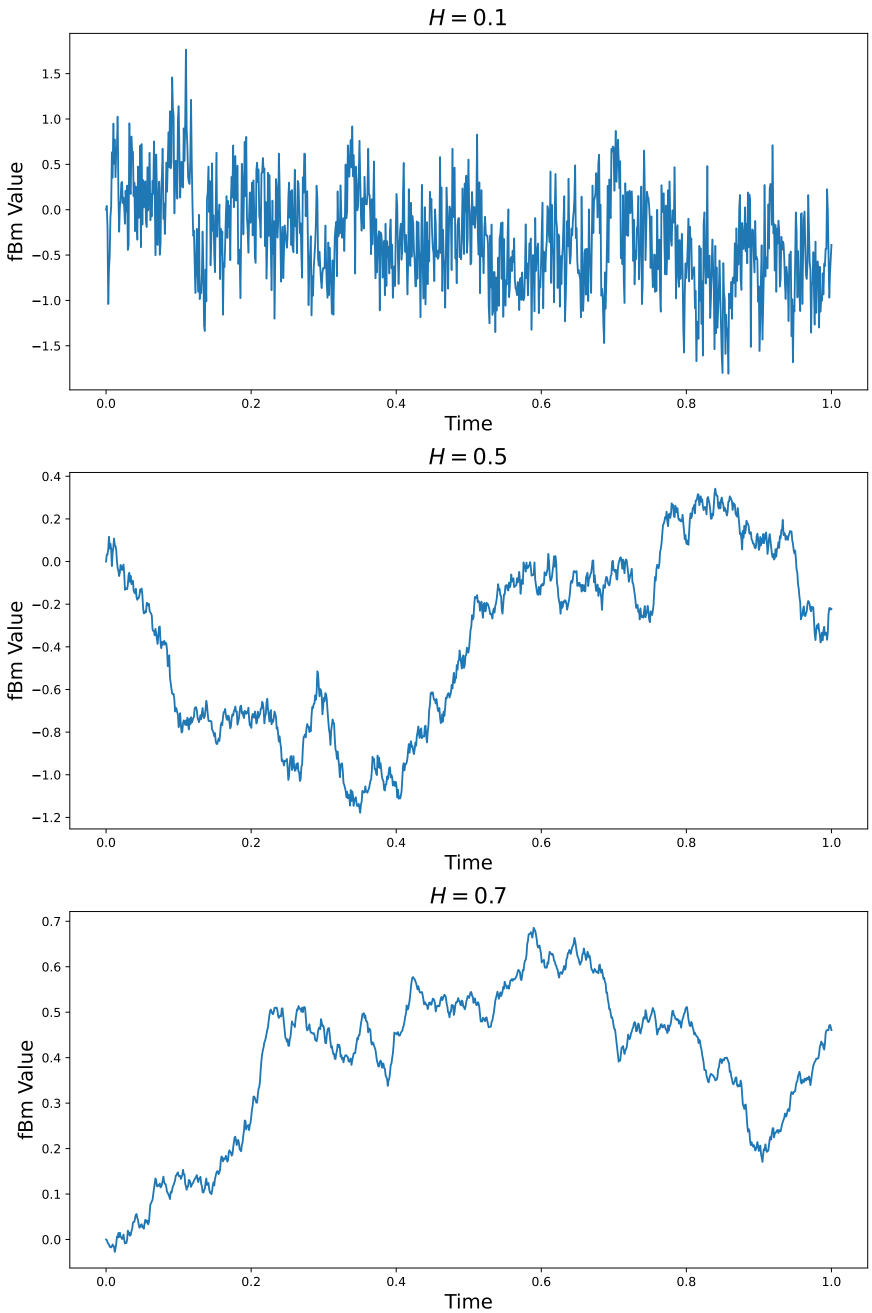

Figure 1: Simulating fractional Brownian motions for $H = 0.1, 0.5, 0.7$.

[You36], Lyons' rough path theory [Lyo98], and Zähle's fractional calculus [Zäh98]. These approaches fix the realization of the fractional Brownian motion, and perform pathwise analysis. The second tool is Malliavin calculus. This calculus was first invented by Malliavin [Mal78] to obtain a probabilistic proof of Hörmander's theorem [Hör67], but it turns out to be also useful to study probabilistic aspects of the fractional Brownian motion [Nua06; Nou12].

However, it is important to note that these tools often yield less precise results compared to the classical Brownian setting. For example, consider the stochastic integral

$$\int_0^T f(B_r^H)\,\mathrm{d}B_r^H \tag{1}$$

for $H \in (1/4, 1)$, and for $H < 1/2$, the integral is understood as a rough integral. Typically, the integral (in multi dimensions) is defined for functions f with Hölder regularity $(1-H)/H$. However, it is natural to suspect that this definition is not optimal, as the Itô integral (1) is well-defined for any bounded measurable f when $H = 1/2$.

Recently, Lê [Lê20] combined the martingale inequality (Burkholder–Davis–Gundy inequality) and Gubinelli's sewing lemma [Gub04] to obtain the *stochastic sewing lemma*. This lemma provides sharp stochastic estimates on stochastic Riemann sums, including the stochastic integral

$$\int_0^T f(B_r^H)\mathrm{d}r,$$

where f can be an irregular function or even a distribution. The stochastic sewing lemma quickly gained recognition for its innovation and has become a central force in the recent development of regularization by noise.

This thesis aims to provide a new perspective on fractional stochastic calculus through the stochastic sewing lemma. Our results are on par with their Brownian counterparts. For instance, we establish the well-definedness of the integral (1) for f of Hölder regularity $(1/(2H) - 1 + \varepsilon)$, for any positive ε. Our main contribution is a novel version of the stochastic sewing lemma, which we call the *shifted* stochastic sewing (Chapter 1). This new version offers the advantage of capturing the asymptotic decorrelation in the stochastic Riemann sums. It can be viewed as a combination of Lê's stochastic sewing and the asymptotic independence introduced by Picard [Pic08]. As applications of the shifted stochastic sewing, we investigate partitions defined by level crossings of fractional Brownian motions (Chapter 2) and study Young and rough differential equations driven by fractional Brownian motions (Chapter 3). Additionally, we derive precise results on regularization by fractional noise for integrable drifts (Chapter 4), which significantly improve upon previous works and align with the results obtained by Krylov and Röckner for the Brownian case [KR05].

In the following sections, we provide more detailed descriptions of each chapter.

Chapter 1: Shifted stochastic sewing

Chapter 1 is the most important part of the thesis, which serves as the foundation for Chapters 2 and 3. The content of this chapter is based on joint work with Nicolas Perkowski.

In the fields of analysis and probability theory, the convergence and the estimate of Riemann sums plays a crucial role. These sums are expressed as

$$\sum_{[s,t]\in\pi} A_{s,t}, \tag{2}$$

where π represents a partition of the interval $[0,T]$. The focus lies on the limit as the mesh size

$$|\pi| := \max_{[s,t]\in\pi} |t-s|$$

tends to 0. The term $A_{s,t}$ is called a germ. For example, when $A_{s,t} := f(s)(t-s)$, we consider a Riemann sum approximation of $\int_0^T f(s)\,ds$. Similarly, when $A_{s,t} := X_s(W_t - W_s)$, where W is a Brownian motion and X is an adapted process, we study the Itô approximation of the stochastic integral $\int_0^T X_r\,dW_r$.

Gubinelli [Gub04], inspired by Lyons' results on almost multiplicative functionals in the theory of rough paths [Lyo98], established the remarkable *sewing lemma*. This lemma states that if the quantity

$$\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}, \quad 0 \le s < u < t \le T,$$

satisfies $|\delta A_{s,u,t}| \lesssim |t-s|^{1+\varepsilon}$ for some $\varepsilon > 0$, then the sums (2) converge. The sewing lemma has proven to be immensely powerful, leading to numerous applications and extensions in the field. Notably, it has been utilized for defining rough integrals, as described in the monographs [Gub04; FH20].

When $(A_{s,t})_{s\le t}$ is random and we aim to prove the convergence of the sums (2), Gubinelli's sewing lemma is often insufficient. For instance, if $A_{s,t} := (W_t - W_s)^2$, the sums converge in $L^m(\mathbb{P})$, $m < \infty$, to the quadratic variation of the Brownian motion. However, we only expect the bound

$$\|\delta A_{s,u,t}\|_{L^m(\mathbb{P})} \lesssim_m |t-s|,$$

and hence we cannot apply the sewing lemma.

In his seminal work, Lê [Lê20] obtained a stochastic version of Gubinelli's sewing lemma. Just as Gubinelli's sewing lemma plays an important role in *pathwise* stochastic calculus, Lê's stochastic sewing lemma does so in *probabilistic* stochastic calculus. In particular, the discovery of the stochastic sewing has significantly advanced the field of regularization by noise.

A concrete statement of the stochastic sewing lemma is as follows. If $(A_{s,t})_{s<t}$ is a stochastic germ adapted to a filtration $(\mathcal{F}_t)$ and if

$$\|\delta A_{s,u,t}\|^2_{L^m(\mathbb{P})} + \|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]\|_{L^m(\mathbb{P})} \leq \Gamma(t-s)^{1+\varepsilon},$$

for $s < u < t$, $m \in [2, \infty)$, $\Gamma \in (0, \infty)$ and $\varepsilon > 0$, then the Riemann sums $\sum_{[s,t]\in\pi} A_{s,t}$, where π is a partition of some fixed interval $[0, T]$, converge in $L^m(\mathbb{P})$ as the mesh size of π tends to 0. The strength of the stochastic sewing lemma lies in the fact that we only need to assume $\left(\frac{1}{2}+\varepsilon\right)$-regularity for $\|\delta A_{s,u,t}\|_{L^m(\mathbb{P})}$, although we also need to consider the regularization effect encoded in the estimate $\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]\|_{L^m(\mathbb{P})} \lesssim (t-s)^{1+\varepsilon}$. Furthermore, if we denote by $\mathcal{A}_T$ the limit of the Riemann sums in the interval $[0, T]$, we have a quantitative bound

$$\|\mathcal{A}_{s,t}\|_{L^m(\mathbb{P})} \lesssim_{m,\varepsilon} \Gamma(t-s)^{\frac{1+\varepsilon}{2}}.$$

That is, we can transfer the estimate of $A_{s,t}$ to that of $\mathcal{A}_{s,t}$.

Sometimes, it is difficult to observe the regularization effect through $\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]\|_{L^m(\mathbb{P})}$. The easiest example is $A_{s,t} = |B^H_t - B^H_s|^{1/H}$, the $1/H$-variation of the fractional Brownian motion B^H. For this example, it is not possible to estimate $\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]$, although the convergence of the Riemann sums (along equipartitions) is well known.

Chapter 1 of the thesis presents an extension of Lê's stochastic sewing (**??**), relaxing the estimate of the conditional expectation $\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]$. We replaced it with

$$\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_v]\|_{L^m(\mathbb{P})} \lesssim (s-v)^{-\alpha}(t-s)^{1+\varepsilon}, \quad v < s < u < t, \quad \alpha < \frac{1}{2} + \varepsilon. \tag{3}$$

Because of this new condition, where the conditioning is shifted from $\mathcal{F}_s$ to $\mathcal{F}_v$, we call this extension the *shifted* stochastic sewing lemma. The case where $\alpha = 0$ and $v = s$ corresponds to Lê's stochastic sewing. The version of the mild shifting, namely the case where $\alpha = 0$ and $v = s - M(t-s)$ for a fixed positive constant M, is obtained by Gerencsér [Ger22]. Our extension allows us to take advantage of the *asymptotic* effect of regularization, inspired by [Pic08].

For the example $A_{s,t} = |B_t - B_s|^{1/H}$, we can prove estimates of the form (3). Additionally, as a more interesting application, we demonstrate the convergence of Itô approximations and Stratonovich approximations under low regularity assumptions, which can be viewed as a simplification and improvement of [Nou12, Theorem 3.5]. More precisely, in Section 1.3 we prove

$$\exists \lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} f(B_s^H)(B_t^H - B_s^H) \quad \text{in } L^m(\mathbb{P}) \text{ for } H \in (1/2,1) \text{ and } f \in L^\infty(\mathbb{R}^d)$$

and with $H \in (1/4, 1/2)$ and $\gamma > \frac{1}{2H} - 1$ we prove

$$\exists \lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} \frac{f(B_s^H) + f(B_t^H)}{2}(B_t^H - B_s^H) \quad \text{in } L^m(\mathbb{P}) \text{ for } f \in C^\gamma(\mathbb{R}^d).$$

Furthermore, the shifted stochastic sewing lemma will be crucially applied in Chapters 2 and 3. The reader however can skip the proof of **??** (Section 1.2 and Section 1.4) without any problem for further reading. The result of Section 1.3 will be used in Chapter 3.

Chapter 2: Level crossings of fractional Brownian motions

In this section, we provide a summary of Chapter 2, which is based on collaborative work with Purba Das, Rafał Łochowski, and Nicolas Perkowski.

We consider a fractional Brownian motion B^H with a Hurst parameter $H \in (0,1)$. It is known that the following convergence holds:

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left| B^H_{\frac{k+1}{2^n}T} - B^H_{\frac{k}{2^n}T} \right|^{1/H} = \mathbb{E}[|B_1^H|^{1/H}]T, \quad \text{a.s.}$$

One of the key objectives of Chapter 2 is to investigate the $(1/H)$-variations along *Lebesgue partitions*, which are random partitions defined by the level crossings of B^H. To construct these partitions, we start with $T_0^n := 0$ and recursively define the stopping times T_k^n by

$$T_k^n := \inf\{t > T_{k-1}^n : |B_t^H - B^H_{T^n_{k-1}}| = 2^{-n}\}.$$

The reader can refer to Figure 2 for an illustration. Each nth Lebesgue partition consists of intervals of the form $[T_{k-1}^n, T_k^n]$ for $k \in \mathbb{N}$ satisfying $T_k^n \le T$. The main objective is to establish the convergence of $(1/H)$-variations along these Lebesgue partitions. In particular, we aim to prove the existence of the limit:

$$\lim_{n\to\infty} 2^{-n/H} \#\{k : T_k^n \le T\}, \tag{4}$$

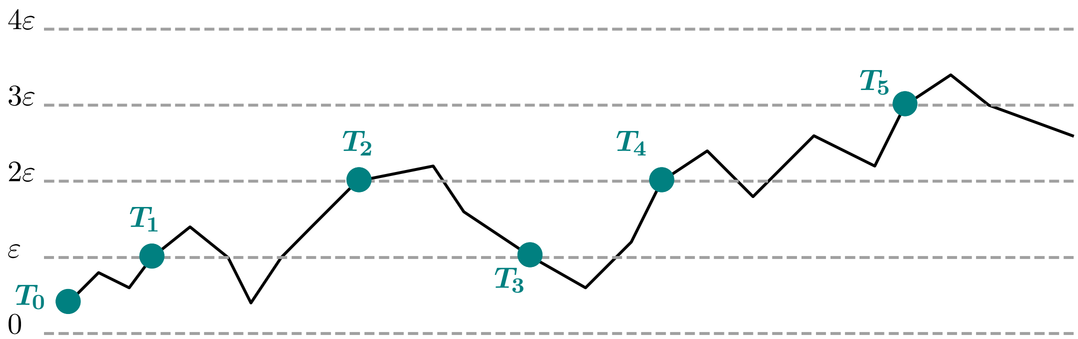

Figure 2: Stopping times T_k^n.

where $\#$ denotes the cardinality.

Additionally, we can count level crossings specifically around the level 0 and investigate the convergence of the number of these crossings towards the local time at 0.

The convergence of (4) and its local time counterpart is well-known for the Brownian case, where the proof relies on martingale or Markovian properties, such as Itô's formula (see, e.g., [RY99]). However, such martingale or Markovian arguments are not applicable when $H \neq 1/2$. Chapter 2 presents a completely different strategy to prove (4) for $H \in (0,1)$ and its local time counterpart for $H < 1/2$. This novel approach resolves a conjecture posed in [CP19].

To demonstrate our strategy, we denote by $K_{s,t}(\varepsilon, w)$ the number of ε-level crossings of the process w in the interval $[s,t]$. Then the limit (4) is equal to

$$\lim_{n\to\infty} 2^{-n/H} K_{0,T}(2^{-n}, B).$$

The key observation is that the family $(K_{s,t}(\varepsilon, B))_{0\le s<t\le T}$ is superadditive and almost subadditive. This leads to the approximation

$$K_{0,T}(2^{-n}, B) \approx \sum_{[s,t]\in\pi_n} K_{s,t}(2^{-n}, B),$$

where π_n is a partition of $[0,T]$ with a mesh size of order $2^{-n/H}$. Hence, we can approximate $K_{0,T}(2^{-n}, B)$ by a stochastic Riemann sum, which can then be estimated by the shifted stochastic sewing. To verify the conditions of the shifted stochastic sewing, the computations will be carried out in the spirit of Picard [Pic08]. The strategy of proving the convergence to the local time follows similar arguments, but due to the lack of stationarity, technicalities dramatically increase.

Our result also poses a very interesting conjecture on whether the limit (4) is equal to that of $(1/H)$-variations along deterministic dyadic partitions. For $H = 1/2$, they are equal. Numerical simulation suggests that they are different for $H \neq 1/2$. If this were indeed true, it would be an interesting manifestation of non-Markovianity.

Chapter 3: Probabilistic Young and rough differential equations

This is the summary of Chapter 3, based on joint work with Avi Mayorcas. In this chapter, we focus on the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t^H, \quad X_0 = x \in \mathbb{R}^{d_1}, \tag{5}$$

where σ is a map valued in the space of $d_1 \times d_2$ matrices, and B^H is a d_2-dimensional fractional Brownian motion with Hurst parameter $H \in (1/3, 1)$. The differential equation is interpreted either as Young's differential equation ($H > 1/2$) or as Lyons' rough differential equation [Lyo98] ($H < 1/2$). Our main result is on the pathwise uniqueness of (5) under a low regularity assumption on σ. Rather than directly stating the result, we provide a background review leading up to this outcome.

For $H = 1/2$, we often apply Itô's theory to study (5); we will discuss the alternative theory of Lyons later. In Itô's theory, there are a few notions of solutions and their uniqueness, among which the most relevant to us is the notion of *pathwise uniqueness*. It says that two solutions, adapted to some filtration making the driving Brownian motion martingale, must be indistinguishable. Hence, pathwise uniqueness is a *probabilistic* concept of uniqueness (despite its name). It is a classical result, as can be found in all textbooks of stochastic calculus, that pathwise uniqueness holds for (5) with $H = 1/2$ provided that σ is Lipschitz. The proof is a consequence of Itô's isometry: for an adapted process Y we have

$$\mathbb{E}\Big[\Big|\int_0^T Y_r \mathrm{d}B_r^{1/2}\Big|^2\Big] = \mathbb{E}\Big[\int_0^T |Y_r|^2 \mathrm{d}r\Big].$$

Itô's isometry is due to the martingale property of the Brownian motion. Since B^H, $H \neq 1/2$, is not a martingale (nor Markovian), Itô's theory is not available for $H \neq 1/2$. Lack of probabilistic tools naturally motivates us to study the SDE (5) *pathwisely*. Based on Young's integration theory, Lyons [Lyo94] showed that the (deterministic) differential equation

$$\mathrm{d}x_t = f(x_t)\mathrm{d}y_t \tag{6}$$

driven by a path y of finite p-variation with $p \in [1, 2)$ has a unique solution provided that f is α-Hölder with $\alpha > p$. Furthermore, [Lyo98] extended the result for $p \in [2, \infty)$, provided that we are additionally given "iterated integrals"

$$\int_s^t \int_s^{r_1} \mathrm{d}y_{r_2} \mathrm{d}y_{r_1}, \int_s^t \int_s^{r_1} \int_s^{r_2} \mathrm{d}y_{r_3} \mathrm{d}y_{r_2} \mathrm{d}y_{r_1}, \ldots,$$

satisfying certain analytic and algebraic conditions. The tuple of y and its (sufficient number of) iterated integrals is called a rough path of y. Later, Coutin and Qian [CQ02] proved that the fractional Brownian motion B^H, with $H > 1/4$, can be naturally lifted to a rough path. Since B^H has finite p-variation for any $p > 1/H$, we see that (5) has a unique solution provided that $\sigma \in C^\gamma$ with $\gamma > 1/H$ and $H \in (1/4, 1)$.

We remark two important differences in Itô's probabilistic theory and Lyons' pathwise theory. One is that the former considers uniqueness among solutions adapted to a given filtration, while the latter considers uniqueness among all solutions satisfying (6), which do not need to be adapted. In other words, the notion of uniqueness is stronger in the pathwise theory, referred to as *path-by-path uniqueness*, following the works of Davie [Dav07; Dav08]. The other difference lies in the regularity assumption on σ. When $H = 1/2$, Itô's theory assumes that σ is only Lipschitz, while Lyons' theory assumes that $\sigma \in C^\gamma$ with $\gamma > 2$. In summary, Itô's theory requires less regularity assumption on σ at the cost of a weaker notion of uniqueness.

Although Itô's theory is not available for $H \neq 1/2$, pathwise uniqueness is a well-defined notion in this setting. Now it is natural to ask if we can prove pathwise uniqueness of (5) for $\sigma \in C^\gamma$ with $\gamma < 1/H$. Our main result in this chapter answers the question affirmatively. That is, under the uniformly elliptic condition ($\sigma\sigma^{\mathrm{T}}$ is non-degenerate), we prove pathwise uniqueness under $\sigma \in C^\gamma$ with $\gamma > \max\{1/(2H), (1-H)/H\}$, as shown in Figure 3.

The proof follows Lê's strategy [Lê20] to prove pathwise uniqueness. In fact, pathwise uniqueness is deduced from a sharp estimate on the stochastic integral

$$\int f(X_r) \mathrm{d}B_r,$$

where X is a path controlled by B and f is a map of low regularity. Specifically, we can define the stochastic integral for $f \in C^\gamma$ with $\gamma > 1/(2H) - 1$. The estimate is proven using stochastic sewing techniques, including the shifted stochastic sewing.

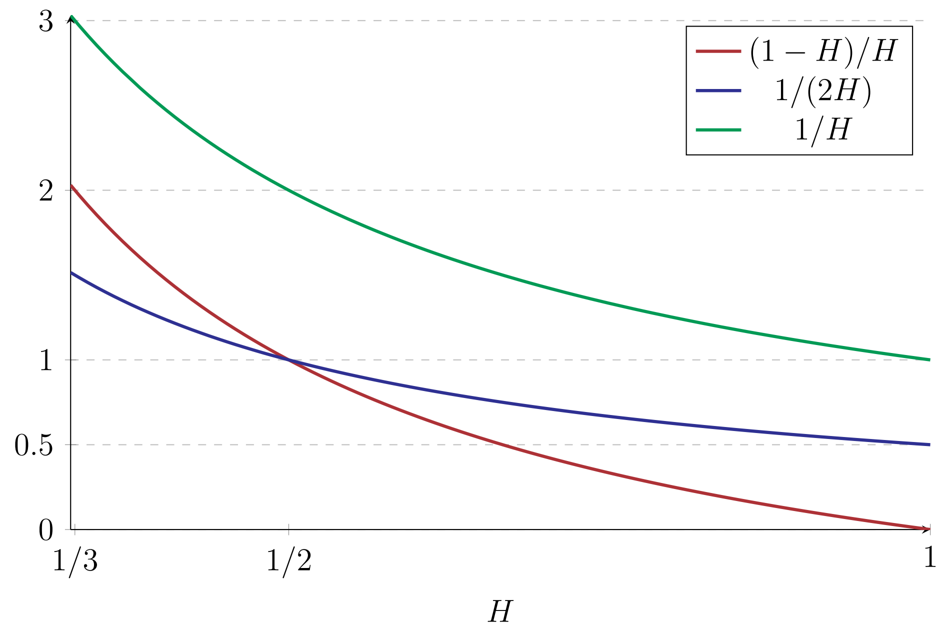

Figure 3: Some graphs of H related to the main result of Chapter 3. Pathwise theory covers $\sigma \in C^\gamma$ with $\gamma > 1/H$ (green), while the result of Chapter 3 says that pathwise uniqueness holds if $\gamma > 1/(2H)$ (blue) and if $\gamma > (1-H)/H$ (red).

Chapter 4: Regularization by fractional noise for integrable drift

This is the summary of Chapter 4, based on joint work with Oleg Butkovsky and Khoa Lê. This chapter is relatively independent as it does not rely on the shifted stochastic sewing. The main result focuses on the strong well-posedness of the fractional SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}B_t^H. \tag{7}$$

The well-posedness is straightforward when b is smooth, but our interest lies in the case of non-smooth b. Here, we consider an integrable drift $b \in L^q([0,T]; L^p(\mathbb{R}^d))$, and our goal is to determine the condition on p, q, d, H for strong well-posedness. The topic discussed in this chapter falls within the highly active field of *regularization by noise*, which will be further explored and reviewed in the following paragraphs.

Ill-posed differential equations sometimes regain well-posedness by introducing noise. For instance, the differential equation $\mathrm{d}X_t = \sqrt{|X_t|}\mathrm{d}t$ may have multiple solutions, while the stochastic differential equation (SDE) $\mathrm{d}X_t = \sqrt{|X_t|}\mathrm{d}t + \mathrm{d}B_t^{1/2}$ has a unique strong solution (strongly well-posed). This phenomenon is known as *regularization by noise*. Recently, there has been growing interest in understanding this phenomenon beyond the Brownian setting; among them is regularization by fractional noise.

In their recent work [GG23], Galeati and Gerencsér introduced the notion of subcriticality for fractional SDEs. Subcriticality refers to the domination of fractional noise under small scales. If X solves the SDE (7), then the scaled process $X_t^{(\lambda)} = \lambda^{-H} X_{\lambda t}$ solves the SDE

$$\mathrm{d}X_t^{(\lambda)} = \lambda^{1-H-\frac{dH}{p}-\frac{1}{q}} b^{(\lambda)}(t, X_t^{(\lambda)})\mathrm{d}t + \mathrm{d}B_t^{(\lambda)},$$

where

$$b^{(\lambda)}(t,x) = \lambda^{\frac{dH}{p}+\frac{1}{q}} b(\lambda t, \lambda^H x), \quad B_t^{(\lambda)} = \lambda^{-H} B_{\lambda t}.$$

It is worth noting that $\|b^{(\lambda)}\|_{L_t^q L_x^p} = \|b\|_{L_t^q L_x^p}$ and $B^{(\lambda)}$ has the same law as B. The domination of the noise at small scales implies that the order of the drift term is smaller than that of the driving noise as λ approaches 0. This leads to the condition

$$1 - H - \frac{dH}{p} - \frac{1}{q} > 0. \tag{8}$$

Therefore, the condition (8) is natural for the solution theory of (7). In fact, the celebrated result by Krylov and Röckner [KR05] proves strong well-posedness for $H = 1/2$ under (8). The main result of Chapter 4 addresses the strong well-posedness of (7) in the

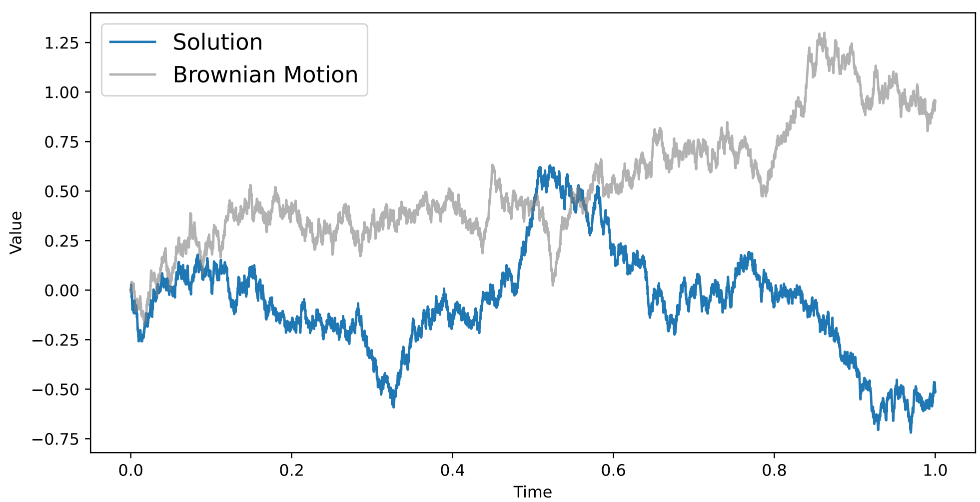

Figure 4: Simulation of the SDE $\mathrm{d}X_t = \sqrt{|X_t|}\mathrm{d}t + \mathrm{d}B_t^{1/2}$ with $X_0 = 0$. The differential equation $\mathrm{d}x_t = \sqrt{|x_t|}\mathrm{d}t$ with $x_0 = 0$ has multiple solutions, but the SDE has a unique strong solution. Heuristically, the solution of the SDE must behave like Brownian motion, which enforces a unique way for the solution to escape the singularity at 0.

fractional case $H < 1/2$. Specifically, we prove strong well-posedness for $H < 1/2$, $p \geq \max\{(1-H)^{-1}, 2dH\}$, and (8). We also consider the case where $p < (1-H)^{-1}$ with an additional condition. Additionally, we establish the stability of (7) with respect to the initial condition and the drift b. Our arguments are based on the stochastic sewing with control [Lê23] and the notion of processes of vanishing mean oscillation introduced in another seminal work by Lê [Lê22].

Reading guide

$$\begin{tikzcd}
\boxed{\text{Section 1.1}} \arrow[d] \arrow[dr] & \boxed{\text{Section 1.3}} \arrow[d] & \\
\boxed{\text{Chapter 2}} & \boxed{\text{Chapter 3}} & \boxed{\text{Chapter 4}}
\end{tikzcd}$$

The relation of each chapter is depicted in the above diagram. The statement of the shifted stochastic sewing appears in Section 1.1, and the result will be used in Chapters 2 and 3. However, the reader can skip its rather involved proof (Sections 1.2 and 1.4) for further reading. We remark that Chapter 4 is essentially independent of the preceding chapters.

Introduction of each chapter includes a section on notation specific to that chapter. Below, we collect the most frequently used notations:

- We use the notation $A := B$ to indicate that A is defined by B.
- The symbol $\mathbb{N}$ represents the set of natural numbers $\{1, 2, \dots\}$, $\mathbb{Q}$ represents the set of rational numbers, and $\mathbb{R}$ represents the set of real numbers.
- We denote by $\mathbf{1}_A$ the indicator function for the set A.
- We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the underlying probability space, which is often implicit. The symbol $\mathbb{E}$ denotes the expectation. We write $\mathbb{E}[\cdot|\mathcal{G}]$ for the conditional expectation given $\mathcal{G}$. We set

  $$\|F\|_{L^m(\mathbb{P})} := \Big(\int_\Omega |F(\omega)|^m \mathrm{d}\mathbb{P}(\omega) \Big)^{\frac{1}{m}}$$

  with usual convention for $m = \infty$.

- The d-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ is represented as $B^H = (B^{H,i})_{i=1}^d$. The components of B^H are independent. In Chapter 2, it takes values in $\mathbb{R}$ ($d = 1$), and in Chapter 3, it takes values in $\mathbb{R}^{d_2}$ ($d = d_2$). We typically use the symbol W to denote the Brownian motion.
- For a given map $f\colon [0,T] \to \mathbb{R}^d$, we write $f_{s,t} := f_t - f_s$.
- The notation $A \lesssim B$ signifies that there exists a constant C depending only on irrelevant parameters such that $A \leq CB$. If we want to emphasize the dependency on $\alpha, \beta, \ldots$, we write $A \lesssim_{\alpha,\beta,\ldots} B$. We often write $C = C(\alpha, \beta, \ldots)$ to emphasize that the constant C depends on $\alpha, \beta, \ldots$.

I hope that you enjoy the reading :)

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