Research statement

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In his review article (20, arXiv:2004.09336), Lorenzo Zambotti wrote his dream of unifying Itô's theory of martingales and *stochastic partial differential equations* (SPDEs). In a broader sense, this dream can be interpreted as developing sophisticated *stochastic* calculus beyond classical martingale, Markovian or Brownian settings. I feel that there is a growing number of works appearing in the direction towards this dream — singular SPDEs, KPZ universality, regularisation by noise, non-Markovian/infinite-dimensional stochastic dynamics, to name just a few.

In Section 1 and 2, I review my previous and ongoing works towards the dream of Zambotti in the context of *Gaussian stochastic calculus*. In Section 3, I describe some of my future plans to establish similar results in the context of SPDEs.

1 Previous research

1.1 Shifted stochastic sewing

In his seminal work, Lê (20) obtained a stochastic version of Gubinelli's sewing lemma. As Gubinelli's sewing lemma plays an important role in *pathwise* stochastic calculus, so does Lê's stochastic sewing lemma in *probabilistic* stochastic calculus. In particular, the discovery of the stochastic sewing has significantly advanced the field of regularisation by noise.

A concrete statement of the stochastic sewing is as follows. If $(A_{s,t})_{s<t}$ is a stochastic germ adapted to a filtration $(\mathcal{F}_t)$ and if the following estimates

$$\|\delta A_{s,u,t}\|^2_{L^m(\mathbb{P})} + \|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]\|_{L^m(\mathbb{P})} \lesssim (t-s)^{1+\varepsilon}, \quad \delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}$$

are satisfied for $s<u<t$, $m \in [2,\infty)$ and $\varepsilon > 0$, then Riemann sums $\sum_{[s,t]\in\pi} A_{s,t}$, where π is a partition of some fixed interval $[0,T]$, converge in $L^m(\mathbb{P})$ as the mesh size of π tends to 0. The strength of the stochastic sewing is that we only need to assume $\left(\frac{1}{2}+\varepsilon\right)$-regularity for $\|\delta A_{s,u,t}\|_{L^m(\mathbb{P})}$, although we also need to see the regularisation effect encoded in the estimate $\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]\|_{L^m(\mathbb{P})} \lesssim (t-s)^{1+\varepsilon}$.

Sometimes, it is difficult to observe the regularisation effect through $\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]\|_{L^m(\mathbb{P})}$. The easiest example is $A_{s,t} = |B^H_t - B^H_s|^{1/H}$, $1/H$-variation of the fractional Brownian motion B^H. For this example, it is not possible to estimate $\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]$, although the convergence of the Riemann sums is well known.

Jointly with Nicolas Perkowski, we obtained an extension of Lê's stochastic sewing, relaxing the estimate of the conditional expectation $\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]$. Namely, we replaced it by

$$\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_v]\|_{L^m(\mathbb{P})} \lesssim (s-v)^{-\alpha}(t-s)^{1+\varepsilon}, \quad v<s<u<t, \quad \alpha < \frac{1}{2}+\varepsilon. \tag{1}$$

The case $\alpha = 0$ corresponds to Lê's stochastic sewing. Our extension allows us to take advantage of *asymptotic* effect of regularisation. For the above example of $A_{s,t} = |B_t - B_s|^{1/H}$, we can prove the estimate of the form (1). As further applications, we proved novel results on fractional Brownian motion: (i) the convergence of Itô approximations and that of Stratonovich approximations with low regularity assumptions, and (ii) the convergence of discretized local time. Moreover, we made the first step to understand probabilistic well-posedness of stochastic Young's differential equation $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B^H_t$, $H > \frac{1}{2}$, with low regularity assumptions on σ. A work towards improving the last result will be discussed later.

1.2 Regularisation by fractional noise for $L^q_t L^p_x$-noise

Ill-posed differential equations sometimes restore well-posedness by adding a noise. For instance, the differential equation $\mathrm{d}X_t = \sqrt{|X_t|}\mathrm{d}t$ may have multiple solutions, while the stochastic differential equation (SDE) $\mathrm{d}X_t = \sqrt{|X_t|}\mathrm{d}t + \mathrm{d}B^{1/2}_t$ has a unique strong solution (strongly well-posed). This phenomenon is called *regularisation by noise*. Recently, there is a growing interest to understand this phenomenon beyond the Brownian setting; among them, regularisation by fractional noise.

In the recent work of Galeati and Gerencsér (22), they introduced the notion of subcriticality for fractional SDEs. If we consider d-dimensional SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}B_t^H, \quad b \in L_t^q L_x^p, \tag{2}$$

then the subcritical condition is $\frac{dH}{p} + \frac{1}{q} < 1 - H$. Therefore, it is conjectured that in this regime strong well-posedness should be established. This conjecture holds for the Brownian case ($H = \frac{1}{2}$), due to a famous result by Krylov and Röckner (05). For the fractional case, $H < \frac{1}{2}$, the best known result was obtained by Lê (20), who proved the strong well-posedness of (2) under $\frac{dH}{p} + \frac{1}{q} < \frac{1}{2} - H, p \geq 2$.

In a joint work with Oleg Butkovsky and Khoa Lê, we improved Lê's previous result, proving the conjecture of Galeati and Gerencsér for (2). More precisely, we proved strong well-posedness under $H < \frac{1}{2}$, $p \geq 2\max\{1, dH\}$ and $\frac{dH}{p} + \frac{1}{q} < 1 - H$. Besides, we proved the stability of (2) with respect to the initial condition and the drift b. Our arguments develop a quite novel methods, based on recent extensions of the stochastic sewing (including the shifted stochastic sewing) and the notion of processes of vanishing mean oscillation introduced by Lê (22).

1.3 Anderson Hamiltonians with singular potentials

Although this topic does not lie in the framework of Gaussian stochastic calculus, I would like to mention it here, as it is one of the best results of mine and is related to a topic discussed in Section 3. A Schrödinger operator $-\Delta + \xi$ with random potential ξ, called Anderson Hamiltonian, has been extensively studied both in mathematics and in physics, since the operator describes movement of a quantum particle in disordered environment. Since the birth of regularity structures, some started to study the Anderson Hamiltonian with random singular potential such as white noise.

Jointly with Willem van Zuijlen, we studied singular Anderson Hamiltonians in the full-fledged framework of the regularity structure. Furthermore, we constructed the integrated density of states for such operators, and investigated its asymptotics in the relation to eigenvalues of the operator.

2 Ongoing research

2.1 Variation and local time along Lebesgue partitions

Joint work with Purba Das, Rafael Lochowski and Nicolas Perkowski. For a fractional Brownian motion B^H with Hurst parameter H, the following convergence is known:

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left| B^H_{\frac{k+1}{2^n}T} - B^H_{\frac{k}{2^n}T} \right|^{1/H} = \mathbb{E}[|B_1^H|^{1/H}]T, \quad \text{a.s.}$$

A similar result holds for deterministic partitions if their diameters decay sufficiently fast. We would like to understand $(1/H)$-variations along *Lebesgue partitions*. They are random partitions defined as follows. For each $n \in \mathbb{N}$ we set $T_0^n := 0$ and inductively

$$T_k^n := \inf\{t > T_{k-1}^n : |B_t^H - B^H_{T^n_{k-1}}| = 2^{-n}\}.$$

Then, the nth Lebesgue partition π_n is given by $\{[T_{k-1}^n, T_k^n] : k \in \mathbb{N}, T_k^n \leq T\}$. The problem is to prove the convergence of $(1/H)$-variations along the Lebesgue partitions, namely,

$$\exists \lim_{n\to\infty} 2^{-n/H} \#\{k : T_k^n \leq T\}. \tag{3}$$

Similarly, we can count level crossings only around the level 0 and consider the limit of the numbers of those level crossings to the local time at 0.

The convergence of (3) and its local time counterpart are only known for the Brownian case, where the proof is based on martingale or Markovian properties such as Itô's formula, which is not available for $H \neq \frac{1}{2}$. We are currently trying to prove (3) and its local time counterpart for $H \neq \frac{1}{2}$. Our ideas of the proof are inspired by the subadditive ergodic theorem and the shifted stochastic sewing. Indeed, we use the sub(super)-additivity of the level crossing counting function, and reduce the problem to estimates on sums of level crossing counting functions in smaller intervals. We then estimate the sums based on the technique of the shifted stochastic sewing.

A very interesting question, which we have no idea on how to prove, is whether the limit is equal to that of $(1/H)$-variations along deterministic dyadic partitions. For $H = \frac{1}{2}$, they are equal. Numerical simulation suggests that they are different for $H \neq \frac{1}{2}$; if this were indeed true, it would be an interesting manifestation of the non-Markovianity.

2.2 Excursions of Gaussian Volterra process

Joint work with Rama Cont. An excursion X is a continuous path with $X_0 = X_T = 0$ but with $X_t \neq 0$ for $t \in (0, T)$. Since an excursion takes either non-negative or non-positive value, it can be used to model some price processes in long or short position. So far, excursions are studied only for Markov processes such as diffusions. However, in view of increasing use of fractional processes in mathematical finance, it is very important to develop a theory of excursions for non-Markov processes such as fractional Brownian motion.

We are trying to develop an excursion theory, in particular defining a natural notion of excursion measures, for some Gaussian Volterra processes including Riemann-Liouville process. Our standing point is that those Gaussian processes can be viewed as a projection of an infinite-dimensional Ornstein-Uhlenbeck process. Based on this observation, we develop an analogous theory of Maisonneuve (75) in infinite dimensions. As a byproduct of our approach, we can establish a new formula of the local time and the hitting time. This can be viewed as a potential theoretic characterisation of the local time for Gaussian Volterra processes.

2.3 Regularisation by noise for diffusion coefficients

Joint work with Avi Mayorcas. Although many works on regularisation by noise consider irregular drifts, it is equally important to consider irregular diffusion coefficients. In this work, we are considering strong well-posedness of $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B^H_t$ for $H \in (\frac{1}{3}, 1)$. This work is still at an early stage, but our current conjecture is that strong well-posedness holds provided $\sigma \in C^\gamma$ with $\gamma > \max\{\frac{1}{2H}, \frac{1-H}{H}\}$ and σ is non-degenerate. In addition to strong well-posedness, we are planning to study the stability, the 1D case and the distribution-dependent case.

In this work, the main theme is to give a probabilistic estimate on $\int_s^t f(X_r)\mathrm{d}B^H_r$ for an irregular f. This may have some applications to slow-fast systems with fractional Brownian motions.

3 Planned research

3.1 Regularisation by noise for SPDEs

Athreya, Butkovsky, Lê and Mytnik (20) obtained a novel result on regularisation by noise for SPDEs. More precisely, they prove strong well-posedness of 1D stochastic heat equation

$$\partial_t u(t, x) = \Delta_x u(t, x) + b(u(t, x)) + \xi(t, x) \tag{4}$$

with space-time white noise ξ, under the condition $b \in B^\alpha_{p,\infty}$ with $\alpha - 1/p \geq -1$. Considering the fact that for a fixed x the process $t \mapsto u(t, x)$ behaves like $B^{1/4}$ (this is precisely the case for $b \equiv 0$), the condition on b should be the same as that for strong well-posedness of $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B^{1/4}_t$. Therefore, we expect that it suffices to assume $b \in B^\alpha_{p,\infty}$ with $\alpha > -1$ and $p \geq 2$, and it is interesting to study further properties of the solution, such as stability. This problem can be viewed as an SPDE version of our work from Section 1.2. Currently, it is not clear how the result of Section 1.2 can be extended to (4); one of the main reasons is lack of Girsanov's theorem for (4).

Another interesting question is to consider spatially correlated noise, especially in higher dimensions. In this direction, it is natural to extend the result of Section 2.3 in the SPDE setting; that is, finding the optimal regularity condition on σ for $\partial_t u = \Delta u + \sigma(u)\xi$. This problem is mentioned as Conjecture 1.6 of Mytnik and Perkins (08).

3.2 SPDEs in random environments

Nice examples of SPDEs with irregular coefficients are provided by SPDEs in random environments. For instance, we can set $b = \eta$ in (4) with η white noise independent of ξ. This is an infinite-dimensional analogue of Brox's diffusion. Systems or processes in random environments often show different behaviours than those without external randomness. Indeed, this is the case for Brox's diffusion as well as random Schrödinger operators discussed in Section 1.3. Both examples show the phenomenon of localization. It is an interesting question to see such effects in SPDE settings.

I am not aware of any previous papers considering Brox's SPDEs. But relevent SPDEs discussed in physics literature, see e.g. Weise (arXiv:2102.01215), are SPDEs with quenched noise, such as quenched stochastic heat equation:

$$\partial_t u = \Delta u + F + \xi(x, u), \tag{5}$$

where $F \in \mathbb{R}$ and $\xi = \xi(x, y)$ $(x, y \in \mathbb{R})$ is a Gaussian noise white in x-variable and correlated in y-variable. The SPDE (5) is expected to exhibit pinning (localization) / depinning (delocalization) transition depending on the value of F. There are very few works to understand this phenomenon mathematically.