Digital Signal Processing

Donia Gamal

Lecture 3

Agenda

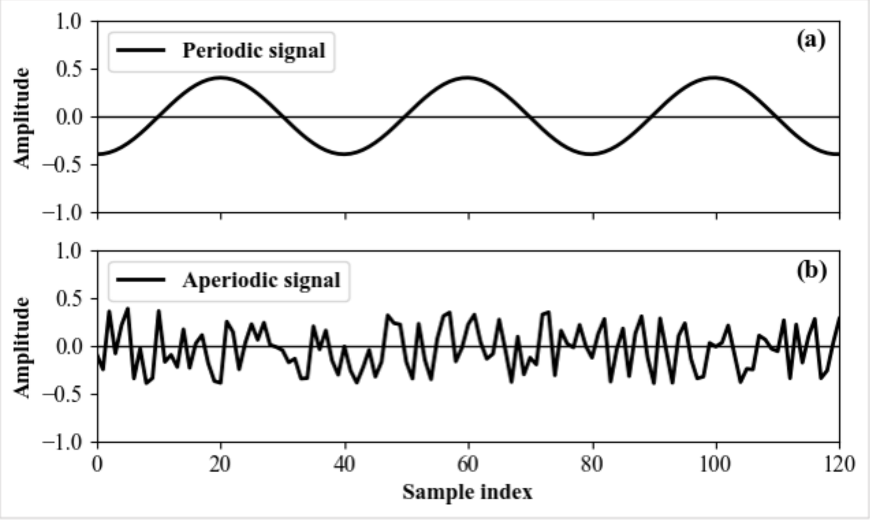
- Periodicity Exponential Signals
- Sampling
- Aliasing Problem
- Methods to Avoid Aliasing
- Types of Sampling
- Exponential Signals
- Importance of Fourier Transform

Periodic and Aperiodic Signals (non-periodic)

A signal is **periodic if it repeats** itself after a fixed period T,

$$x(t) = x(t + nT) \qquad \textit{for all } t$$

$$x(t) = x(t+T) = x\,(t+2T) = x(t-3T)$$

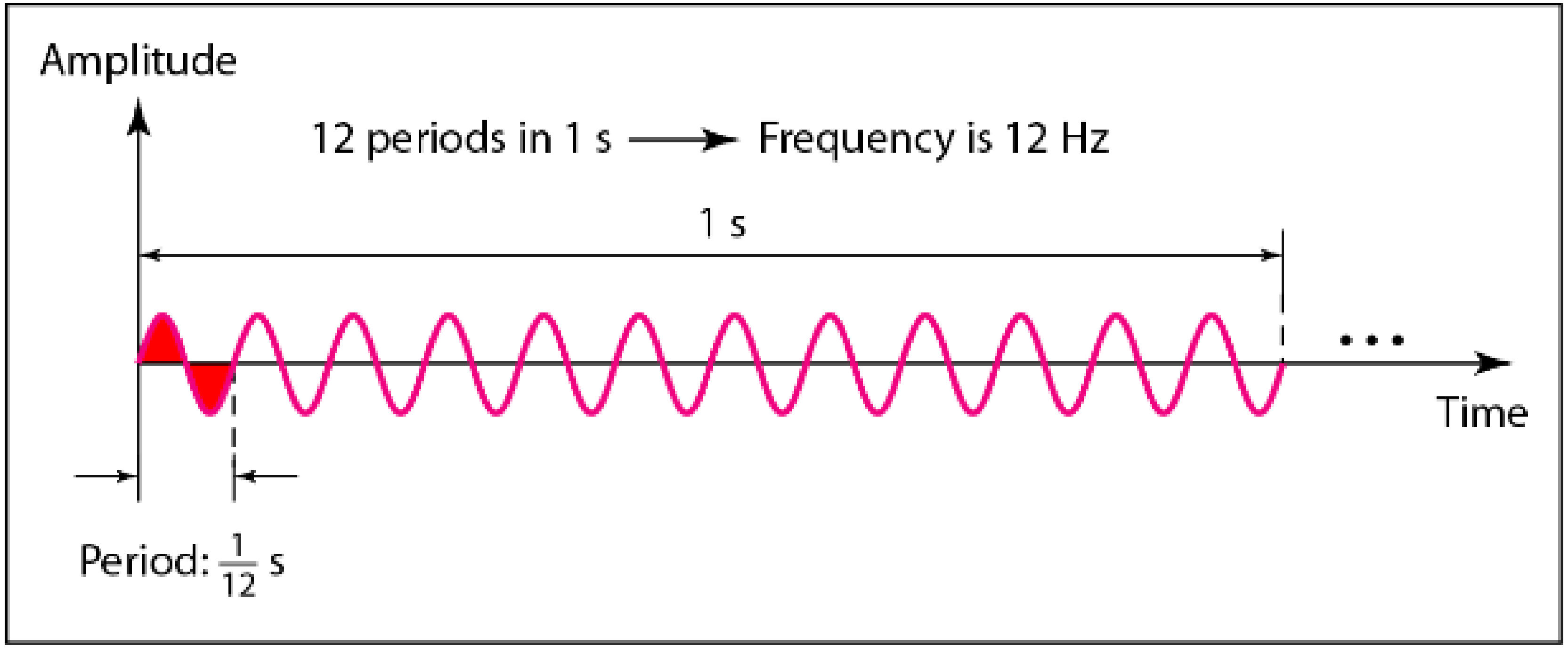


A Sin(t) and Cos(t) signals are periodic.

T is the fundamental period of the signal which is the minimum positive period of interval.

Any analog signal has to be a **periodic signal**.

Discrete Sinusoidal Signal



a. A signal with a frequency of 12 Hz

Discrete Sinusoidal Signal

Two Signals with two different Frequencies could meet at the same points?

$\Omega_k = \Omega_o + 2\pi k$

nusoids $x[n] = A\cos(\Omega n + \phi)$

$$A\cos(\Omega n + \phi) = A\cos(\Omega(n + N) + \phi)$$

K & N are Integers

$$N = \frac{2\pi k}{\Omega}$$

$$\Omega = \frac{2\pi k}{N} = 2\pi \frac{k}{N} = \pi \cdot \underbrace{\boxed{\frac{2k}{N}}}_{RATIONAL}$$

$\Omega \equiv \omega$

Capital Letter in Greek while ω is small letter in Greek

Therefore, a discrete-time sinusoid is periodic if its radian frequency Ω is a rational multiple of π.

Otherwise, the discrete-time sinusoid is non-periodic.

Sampling (Alias)

If we have 2 Analog signals A and B and we need to choose sampling frequency f_s

Analog

Digital

Aliasing (Sampling Ambiguity)

A

$$f_{As} = \frac{1}{4}$$

B

$$f_{Bs} = \frac{2}{8} = \frac{1}{4}$$

If we reconstructed Signal B from f_{Bs} , it will give us signal A.

Sinusoidal Signal

Fundamental Range for ω is defined as:

$$0 < \omega < 2\pi$$

$$0 < \cancel{2\pi} f < \cancel{2\pi}$$

$$0 < f < 1$$

$$\mathbf{0 - \frac{1}{2} < f < 1 - \frac{1}{2}}$$

As we agreed f is a fraction

$$-\frac{1}{2} < f < \frac{1}{2}$$

Sampling

If x(t) is band limited signal with a frequency equal to f_m.

Then, it can be sampled by a sampling frequency rate f_s

Sampling Time

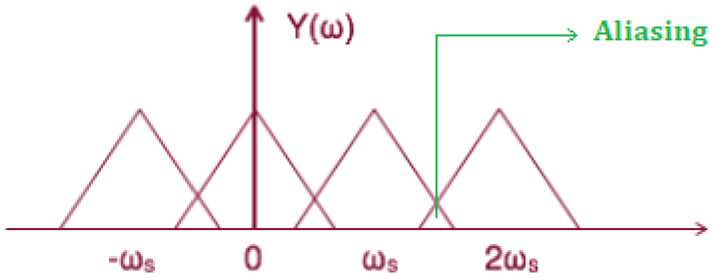
If $f_s \geq 2f_{max}$ *then we can recover* $x(t)$ *without loss of data.*

$$f_s \geq 2f_{max} \quad \rightarrow \quad \frac{1}{T_s} \geq 2f_{max} \quad \rightarrow \quad T_s \leq \frac{1}{2f_{max}}$$

Nyquist Rate is the minimum rate at which the signal can be sampled = $2f_{max}$

Sampling (Alias)

The **sampling rate** is less than the Nyquist rate, making it impossible to extract the original signal from the sampled signal



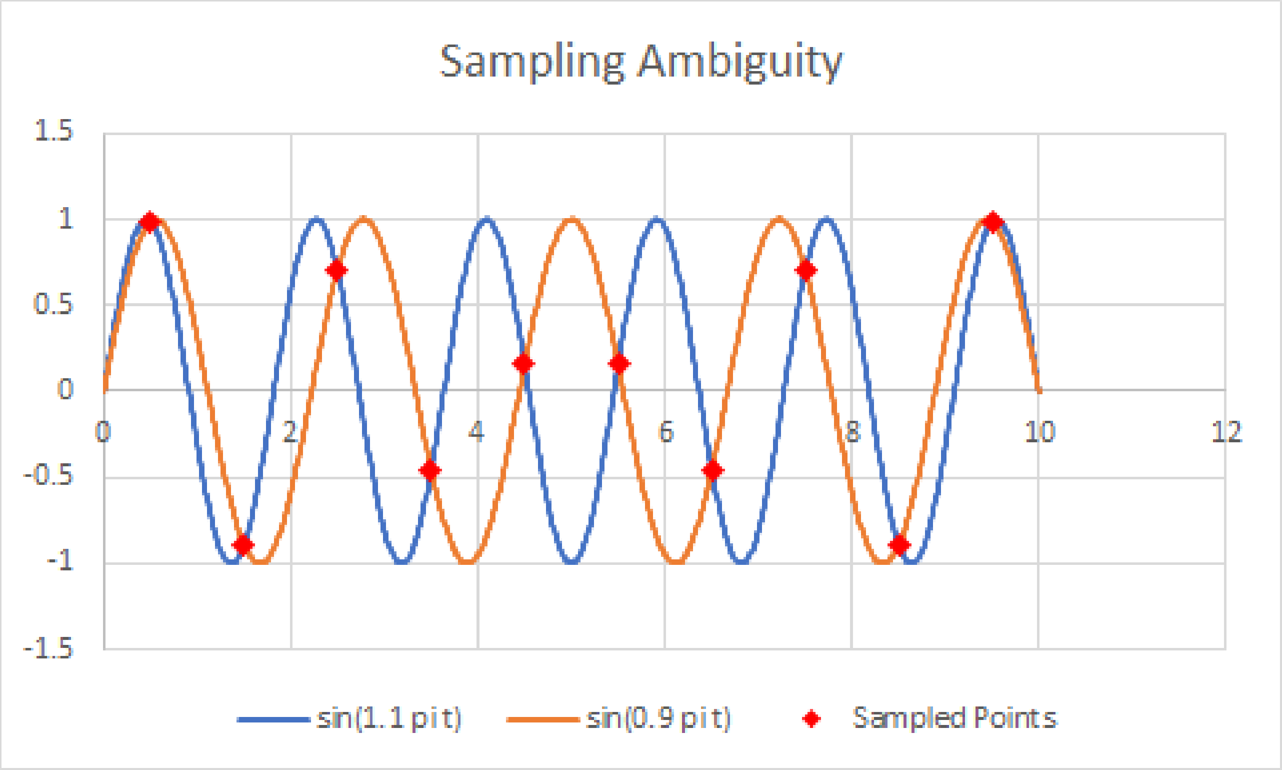
$f_s < 2f_m$
under sampling

It occurs when a high-frequency signal is represented at a lower frequency.

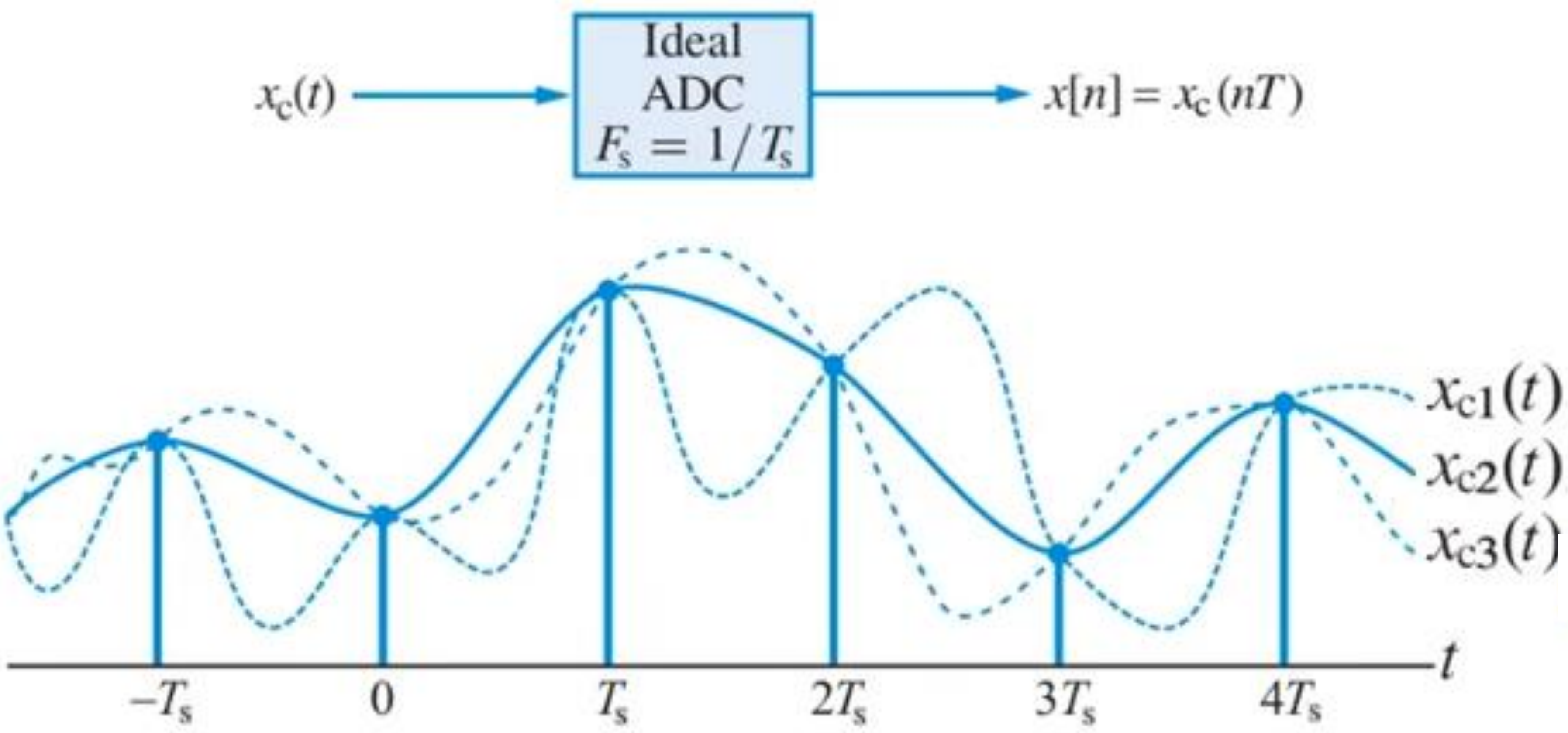
Sampling (Alias)

Aliasing means that two or more signals have the same sampling frequency f_s.

It occurs when the sampling rate is insufficient and fails to capture the signal properly.

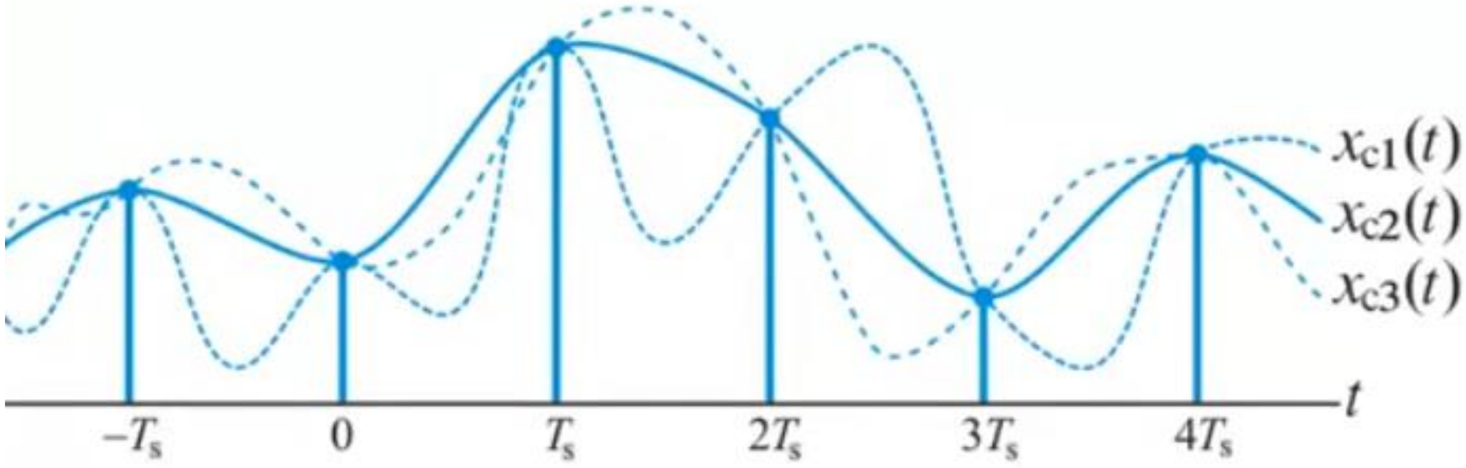


Sampling (Alias)



Sampling (Alias)

- Question: are the samples $x[n]$ sufficient to uniquely describe the original continuous-time signal $x_c(t)$?
- Question: If so, How can $x_c(t)$ be reconstructed from $x[n]$?



- As illustrated by this figure, there is (in this case) an infinite number of signals that can generate the same set of samples.
- Need some constraint on the behavior of the continuous-time signal.

Avoid Aliasing

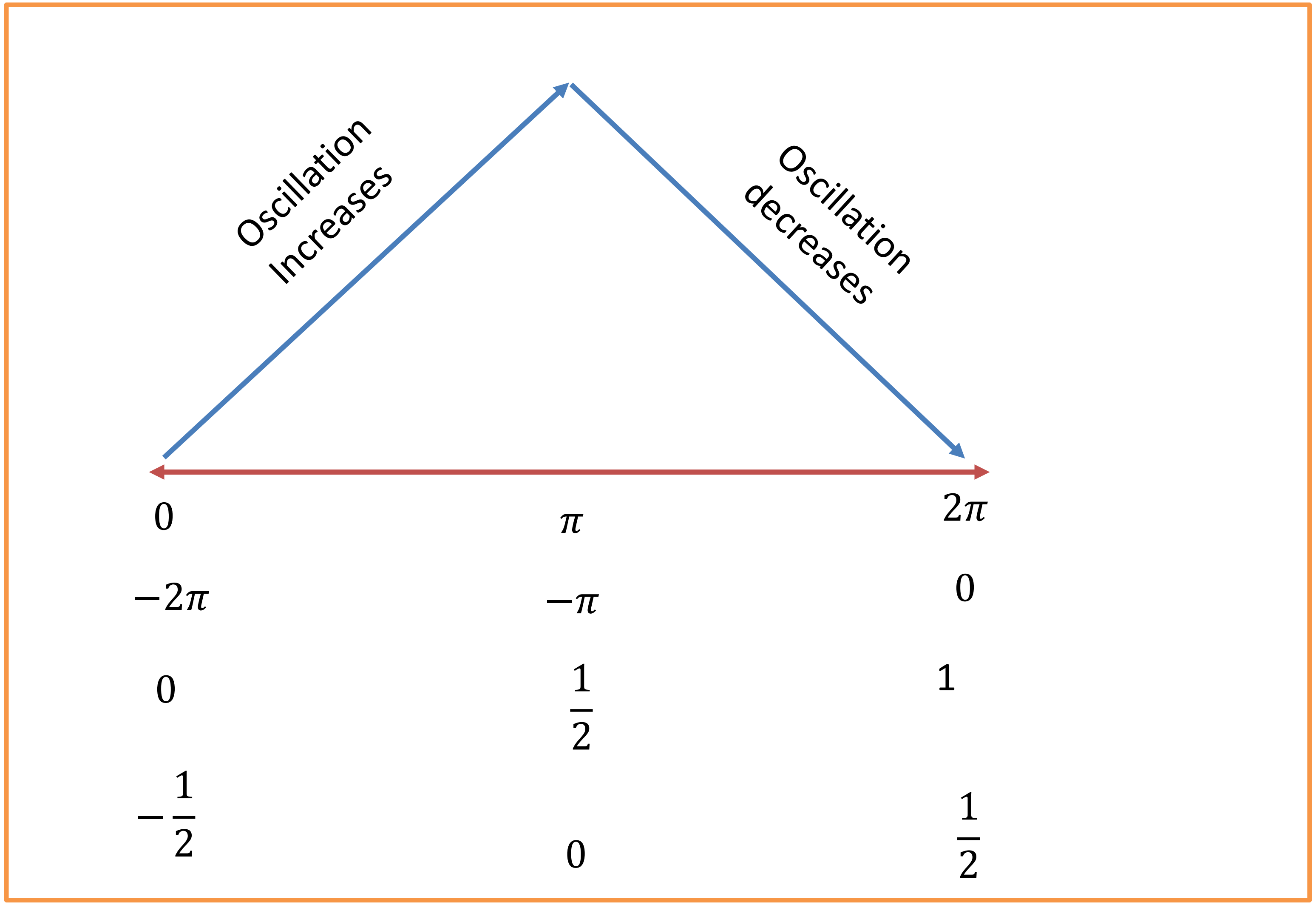
- Increase the number of samples for each signal to differentiate it from others.

- This means increasing the sampling Frequency (Use Nyquist Sampling Rate or higher).

- Each analog signal should be mapped to a unique sampling frequency than others.

Avoid Aliasing

$$-\frac{1}{2} < f < \frac{1}{2}$$



Avoid Aliasing

- Increase the number of samples for each signal to differentiate it from others.

$$0 < f_s < \frac{1}{2}$$

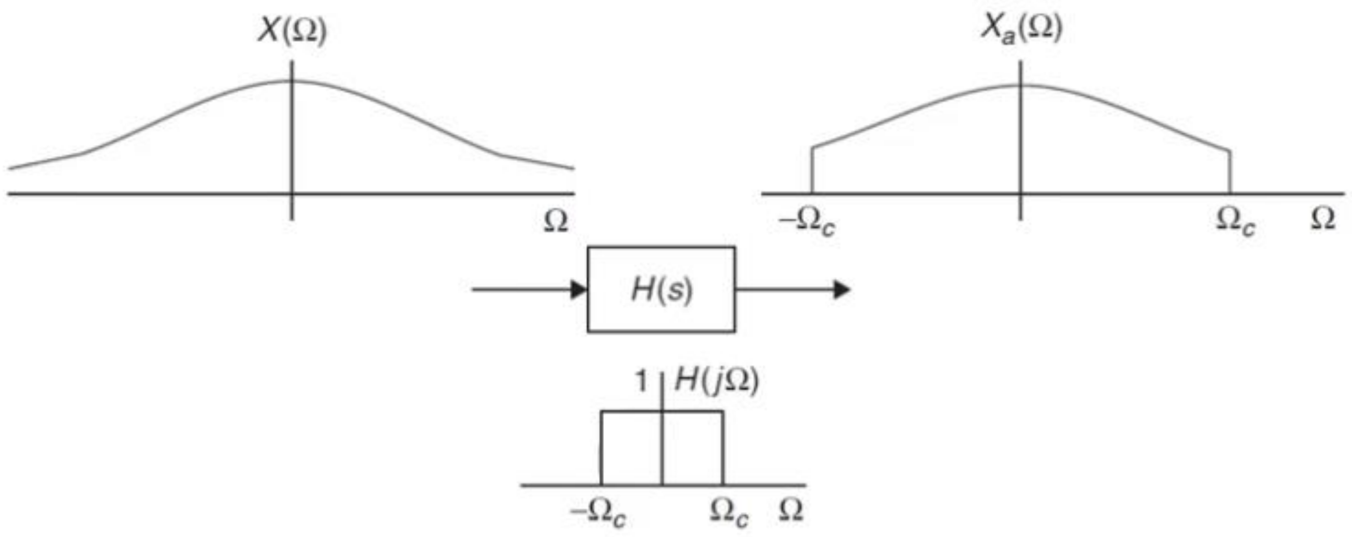
- This means increasing the sampling Frequency.

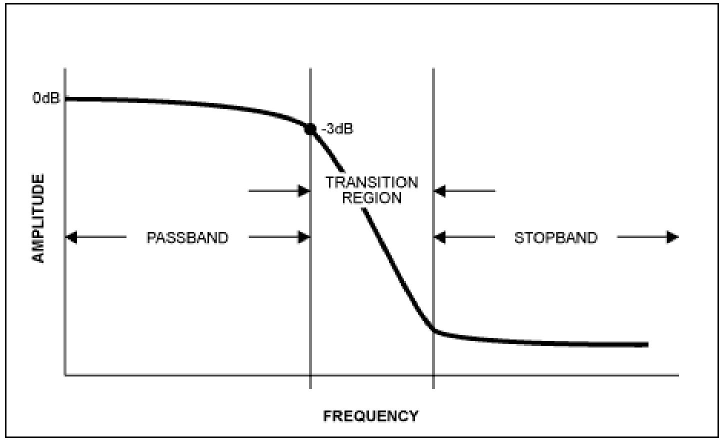
- Each analog signal should be mapped to a unique sampling frequency than others.

Avoid Aliasing

- Or, by Using Anti-Aliasing Filter

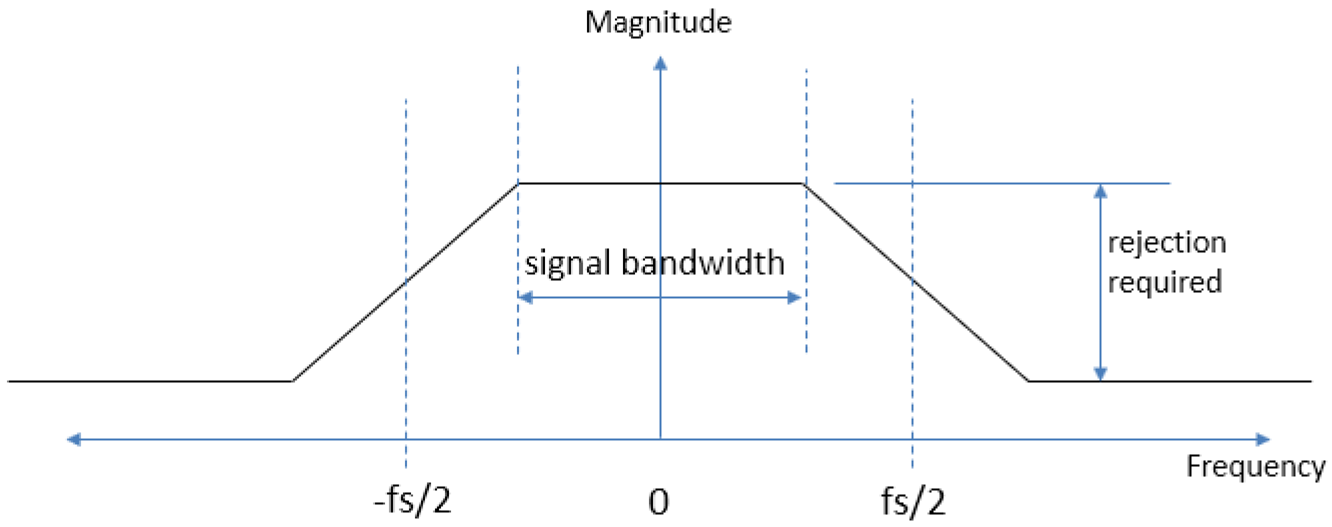
- Anti-aliasing filter: a low-pass filter applied to the input signal to make sure that the signal to be sampled has a limited bandwidth
  - Applied in all practical analog-to-digital converters

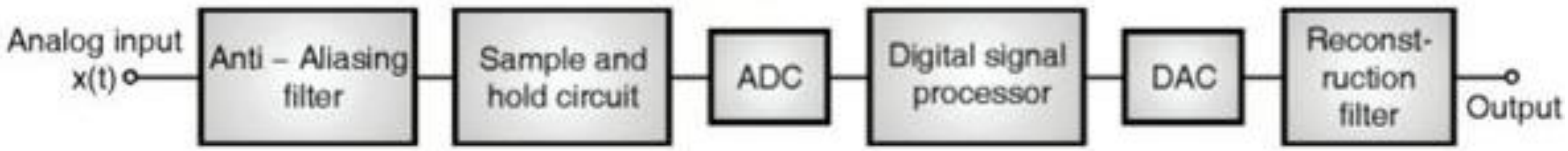




Avoid Aliasing

Anti-Alias Filter Design





Anti-Aliasing Filter

Working Anti-Aliasing Filter

1. **Filtering High-Frequency Components:** The anti-aliasing filter is used to remove high-frequency components in analog signals that exceed the Nyquist frequency, which is half the sampling rate of the ADC. If these high-frequency components are not filtered, they will cause aliasing which results in incorrect information.

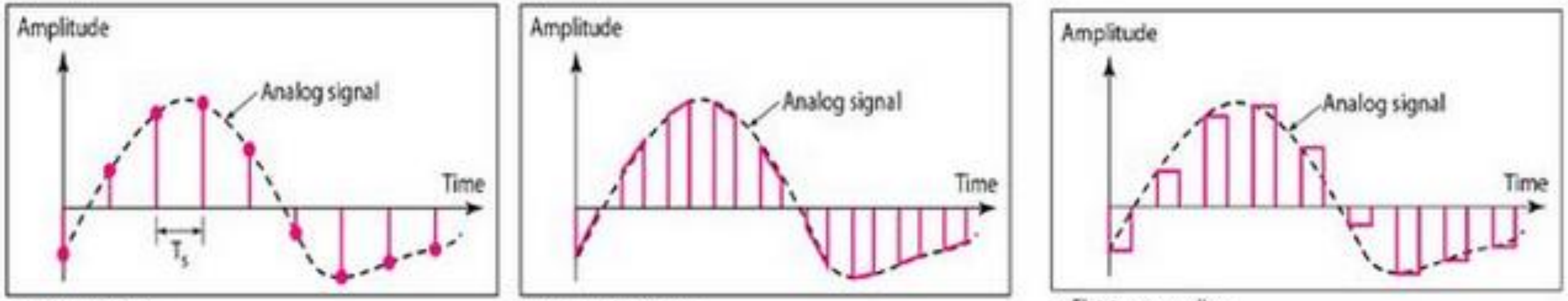
2. Preventing Aliasing: The anti-aliasing filter ensures that only the desired frequency is represented in the digital signal by attenuating the high-frequency components.

3. Improved Signal Quality: It improves signal quality and allows for more accurate data gathering. It helps to retain the original signal's integrity and decreases the possibility of errors in later digital processing.

Methods of Sampling

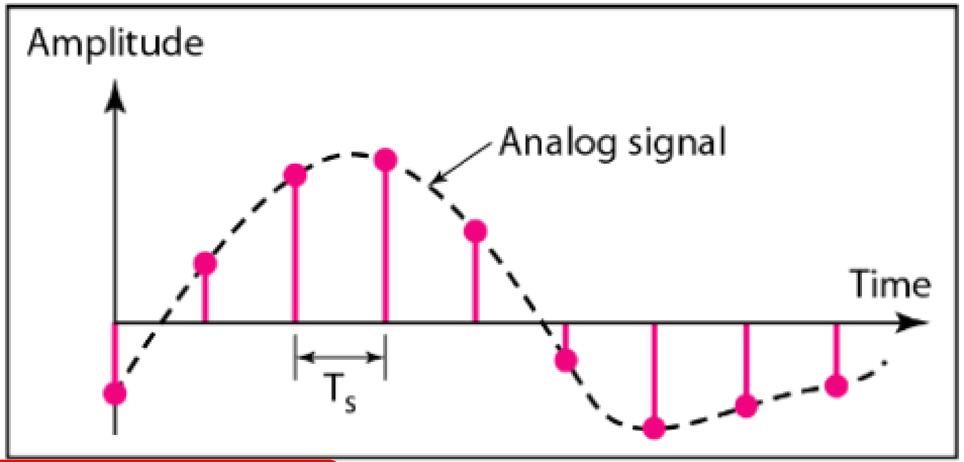
- <u>There are 3 sampling methods:</u>

- Ideal - an impulse at each sampling instant.
- Natural - a pulse of short width with varying amplitude.
- Flattop - sample and hold, like natural but with single amplitude value.

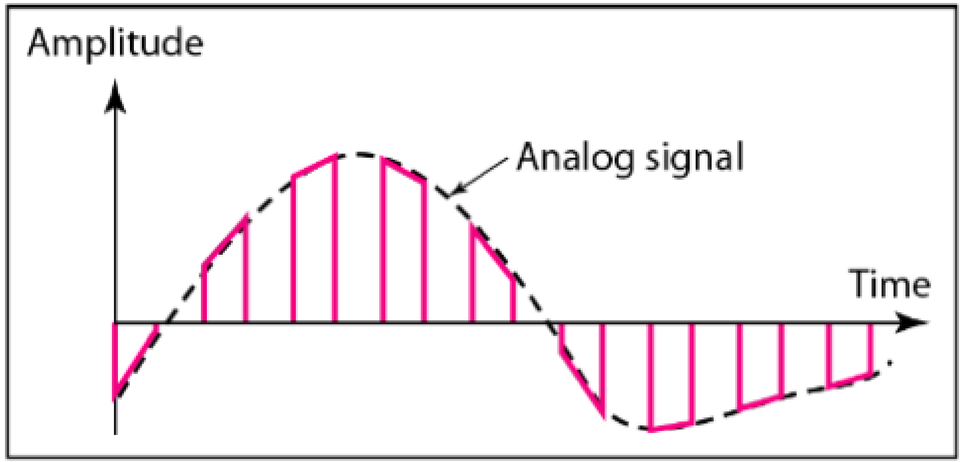


a. Ideal sampling   b. Natural sampling   c. Flat-top sampling

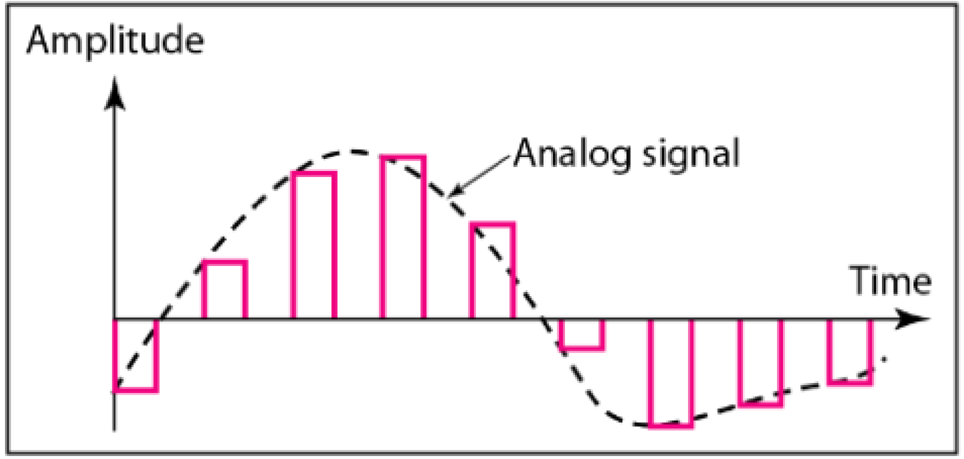
Methods of Sampling



a. Ideal sampling



b. Natural sampling

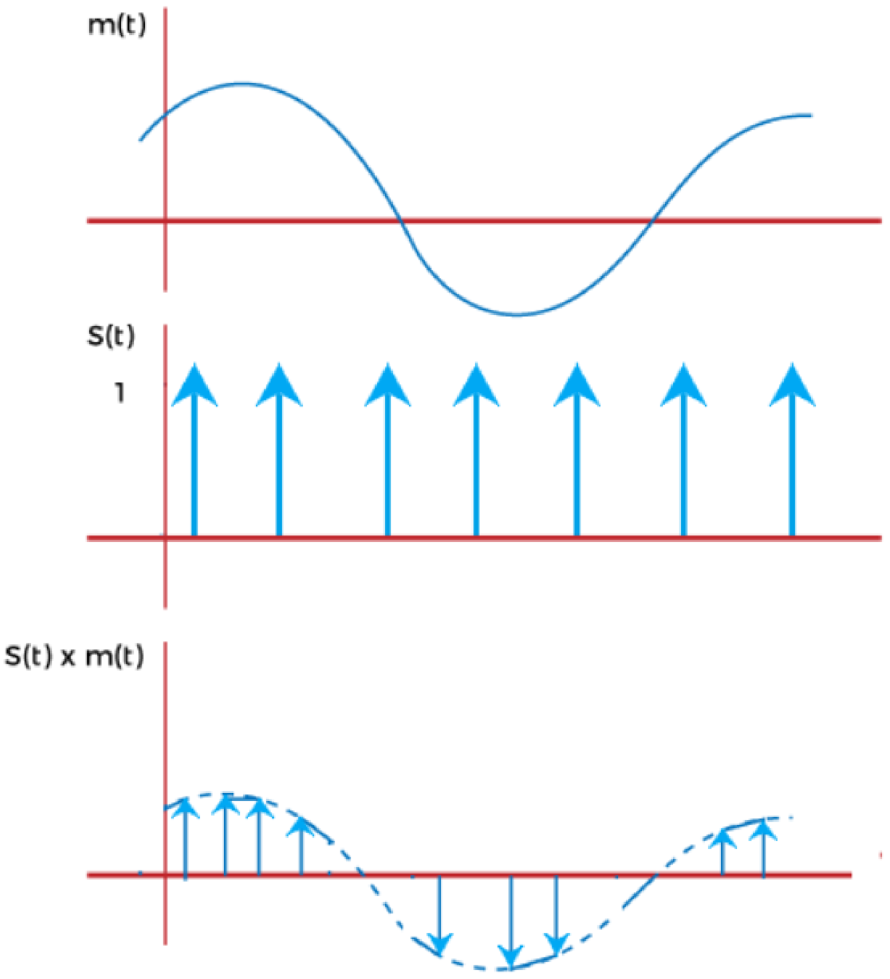


c. Flat-top sampling

Ideal Sampling

Ideal sampling is also known as instantaneous sampling or impulse sampling.

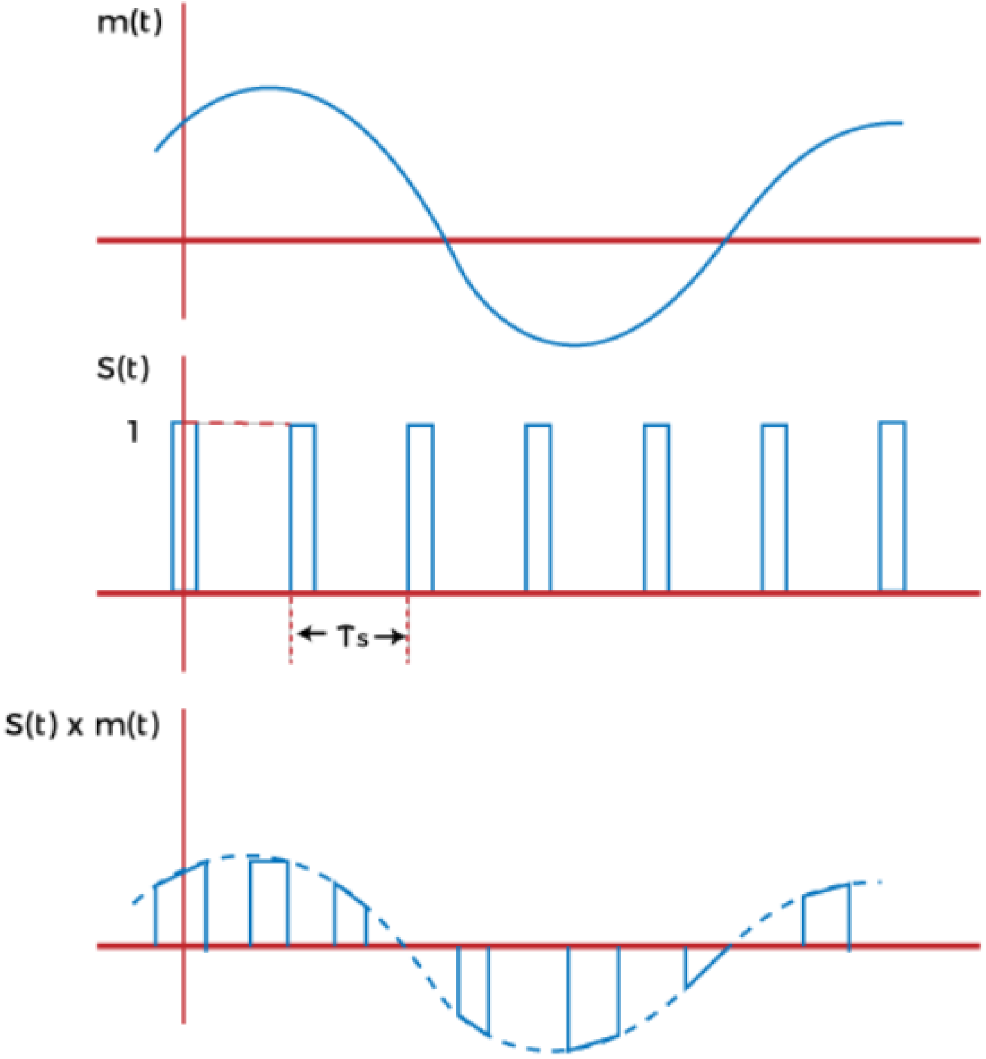
The sampling process multiplies the input signal and the carrier signal, which is present in the form of a train of pulses.



Natural Sampling

Natural Sampling is considered an efficient multiplexing method in Pulse Amplitude Modulation.

Very Hard in computation.

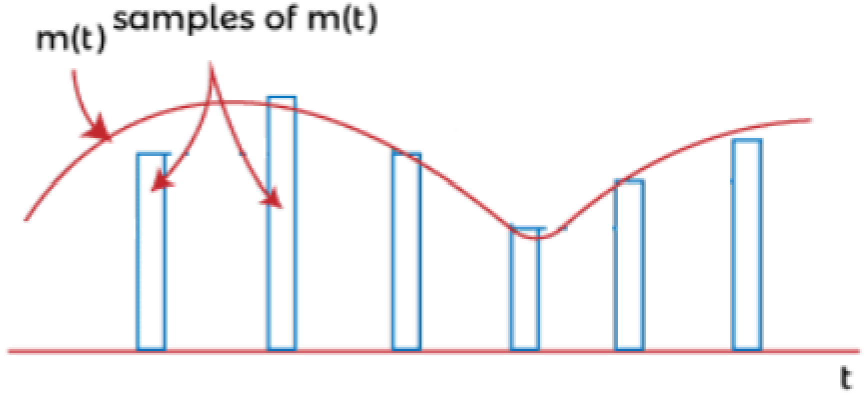


Flat-top Sampling

The design and reconstruction of flat-top sampling is easier than the natural sampling process.

The pulses in the flat-top sampling method are in a flat shape at the top and are held at a constant height (Amplitude).

It has the highest error rate.

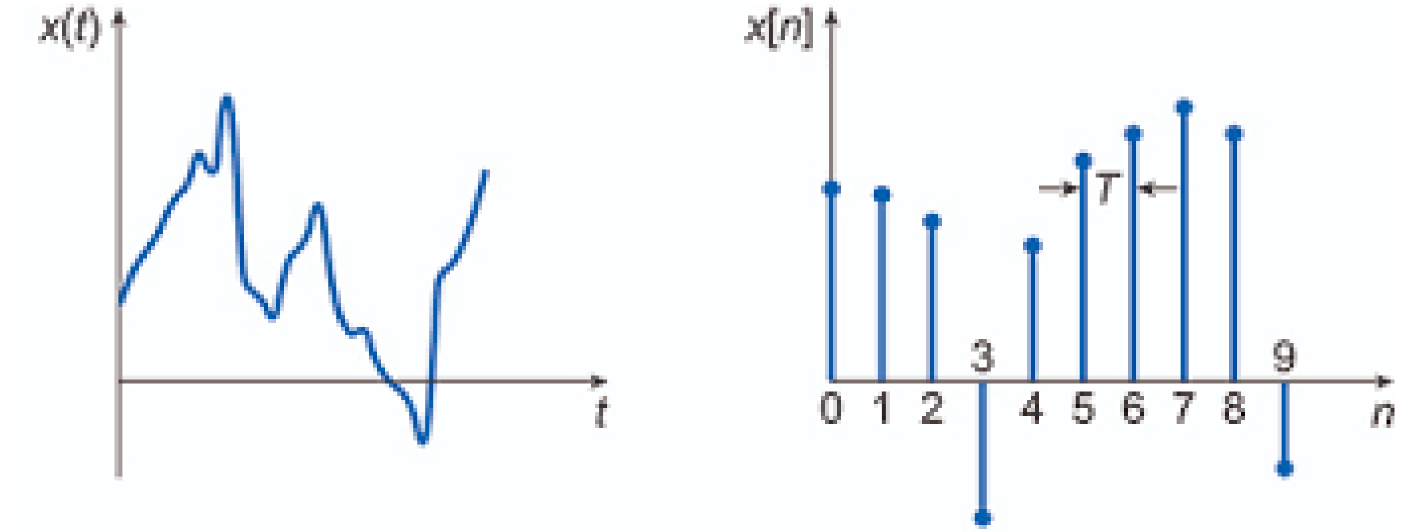


Ideal (uniform) Sampling

What is the relation between F (analog frequency) & f (discrete frequency) ?

$$F_s = \frac{1}{T_s} \qquad T_s = \frac{1}{F_s}$$

$$t = nT_s = \frac{n}{F_s}$$



(a) (b)

Sampling rate or sampling frequency defines the number of samples per second (or per other unit) taken from a continuous signal to make a discrete or digital signal.

Relative Frequency (Normalized Frequency)

Sinusoidal Continuous Signal

$$x_a(t) = A\cos(\Omega t + \theta) = A\cos(2\pi F t + \theta), \quad t \in \mathbb{R}$$

- analog signal, $\because -A \leq x_a(t) \leq A$ and $-\infty < t < \infty$
- A = amplitude
- Ω = frequency in rad/s
- F = frequency in Hz (or cycles/s); note: $\Omega = 2\pi F$
- θ = phase in rad

Relative Frequency (Normalized Frequency)

Sinusoidal Discrete Signal

$$x(n) = A\cos(\omega n + \theta) = A\cos(2\pi f n + \theta), \quad n \in \mathbb{Z}$$

- discrete-time signal (not digital), $\because -A \leq x_a(t) \leq A$ and $n \in \mathbb{Z}$
- $A =$ amplitude
- $\omega =$ frequency in rad/sample
- $f =$ frequency in cycles/sample; note: $\omega = 2\pi f$
- $\theta =$ phase in rad

Relative Frequency (Normalized Frequency)

The sampling rate is defined as the number of samples taken per second from a continuous signal for a finite set of values.

$$X_a(t) = A\cos(2\pi F t + \Theta)$$ **Analog Sinusoidal**

$$X_a(nT_s) = A\cos(2\pi F n T_s + \Theta) = A\cos(\frac{2\pi F n + \Theta}{F_s}) = A\cos(2\pi n f + \Theta)$$

$$\mathrm{f} = \frac{F}{F_s}$$

f digital frequency, F is analog frequency and F_s is frequency sampling

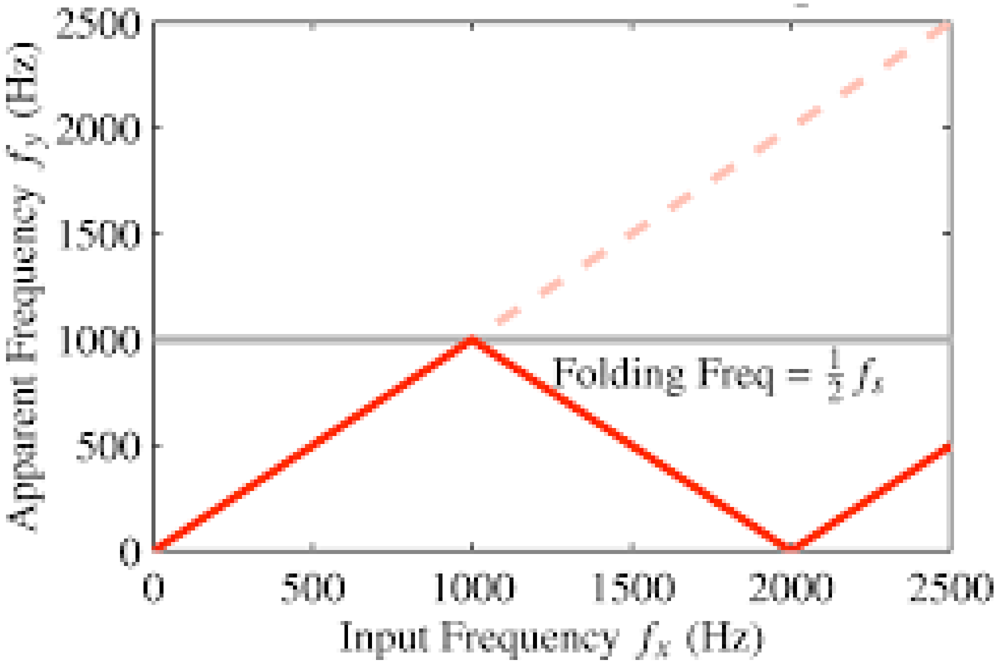
Relative Frequency (Normalized Frequency) Range

$$-\frac{1}{2} < f < \frac{1}{2}$$

$$-\frac{1}{2} < \frac{F}{F_S} < \frac{1}{2}$$

$$-\frac{1}{2} F_S < F < \frac{1}{2} F_S$$

Folding Frequency => $F_{fold} = \frac{F_S}{2}$



$$\Rightarrow \quad F < \frac{F_S}{2} = \frac{1}{2T_S} \quad \Rightarrow \quad F_S \geq 2F$$

Example

Find Digital analog for X1(n) and X2(n) for the following while both were sampled at $F_S = 40$:

$$x1 = \cos 2\pi(10t)$$
$$x2 = \cos 2\pi(50t)$$

$$\boxed{f = \frac{F}{F_S}}$$

$$x_1(n) = \cos 2\pi\left(\frac{10}{40}\right)n = \cos \boxed{\frac{\pi}{2}}n$$
$$x_2(n) = \cos 2\pi\left(\frac{50}{40}\right)n = \cos \boxed{\frac{5\pi}{2}}n$$

$$-\frac{1}{2} < f < \frac{1}{2}$$

Exponential Signal

The Exponential signal is defined as one of the elementary signals.

It has two types:

1. Real Exponential Signals.
2. Complex Exponential Signals

The standard signals are as follows:

1. Step signal
2. ramp signal
3. parabolic signal
4. impulse signal
5. Sinusoidal signal
6. $\boxed{\text{Exponential signal}}$

Real Exponential Signal

- A real exponential signal is defined as

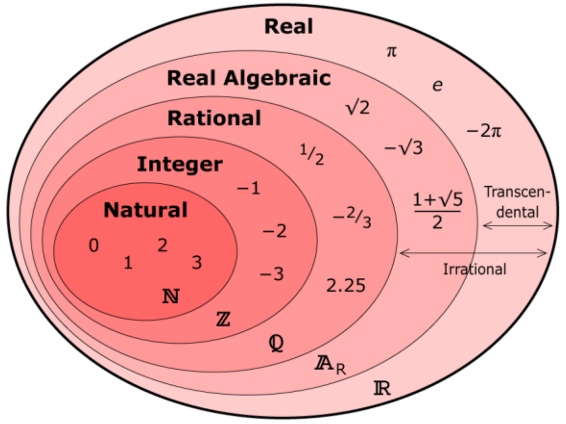
In continuous time, real exponentials are typically expressed in:

$$x(t) = Ae^{\sigma t}$$

whereas in discrete time they are typically expressed in the form:

$$x(t) = Ae^{n\sigma T}$$

Where A is the Amplitude; A and σ is a real number.



- Depending on the value of "σ" the signals will be different.

- If *"σ"* is positive the signal *x(t)* is a growing exponential and if *"σ"* is negative then the signal *x(t)* is a decaying exponential.

- For σ=0, signal *x(t)* will be constant.

Continuous Real Exponential Signal

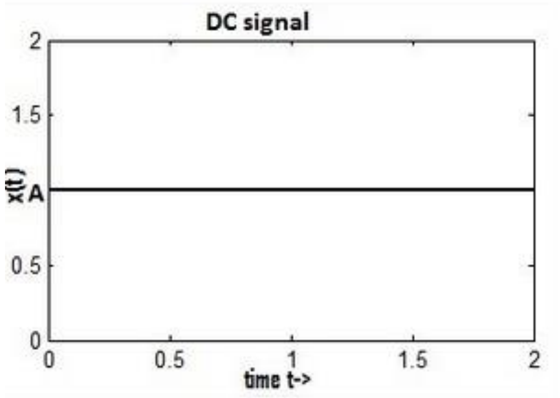


Fig.10(a) A dc signal

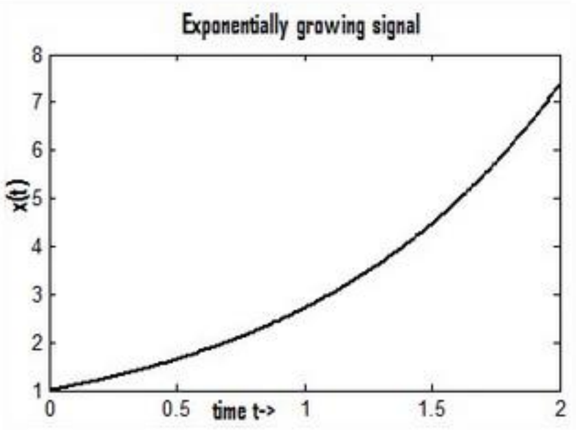


Fig.10(b) Exponentially growing signal

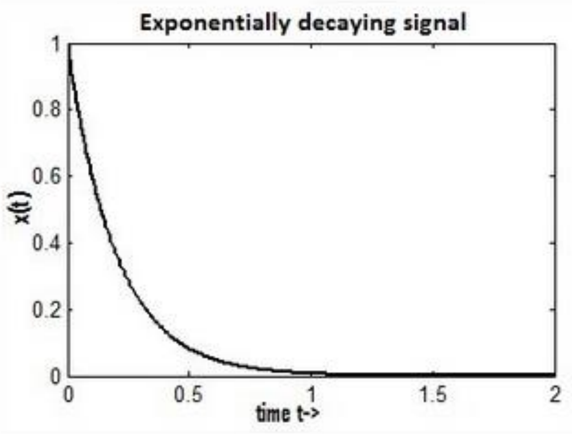
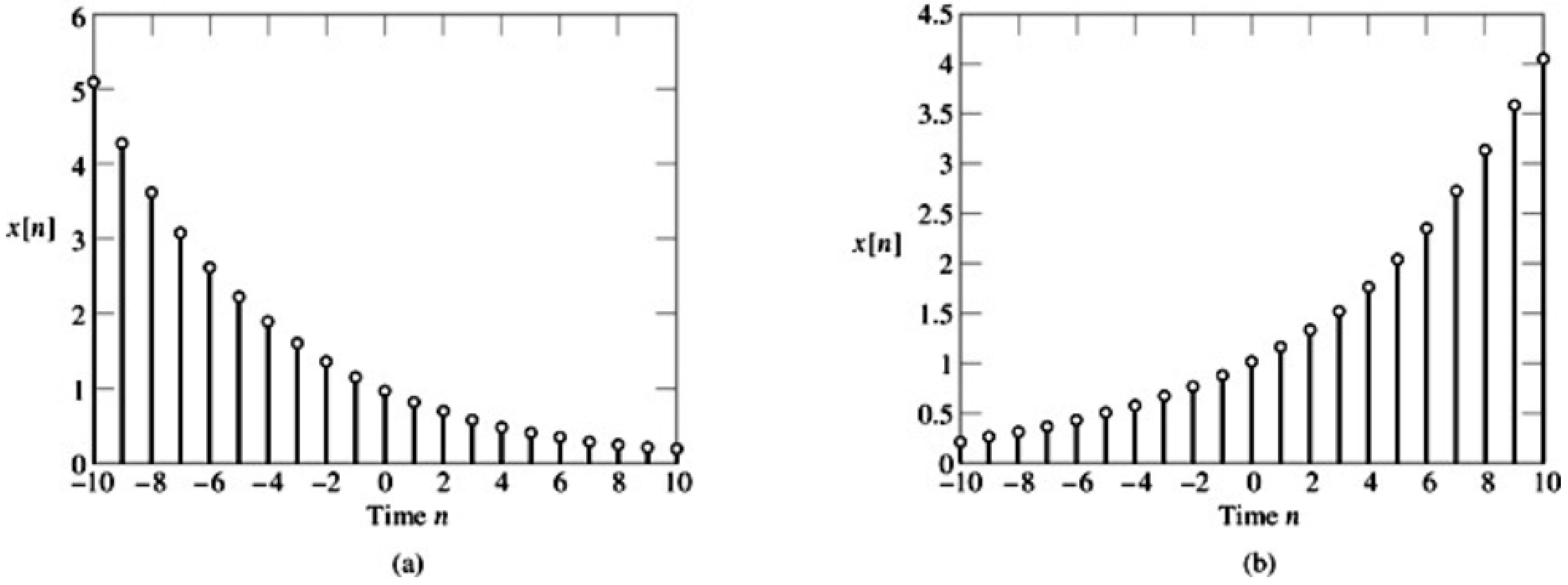


Fig.10(c) Exponentially decaying signal

Discrete Real Exponential Signal



(a) Decaying exponential form of discrete-time signal. (b) Growing exponential form of discrete-time signal.

Complex Exponential Signal

It depends on exponential function.

Exponential Function (e) has a special **Derivative.**

$$\frac{d}{dx} e^x = e^x$$

Euler Formula

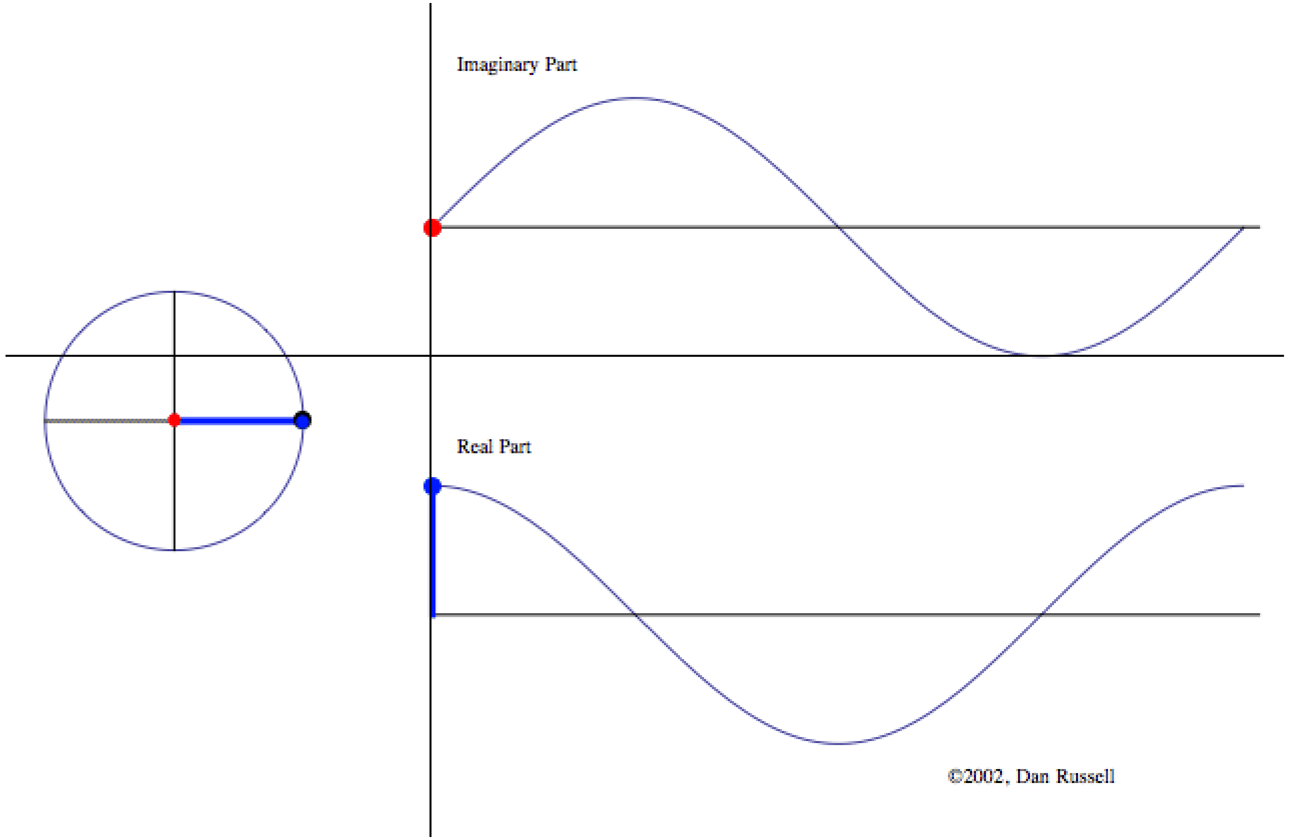
$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \quad \text{Euler's relation}$$

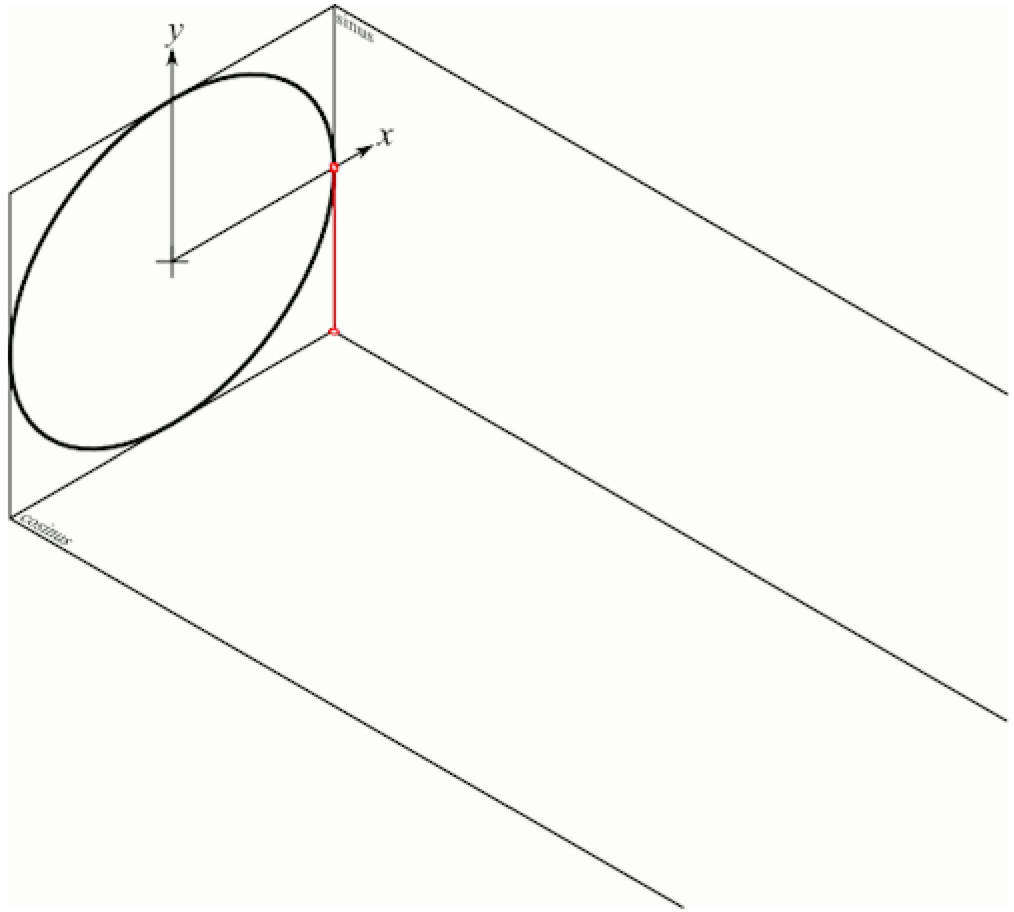
$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

$$\text{where } j \triangleq \sqrt{-1}$$

Complex Exponential Signal





Complex Exponential Signal

Continuous-time: $A\ e^{j(\Omega t+\theta)} = A\ e^{j(2\pi F t+\theta)}$

Discrete-time: $A\ e^{j(\omega n+\theta)} = A\ e^{j(2\pi f n+\theta)}$

Periodicity: Continuous-time Complex Exponential Signals

$$
\begin{aligned}
x(t) &= x(t+T),\ T \in \mathbb{R}^+ \\
A\ e^{j(2\pi Ft+\theta)} &= A\ e^{j(2\pi F(t+T)+\theta)} \\
e^{j2\pi Ft} \cdot e^{j\theta} &= e^{j2\pi Ft} \cdot e^{j2\pi FT} \cdot e^{j\theta} \\
1 &= e^{j2\pi FT} \\
e^{j2\pi k} = 1 &= e^{j2\pi FT},\quad k \in \mathbb{Z} \\
T &= \frac{k}{F}\ k \in \mathbb{Z}
\end{aligned}
$$

Periodicity: Discrete-time Complex Exponential Signals

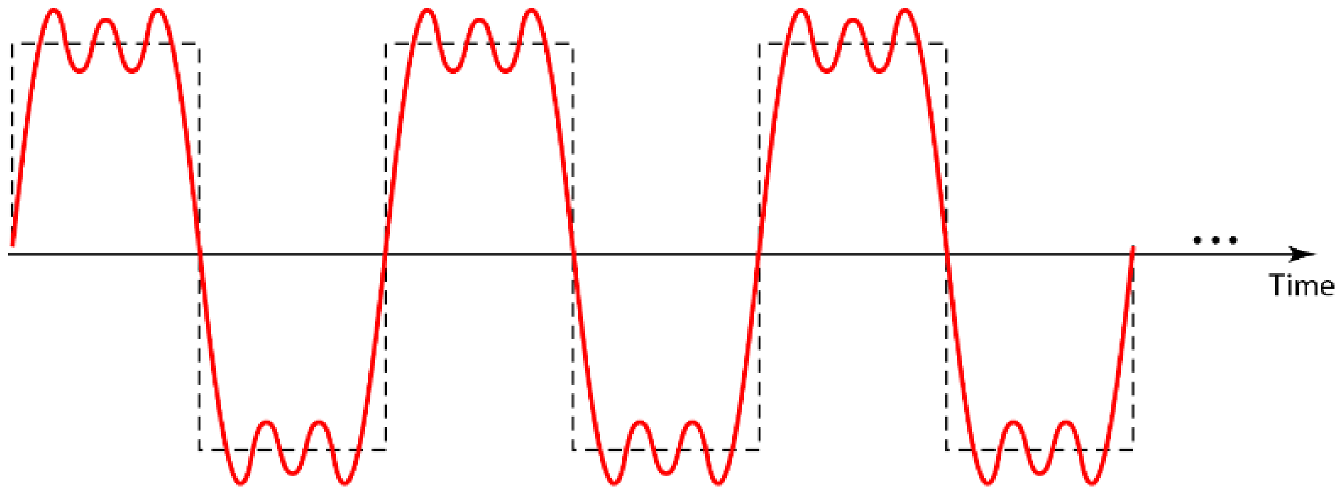
$$
\begin{aligned}
x(n) &= x(n+N),\ N \in \mathbb{Z}^+ \\
A\ e^{j(2\pi fn+\theta)} &= A\ e^{j(2\pi f(n+N)+\theta)} \\
e^{j2\pi fn} \cdot e^{j\theta} &= e^{j2\pi fn} \cdot e^{j2\pi fN} \cdot e^{j\theta} \\
1 &= e^{j2\pi fN} \\
e^{j2\pi k} = 1 &= e^{j2\pi fN},\quad k \in \mathbb{Z} \\
f &= \frac{k}{N}\ k \in \mathbb{Z}
\end{aligned}
$$

Importance of Fourier Transfer

- Simple sin (sinusoidal) waves have many applications in daily life.
- We can send a single sine wave to carry electric energy from one place to another. For example, the power company sends a single sine wave with a frequency of 60 Hz to distribute electric energy to houses and businesses.
- As another example, we can use a single sine wave to send an alarm to a security centre when a burglar opens a door or window in the house.

Importance of Fourier Transfer

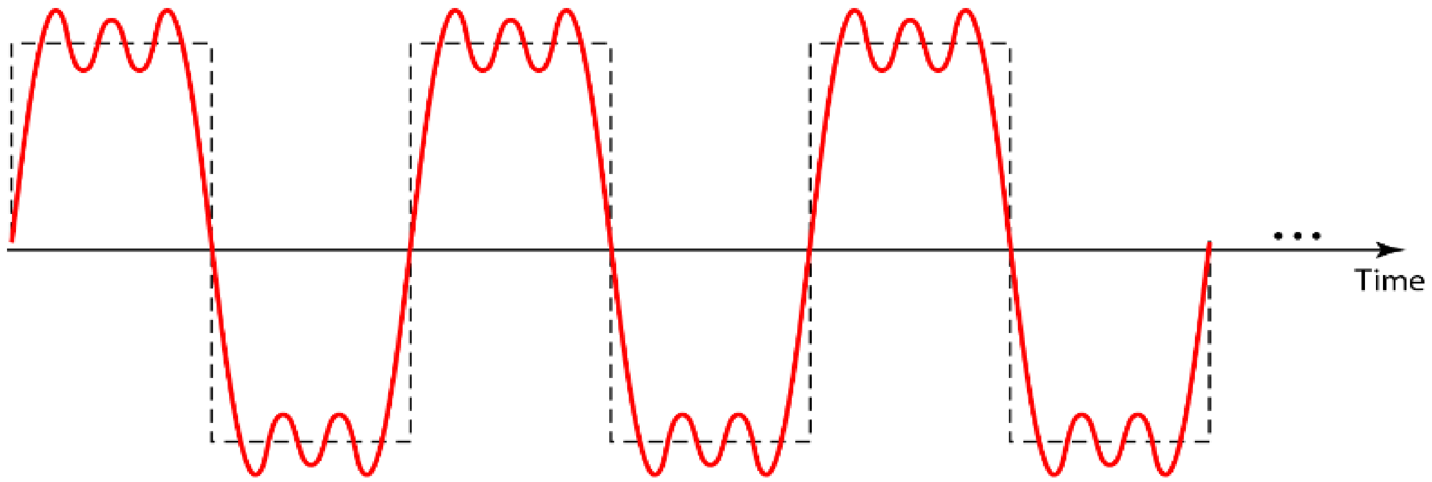
- If we had only one single sine wave to convey a conversation over the phone, it would make no sense and carry no information. We would just hear a buzz.
- So we transfer information through composite signals



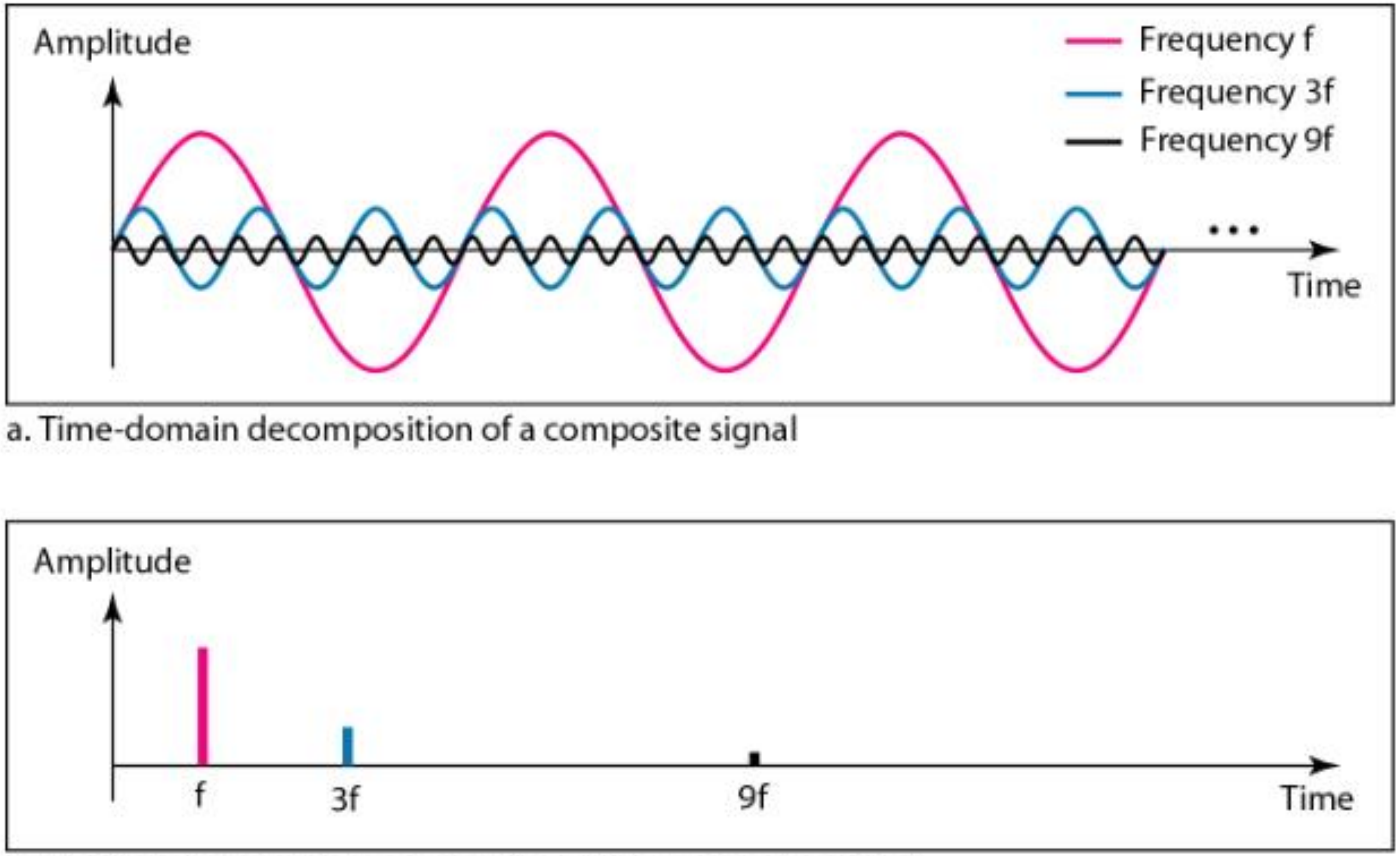
Importance of Fourier Transfer

According to Fourier analysis, any composite signal is a combination of simple sine waves with different frequencies, amplitudes, and phases.

A composite signal can be periodic or non periodic. A periodic composite signal can be decomposed into a series of simple sine waves with discrete frequencies — frequencies that have integer values (1, 2, 3, and so on). A non periodic composite signal can be decomposed into a combination of an infinite number of simple sine waves with continuous frequencies, frequencies that have real values.



Decomposition of a composite periodic signal in the time and frequency domains



b. Frequency-domain decomposition of the composite signal

Any Question?
Thank you