Digital Signal Processing

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Lecture 4

Agenda

- Even and Odd Signals
- Decomposition of Even and Odd Signals
- Fourier Series
- Fourier Transform

Periodic Signal

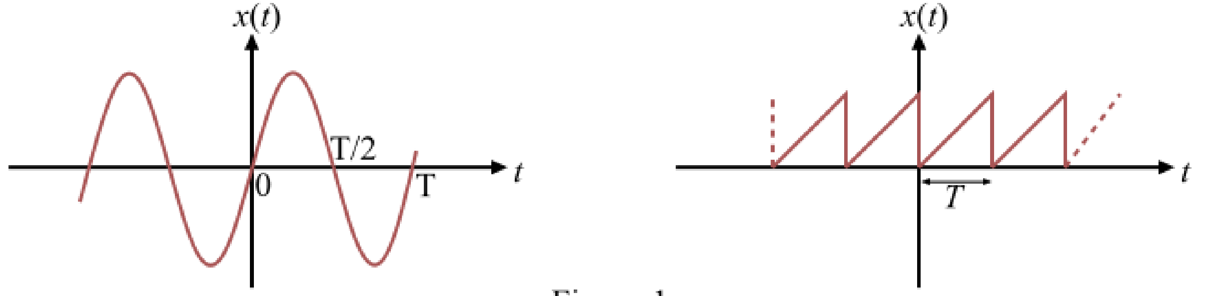


Figure-1

A continuous-time signal x(t) is said to be periodic if and only if

$$x(t+T) = x(t) \qquad \text{for} -\infty < t < \infty$$

Where T represents the time period of the periodic signal. Also, it is known as the **fundamental time period** of the signal and is denoted by (T_o).

$$\omega = 2\pi f$$

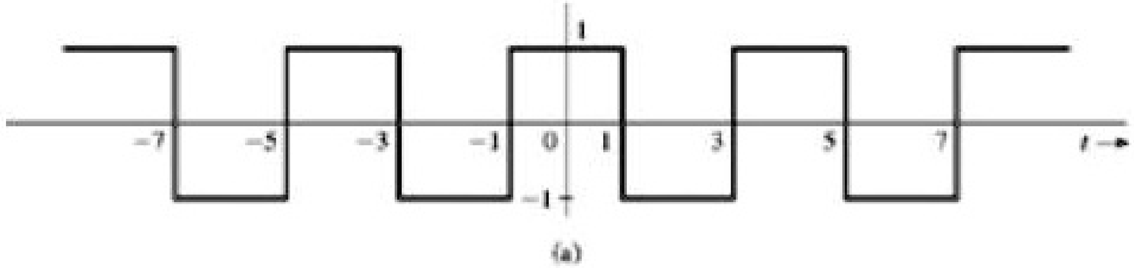
$$T = \frac{2\pi}{f}$$

$$x(t+mT) = x(t)$$

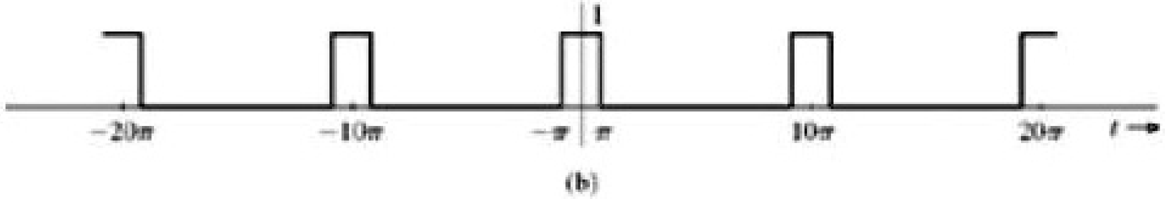
Where, m is an integer. This means if the definition is satisfied for $T = T_0$, then it is also satisfied for $T = 2T_0$, $T = 3T_0$ … and so on with T_0 as the fundamental time period.

Periodic Signal

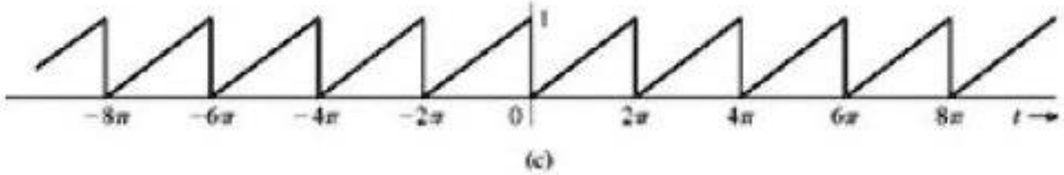
$f(t) = f(t+4)$



$f(t) = f(t+9\pi)$



$f(t) = f(t+2\pi)$



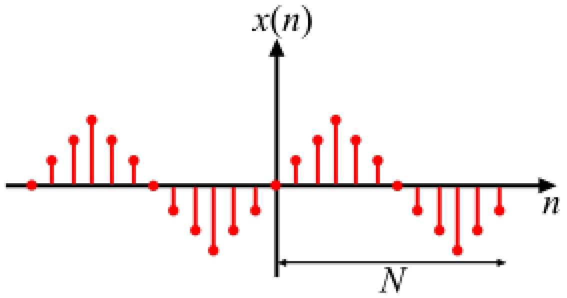
Discrete Time Periodic Signal

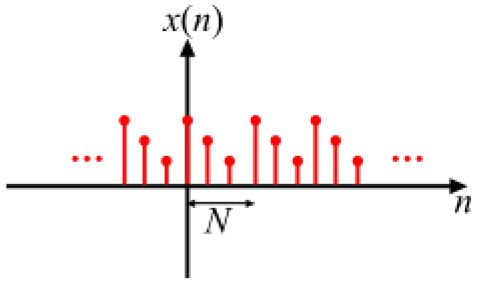
A discrete-time signal x(n) is said to be periodic if it satisfies the following condition

$$x(n) = x(n + N); \quad \text{for all integers } n$$

The fundamental time period is N

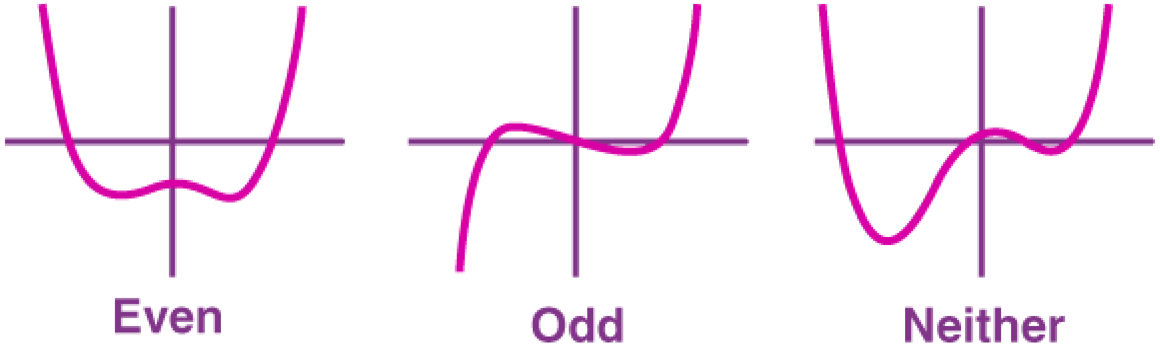
$$T = \frac{2\pi}{\omega} \qquad \omega = \frac{2\pi}{N}$$





Even and Odd Signals

- Signals can be classified as either Even, Odd, or neither.



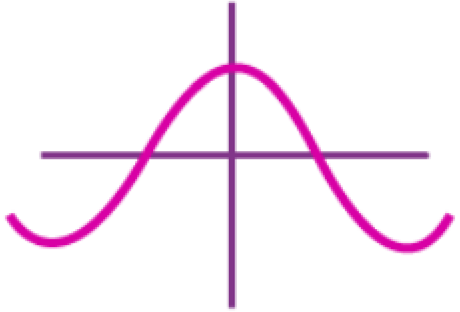
Even Signals

The Continuous-time signal is said to be even if f(−t) = f(t)
The discrete-time signal is even when if f[−n]=f[n]

- Example:

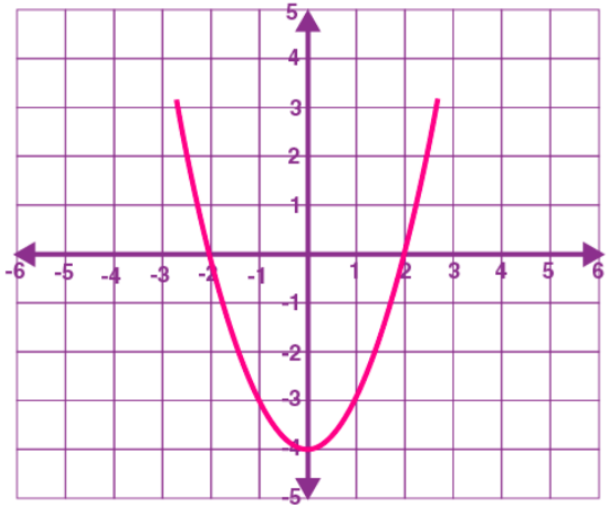
  $\cos(-x) = \cos(x)$

  $f(x) = x^4 - 4$
  $f(-x) = (-x)^4 - 4 = x^4 - 4$



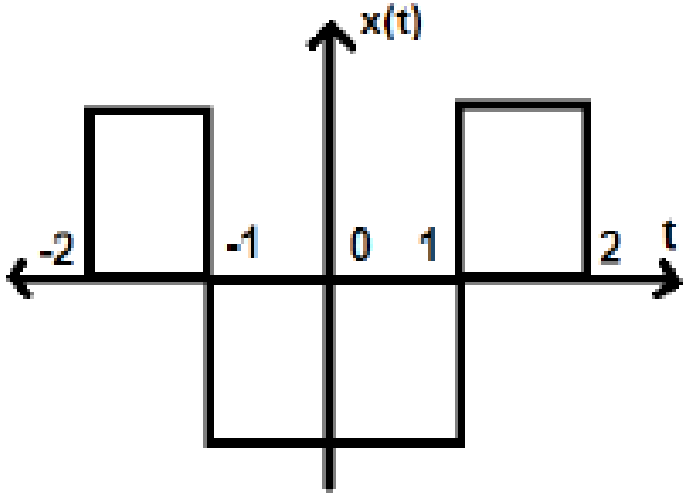
cos x (even)

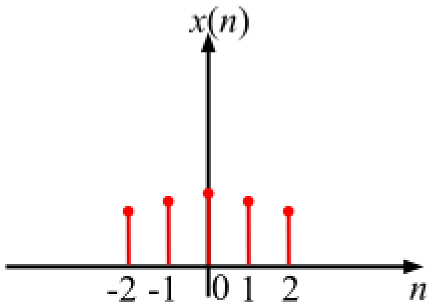
Non-periodic



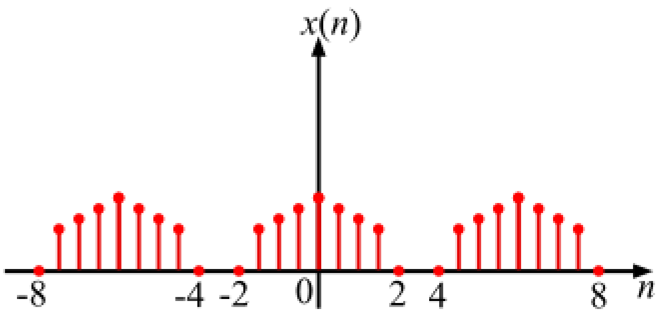
Even Signals

- A Signal is said to be even if f(−t) = f(t)





Periodic



Odd Signals

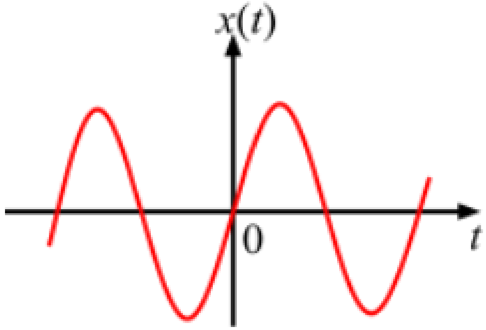
The Continuous-time signal is said to be odd if f(−t) =- f(t)

The discrete-time signal is odd when if f[n]=-f[n]

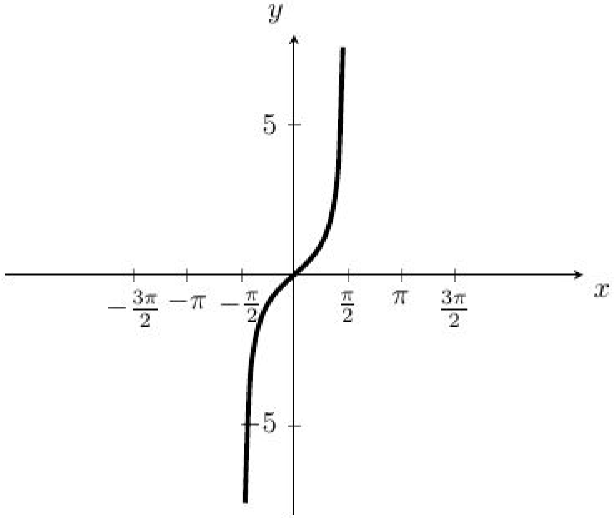
Non-
periodic

Example:

sin(-x) = -sin(x)

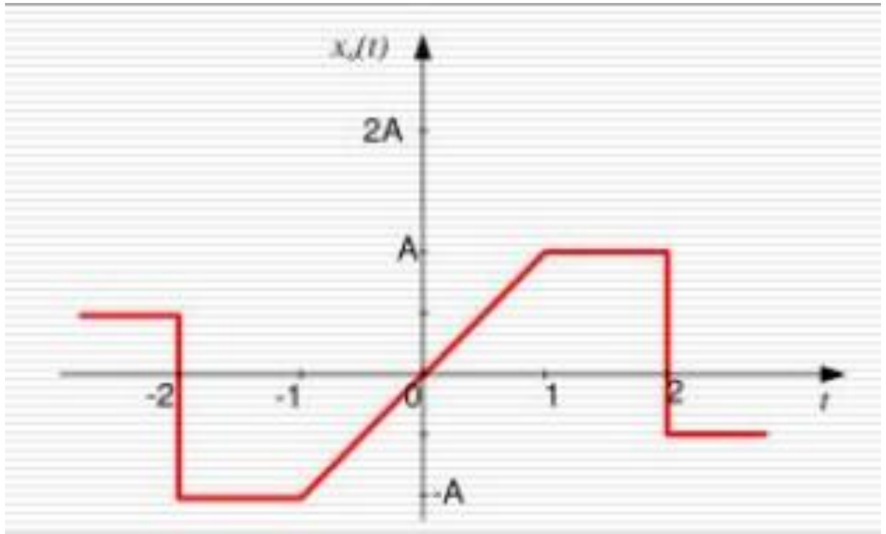


tan(-x)=-tan(x)

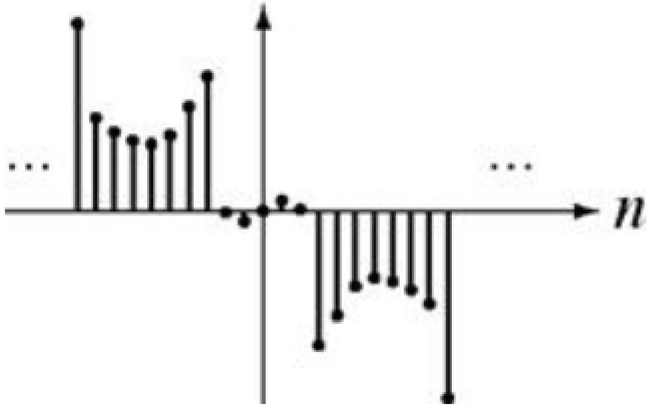


Odd Signals

- A Signal is said to be odd if $f(-t) = -f(t)$

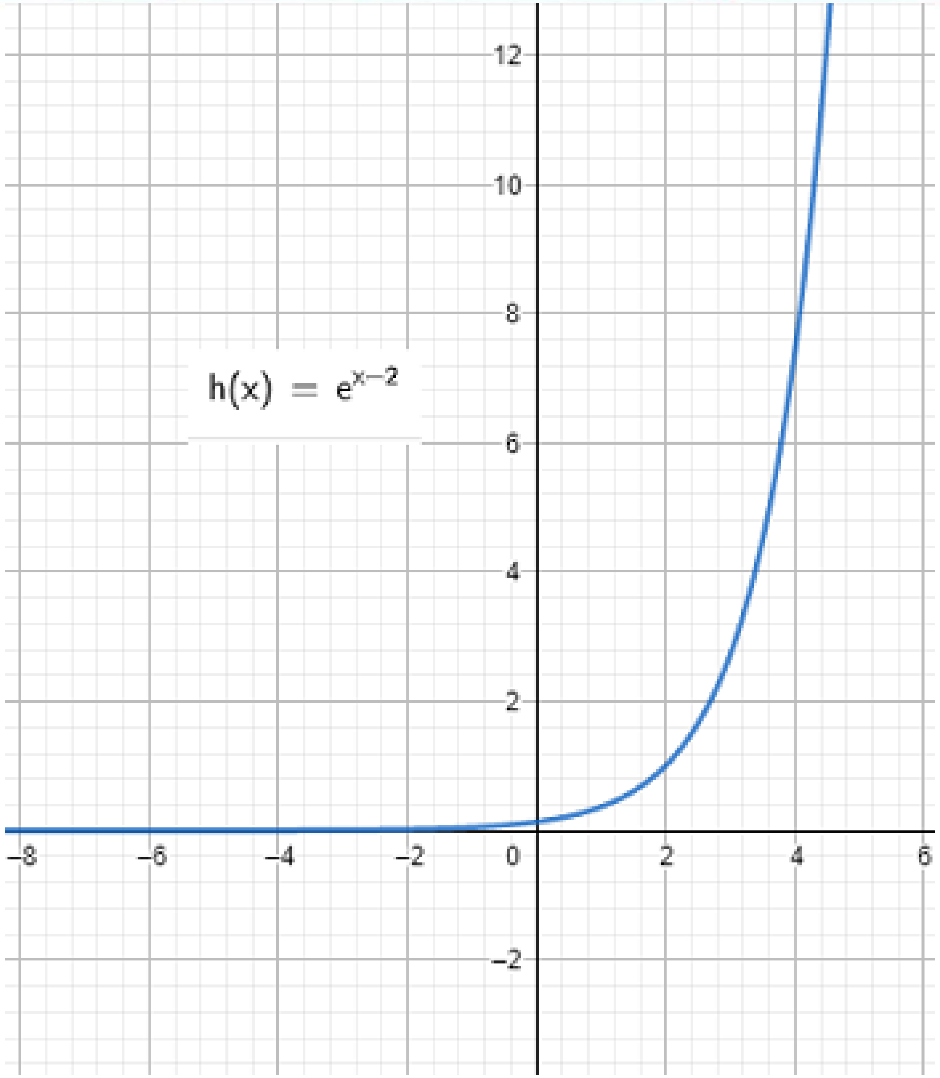


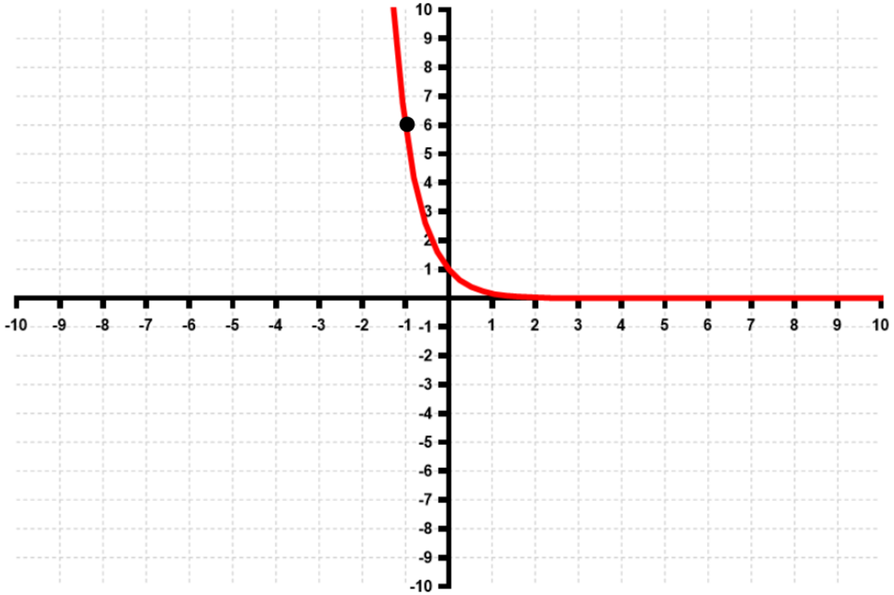
Periodic



Neither Even or Odd Signals

f(x)=e^{x}
f(-x) = e^{-x}





Decomposition of Even and Odd Signals

Any Signal can be presented as the sum of an even and an odd function

$$f(x) = f_{even}(x) + f_{odd}(x) \quad \longrightarrow \quad (1)$$

$$f(-x) = f_{even}(-x) + f_{odd}(-x)$$

$$f(-x) = f_{even}(x) - f_{odd}(x) \quad \longrightarrow \quad (2)$$

$$f(x) + f(-x) = 2\, f_{even}(x) \qquad f(x) - f(-x) = 2\, f_{odd}(x)$$

$$f_{even}(x) = \frac{f(x) + f(-x)}{2} \qquad f_{odd}(x) = \frac{f(x) - f(-x)}{2}$$

Fourier Representation

- *Joseph Fourier* developed a technique for analyzing non-sinusoidal waveforms applicable.

- A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of sines and cosines.

- Many times all the information available in the time domain is not sufficient for the analysis, for this reason, we have to transform the signal into the frequency domain to extract more information about the signal.

- In the Fourier series the periodic signal is decomposed into related sinusoidal functions.

Fourier Representations for Signals

Time Property	*Periodic*	*Nonperiodic*
Continuous (t)	Fourier Series (FS)	Fourier Transform (FT)
Discrete [n]	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)

Fourier Series Representation

- There are two types of Fourier series representations, both are equivalent to each other.

- We will focus on the Trigonometric Form of the Fourier Series

Trigonometric Form of Fourier Series

$$f(t)= \frac{a_0}{2} + a_1 \cos \omega_\circ t + a_2 \cos 2\,\omega_\circ t + \dots + b_1 \sin \omega_\circ t + b_2 \sin 2\,\omega_\circ t + \dots$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\,\omega_\circ t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_\circ t$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n 2\pi f t + \sum_{n=1}^{\infty} b_n \sin n\, 2\pi f t$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \frac{2\pi}{T} t + \sum_{n=1}^{\infty} b_n \sin n\, \frac{2\pi}{T} t$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$$

$$\boxed{L = \frac{T}{2}}$$

Trigonometric Form of Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

f(t) is a given periodic function either even, odd, or neither.

n represents the counter of sines or cosines.

T is the periodic interval.

$$\boxed{L = \frac{T}{2}}$$

The set of constants a_0, a_n, b_n, n=1,2,... are called the **Fourier coefficients** will be **evaluated**.

Trigonometric Form of Fourier Series

One period

$$a_\circ = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$L = \frac{T}{2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt$$

Trigonometric Form of Fourier Series

$$\int_{-a}^{a} odd\ function = 0$$

$$\int_{-a}^{a} even\ function = 2\int_{0}^{a} even\ function$$

- The product of two even functions is even, and the product of two odd functions is even.

- The product of an even function and an odd function is an odd function.

Trigonometric Form of Fourier Series

$$a_\circ = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

Cosine is even fn

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi \mathrm{t}}{L}dt$$

Sine is odd fn

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi \mathrm{t}}{L}dt$$

Trigonometric Form of Fourier Series

f(t) is even	f(t) is odd	f(t) is neither
$b_n = 0$, $a_\circ, a_n$ *to be evaluated*	$a_\circ = 0$, $a_n = 0$, b_n *to be evaluated*	All to be evaluated

Trigonometric Form of Fourier Series Example

Example:

$$f(t) = t \qquad -\pi < t < \pi$$
$$f(t + 2\pi) = f(t)$$

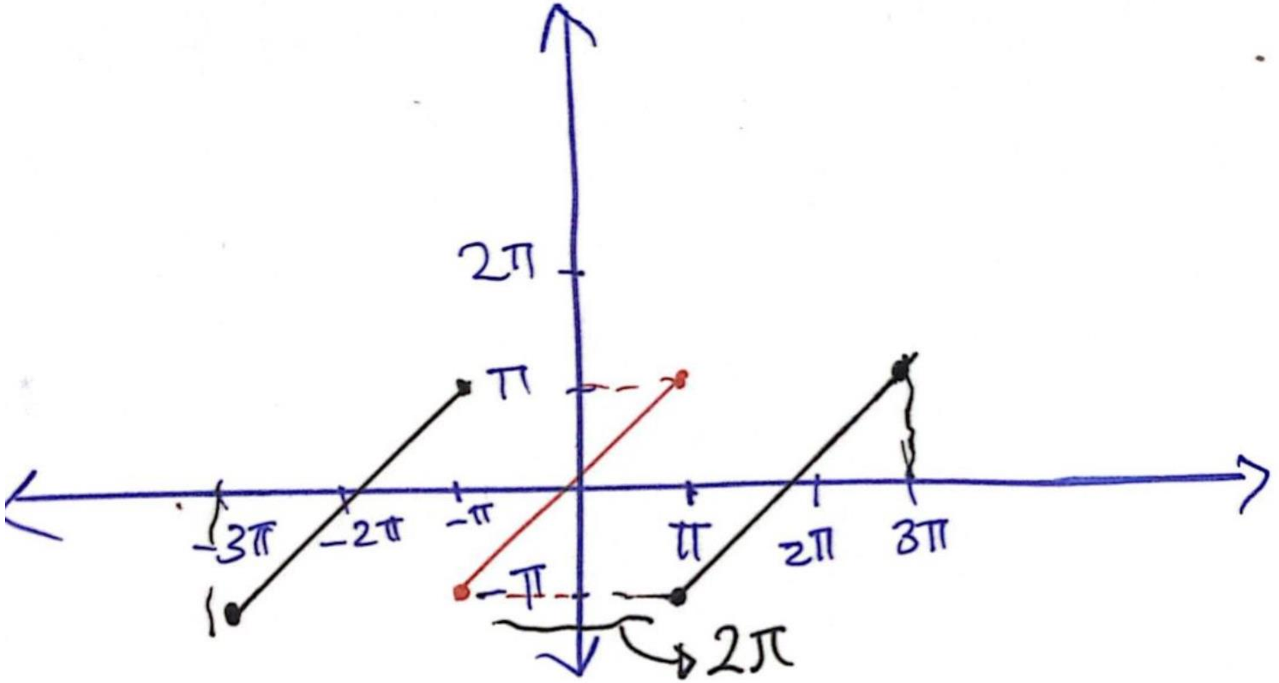
Find Fourier series expansion

Solution

$$T = 2\pi \quad \to \quad L = \pi$$

Odd function then we need to calculate b_n only.

$a_\circ = a_n = 0$



Trigonometric Form of Fourier Series Example

Example:

$$f(t) = t \qquad -\pi < t < \pi$$
$$f(t+2\pi) = f(t)$$

Find Fourier series expansion

Solution

$$b_n = \frac{1}{\pi}\int_{-L}^{L} t.\sin\frac{n\pi \mathrm{t}}{\pi}\,dt$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} t.\sin n\mathrm{t}\,dt$$

$$b_n = \frac{-2}{\pi n}\, tCos(nt)\,|_0^{\pi}$$

$$b_n = \frac{-2}{2\pi n}\, t^2 Cos(nt)\,|_0^{\pi}$$

$$b_n = \frac{-1}{\pi n}\,(\pi^2 Cos(n\pi) - \pi^2 \mathrm{Cos}(0))$$

Trigonometric Form of Fourier Series Example

$b_n = \frac{-1}{\pi n}\ (\pi^2 Cos(n\pi) - \pi^2 \text{Cos}(0))$

$b_n = \frac{-\pi}{n}\ (Cos(n\pi) - \text{Cos}(0))$

$b_n = \frac{-\pi}{n}\ (Cos(n\pi) - 1)$ for n = 1,2,3,4,....

Cos(n π) = -1, 1, -1,1

$b_n = \frac{-\pi}{n}((-1)^n - 1)$

f(t) = $\sum_{n=1}^{\infty} \frac{-\pi}{n}((-1)^n - 1)\ \sin(nt)$ = 2 π, 0, $\frac{2}{3}\pi$,0, $\frac{2}{5}\pi$,...

Trigonometric Form of Fourier Series Example

If we asked for half range of periodic which is L for cosine part only or sine part only.

Then Sine will be treated as an odd function.

Then Cosine will be treated as an even function.

And T will equal L.

Fourier Transform

- It converts from time domain to frequency domain.

- The difference between the Fourier transform and the Fourier series is that the Fourier transform is applicable for non-periodic signals, while the Fourier series is applicable to periodic signals.

Fourier Transform

- If it’s neither even or odd function

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

- If it’s an even function

$$f(\omega) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) Cos(\omega t)\, dt$$

- If it’s an odd function

$$f(\omega) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) Sin(\omega t)\, dt$$

Your Turn find the Fourier Transform

Find the Fourier transform for f(t) = e^{-at} for t>0

Any Question?
Thank you