Digital Signal Processing

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Lecture 5

Agenda

- Complex form of Fourier Series
- Fourier Transform Example
- Discrete Fourier Series
- Discrete Fourier Transform
- Inverse Discrete Fourier Transform

Fourier Representations for Signals

Time Property	*Periodic*	*Nonperiodic*
Continuous (t)	Fourier Series (FS)	Fourier Transform (FT)
Discrete [n]	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)

Trigonometric Form of Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

The set of constants a_0, a_n, b_n,n=1,2,... are called the **Fourier coefficients** will be **evaluated**.

$$L = \frac{T}{2}$$

$$a_\circ = \frac{1}{L}\int_{-L}^{L} f(t)dt \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt$$

Complex form of Fourier Series

Instead of trigonometric functions $\cos$ and $\sin$, we can use complex exponential functions.

Euler Formula

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \quad \text{Euler's relation}$$

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

where $j \triangleq \sqrt{-1}$

Notice that sine function is odd signal and cosine function is **even signal**.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\,\omega_\circ t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_\circ t$$

Complex form of Fourier Series

Instead of trigonometric functions cos and sin, we can use complex exponential functions.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\,\omega_\circ t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_\circ t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\,\omega_\circ t + b_n \sin n\,\omega_\circ t$$

$$= \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} c_n (\cos n\,\omega_\circ t + \sin n\,\omega_\circ t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_\circ t}$$

c_n **Fourier coefficient** will be **evaluated**.

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-jn\omega_\circ t} dt}$$

Fourier Transform

- If it's neither even or odd function

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

- If it's an even function (Fourier Cosine Transform)

$$f(\omega) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) Cos(\omega t)\, dt$$

- If it's an odd function (Fourier Sine Transform)

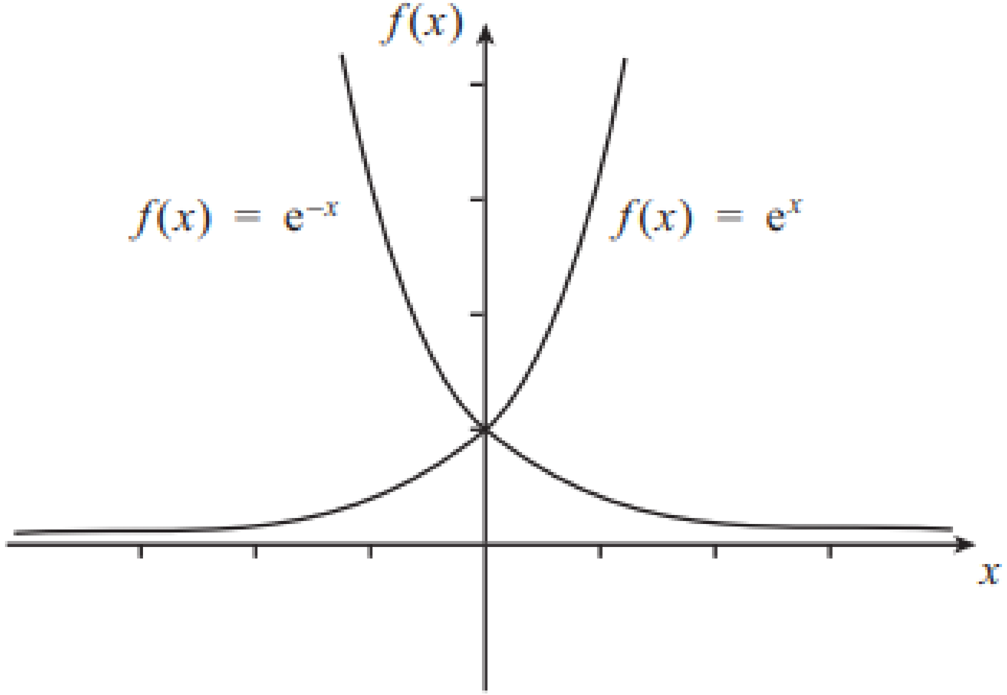
$$f(\omega) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) Sin(\omega t)\, dt$$

Absolute Value of Complex Numbers Recall

$$|a + jb| = \sqrt{a^2 + b^2}$$

$$|3 + j4| = \sqrt{3^2 + 4^2} = 5$$

Exponential Function Recall

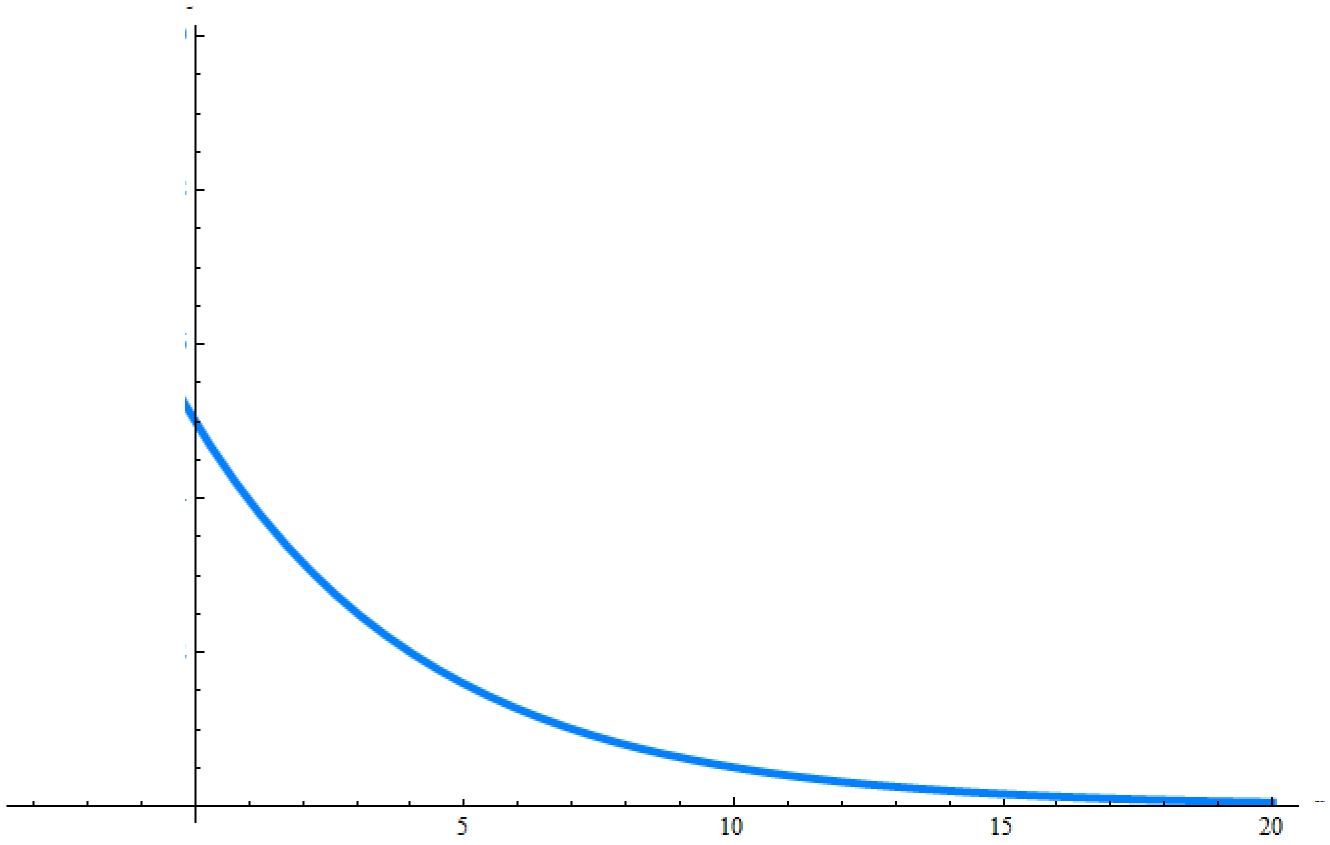


$$\int e^x dx = e^x + C$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

Find the Fourier Transform

Find the Fourier transform for f(t) = e^{-at} for t≥0



Even or Odd ?

Neither Even nor Odd

Fourier Transform (FT)

- If it’s neither even or odd function

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

- If it’s an even function

$$f(\omega) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) Cos(\omega t)\, dt$$

- If it’s an odd function

$$f(\omega) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) Sin(\omega t)\, dt$$

Find the Fourier Transform (FT)

Find the FT for f(t) = e^{-at} for t≥0

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{f(t)} e^{-j\omega t}\, dt$$

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{0}}^{\infty} \boxed{e^{-at}} e^{-j\omega t}\, dt$$

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-at+(-j\omega t)}\, dt = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \Big|_{0}^{\infty} = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-(a+j\omega)\infty}}{-(a+j\omega)} - \frac{e^{-(a+j\omega)0}}{-(a+j\omega)} \right) = \boxed{\frac{1}{\sqrt{2\pi}} \left(\frac{1}{(a+j\omega)} \right)}$$

Find the Fourier Transform (FT)

Find the Fourier transform for f(t) = e^{-at} for t≥0

$$f(\omega) = = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a + j\omega}\right)$$

$$a + j\omega \Rightarrow |\,\mathrm{a} + \mathrm{j}\omega\,| = \sqrt{a^2 + \omega^2}$$

$$f(\omega) = = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{a^2+\omega^2}}\right) = \boxed{\frac{1}{\sqrt{a^2+\omega^2}}}$$

Discrete Time Signals Recall

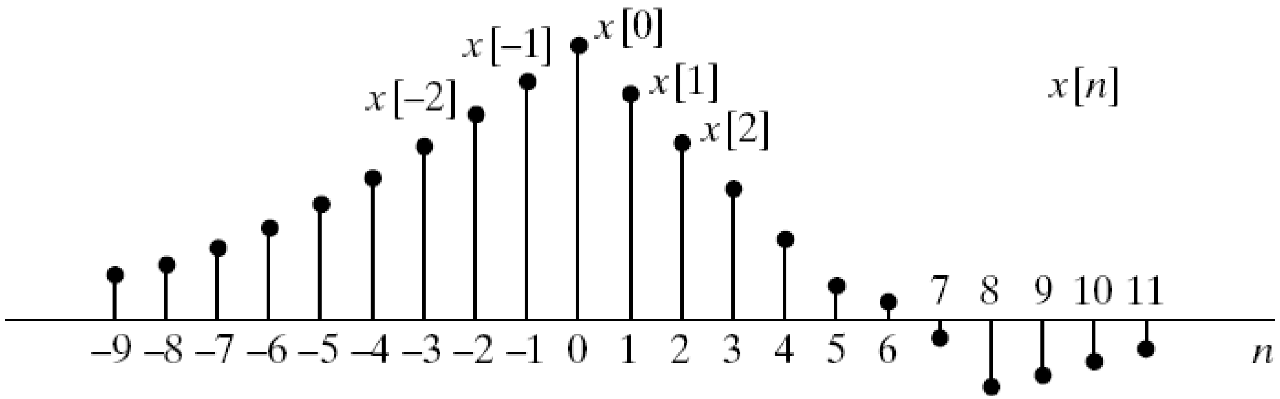
- Discrete-time signals are represented as

$$x = x[n], \quad -\infty < n < \infty, \quad n\text{: integer}$$

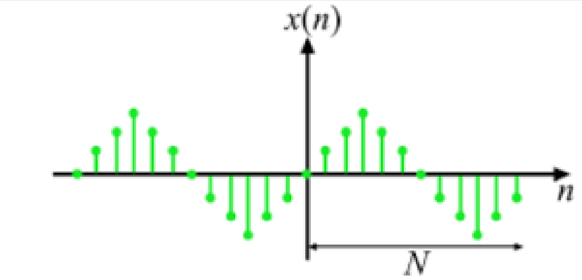
- In sampling of an analog signal $x_a(t)$:

$$x[n] = x_a(nT_s), \quad T\text{: }sampling\ period$$

Periodic Frequency Sampling: $\frac{2\pi}{N}k$



Discrete Time Fourier Series (DTFS)



Continuous Fourier Series	Discrete Fourier Series
$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_\circ t}$	$f[k] = \sum_{n=0}^{N-1} c_k e^{j\Omega_\circ n}$
$c_n = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-jn\omega_\circ t} dt$	$=\sum_{n=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$
	$c_k = \frac{1}{N}\sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}kn}$

Discrete Time Fourier Series (DTFS)

Example: find DTFS of the Periodic signal x[n] given by	Discrete Fourier Series
$X[n] = (\frac{5}{4})^n \; where \; 0 \le n \le 6$ Answer: N = 6 – (0) + 1=7 $c_k = \frac{1}{7}\sum_{n=0}^{6}(\frac{5}{4})^n e^{-j\frac{2\pi}{7}kn} = \frac{1}{7}\sum_{n=0}^{6}\left(\left(\frac{5}{4}\right)e^{-j\frac{2\pi}{7}k}\right)^n$	$f[n] = \sum_{n=0}^{N-1} c_k e^{j\Omega_o n}$ $= \sum_{n=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$ $c_k = \frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-j\frac{2\pi}{N}kn}$

Discrete Time Fourier Series (DTFS)

Summation of Geometry Series

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

$$e^{i2\pi x} = \left(e^{i2\pi}\right)^x = 1^x = 1$$

Discrete Time Fourier Series (DTFS)

Example: find DTFS of the Periodic signal x[n] given by	Discrete Fourier Series

$$X[n] = \left(\frac{5}{4}\right)^n \; where \; 0 \le n \le 6$$

Answer:

N = 6 – (0) + 1= 7

$$c_k = \frac{1}{7}\sum_{n=0}^{6}\left(\frac{5}{4}\right)^n e^{-j\frac{2\pi}{7}kn} = \frac{1}{7}\sum_{n=0}^{6}\left(\left(\frac{5}{4}\right)e^{-j\frac{2\pi}{7}k}\right)^n =$$

$$\frac{1}{7}\left(\frac{1-\left(\frac{5}{4}e^{-j\frac{2\pi}{7}k}\right)^7}{1-\left(\frac{5}{4}e^{-j\frac{2\pi}{7}k}\right)}\right) = \frac{1}{7}\left(\frac{1-\left(\frac{5}{4}\right)^7\cancel{(e^{-j2\pi k})}}{1-\left(\frac{5}{4}e^{-j\frac{2\pi}{7}k}\right)}\right) = \frac{1}{7}\left(\frac{1-\left(\frac{5}{4}\right)^7}{1-\left(\frac{5}{4}e^{-j\frac{2\pi}{7}k}\right)}\right)$$

Discrete Fourier Series:

$$f[k] = \sum_{n=0}^{N-1} c_k e^{j\Omega_o n}$$

$$= \sum_{n=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}kn}$$

Discrete Time Fourier Series (DTFS)

Example: find DTFS of the Periodic signal x[n] given by	Discrete Fourier Series
$X[n] = (\frac{5}{4})^n \; where \; 0 \le n \le 6$ Answer: N = 6 – (0) + 1= 7 $c_k = \frac{1}{7}\left(\frac{1-\left(\frac{5}{4}\right)^7}{1-\left(\frac{5}{4}e^{-j\frac{2\pi}{7}k}\right)}\right)$ F[k] = $\sum_{n=0}^{6} c_k e^{j\frac{2\pi}{7}kn} = \sum_{n=0}^{6} \frac{1}{7}\left(\frac{1-\left(\frac{5}{4}\right)^7}{1-\left(\frac{5}{4}e^{-j\frac{2\pi}{7}k}\right)}\right)e^{j\frac{2\pi}{7}kn}$	$f[k] = \sum_{n=0}^{N-1} c_k e^{j\Omega_o n}$ $= \sum_{n=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$ $c_k = \frac{1}{N}\sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}kn}$

Discrete Time Fourier Transform (DTFT)

$$f(k) = \sum_{n=0}^{N-1} f[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} f[n]e^{-j\frac{2\pi}{N}kn}$$

Discrete Time Fourier Transform (DTFT) Example

- **Given the signal:**

$$x[0] = 1,\ x[1] = 2,\ x[2] = 2,\ x[3] = 1$$
$$x[n] = 0 \text{ otherwise } \rightarrow \mathbf{x} = [1,2,2,1]$$

$$
\begin{aligned}
X_k &= \sum_{n=0}^{3} x[n]e^{-j2\pi kn/4}, \quad k = 0,\ 1,\ 2,\ 3 \\
&= x[0]e^{-j2\pi k(0)/4} + x[1]e^{-j2\pi k(1)/4} + x[2]e^{-j2\pi k(2)/4} + x[3]e^{-j2\pi k(3)/4} \\
&= 1 + 2e^{-j\pi k/2} + 2e^{-j\pi k} + 1e^{-j\pi 3k/2}, \quad k = 0,\ 1,\ 2,\ 3 \\
&= \left[1 + 2\cos\left(\frac{-\pi k}{2}\right) + 2\cos(-\pi k) + \cos\left(\frac{-3\pi k}{2}\right)\right] \\
&\quad + j\left[-2\sin\left(\frac{\pi k}{2}\right) - 2\sin(\pi k) - \sin\left(\frac{3\pi k}{2}\right)\right]
\end{aligned}
$$

$$
X_k = \begin{cases} 6 & k = 0 \\ -1 - j & k = 1 \\ 0 & k = 2 \\ -1 + j & k = 3 \end{cases}
$$

Inverse DFT

- **The inverse transform follows from the DT Fourier Series:**

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad n = \ 0,\ 1,\ \ldots,\ N-1$$

Inverse DFT

$$Given\ X_k = \begin{cases} 6 & k = 0 \\ -1 - j & k = 1 \\ 0 & k = 2 \\ -1 + j & k = 3 \end{cases}$$

$$x[n] = \frac{1}{4}\left[X_0 + X_1 e^{j2\pi kn/4} + X_2 e^{j2\pi kn/4} + X_3 e^{j2\pi kn/4}\right]$$

$$x[0] = \frac{1}{4}[X_0 + X_1 + X_2 + X_3] = \frac{1}{4}[6 - 1 - j - 1 + j] = \frac{4}{4} = 1$$

$$x[1] = \frac{1}{4}\left[X_0 + X_1 e^{j\pi/2} + X_2 e^{j\pi} + X_3 e^{j3\pi/2}\right] = \frac{1}{4}[6 + (-1 - j)j + (0)(-1) + (-1 + j)(-j)]$$

$$= \frac{1}{4}[6 - j + 1 + j + 1] = 8/4 = 2$$

$$x[2] = \frac{1}{4}\left[X_0 + X_1 e^{j\pi} + X_2 e^{j2\pi} + X_3 e^{j3\pi}\right] = \frac{1}{4}[6 + (-1 - j)(-1) + (0) + (-1 + j)(-1)]$$

$$= \frac{1}{4}[6 + 1 + j + 1 - j] = 8/4 = 2$$

$$x[3] = \frac{1}{4}\left[X_0 + X_1 e^{j3\pi/2} + X_2 e^{j3\pi} + X_3 e^{j18\pi/4}\right] = \frac{1}{4}[6 + (-1 - j)(-j) + (0) + (-1 + j)(j)]$$

$$= \frac{1}{4}[6 + j - 1 - j - 1] = 4/4 = 1$$

Any Question?
Thank you