Afterword: Acceleration 29

29.1 Can there be acceleration in SR?

Let us briefly recapitulate a few insights we have presented in this book. We recognized that an accelerated material body differs from any inertial reference observer, and there is no relativity principle that applies to accelerated motion:

- It is the traveling-accelerated 'twin' who upon rejoining her base notices a shorter increment of proper time measured by her clock compared to the proper time ticking in the base station;
- It is the train departing a station that will upon reaching relativistic speed fit into the usually not long enough tunnel.

The consistency of any changes of the body properties with the Lorentz coordinate transformation emerges when we implement the measurement process appropriate for the circumstance. We recall that the spatial separation between two events measured at equal time in the observer's frame of reference agrees with the Lorentz-FitzGerald contracted body that can be placed between these events, see Chap. 9. Similarly, we can measure that a moving clock with $\Delta x' \neq 0$ ticks slower by observing it from a fixed $\Delta x = 0$ base location, see Sect. 12.3.

Upon making these statements we need to recognize that the theory of special relativity does not incorporate explicitly the notion of accelerated material bodies. Acceleration is added in when we postulate the existence of Lorentz force. Introduction of this ad-hoc force is on first sight inconsistent with two accomplished theoretical frameworks, general relativity (GR) and quantum mechanics (QM), which both are foundational theories without acceleration. Here is how this happens:

GR: By recognizing the proportionality of inertial and gravitational mass (equivalence principle), Einstein could describe the force of gravity as quasi-inertial: a free-falling observer in a gravity field does not locally experience any force at all. In a second step,

J. Rafelski, *Relativity Matters*,
DOI 10.1007/978-3-319-51231-0_29

Einstein's GR equations introduce space-time deformation by the action of the energy-momentum of all particles, allowing this back-reaction modification of the space-time geometry. The free-falling particles follow trajectories in a curved space-time showing the properties we associate with motion executed according to the effect of gravity.

Note that the force of gravity we feel is due to the 'surface normal' force that keeps us from free-falling. We therefore report being subject to Earth's force. In classical point-particle model of matter all particles are free-falling and there is no force.

QM: Quantum mechanics describes particles on an elementary scale. The electron in a quantum orbit is stationary, and not accelerated; hence there is no continuous emission of radiation; an electron does not spiral into the nucleus. A modification needed to assure the stability of atoms against acceleration induced radiation motivated the discovery of the atomic quantum theory in the first place. To clarify, when an excited atomic state decays, emitting a photon, this situation is like an unstable particle undergoing a radioactive decay, and not a result of continuous 'synchrotron' radiation due to accelerated motion.

29.2 Evidence for acceleration

Theories are built without acceleration and yet our daily experience contradicts this. One is justified in asking if there is acceleration at all – and this is synonymous with the question – *how do we know if a body is accelerated?* This is not a new question. Newton explored the meaning of acceleration in his widely quoted water bucket experiment and more than 300 years later this is still a good place to start. Consider a suspended rotating water bucket – at first the water in the bucket does not rotate with the bucket. When the water begins to rotate, the surface of the water becomes concave.

Newton concluded that by observing the curving of water surface in a rotating bucket, one can determine that the bucket is being rotated and not the rest of the world around the bucket. If you have doubts about Newton's conclusions, place a water bucket in the immobile center of a merry-go-round and while enjoying a ride see if the water surface turns concave. However, some will still claim that this demonstration is flawed since the experiment was not carried out in an inertial frame in empty space.

For this reason Newton supplemented a real bucket experiment with what we today call 'thought experiments'. He considered two rocks at the end of a rope in empty space, far from everything. When they rotate around, the rocks will create an outward force pulling the rope tight. In empty space there is nothing to define and measure the rotation as occurring with respect to, except for space itself. Since we expect the rope will record a tension, space itself provides a reference. Such (thought) experiments led Newton to propose the existence of absolute space.

This was the situation when about 150 years ago Ernst Mach[16] returned to the question raised by Newton. Mach proposed the 'Universe at rest', defined by the fixed stars, to be Newton's absolute space. Adapting these ideas to the post-relativity context of this book we would say that we always measure acceleration against any known inertial frame, and the fixed star frame is a suitable inertial reference frame. Therefore choice of a preferred *inertial* frame such as that proposed by Mach is not in conflict with special relativity. This differs from a preferred origin in Newton's absolute space required in the heliocentric 'Weltbild' from the age of Newton, a notion that is manifestly incompatible with SR.

The choice of an inertial frame made by Mach makes good sense for the merry-go-round experiment in space, far from anything. By looking at starlight we can determine who is rotating and who is not. Now the entire Universe dictates the answer to who is accelerated. However, this also prompts us to reflect on the question: "How does the water bucket know about the universal inertial frame?" Especially in a book where we often check if a causal sequence of events is guaranteed, it seems doubtful that this information is transferred from distant locations of fixed stars, 'informing' the water bucket to curve its surface. This information about who or what is accelerated has to be available locally. Within the laws of classical physics this is a more difficult to fulfill requirement compared to quantum physics where we have accepted that the quantum vacuum state is structured, a topic that is now of importance but transcends this book's scope.[17]

One can see the situation as follows: the same quantum vacuum state properties that assure quark confinement, and predominantly define the inertial mass of matter, also define locally at a very microscopic scale the inertial frame of reference. In this way the structured quantum vacuum supplants Einstein's view of the universal relativistically invariant æther we have described in this book's first pages. Taking this argument further, the reader should wonder in what way the gravitational radiation friction we introduce below relies on concepts transcending the ideas of free-falling mass, in that we accept the universal presence of a class of reference frames we call inertial that allow an objective characterization of motion as being non-inertial.

The author of this book has not found one student who would disagree with the fact that acceleration exists. But which experiment shows this unequivocally? All agree that charged particles in inertial motion cannot emit radiation. The argument that will be developed in these pages is that emission of radiation by charged particles is the experimental evidence for presence of accelerated motion. In fact I will first argue that emission of gravitational radiation by accelerated massive bodies provides decisive evidence in recognizing that local acceleration exists. I turn to gravity first since everybody is in agreement that gravitational phenomena are in the classical domain that this book addresses. For the in-

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[16]Ernst Mach (1838–1916), Professor at Graz, Salzburg, Prague (for most of his life), and Vienna; remembered for Mach number, shock waves, and Mach's principle.

[17]For a generally accessible discussion see J. Rafelski and B. Müller, *The Structured Vacuum: Thinking about Nothing*, Harri Deutsch (Frankfurt 1985); hard copy edition out of print, see E-republication in 2006 at http://www.physics.arizona.edu/~rafelski/Books/StructVacuumE.pdf.

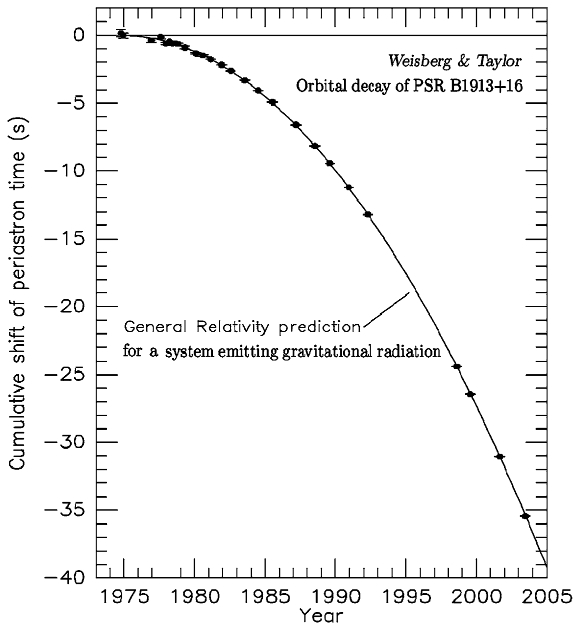

Fig. 29.1 30 years of time signal modification of binary Hulse-Taylor pulsar B1913+16 compared to the GR prediction. Adopted from Ref. 19

terested reader we refer to discussion of the contextual story about gravitational radiation emission presented recently by Poisson and Will.[18]

Today we have convincing experimental evidence that gravitational radiation friction exists:

i) Based on long term analysis of relativistic radio pulsars in binary systems there is agreement that the orbit change observed is a consequence of radiative energy loss. The experimental data showing how excess orbital phase (relative to an unchanging orbit) has accumulated[19] is displayed in Fig. 29.1 based on 30 years of observation of a relativistic binary pulsar B1913+16. This binary system will spiral inward and crash in about 250 million years.

ii) In February 2016 the LIGO-VIRGO collaboration announced the event GW150914, the first detection[20] of gravitational waves emitted during the last stage of inspiraling in a very massive binary – it seems there is no sufficient data evidence to call the compact

[18] For detailed discussion see box 11.2, pp. 553/4 and related material in E. Poisson and C.M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*, Cambridge University Press (2014).

[19] J.M. Weisberg, J.H. Taylor, "Relativistic Binary Pulsar B1913+16: Thirty Years of Observations and Analysis," arXiv:astro-ph/0407149 (2004); J.M. Weisberg, D.J. Nice, and J.H. Taylor, "Timing Measurements of the Relativistic Binary Pulsar PSR B1913+16," *Astrophys. J.* **722**, 1030 (2010).

[20] B.P. Abbott, et al., LIGO and VIRGO Scientific Collaborations, "Observation of Gravitational Waves from a Binary Black Hole Merger," *Phys. Rev. Lett.* **116**, 061102 (2016).

objects in the event GW150914 (September 14, 2015 event) a black hole.[21] A second GW151226 event confirms the evidence for compact massive object mergers.[22]

Here are a few important insights that we can draw from these experiments: i) Acceleration (just like deceleration) cannot be relative for if it were, how could gravitational radiation friction arise? ii) Any two orbiting objects in space have a well defined acceleration with regard to the inertial Universe. iii) Emission of gravitational radiation experiments demonstrate that tiny radiation friction effects accumulate. iv) Radiation friction can further reinforce the deceleration radiation friction, with the final ultra-strong collapse becoming observable over a cosmic scale distance.

In summary, the insights about SR and acceleration are:

1. Acceleration can be measured against any inertial reference frame. The cosmic reference frames, such as Mach's fixed star frame of reference, or microwave background (CMB) frame – *i.e.* the frame in which the CMB spectrum is isotropic – are usually considered.
2. The 4-vector of acceleration Sect. 22.2 provides a local invariant measure describing the magnitude of the spacelike acceleration. This allows all inertial observers to agree to a common way of establishing the magnitude of acceleration. Any inertial observer recognizes the accelerated observer to be different. There is no relativity of acceleration.
3. Acceleration is defined at the body location in space-time, in a process that manifestly does not depend on the global Universe reference frame. The structured quantum vacuum frame provides the required local point of reference. This is a more general reference point compared to the cosmic reference frames: Just like in Einstein's æther, the concept of local velocity cannot be associated with the structured quantum vacuum reference frame.[23]

29.3 Strong acceleration

In Sect. 22.2 we introduced 'small acceleration' into SR comparing to natural unit-1 strong acceleration Eq. (22.4). Since we just considered gravitational radiation friction it is natural to ask when and how can such acceleration arise, beginning with Newton's force. Since this force increases as distance decreases we need some very short and elementary unit of

[21] V. Cardoso, E. Franzin, and P. Pani, "Is the gravitational-wave ringdown a probe of the event horizon?" *Phys. Rev. Lett.* **116**, 171101 (2016).

[22] B.P. Abbott, et al., LIGO and VIRGO Scientific Collaborations, "GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence," *Phys. Rev. Lett.* **116**, 241103 (2016).

[23] It is generally believed that dominant component in the gravitating energy of the Universe is originating in the vacuum structure. This 'dark' energy cannot thus be moved and/or concentrated.

length. The shortest known elementary length is the Planck length

$$\ell_\mathrm{P} = \sqrt{\frac{\hbar G_\mathrm{N}}{c^3}} \equiv 1.616210^{-35}\,\mathrm{m}\,, \tag{29.1}$$

twenty orders of magnitude below the proton size. Today it is thought that near ℓ_P, gravity connects with quantum physics. Planck noted introducing ℓ_P in an appendix[24] that such a new elementary scale is of interest as it arises in consequence of the introduction of the quantum of radiation $\hbar$ and connects with Newton's Gravity.

We consider the Newton gravity self-acceleration by any particle of mass m at the Planck distance ℓ_P:

$$a_\mathrm{P} = \frac{G_\mathrm{N} m}{\ell_\mathrm{P}^2} = mc^2\frac{c}{\hbar} \equiv a_\mathrm{cr}\,. \tag{29.2}$$

We note the cancellation of Newton's constant G_N and recognize a_P to be the unit acceleration a_cr we have introduced in Eq. (22.4). Thus where quantum and gravity phenomena are expected to meet, a unit-1 acceleration is present. In this sense the unit-1 acceleration is connected to Planck's natural scales but it does not require G_N. Moreover it is in the realm of the possible that when we achieve unit-1 acceleration, some new phenomena appear, reminding us of the deeper connection that Eq. (29.2) brings to mind.

Thus there is profound foundational interest in achieving unit-1 acceleration. An electrical field required to generate this critical acceleration for the lightest elementary particle, the electron, would have the so-called 'Schwinger critical' field strength

$$E_\mathrm{cr} = \frac{m a_\mathrm{cr}}{e} = \frac{m^2c^3}{e\hbar} = \frac{mc^2}{e\lambda\!\!\!^{-}_\mathrm{C}} = 1.323\times 10^{18}\,\frac{\mathrm{V}}{\mathrm{m}}\,. \tag{29.3}$$

To evaluate the value E_cr in SI units *i.e.* V/m we insert for the elementary constants in Eq. (29.3) the SI values m[kg], c[m/s], e[C], $\hbar$[Js] and compute (note 'C' stands here for 'Coulomb'). However it is by far simpler to remember that $mc^2 = 0.511\,\mathrm{MeV}$ and to cancel the e between numerator and denominator. The numerical coefficient in Eq. (29.3) is recognized as 511/386, the last number being $\lambda\!\!\!^{-}_\mathrm{C}$ in fm, while the power comes from remembering that 'M' stands for mega $= 10^6$ and that fm $= 10^{-15}$ m. The corresponding critical magnetic field is obtained dividing the SI value of E_cr by $c = 3\times 10^8$ m/s to find the SI value of the magnetic field, see Insight on page 318

$$B_\mathrm{cr} \equiv \frac{m a_\mathrm{cr}}{ce} = \frac{m^2c^2}{e\hbar} = 4.414\times 10^9\,\mathrm{T}\,. \tag{29.4}$$

At this point we note that there is actually a hierarchy of critical accelerations. For example we could ask what force do we need to apply to rip apart an atom? The atom size

[24] M. Planck, "Über irreversible Strahlungsvorgänge," (translated: "On irreversibility of radiation processes") *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin* **5**, 440 (1899), see last page 480.

described by the Bohr radius is larger than the Compton wavelength $\lambda\!\!\!^{-}_{\rm C}$, the quantum electron size, by a factor corresponding to the inverse fine-structure constant, $\alpha^{-1} \simeq 137.036$. Thus the acceleration, and hence the field required to 'critically' accelerate atoms, is of the magnitude

$$E_{\rm cr}^{\rm Atom} = \frac{m\alpha a_{\rm cr}}{e} = \frac{m^2c^3}{e\hbar} = \frac{mc^2}{e(\alpha^{-1}\lambda\!\!\!^{-}_{\rm C})} = 0.965 \times 10^{16}\,\frac{\rm V}{\rm m}\,, \tag{29.5}$$

and

$$B_{\rm cr}^{\rm Atom} \equiv \frac{m\alpha a_{\rm cr}}{ce} = \frac{m^2c^2}{e\hbar} = 3.208 \times 10^7\,{\rm T}\,. \tag{29.6}$$

On the other hand fields that can rip apart an elementary particle must address distances at the scale of classical electron radius r_e, a length scale that is by the same factor $\alpha^{-1} \simeq 137.036$ smaller compared to $\lambda\!\!\!^{-}_{\rm C}$, requiring stronger fields

$$E_{\rm cr}^{\rm Part} = \frac{m\alpha^{-1}a_{\rm cr}}{e} = 1.813 \times 10^{20}\,\frac{\rm V}{\rm m}\,, \qquad B_{\rm cr}^{\rm Part} = 6.05 \times 10^{11}\,{\rm T}\,. \tag{29.7}$$

Achieving such fields may be impossible by laboratory devices including pulsed lasers as on the way we encounter vacuum instabilities related to the critical fields as presented in Eq. (29.3). One can imagine that a magnetized neutron star, a 'magnetar', could offer an astrophysical laboratory where such magnetic fields Eq. (29.7) are possibly present, and thus protons and neutrons are ripped apart into quarks and gluons. Alternatively, in the context of strong interaction physics and the formation of quark-gluon plasma in relativistic heavy ion collisions, this estimate produces an order of magnitude for the required fields in the context of 'chromodynamics'.

The Schwinger critical electric field Eq. (29.3) characterizes in quantum electrodynamics the condition when spontaneous pair production of particles of mass m is so abundant that the field is rapidly neutralized. Said differently, the field energy 'materializes' – achieving such an electric field in static condition in the laboratory is thus, in principle, not possible. However the equivalent static critical magnetic field can be in principle created.

The laboratory acceleration achieved today is unobservable viewed from the perspective of the scale considerations above: the largest bending magnetic field in accelerators is typically only a few Tesla, and certainly in the foreseeable future below 10 s of Tesla. The laboratory bending magnetic field is 8 orders of magnitude below the critical acceleration field Eq. (29.4). The only reason that we can measure and actually in certain circumstances use radiation friction is that we accumulate a tiny effect on an elementary scale over a macroscopic time and distance scale. What this teaches us is that the perturbative description of radiation friction works well, and we return to this situation in following section.

Let us, however, mention a few experimental environments in which it is perhaps possible to achieve and study critical acceleration today and in the near future:

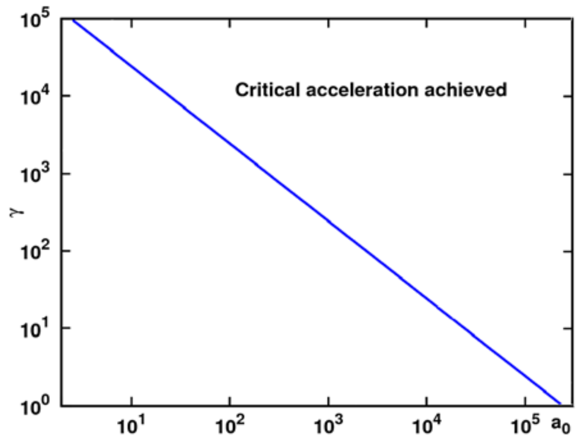

Fig. 29.2 Collision of a high energy $E = m_e c^2 \gamma$ electron with an intense laser pulse with normalized amplitude a_0 and frequency $\hbar\omega = 4.8\,\mathrm{eV}$. Above the line the electron experiences critical acceleration – the central domain of the figure is where present day technology makes experiments possible

Supercharged (quasi) nuclei: Compared to the field strength we discussed in the context of the Born-Infeld model of the electron in Sect. 28.3, the value E_{cr} Eq. (29.3) appears small, and even smaller compared to fields available at the nuclear surface, see Eq. (28.38). However, atomic electrons are not near to the nuclear surface and are delocalized over the atomic Bohr radius. Moreover, the Schwinger field at the nuclear surface is found only in a small volume. Clearly, just reaching E_{cr} is not a key criterion for the appearance of new physics phenomena. We ask the question: what nuclear charge Ze is needed so that at distance $\lambdabar_C$ the field becomes critical? This condition can be also written as

$$\frac{Ze}{\lambdabar_C^2} = E_{\mathrm{cr}} = \frac{m^2c^3}{e\hbar} \rightarrow Z\alpha = 1\,, \quad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036}\,. \tag{29.8}$$

Thus if we can assemble a nuclear charge of $Z > 137$ we could test in an atomic physics experiment the condition of critical acceleration. This situation has been thoroughly explored and described in the context of relativistic quantum mechanics.[25] Since the nuclei have finite size, the actual value of 'critical charge' $Z_{\mathrm{cr}} \simeq 171$ is recognized. At this point the quantum vacuum, the ground state, becomes charged. The experimental realization of this situation is in the study of slow, $v \ll c$, large nuclei called 'heavy ion' collisions, such that $Z_1 + Z_2 > Z_{\mathrm{cr}}$. The study of positron production in such heavy ion collisions carried out in the 1980s was hampered by large nuclear reaction backgrounds.

Ultra-Intense laser pulse collision with relativistic electrons: It has been noted that when an 'observer' riding a relativistic electron is hit by a laser pulse, the required characteristics of supercritical force can occur. In Fig. 29.2 the domain where the required conditions are achieved[26] is located above the line. We need a combination of a) a rela-

[25]See: W. Greiner, B. Müller and J. Rafelski, *Quantum Electrodynamics of Strong Fields*, Springer (Heidelberg, New York, 1985).

[26]Adapted from: J. Rafelski and L. Labun, "Critical Acceleration and Quantum Vacuum," *Modern Phys. Lett. A* **28**, 1340014 (2013).

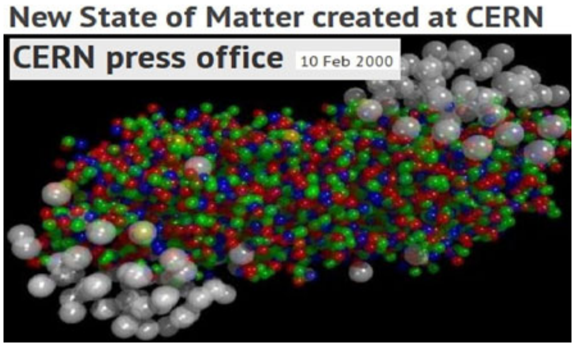

Fig. 29.3 CERN announces the discovery of quark-gluon plasma in a press-release on 10 February 2000 press.web.cern.ch/press-releases/2000/02/new-state-matter-created-cern

tively large electron Lorentz factor γ, with b) intense laser pulses with a large factor a_0, see exercise IX–8. What comes handy is that such intense laser pulses are capable of accelerating electrons[27] to energies in the GeV range ($\gamma > 1000$). Thus one can envisage in an experimental environment the collisions of laser pulses with laser pulse generated electron pulses. Laser pulse laboratories where the required conditions and experimental beam lines are prepared are at the time of writing under construction.[28] At these facilities experimental study of a radiation reaction effect will be possible, a physics phenomenon we return to discuss in a qualitative manner in Sect. 29.4 – a charged particle interacting with strong EM-fields experiences 'vacuum friction', dissipating its energy in form of radiation, a foundational consistent description of this novel physics frontier has not been achieved.[29]

Ultrarelativistic heavy ion collisions: We have established in exercise VIII–14 on page 307 that the 4-acceleration magnitude is

$$a \equiv c\,\frac{dy_p}{d\tau}\,, \quad \text{or} \quad a \equiv c\,\frac{dy_p}{dt}\cosh y_p\,. \tag{29.9}$$

[27]Review Article: Tae Moon Jeong and Jongmin Lee "Femtosecond petawatt laser," *Ann. Phys.* (Berlin) **526**, 157–172 (2014), doi:10.1002/andp.201300192.

[28]Page 71 of *European Strategy Forum Report 2016 on Research Infrastructures*, prepared by the StR-ESFRI project. To quote: "the Extreme Light Infrastructure (ELI) is a Research Infrastructure of Pan-European interest for experiments on extreme light-matter interactions at the highest intensities, shortest time scales and broadest spectral range. ELI is based on three sites (known as pillars, located in the Czech Republic, Hungary and Romania) with complementary scientific profiles, and the possible implementation of a fourth pillar, the highest intensity pillar, dependent on on-going laser technology development and validation. The fourth pillar laser power is expected to exceed that of the current ELI pillars by another order of magnitude, allowing for an extended scientific program in particle physics, nuclear physics, gravitational physics, nonlinear field theory, ultrahigh-pressure physics, astrophysics and cosmology (generating intensities exceeding 10^{23} W/cm^2)."

[29]Y. Hadad, L. Labun, J. Rafelski, N. Elkina, C. Klier and H. Ruhl, "Effects of Radiation-Reaction in Relativistic Laser Acceleration," Phys. Rev. D **82**, 096012 (2010).

When big nuclei collide head-on in a relativistic heavy ion collider experiment, the duration time of the collision in the laboratory frame is $dt \to \Delta t = R/c$, where R is the nuclear radius. This is the maximum time; the actual time dt could be a fraction of Δt, but we want to be conservative in estimating the value of a

$$a \simeq c\,\frac{\Delta y_p}{\Delta t}\cosh\Delta y_p = a_{\mathrm{RHIC}}\,\Delta y_p \cosh\Delta y_p\,, \tag{29.10}$$

where the scale on which we measure acceleration in these collisions is

$$a_{\mathrm{RHIC}} \equiv \frac{c}{\Delta t} = \frac{c^2}{R} \simeq 2\times 10^{31}\,\frac{\mathrm{m}}{\mathrm{s}^2}\,. \tag{29.11}$$

This result is by a factor 100 greater than the values shown in Eq. (22.5) and corresponds to fields we also obtained in a somewhat different manner, see Eq. (29.7). As these considerations suggest when nucleons from the incoming nucleus are stopped so that rapidity shifts significantly from a value prior to collision, critical acceleration must have been achieved, ripping nucleons apart. In such process there is creation of a large particle multiplicity. This finding coincides with the formation of quark-gluon plasma (QGP) at CERN in the year 2000, see Fig. 29.3. The abundant particle production accompanying achievement of critical acceleration is probably the cause of formation of this new state of matter.

29.4 EM radiation from an accelerated particle

We presented in Fig. 29.1 the effect of gravitational radiation friction: since the binary system spirals in as energy is carried out by gravitational radiation, one can say the motion is damped by radiation friction. Similarly, a non-inertial moving charged particle will emit EM radiation, lose energy and thus be subject to radiation vacuum-friction effect. The emitted EM radiation is in a particular context of radial circular motion called synchrotron radiation. Synchrotron radiation reminds us of the GR radiation and is a parasitic friction effect in particle accelerators.

On the other hand it can be greatly enhanced in special devices in order to make the emitted radiation intense enough to become a research tool. There are many practical applications of intentional intense synchrotron radiation. This is a very well studied domain of physics with diverse literature and large laboratories participating.[30]

Our objective here is different: we are interested in understanding the influence of radiation emission on the motion of a charged accelerated particle. Radiation emission has the characteristics of a friction force in that a particle loses momentum and energy. While in nonrelativistic dynamics the friction force grows (Newton) with velocity, in the relativistic dynamics of a charged particle due to the relativity principle, the radiation friction effect cannot depend on velocity; it depends on acceleration.

[30]For a review see: S. Mobilio, F. Boscherini, C. Meneghini (eds.) *Synchrotron Radiation: Basics, Methods and Applications*, Springer (Heidelberg, New York, 2015).

To describe the reaction force that an accelerated charged particle experiences as a first step we will consider the vacuum-friction effect in the instantaneous particle rest-frame $\vec{v}(t) = 0$ where

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \dot{\vec{v}} \neq 0\,. \tag{29.12}$$

As a second step we will use the result we obtain to propose a covariant reaction force which reduces in the particle rest-frame to the result obtained. We are not using in this process a Lorentz transformation to shift from the particle rest-frame to an inertial reference frame since the particle frame is non-inertial. Therefore the transition from the accelerated and instantaneous particle rest-frame to the observer inertial frame does not need to have the format of a LT. Our procedure is an elaboration of a century old guessing-game and the outcome is not entirely satisfactory.

The key insight is that aside from the point charge $\rho \equiv e\delta^3(\vec{x})$, in the presence of acceleration, there is another source of Maxwell equations, *i.e.* the time derivative of the current density

$$\frac{d\vec{j}}{dt} = \frac{d(\vec{v}\rho)}{dt} = \vec{a}\delta^3(\vec{x})\,. \tag{29.13}$$

We restrict ourselves to the instantaneous rest-frame $\vec{v} = 0$, allowing $\vec{a} \neq 0$. The question that arises is if we should also introduce higher derivatives of $\vec{v}$ as independent sources. This is not the case since when solving the Maxwell equations in the instantaneous rest-frame $\vec{v} = 0$, we do not need to restrict $\vec{a}$ to be time independent.

Taking the time derivative of Maxwell's equation (27.11) and replacing the time derivative of $\vec{\mathcal{B}}$ using Maxwell's equation (27.28), we obtain in the particle rest-frame

$$-\frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2} + \vec{\nabla} \times \frac{\partial\vec{\mathcal{B}}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2} - \vec{\nabla} \times (\vec{\nabla} \times \vec{\mathcal{E}}) = \frac{1}{\epsilon_0 c^2}\,\vec{a}\,\delta^3(\vec{x})\,, \tag{29.14}$$

which can be written as

$$-\frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2} + \vec{\nabla}^2\vec{\mathcal{E}} = \frac{1}{\epsilon_0 c^2}\vec{a}\delta^3(\vec{x}) + \vec{\nabla}(\vec{\nabla} \cdot \vec{\mathcal{E}})\,. \tag{29.15}$$

Maxwell's Eq. (27.10), restated here for convenience, is

$$\vec{\nabla} \cdot \vec{\mathcal{E}} = \frac{e}{\epsilon_0}\delta^3(\vec{x})\,. \tag{29.16}$$

Since both sources of $\vec{\mathcal{E}}$ in Eq. (29.15) and Eq. (29.16) are not explicitly dependent on time, we can ignore the partial time derivative in Eq. (29.15). The electric field that solves Eq. (29.15) and Eq. (29.16) contains in addition to the usual radial electric field $\propto 1/r^2$ another orthogonal component, which varies like $1/r$. This is so since to solve Eq. (29.15) we need to take advantage of

$$\vec{\nabla}^2\frac{a}{r} = -4\pi\,a\,\delta^3(\vec{x})\,.$$

We introduce the radial unit vector $\hat{r}$ and the acceleration unit vector $\hat{a}$ by $\dot{\vec{v}} = \vec{a} = a\hat{a}$. Since both Eq. (29.14) and Eq. (29.16) are linear we decompose the solution into radial and 'rotational' parts

$$\vec{\mathcal{E}} = \hat{r}\mathcal{E}_c + \vec{\mathcal{E}}_a\,, \qquad \hat{r}\cdot\vec{\mathcal{E}}_a = 0\,, \tag{29.17}$$

and hence

$$\vec{\mathcal{E}}_a = \left(\hat{r}\,(\hat{r}\cdot\hat{a}) - \hat{a}\right)\mathcal{E}_a = \hat{r}\times(\hat{r}\times\hat{a})\mathcal{E}_a\,, \tag{29.18}$$

where we used the 'bac-cab' rule backwards. Note that the $\hat{r}\mathcal{E}_c$ component in $\vec{\mathcal{E}}$ vanishes when inserted into Eq. (29.15), while the $\vec{\mathcal{E}}_a$ component vanishes when inserted into Eq. (29.16). Hence, the solution in the instantaneous rest-frame of a uniformly accelerated charge e is

$$\boxed{\vec{\mathcal{E}} = \frac{e}{4\pi\epsilon_0}\left(\frac{\hat{r}}{r^2} + \frac{a}{c^2}\,\frac{\hat{r}\times(\hat{r}\times\hat{a})}{r}\right)\,.} \tag{29.19}$$

The result Eq. (29.19) is identical to the one following when taking the $\vec{\beta} = \vec{v}/c = 0$ limit of Liénard-Wichert fields, obtained by differentiating the potential presented below in exercise XI–6 and keeping only first time derivative terms in $\vec{v}$ for the case of a moving spherically symmetric charge distribution. A systematic expansion of the Liénard-Wichert potentials exploring multipole moments accompanied by higher order derivatives of the velocity has been presented by Kijowski and collaborators.[31]

Since both contributions in Eq. (29.17) are orthogonal, the square of the field comprises these two terms without an interference term

$$\vec{\mathcal{E}}^{\,2} = \mathcal{E}_c^{\,2} + \vec{\mathcal{E}}_a^{\,2} = \frac{e^2}{(4\pi\epsilon_0)^2}\left(\frac{1}{r^4} + \frac{a^2}{c^4\,r^2}(\hat{r}\times\hat{a})^2\right)\,. \tag{29.20}$$

Note that the radiation term, $\vec{\mathcal{E}}_a^{\,2}$ in Eq. (29.19), decreases with growing r as $1/r^2$.

We next determine the magnetic field in the instantaneous rest-frame from the requirement that the mass density of the field surrounding a charged particle as described by the invariant Eq. (28.13) cannot depend on the force we apply to the charged particle: this is the present day theoretical paradigm. Inspecting Eq. (28.13) we realize that this means that $\vec{\mathcal{B}}$ must be orthogonal to both parts of the electric field $\mathcal{E}$, Eq. (29.19), assuring $\vec{\mathcal{E}}\cdot\vec{\mathcal{B}} = 0$. At the same time $c^2\vec{\mathcal{B}}^{\,2}$ must compensate exactly in $\vec{\mathcal{E}}^{\,2}$, Eq. (29.20), the second radiation term. Thus the magnitude of the magnetic field $c\mathcal{B}$ must be the same as that of the radiation term in the electric field, and to be orthogonal to both parts of $\vec{\mathcal{E}}$ we must have $\vec{\mathcal{B}} \propto \hat{r}\times\hat{a}$.

[31] J. Kijowski and M. Kościelecki, "Asymptotic Expansion of the Maxwell Field in a Neighborhood of a Multipole Particle," *Acta Phys. Pol. B* **31**. 1675 (2000); and "Algebraic Description of the Maxwell Field Singularity in a Neighborhood of a Multipole Particle," *Rep. in Math. Phys.* **47**, 301 (2001); J. Kijowski and P. Podleś, *J. Geometry and Physics* **59**, 693 (2009).

Thus the $c\vec{\mathcal{B}}$-field is determined except for the sign, to read

$$\boxed{c\vec{\mathcal{B}} = \hat{r} \times \vec{\mathcal{E}}_a = -\frac{e}{4\pi\epsilon_0 c^2}\frac{a}{r}(\hat{r}\times\hat{a})\,, \quad \vec{\mathcal{E}}\cdot\vec{\mathcal{B}} = 0\,, \quad \vec{\mathcal{E}}_a^{\,2} = c^2\vec{\mathcal{B}}^{\,2}\,.} \tag{29.21}$$

In the instantaneous rest-frame of the particle we find for the Poynting vector Eq. (28.2)

$$\boxed{\vec{S} = c^2\epsilon_0\vec{\mathcal{E}}\times\vec{\mathcal{B}} = \frac{e^2}{4\pi\epsilon_0 c^3}\frac{a^2}{4\pi r^2}\,(\hat{r}\times\hat{a})^2\,\hat{r}\,,} \tag{29.22}$$

where we used

$$\big((\hat{r}\times\hat{a})\times\hat{r}\times(\hat{r}\times\hat{a})\big) = \hat{r}\,(\hat{r}\times\hat{a})^2 - \hat{r}\times\hat{a}\,[(\hat{r}\times\hat{a})\cdot\hat{r}] = \hat{r}\,(\hat{r}\times\hat{a})^2\,. \tag{29.23}$$

The magnitude of the Poynting vector Eq. (29.22) has angular dependence. By choosing the z-axis to align with the direction of $\vec{a}$, we see that $(\hat{r}\times\vec{a})^2 = \sin^2\theta$; the maximum of emission is transverse to $\vec{a}$. $\vec{S}$ allows us to determine directly the equivalent Newton force that represents momentum loss due to the emitted radiation momentum. In a second step, upon casting this effective force into the covariant format, we find also the energy loss.

29.5 EM radiation reaction force

We seek a radiation reaction friction force $\vec{\mathcal{F}}_{\text{rad}}$ which satisfies the work-energy theorem condition

$$\int \vec{\nabla}\cdot\vec{S}\,d^3x + \int \vec{v}\cdot\vec{\mathcal{F}}_{\text{rad}}\,dt = 0\,. \tag{29.24}$$

The first term is the radiation power of the accelerated particle and the second term is the work done by the effective radiation reaction force $\vec{\mathcal{F}}_{\text{rad}}$ we seek to identify. Using Gauss's theorem to rewrite the left-hand side

$$\int d^3x\,\vec{\nabla}\cdot\vec{S} = \int d^2\vec{A}\cdot\vec{S} = \frac{e^2}{4\pi\epsilon_0 c^3}\frac{2a^2}{3}\,, \tag{29.25}$$

where we used Eq. (29.22) and have encountered the angular average $\langle\sin^2\theta\rangle = 2/3$.

We can now identify the effect of the radiation reaction force $\vec{\mathcal{F}}_{\text{rad}}$

$$\int \vec{v}\cdot\vec{\mathcal{F}}_{\text{rad}}\,dt \equiv -\frac{2}{3}\frac{e^2}{4\pi\epsilon_0 c^3}\int \vec{a}^{\,2}\,dt\,. \tag{29.26}$$

To obtain the explicit form of $\vec{\mathcal{F}}_{\text{rad}}$ in the instantaneous rest-frame we integrate by parts using $\vec{a} = d\vec{v}/dt$

$$\int \vec{v}\cdot\vec{\mathcal{F}}_{\text{rad}}\,dt = \frac{2}{3}\frac{e^2}{4\pi\epsilon_0 c^3}\int \vec{v}\cdot\frac{d\vec{a}}{dt}\,dt + \frac{2}{3}\frac{e^2}{4\pi\epsilon_0 c^3}\,\vec{v}\cdot\frac{d\vec{a}}{dt}\bigg|_{t_-}^{t_+}\,. \tag{29.27}$$

The last surface term vanishes for the initial and final integration boundaries outside of the acceleration domain, and for particles subject to periodic stationary motion. We thus find

$$\int \vec{v} \cdot \left(\vec{\mathcal{F}}_{\rm rad} - \frac{2}{3} \frac{e^2}{4\pi\epsilon_0 c^3} \frac{d\vec{a}}{dt} \right) dt = 0\,. \tag{29.28}$$

Since this relation holds for an arbitrary applied force causing the acceleration we obtain the general result for the 3-vector format of the radiation reaction force in the instantaneous rest-frame $\vec{v} = 0$

$$\boxed{\vec{\mathcal{F}}_{\rm rad} = m\tau_0 \frac{d\vec{a}}{dt}\,. \quad \tau_0 = \frac{2}{3} \frac{e^2}{4\pi\epsilon_0 m c^3} = \frac{2r_e}{3c} = 0.626 \times 10^{-23}\ \mathrm{s}\,,} \tag{29.29}$$

where the value of the time constant τ_0 is stated for an electron.

It is important to note that Eq. (29.29) predicts that a uniformly accelerated charge $\vec{a} =$ Const. does not experience a radiation reaction force in its rest-frame. This is one among many challenging radiation reaction puzzles whose solution remains elusive.[32] This author would argue that the effect of radiation emission is restored upon introduction of a yet-to-be formulated acceleration-adapted Lorentz-like transformation from accelerated into an inertial frame. A 'good' consequence of Eq. (29.29) is that an electron 'left sitting on a table' and thus for all laboratory observers sitting still in its instantaneous rest-frame but subject to constant normal force that prevents free-fall will not emit radiation we can observe.

The appearance of a higher derivative of acceleration in Eq. (29.29) has unpalatable consequences. Newton's equation for a free particle not subject to noticeable external force now has two independent solutions

$$m\frac{d\vec{v}}{dt} = \vec{\mathcal{F}}_{\rm rad} = m\tau_0 \frac{d^2\vec{v}}{dt^2}\,, \rightarrow \ \vec{v} = \vec{v}_0\,, \quad \& \quad \vec{v}_{\rm run}(t) = \vec{v}_0\, e^{t/\tau_0}\,. \tag{29.30}$$

The removal of the remnants of the 'runaway' solution $\vec{v}_{\rm run}(t)$ which shows that a particle can start in its rest-frame and accelerate its motion is the goal of anyone attempting to incorporate radiation reaction into charged particle dynamics. Both in nonrelativistic and relativistic physics in order to allow only physical particle trajectories we need to prescribe additional boundary conditions capable of suitably restricting motion. This turns out to be in general a procedure that requires causality violation: we must know the motion at the future time, typically τ_0 ahead, to assure that no runaway occurs.

[32] The reader interested in historical evolution of ideas about the radiation reaction force will gain additional insight from: Fritz Rohrlich, *Classical Charged Particles*, 3rd Edition, (World Scientific, Singapore, 2007); and: V.L. Ginzburg, *Theoretical Physics and Astrophysics*, 1st Edition (Pergamon Press, Oxford, 1979) (translation by D. Ter Haar from the original Russian edition of 1974).

We now generalize the radiation reaction force Eq. (29.29) to the relativistic domain and obtain the covariant form

$$\boxed{\mathcal{F}^{\mu}_{\text{rad}} \equiv m\tau_0 \left(g^{\mu\nu} - \frac{u^{\mu}u^{\nu}}{c^2} \right) \frac{d^2 u_{\nu}}{d\tau^2} \,, \qquad u \cdot \mathcal{F}_{\text{rad}} = 0 \,.} \tag{29.31}$$

To assure the nonrelativistic limit we replaced the second derivative with respect to time by the second derivative with respect to proper times. The term in parentheses assures $u \cdot \mathcal{F}_{\text{rad}} = 0$, securing the $u^2 = c^2$ constraint.

To verify this form of the force we study in an explicit manner the properties of the instantaneous rest-frame, beginning with 4-acceleration as given in Eq. (22.13). We also need its derivative seen in Eq. (29.31) for which the leading terms are obtained by differentiating Eq. (22.13)

$$\frac{d\,a^{\mu}}{d\tau} \underset{c\to\infty}{\longrightarrow} \left\{ \frac{\vec{a}^{\,2}}{c} + \frac{\vec{v}}{c} \cdot \frac{d\vec{a}}{dt} \,, \, \frac{d\vec{a}}{dt} \right\} , \qquad u_{\mu} \frac{d\,a^{\mu}}{d\tau} \underset{c\to\infty}{\longrightarrow} \vec{a}^{\,2} \,. \tag{29.32}$$

The last term in the brackets shows the correspondence $\mathcal{F}^{\mu}_{\text{rad}}$ to $\vec{\mathcal{F}}_{\text{rad}}$ in Eq. (29.29); all other terms are higher orders in $1/c$, as we explained to motivate the form of Eq. (29.31).

A new result related to the energy-work theorem emerges when we consider the time-like component of Eq. (29.31)

$$\frac{dp^0}{d\tau} = \mathcal{F}^0_{\text{rad}} = m\tau_0 \left(\frac{\vec{a}^{\,2}}{c} + \frac{\vec{v}}{c} \cdot \frac{d\vec{a}}{dt} - \frac{c}{c^2} \vec{a}^{\,2} \right) . \tag{29.33}$$

Upon cancellation of one term on the right and multiplication with c we find, using $cp^0 = E$,

$$\frac{dE}{dt} = m\tau_0 \vec{v} \cdot \frac{d^2\vec{v}}{dt^2} \,. \tag{29.34}$$

To finish we replace the term on the right in Eq. (29.34) by $d(\vec{v} \cdot \vec{a})/dt - \vec{a}^{\,2}$ and move the total time derivative component to the left to obtain

$$\frac{d(E - m\tau_0 \vec{v} \cdot \vec{a})}{dt} = -m\tau_0 \vec{a}^{\,2} \,, \tag{29.35}$$

where the right-hand side is the Larmor radiation power. In the instantaneous rest-frame the extra velocity dependent term on the left, a velocity dependent correction to particle energy, vanishes. Thus we find the work-energy theorem based on the 3-force $\vec{\mathcal{F}}_{\text{rad}}$ in Eq. (29.29): on the right we see that the energy loss of the particle is just the well known Larmor radiation power.

We can now justify the sign in Eq. (29.21) which was chosen to obtain the energy loss for the accelerated particle; the Larmor radiated power result Eq. (29.35) for energy loss is usually presented without signs, as an absolute value. In our study the sign was not determined but chosen, as commented below Eq. (29.21). One often sees remarks that the Larmor radiated power Eq. (29.35) is valid only in the 'radiation zone'; that is, far from the location of the charge. However, as obtained here, it is evident that Eq. (29.19) and Eq. (29.21) are exact; there is no need to be distant from the radiating particle – the amount of emitted radiation is the same near and far from the location of the charge of the particle as reported by an observer in the instantaneous rest-frame.

The radiation reaction 4-force Eq. (29.31) can be by simple algebraic manipulation cast into the Lorentz-Abraham-Dirac (LAD) equation written in the following form by Dirac:[33]

$$\frac{d(mu^\mu)}{d\tau} = (eF^{\mu\nu}_{\mathrm{ext}} + eF^{\mu\nu}_{\mathrm{rad}})u_\nu, \qquad \boxed{F^{\mu\nu}_{\mathrm{rad}} = \frac{m}{e}\frac{\tau_0}{c^2}(\ddot{u}^\mu u^\nu - u^\mu \ddot{u}^\nu)\,,} \tag{29.36}$$

where $F^{\beta\alpha}_{\mathrm{ext}}$ is a prescribed external field. We use the notation $\ddot{u} = d^2u/d\tau^2$. Equation (29.36) as a direct transcription of Eq. (29.31) using $u^\nu u_\nu = c^2$.

Finally we note the third format of this force that arises when in Eq. (29.31) we substitute using

$$u^2 = c^2 \;\rightarrow\; u\cdot\dot{u} = 0 \;\rightarrow\; \frac{d}{d\tau}u\cdot\dot{u} = 0 \;\rightarrow\; \boxed{\dot{u}\cdot\dot{u} + u\cdot\ddot{u} = 0\,,} \tag{29.37}$$

to obtain

$$\boxed{\mathcal{F}^\mu_{\mathrm{rad}} \equiv m\tau_0\left(\frac{d^2u^\mu}{d\tau^2} + u^\mu\frac{du_\nu}{cd\tau}\frac{du^\nu}{cd\tau}\right)\,.} \tag{29.38}$$

The closely related three forms of radiation reaction force, Eq. (29.31), Eq. (29.36) and Eq. (29.38), retain the 'runaway' difficulty associated with the radiation reaction force originating in time derivative of acceleration, as we noted in the nonrelativistic reaction force Eq. (29.30). Without entering into detailed discussion and extensive references on this subject we note that a textbook integral equation reformulation[34] shows that the price of removing the runaway solution is violation of causality for time interval of order τ_0.

29.6 Landau-Lifshitz radiation force model

Another well known modification of dynamical equations was developed in the Landau-Lifshitz (LL) text in theoretical physics.[35] There are three different presentations of the model, but it is hard to find a statement that this model differs from LAD radiation friction force. Since LAD is not a foundational model, one can view LL as a method of improving

[33] P.A.M. Dirac, "Classical theory of radiating electrons," *Proc. Roy. Soc. A* **167**, 148 (1938).

[34] See Sect. 21.11 in W.K.H. Panofski and M. Phillips, *Classical Electricity and Magnetism*, Addison-Wesley (Reading, MA, 1962).

[35] The treatment of radiation reaction and its discussion varies from edition to edition, tracking how authors evolved their point of view: L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1962, 1975, 1989).

on the LAD approach solving the problem of runaway solutions and causality violation. Therefore a short derivation with a clarification of how and where LL differs from LAD will offer a useful complement to other presentations.

In the LL approach the problematic third proper time derivative of the position of a particle appearing in Eq. (29.36) is replaced by an expression obtained differentiating the covariant Lorentz force equation

$$\ddot{u}^\mu = \frac{d}{d\tau}\left(\frac{e}{m}F^\mu_\delta u^\delta\right) = \frac{e}{m}u^\gamma\partial_\gamma F^{\mu\nu}u_\nu + \frac{e}{m}F^\mu_\delta \dot{u}^\delta = \frac{e}{m}u^\gamma\partial_\gamma F^{\mu\delta}u_\delta + \left(\frac{e}{m}\right)^2 F^{\mu\delta}F_{\delta\gamma}u^\gamma\,, \tag{29.39}$$

where if and when $F^{\mu\nu} = F^{\mu\nu}_{\text{ext}} + F^{\mu\nu}_{\text{rad}}$ no approximation is being made. Using Eq. (29.39) in the definition of $F^{\mu\nu}_{\text{rad}}$ in Eq. (29.36) we find for the radiation friction force

$$K^\mu_{\text{rad}} \equiv eF^{\mu\nu}_{\text{rad}}u_\nu = e\tau_0 u^\gamma\partial_\gamma F^{\mu\delta}u_\delta + e\tau_0\frac{e}{m}\left(F^{\mu\delta}F_{\delta\gamma}u^\gamma - u_\nu F^{\nu\delta}F_{\delta\gamma}u^\gamma u^\mu\right)\,. \tag{29.40}$$

Considering that $F^{\mu\nu} = F^{\mu\nu}_{\text{ext}} + F^{\mu\nu}_{\text{rad}}$, K^μ_{rad} continues to implicitly contain $\ddot{u}$ terms. The resulting LL equation of motion with simplifying index modifications is

$$m\dot{u}^\mu = eF^{\mu\nu}u_\nu + e\tau_0\left\{u^\gamma\partial_\gamma F^{\mu\delta}u_\delta + \frac{e}{m}\left(g^{\mu\gamma} - \frac{u^\mu u^\gamma}{c^2}\right)F_{\gamma\beta}F^\beta_\delta u^\delta\right\}\,. \tag{29.41}$$

As constructed by inserting Eq. (29.39) in Eq. (29.36) the radiative friction force satisfies $u_\mu K^\mu_{\text{rad}} = 0$, assuring that the modified Lorentz force equation will always produce solutions with $u^2 = c^2$.

It seems that as long as we use on the right-hand side of Eq. (29.40) $F = F_{\text{ext}} + F_{\text{rad}}$, LL and LAD are equivalent. However, the LL equation, as is in use, is never an exact rendition of LAD, which is not surprising, as LL faces no causality and runaway challenges. The reason is that there are in principle two ways to proceed, and both render LL nonequivalent to LAD:

1) We solve simultaneously the LL equation and the Maxwell equations for the fields F, particle velocity u and position x. That means that we incorporate in the field F all radiation effects for a self-consistently determined trajectory rather than adopting the simplifying format of the Larmor radiation formula formulation which leads to LAD. In this procedure, the solution we would obtain differs from the approximation inherent in the LAD form of response to emitted radiation. Moreover, this approach will encounter challenges related to the infinite EM energy, see Sect. 28.2; and, one cannot be sure to find in this approach causal non-runaway solutions.
2) To avoid solving Maxwell equations again, and in order to preserve the consistency with LAD, we insert into the LL force explicitly $F^{\mu\nu} = F^{\mu\nu}_{\text{ext}} + F^{\mu\nu}_{\text{rad}}$. Since $F^{\mu\nu}_{\text{rad}}$ contains $\ddot{u}^\mu$, we need to repeat the procedure, using Eq. (29.39) to eliminate $\ddot{u}$. This iterative procedure generates a series with terms in powers of $\tau_0(d/d\tau)$ and/or equivalently $r_e(\partial/\partial x)$. However, in addition to classical series there are quantum effects to consider

which scale with $\lambda\!\!\!^{-}_C(\partial/\partial x) = \alpha^{-1} r_e(\partial/\partial x)$. These are in general more relevant compared to the second classical step in the iterative approach to LL. Therefore, it is more appropriate to truncate the LL iteration scheme after the leading term. This means that on the right-hand side in Eq. (29.40) we set $F \simeq F_{\text{ext}}$ adopting to the lowest order in τ_0 friction force

$$\boxed{K^{\mu}_{\text{rad}\,(1)} \equiv e\tau_0 u^{\gamma}\,\partial_{\gamma}\,F^{\mu\delta}_{\text{ext}} u_{\delta} + e\tau_0 \frac{e}{m}\left(g^{\mu\gamma} - \frac{u^{\mu}u^{\gamma}}{c^2}\right) F^{\text{ext}}_{\gamma\beta}\,F^{\beta}_{\text{ext}\,\delta}\,u^{\delta} + \cdots .} \tag{29.42}$$

We now read the LL equation of motion Eq. (29.41) as used in approximation $F \simeq F_{\text{ext}}$. This is how LL is in general used in study of radiation friction force.

What this means is that the radiation field does not impact the particle motion as described by the LL dynamical equation. LL can be seen as a LAD motivated radiation friction force approximation where fields instead of higher proper time derivatives are introduced. Neither LAD nor LL are exact renditions of the effect of radiation friction as, for example, neither contains back-reaction of the radiation field on particle motion.

We now describe the strong effect of radiation friction where an electron collides head-on with a laser plane wave light-front. The radiation friction is introduced using the LL force. For comparison we consider the solution for the case of the Lorentz force (LF). The LF part coincides, with a change in initial conditions, with the LF solution we presented in Sect. 24.3. For the case of LL friction, the solutions were obtained semi-analytically in Ref. 29. The results we adapt from this reference consider the case of an electron with an initial energy $E_e = 511\,\text{MeV}$ *i.e.* $\gamma = 1000$, hitting a circular polarized (CP) plane wave light-front with an amplitude $a_0 = 100$ and wavelength $\lambda = 942\,\text{nm}$.

In Fig. 29.4 we see the behavior of the particle velocity vector. The particle moves initially exactly counter to the positive sense of the propagation of the laser pulse which explains the negative initial value of v_z in the left part of Fig. 29.4. On the right in Fig. 29.4 we see transverse velocity v_x, which begins with the particle impact on the plane wave front. For the LF case we see that the plane wave modulates the motion to a minor degree. When LL friction is introduced, v_z rapidly decreases. After the particle moves across 10 wavelengths, all of the motion is either attenuated and/or transferred from longitudinal to transverse direction; the value of the Lorentz factor γ (not shown) drops from 1000 to 100, ultimately the longitudinal motion reverses. If the calculation were to continue we would find the solution applicable to the particle surfing a plane wave as was described in Sect. 24.3.

The question why there is such a significant difference in the LL response compared to LF relates to the condition of critical acceleration that has been exceeded. In Fig. 29.5 the invariant acceleration $a = \sqrt{-a_{\mu}a^{\mu}}$ that is achieved in collision of light-front with electron beam in the LL case is seen to exceed unity. We see that once the acceleration drops below critical value, the radiation response diminishes, and it fades out when $a < 0.2$. Note that without radiation friction in the LF case the particle also experiences a large acceleration,

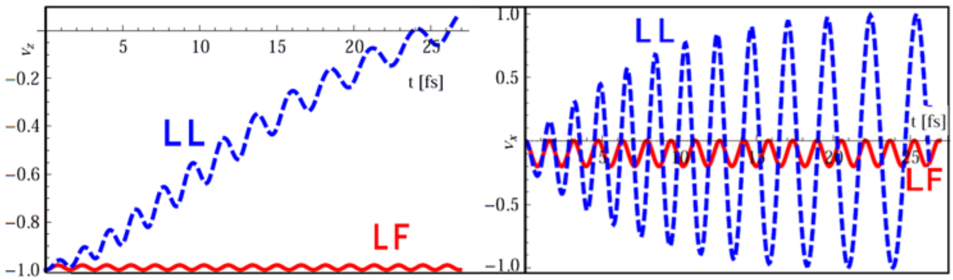

Fig. 29.4 The longitudinal velocity v_z and the transverse velocity v_x (both in units of c) are shown as functions of time measured in fs (10^{-15} s) for the case of an electron with an initial energy $E_e = 511\,\text{MeV}$, *i.e.* $\gamma = 1000$ hitting a circular polarized (CP) light-front with amplitude $a_0 = 100$ and wavelength $\lambda = 942\,\text{nm}$. The solid red line is for electron motion subject to the LF (Lorentz force) case see Sect. 24.3, while the dashed (blue) line gives the deceleration according to the LL (Landau-Lifshitz force). Adopted from Ref. 29

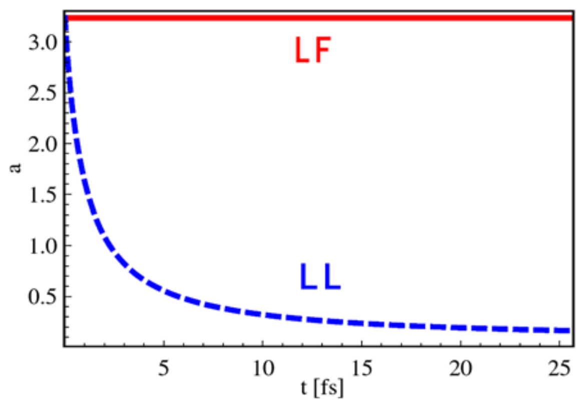

Fig. 29.5 Electron invariant acceleration $a = \sqrt{-a_\mu a^\mu}$ in natural units of critical acceleration, see Eq. (22.5), is shown as a function of time measured in fs (10^{-15} s) for the case of an electron with an initial energy $E_e = 511\,\text{MeV}$, *i.e.* $\gamma = 1000$ hitting a circular polarized laser plane wave with amplitude $a_0 = 100$ and wavelength $\lambda = 942\,\text{nm}$. The solid red line is for electron motion subject to the LF (Lorentz force) case, while the dashed (blue) line gives the deceleration according to the LL (Landau-Lifshitz force). Adopted from Ref. 29

but in the absence of radiative energy loss the charged particle motion remains, on the time scale considered, unaffected.

The strong response when radiation friction force is included is due to radiation power depending on acceleration (squared). When friction is large, (in this case) deceleration is large, which feeds back into the radiative loss, and the radiative loss runs away in producing a gigantic energy loss effect. We should keep in mind that the figure time resolution time scale is 10^{-15} s, while the expansion parameter for the radiation response is

$\tau_0 = 0.63 \times 10^{-23}$ s. Thus what we see in Fig. 29.5 is the outcome of many much smaller reaction processes.

29.7 Caldirola radiation reaction model

The radiation reaction force is obtained under the assumption of radiation emission by a locally accelerated particle. Since the radiation reaction force incorporates a higher derivative, one can argue that in a more consistent non-perturbative study a greater nonlocality could emerge. A proposal was made by Caldirola[36] to generalize the Newton (inertial) part of the Lorentz force

$$K^\mu = m\frac{d\,u^\mu(\tau)}{d\tau} \;\to\; K_{\mathrm{C}}^\mu\,, \tag{29.43}$$

to a non-local form

$$\boxed{K_{\mathrm{C}}^\mu \equiv m\left(g^{\mu\nu} - \frac{u^\mu(\tau)u^\nu(\tau)}{c^2}\right)\frac{u_\nu(\tau) - u_\nu(\tau - 2\tau_0)}{2\tau_0}\,.} \tag{29.44}$$

The Caldirola model keeps the usual form of the Lorentz force linear in fields and a non-local inertial term is introduced. We keep in mind the difference of the Caldirola approach with the LL approach to improve radiation reaction: the LL equation modifies the field side of the covariant Lorentz force equation, while the Caldirola approach modifies the inertia side. Therefore, these two approaches are non-equivalent.

Even though at first look the simplicity of the Caldirola force Eq. (29.44) is surprising, we soon realize that such a combination of the Newton form of inertial force with the LAD radiation reaction into a single and simple term is possible since these terms contain $\dot{u}$ and $\ddot{u}$ and both forces have to assure $u^2 = c^2$. Expanding Eq. (29.44) in powers of τ_0, we find

$$\frac{u_\nu(\tau) - u_\nu\,(\tau - 2\tau_0)}{2\tau_0} = \dot{u}_\nu - \tau_0\ddot{u}_\nu + \cdots\,. \tag{29.45}$$

This shows in the lowest non-vanishing order the usual inertial force using $u^\nu\dot{u}_\nu = 0$. In the next order in τ_0, the LAD radiation reaction is recovered.

Multiplying Eq. (29.44) from the left by $u_\mu(\tau)$ we see that $u_\mu K_{\mathrm{C}}^\mu$ and it seems that Caldirola solutions are, as desired, constrained to $u^2 = c^2$. Moreover, the inertial term appears not to suffer from the LAD solution difficulty of acausal and/or runaway solutions: the past value of the 4-velocity of the particle determines its present 4-velocity, and the effect of the radiation reaction arises from the inertia nonlocality.

[36] P. Caldirola, "A Relativistic theory of the classical electron," *Riv. Nuovo Cim.* **2N13**, 1 (1979); P. Caldirola, G. Casati and A. Prosperetti, "On the classical theory of the electron," *Il Nuovo Cim.* **43**, 127 (1978).

However, the effect of the introduction of a discrete proper time step in the characterization of particle motion is at this time poorly understood. For example, this approach could be inconsistent with the principles of special relativity. And as we will illustrate, the principles we developed for differential form of dynamics in order to assure $u^2 = c^2$ can be misleading when we deal with non-local dynamical extension of equations of motion. To see that this is true we consider Eq. (29.44) for the field-free $F^{\mu\nu} = 0$ case. A general solution is a periodic function which can be written as

$$u_\nu(\tau) = u_\nu^{(0)} \frac{\sin(\tau\, 2n\pi/2\tau_0 + \phi)}{\sin\phi}\,, \qquad u^{(0)\,2} = c^2\,. \tag{29.46}$$

For $n = 0$, Eq. (29.46) is the desired inertial motion solution. However, the nonlocality of Eq. (29.44) also allows countable infinite set of periodic solutions characterized by $n \neq 0$ with $\phi \to \phi_n \neq 0$. We find

$$u_\nu(\tau)\dot{u}^\nu(\tau) = n\,\frac{\pi c^2}{2\tau_0}\,\frac{\sin(\tau\, 2n\pi/\tau_0 + 2\phi_n)}{\sin\phi_n}\,. \tag{29.47}$$

Only the case $n = 0$ satisfies the constraint $u \cdot \dot{u} = 0$. We thus learn that there is simultaneous presence of a physical solution $n = 0$ along with a countable infinite set $n \neq 0$ of non-physical solutions.

Generalizing this finding we have shown that in order to solve the Caldirola form of radiation reaction we must constrain each proper time step to the one solution satisfying $u \cdot \dot{u} = 0$. Such constraint implies lack of causality at the level of the Caldirola proper time step $\Delta t = 2\tau_0$ time increment. This new difficulty replaces the runaway problem of the LAD radiation reaction, we have passed from two in the case of LAD to infinitely many in the case of Caldirola bounded unphysical solutions. We conclude that the Caldirola form of radiation reaction needs further study and exploration, steps beyond the scope of this book.

29.8 Unsolved radiation reaction

To close this review of efforts to improve the Lorentz force as summarized in Table 29.1, we found that by accounting for the radiation friction effect we are led to the LAD modification of the Lorentz force. This effective dynamical equation contains spurious solutions which can be eliminated at the cost of violating causality. Two ad-hoc improvements (among a few others) of LAD have been described that are opposite to each other in that each of these considers a change in one of the ‘sides’ of the Lorentz force equation.

1) The LL model of radiation friction amounts to a modification of the field-particle force (Lorentz force). However, an appropriate fundamental action that addresses the proposed modification has not been found. This is so since the LL model invokes particle motion that is not conservative, containing a friction effect: the transfer of energy from

Table 29.1 Models of radiation reaction extensions of the Lorentz force

Maxwell-Lorentz	$\mathbf{m\dot{u}}^{\mu} = \mathbf{eF}^{\mu\nu}\mathbf{u}_{\nu}$
LAD[33]	$\mathbf{m\dot{u}}^{\mu} = \mathbf{eF}^{\mu\nu}\mathbf{u}_{\nu} + m\tau_0 \left[g^{\mu\nu} - \frac{u^{\mu}u^{\nu}}{c^2} \right] \ddot{u}_{\nu}\ ,\ \tau_0 = \frac{2}{3}\frac{e^2}{4\pi\epsilon_0 mc^3}$
Landau-Lifshitz[35]	$\mathbf{m\dot{u}}^{\mu} = \mathbf{eF}^{\mu\nu}\mathbf{u}_{\nu} + e\tau_0 \left\{ u^{\gamma}\partial_{\gamma}F^{\mu\delta}u_{\delta} + \frac{e}{m}\left(g^{\mu\gamma} - \frac{u^{\mu}u^{\gamma}}{c^2} \right) F_{\gamma\beta}F^{\beta}_{\delta}u^{\delta} \right\}$
Caldirola[36]	$\mathbf{0} = \mathbf{eF}^{\mu\nu}(\tau)\,\mathbf{u}_{\nu}(\tau) - m\left[g^{\mu\nu} - \frac{u^{\mu}(\tau)u^{\nu}(\tau)}{c^2} \right] \frac{u_{\nu}(\tau) - u_{\nu}(\tau - 2\tau_0)}{2\tau_0}$

the particle to the emitted radiation that is not tracked. These LL non-conservative particle dynamics can therefore be studied only at the level of the proposed equation of motion.

2) For the Caldirola proposed ad-hoc modification of the inertial term an action principle from which the dynamics could be derived has not been obtained. We have seen an infinite set of solutions to the Caldirola radiation reaction Eq. (29.44) and a similar causality problem as is present in the LAD formulation.

The framework we have developed in this book comprises the present paradigm of two distinct interacting dynamical equations characterizing particle dynamics on one hand, and EM-field dynamics on the other. These are two independent physics concepts with some unresolved details as we have seen while exploring the inertial part of the particle action in Sects. 25.2, 25.3, and the EM field energy in Sect. 28.2. We do not yet have a theoretical framework extending SR to critical acceleration domain where there is a closer relation of particle and fields dynamics as we discussed in exercise XI–11 on page 426.

The radiation reaction difficulties we are discussing relate to the absence of the proper understanding of acceleration. This conjecture is supported by an explicit computation showing that at condition of critical acceleration the radiation reaction correction exceeds in magnitude the Lorentz force, see Ref. 29 and Sect. 29.6. What happens when acceleration approaches and exceeds the critical value is a research frontier. We do not actually know how particles and fields behave in the context of strong acceleration. Future progress depends on experimental guidance in regard to particle and field dynamics in the presence of strong acceleration.

Discussion XI-1 – **The acceleration frontier**

Topic: Are there novel research opportunities at the radiation reaction and acceleration physics frontier?

Student: Everyone I spoke with who has studied electromagnetism in depth has misgivings about EM theory because of the radiation reaction problem.

Simplicius: My friends say we must remember quantum physics. Electrons in atoms do not radiate. My friends have suggested that the radiation reaction problem is a classical problem resolved in quantum theory.

Professor: Please ask your friends about cosmic binaries emitting gravitational radiation. We cannot claim today that all classical physics phenomena originate in the quantum physics realm. In many cases such claims are without merit.

Simplicius: Can I tell if radiation is emitted by a quantum, or by a classical electron?

Professor: A quantum transition often produces discrete energy when an excited quasi-stable state transits, emitting a photon. Such a quantum jump has little to do with classical continuous radiation.

Student: Could a quantum transition that bridges a highly excited state with many, many $\hbar$, leading to a much lower level, be similar to classical radiation?

Professor: Performing a quantum calculation, you find that radiative transitions across many $\hbar$ are extremely rare.

Simplicius: Meaning that there is no classical radiation limit for an electron sitting in a quantum orbit? So how do we get synchrotron radiation out of electrons such as those circulating in a macroscopic storage ring?

Professor: That synchrotron electron has such a crazy large number of $\hbar$ that the idea of a quantum state model is unthinkable. We are no longer in a quantum jump condition.

Student: Seeking online I have not found a convincing source of wisdom on how to connect classical and quantum radiation schemes.

Professor: We know how to increase classical radiated power. It grows with acceleration (squared). Furthermore there is no limit to this radiation friction. The stronger the acceleration, the more friction we see as the particle radiates more strongly.

Simplicius: Are you saying the particle comes to rest?

Professor: There is no absolute rest in SR; what the particle does is decelerate towards some inertial motion condition. Since we measure the radiation power in some frame of reference, in that frame of reference the radiation friction diminishes as the reservoir of energy and power emitted diminish.

Simplicius: However, you said that an ultrarelativistic particle can decelerate in an ultra-short distance because the radiation reaction helps!

Professor: Yes, one sees this when solving the radiation reaction improved Lorentz force equations (loc. cit. Ref. 29). There is runaway in the radiation friction effect; the more the particle decelerates by radiation, the more energy is radiated, the more the particle is decelerated, and so on. I hope more physics awaits to be discovered that relates to radiation friction. The old idea has been that EM-force unifies with gravity. Many people have looked for this, Weyl, Kaluza, Klein, Infeld, Einstein, to name a few. Yet no scheme has gained widespread acceptance.

Simplicius: Browsing the web to see how people think about particle motion and gravity phenomena I found that collisions at the CERN large hadron collider (LHC) produce black holes.[37] I found a Wikipedia entry[38] describing how particle physicists had to respond to concerns that high energy particle collisions could end the world.

Student: I would think that the formation of black holes is a pretty radical gravitational effect. I do not know anyone expecting black holes to accompany strong acceleration phenomena. But speaking of experiments, are there other manifestations of strong EM acceleration phenomena offering connection to gravity?

Professor: We believe that in quantum field theory context strong fields equivalent to strong acceleration imply instability, fields decaying into particle pairs. Such instability could imply the need for a modification of the theoretical context. Since strong acceleration appears in closer inspection much like a hidden Planck scale, finding a way to explore inertia response to extreme acceleration conditions could be an experimentally challenging and promising objective which in my opinion will see much attention in coming years. It could be that the study of EM radiation friction could help uncover a relation between GR and EM which many have been seeking for more than a century. Thus to paraphrase those who work at the energy and intensity frontiers, I must say that in the foreseeable future we will also be performing experiments in order to explore the acceleration frontier.

Simplicius: Is this the end of this discussion and the book?

Student: I think the exploration of phenomena related to relativistic particle dynamics in presence of strong fields has just begun its second life. So it is the end of the book…

Professor: …and the beginning of a research program. Thank you all for reading these pages. Let us hope that by the time the next edition of this book is being prepared there will be aside of corrections and corroborations truly new research material that can be included, shining a bright light on the acceleration frontier.

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[37] S. Dimopoulos, G.L. Landsberg, "Black holes at the LHC," *Phys. Rev. Lett.* **87**, 161602 (2001).

[38] *Safety of high-energy particle collision experiments,* https://en.wikipedia.org/wiki/Safety_of_high-energy_particle_collision_experiments.