7 Properties of the Lorentz Coordinate Transformation

7.1 Relativistic addition of velocities

Case of two parallel velocities

An observer in frame S' passes with velocity $\vec{v}$ in the x-direction with respect to another observer located in frame S. An object is seen by the observer in frame S to move at a constant velocity $\vec{u}$, as depicted in Fig. 7.1. What velocity is measured by the observer in S'? In the nonrelativistic case, the usual answer is obtained from the Galilean coordinate transformation Eq. (2.2)

$$\vec{u}\,' = \vec{u} - \vec{v}\ . \tag{7.1}$$

However, this Galilean law of addition of velocities does not respect the requirement that a material body velocity cannot be greater than light velocity.

To obtain the relativistic form of velocity addition we need to remember precisely how velocity relates to position $\vec{r}$ and time t; in frame S we have

$$\vec{u} = \frac{d\vec{r}}{dt}\ , \tag{7.2}$$

where both space and time coordinates are subject to the Lorentz coordinate transformation $\vec{u}\,' = d\vec{r}\,'/dt'$ rather than $d\vec{r}\,'/dt$.

In what follows we orient the coordinate system S' such that the velocity vector $\vec{v}$, Eq. (7.1) always points along the x-axis,

$$\vec{v} \equiv v\hat{i}\ . \tag{7.3}$$

Given this choice the body velocity vector $\vec{u}$ cannot be constrained; it can point in any arbitrary direction.

We first consider the special case of $\vec{u}$ being parallel to $\vec{v}$. To obtain the transformed velocity we employ the Lorentz coordinate transformation in the infinitesimal form, see

J. Rafelski, *Relativity Matters*,
DOI 10.1007/978-3-319-51231-0_7

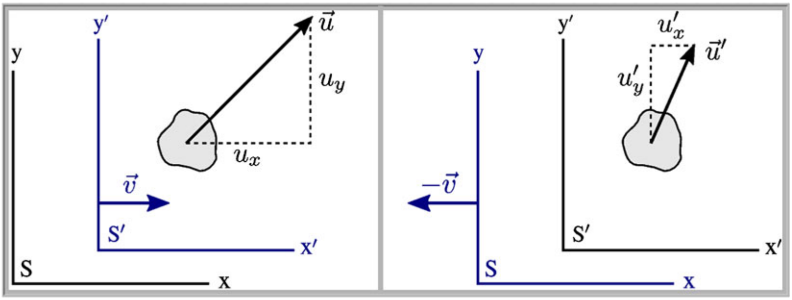

Fig. 7.1 Transformation of velocities: $\vec{u}$ is the velocity in the frame of reference S; $\vec{u}'$ is the velocity in the frame of reference S' that is moving with velocity $\vec{v}$ relative to S. See also Fig. 6.1

Exercise III–3 on page 82

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma}{\gamma}\,\frac{dx - v dt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}\,. \tag{7.4}$$

We recognize $\frac{dx}{dt}$ as u_x, resulting in the relativistic velocity addition relation

$$\boxed{u'_x = \frac{u_x - v}{1 - u_x v/c^2}\,, \qquad u'_y = u'_z = 0\,, \qquad \vec{u} \parallel \vec{v} = v\hat{i}\,.} \tag{7.5}$$

The Galilean relation Eq. (7.1) is corrected by the denominator, which enforces the limit on maximum allowed speed.

Exercise III–5: Checking relativistic velocity addition, case $\vec{u} \parallel \vec{v}$

Consider the value of u'_x when $u = 0.9c$ and $v = \pm 0.8c$

Solution

We will be using the SR result Eq. (7.5) and comparing to the Galilean coordinate transformation Eq. (7.1). For $v = -0.8$, the Galilean coordinate transformation predicts a speed that is 70 % higher than the speed of light. However the SR result is

$$u'_x = \frac{0.9c + 0.8c}{1 + 0.9 \cdot 0.8} = \frac{1.7c}{1.72} = 0.9884c\,,$$

which comes close to speed of light but clearly respects the limit.

For the case $v = 0.8$ the Galilean coordinate transformation result Eq. (7.1) is that the particle has become nearly nonrelativistic moving with 10 % of c. However, the situation is different for the SR case Eq. (7.5); we find

$$u'_x = \frac{0.9c - 0.8c}{1 - 0.9 \cdot 0.8} = \frac{0.1c}{0.28} = 0.36c\,.$$

Thus the motion remains closer to relativistic as compared to the Galilean coordinate transformation case.

End III–5: Checking relativistic velocity addition, case $\vec{u} \parallel \vec{v}$

Exercise III–6: Relativistic relative approach speed

An observer on Earth sees two rockets approach each other with equal and opposite velocities $u_\pm = \pm|u| = \pm 0.6c$. For this observer, the distance between the rockets diminishes according to relative speed $u_+ - u_- = 1.2c$. Thus the distance between the ships diminishes with a speed that exceeds the speed of light! Does this situation violate relativity? Consider an observer riding along in one of the rockets. What velocity of the other rocket does he report?

Solution

Relativity only requires that c be the maximum speed for light and for physical bodies. The relative speed at which coordinate separation between the rockets diminishes as recorded by a third arbitrary observer has nothing to do with the relative speed of two bodies measured by an observer riding one of the rockets. The whole point of the (special) theory of *relativity* is that there is no physical relevance to such an arbitrary third observer.

For an observer riding in one of the rockets, the relative velocity corresponds to the velocity of the other rocket as observed from this rocket. To find this velocity we must appropriately add individual velocities. For simplicity we consider the observer traveling at u_-. The transformation of the velocity u_+ of the other rocket into this frame is

1 $$u'_+ = \frac{u_+ - u_-}{1 - u_+ u_-/c^2} = \frac{2|u|}{1 + |u|^2/c^2} \,.$$

With $|u| = 0.6\,c$, we find that the relative velocity given by Eq. 1 is less than the speed of light: $u'_+ = 15c/17 = 0.88c$.

End III–6: Relativistic relative approach speed

Exercise III–7: Shuttlecraft rescue

Star Wars scene: The Insurrection base reports to its shuttle traveling with $u_\mathrm{s} = 0.3c$ that it is being chased by a rocket following it with $u_\mathrm{r} = 0.6c$, and their rescue spaceship is approaching from the opposite direction traveling with the velocity $u_\mathrm{S} = -0.6c$; all velocities are parallel. What relative speeds are registered in the shuttlecraft? Are these consistent with exercise III–6?

Solution

Let us denote by subscript 'S' = quantities in the spaceship frame and by subscript 's' = quantities in the shuttle frame. We boost to the shuttle frame. The spaceship approaches the shuttle with relative velocity

1 $$v_{\mathrm{Ss}} = \frac{u_\mathrm{S} - u_\mathrm{s}}{1 - u_\mathrm{S} u_\mathrm{s}/c^2} = \frac{-0.9c}{1 + 0.18} = -0.763c \,.$$

However, the rocket is gaining on the shuttle with

2 $$v_{\rm rs} = \frac{u_{\rm r} - u_{\rm s}}{1 - u_{\rm r} u_{\rm s}/c^2} = \frac{0.3c}{1 - 0.18} = 0.366c\ ,$$

so rescue is possible but not assured. We check if the shuttlecraft crew computations are correct by boosting with $v = v_{\rm rs} = -0.366c$ from shuttle frame to the rocket frame of reference. This gives the rocket observed velocity of the spaceship, independent of what the shuttle is doing:

3 $$v_{\rm Sr} = \frac{u_{\rm Ss} - v_{\rm rs}}{1 - u_{\rm Ss} v_{\rm rs}/c^2} = \frac{-0.763c - 0.366c}{1 + 0.763 \cdot 0.366} = \frac{-1.129c}{1 + 1.28} = -0.88c\ .$$

This is the expected result

4 $$v_{\rm Sr} = \frac{u_{\rm S} - u_{\rm r}}{1 - u_{\rm r} u_{\rm S}/c^2} = \frac{-0.6c - 0.6c}{1 + 0.36} = -0.88c\ ,$$

as we already evaluated in exercise III–6. It is rather complicated to show the above result analytically. This becomes much easier once additional tools are developed and we return therefore to this problem once more in exercise III–18 on page 112.

End III–7: Shuttlecraft rescue

Case of two arbitrary velocities

In the general case where the direction of $\vec{u}$ is arbitrary it is helpful to decompose $\vec{u}$ into its component u_x parallel to the relative velocity $\vec{v}$, Eq. (7.3), and the orthogonal components u_y and u_z:

$$\begin{aligned} \vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \vec{u} &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}\ . \end{aligned} \tag{7.6}$$

Likewise we decompose $\vec{u}\,'$ observed in frame S', remembering that the x'- and x-axes are parallel:

$$\vec{u}\,' = \frac{dx'}{dt'}\hat{i} + \frac{dy'}{dt'}\hat{j} + \frac{dz'}{dt'}\hat{k} = u'_x\hat{i} + u'_y\hat{j} + u'_z\hat{k}\ . \tag{7.7}$$

Even though $dy' = dy$ and $dz' = dz$, we see dt' in the denominator and thus all three velocity components transform; *i.e.*, both the parallel *and* orthogonal components of the velocity are modified when we transform coordinates. Carrying this out we note that the first form of Eq. (7.5) remains valid while for u'_y and u'_z we find

$$u'_y = \frac{dy'}{dt'} = \frac{dy\sqrt{1-(v/c)^2}}{dt - \frac{v}{c^2}dx} = \frac{u_y\sqrt{1-(v/c)^2}}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{u_y\sqrt{1-(v/c)^2}}{1 - \frac{vu_x}{c^2}}\,,$$
$$u'_z = \frac{dz'}{dt'} = \frac{u_z\sqrt{1-(v/c)^2}}{1 - \frac{vu_x}{c^2}}\,. \tag{7.8}$$

In the last step we are recognizing $dx/dt = u_x$.

To summarize, for $\vec{v}$ along x-axis, see Eq. (7.3), the relativistic velocity addition equations are for parallel and orthogonal addition, respectively

$$\boxed{u'_x = \frac{u_x - v}{1 - u_x v/c^2}\,, \qquad \vec{v} = v\hat{i}\,,} \tag{7.9a}$$

$$\boxed{u'_y = \frac{u_y}{1 - u_x v/c^2}\sqrt{1 - v^2/c^2}\,,} \quad \boxed{u'_z = \frac{u_z}{1 - u_x v/c^2}\sqrt{1 - v^2/c^2}\,.} \tag{7.9b}$$

Exercise III–8: Relative rocket motion

An observer on Earth sees two rockets traveling with the velocities $\vec{u}^{\,\pm}$. What is the relative speed of the rockets?

Solution

We employ the result Eq. (7.9a). We align the coordinate x-axis with vector $\vec{u}^{\,-}$ and position the observer on this rocket '$-$'. Upon the Lorentz coordinate transformation into rocket '$-$'; that is, $v = u_x^-$ Eq. (7.9a), we find the relative velocity vector is $\vec{u}^{\,r} = \vec{u}^{\,\prime +}$

$$u^r_x = \frac{u^+_x - u^-_x}{1 - u^+_x u^-_x/c^2}\,,$$
1 $$u^r_y = \frac{u^+_y}{1 - u^+_x u^-_x/c^2}\sqrt{1 - (u^-_x/c)^2}\,,$$
$$u^r_z = \frac{u^+_z}{1 - u^+_x u^-_x/c^2}\sqrt{1 - (u^-_x/c)2}\,.$$

This is the relative velocity vector expressed in a coordinate system of which the orientation of the x-axis is parallel to the laboratory rocket '$-$' velocity vector. Thus u^r_x is what the observer in rocket '$-$' sees as the velocity of rocket '$+$' in approach, while u^r_y and u^r_z are components that are normal to the approach axis.

End III–8: Relative rocket motion

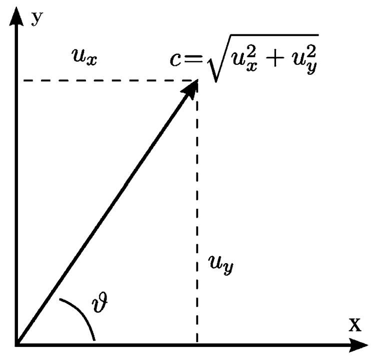

Fig. 7.2 Photon moving in S at angle ϑ to the x-axis. See exercise III–9

Exercise III–9: Relativistic forward projection

We consider two sources of photons S and S'. S is at rest with respect to the laboratory observer while S' moves with speed $v' = 0.6c$ relative to S along the x-axis. In the rest-frame of both systems one measures an angle of emission $\vartheta = 60°$ between the direction of movement of the photon and the direction of movement of S' as depicted in Fig. 7.2. What is the angle of emission that the laboratory observer reports for the photon originating in the moving source S'?

Solution

Geometry: the source S' moves along the x-axis, which is chosen to be parallel to the x'-axis; the movement of the photon is confined to the xy-plane, Fig. 7.2. The y and y' axes are also parallel.

The velocity of the photon in system S is therefore

1 $$\vec{v} = c\cos 60°\hat{i} + c\sin 60°\hat{j} = u_x\hat{i} + u_y\hat{j}\,;$$

that is, $u_x = 0.5c$, $u_y = 0.866c$.

We calculate the component u'_x for the photon emitted by moving source using the addition of velocity theorem,

2 $$u'_x = \frac{u_x + v'}{1 - u_x v'/c^2}\,.$$

Inserting $u_x = 0.5c$ and $v' = 0.6c$, one obtains $u'_x = 0.846c$.

To obtain u'_y we consider the requirement

3 $$c = \sqrt{u'^2_x + u'^2_y}\,,$$

see exercise III–11 on page 96. Therefore

4 $$u'_y = \sqrt{c^2 - u'^2_x} = \sqrt{1 - 0.846^2}\,c = 0.533c\,.$$

We recall that this transverse to motion component of velocity of light is changed solely due to the transformation of laboratory time.

For the angle ϑ' between the direction of the photon and the x-axis of the observer at rest in system S', we then have

5 $$\tan\vartheta' = \frac{u'_y}{u'_x} = \frac{0.533c}{0.846c} = 0.630\,,$$

and therefore: $\vartheta' = 32.2°$.

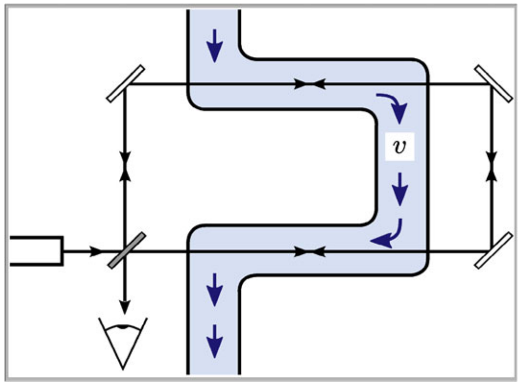

Fig. 7.3 Illustration of a table-top interference experiment to measure the Fresnel's drag coefficient: one observes the change in the interference between light carried with and against the (changing) flow velocity v of a fluid, see exercise III–10

We performed this computation for photons, but the principles apply equally to the emission of all relativistic particles. Therefore this exercise shows an important phenomenon often observed in particle physics experiments, namely that particles are relativistically focused pointing forward along the direction of motion of the source. Such a situation arises for example when a fast cosmic particle hits the upper atmosphere. Particles are produced in a rest-frame intrinsic to the reaction which moves at a high speed in the same direction as the primary cosmic ray. Our computation explains why the secondary particles observed in the Earth frame are focused along the direction of the primary cosmic particle. The study of light aberration leads to a related and more general result, see exercise III–13 on page 103. A complete discussion of the relativistic focusing effect is presented in exercise VI–6 on page 199.

End III–9: Relativistic forward projection

Exercise III–10: Fresnel drag coefficient

The speed of light in a medium with a refraction index $n > 1$ is known to be $\tilde{c} < c$. Furthermore, $\tilde{c}$ is dependent on the velocity v of the medium relative to a (laboratory) observer. Obtain the lowest correction in v: $\tilde{c}(v) = \tilde{c}_0 + v f(n)$ as measured by the laboratory observer, where f is the Fresnel drag coefficient (Fresnel,[2] 1818).

Solution

The speed of light is reduced by the index of refraction n in a stationary medium. It is known that the Maxwell equations allow one to deduce the relation

1 $$\tilde{c}_0 = \frac{c}{n} \,.$$

We wish to understand what happens when the fluid medium as shown in Fig. 7.3 is set in motion where v is medium velocity relative to the lab frame, and the beam of light is propagating parallel or antiparallel to v.

[2]Augustin-Jean Fresnel (1788–1827), luminary French physicist and founder of the theory of wave optics.

There is a fluid comoving observer $S(v)$, moving with $\pm v$ with respect to the laboratory observer S, for whom the light can be traveling through a stationary medium in each of the arms of interferometer seen in Fig. 7.3. To find the speed of light $\tilde{c}(v)$ as measured by the observer in the lab frame, we can thus use the addition of velocities theorem, Eq. (7.9a):

2 $$\tilde{c}(v) = \frac{\tilde{c}_0 + v}{1 + \tilde{c}_0 v/c^2} = \frac{c/n + v}{1 + v/(nc)} = \left(\frac{c}{n}\right)\left(\frac{1 + nv/c}{1 + v/(nc)}\right) .$$

Any velocity that can be achieved in a laboratory satisfies $v \ll c$, thus we can expand:

3 $$\begin{aligned}\tilde{c}(v) &\simeq \frac{c}{n}\left(1 + \frac{nv}{c}\right)\left(1 - \frac{v}{nc}\dots\right) \simeq \frac{c}{n}\left(1 + \frac{nv}{c}\left(1 - \frac{1}{n^2}\right)\right)\\ &\simeq \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right) .\end{aligned}$$

We see that Eq. 3 applies to both cases, when v is positive and negative.

The Fresnel drag coefficient is therefore

4 $$\boxed{f = 1 - \frac{1}{n^2} .}$$

For the special case of a vacuum ($n = 1$), we find that $f = 0$, confirming the drag limit in the vacuum where the speed of light is independent of the motion of the observer.

Further reading: The first experimental demonstration of Fresnel drag was carried out by Fizeau in 1851, and much experimental and theoretical work followed in the second half of 19th century. Fresnel drag was a cornerstone precursor to the development of SR; the concept of æther motion does not enter present discussion, a topic that has been much a part of Fresnel drag experiments prior to Einstein's development of SR. The solution of Fresnel drag using SR principles is due to Max von Laue.[3]

End III–10: Fresnel drag coefficient

Exercise III–11: Maximum speed for arbitrary $\vec{u}$

An object moves at a velocity of magnitude u ($u < c$) in an arbitrary direction as observed in frame S. Show that the velocity u' observed in frame S' moving relative to S at velocity v along the x-axis is less than the speed of light, $u' < c$.

Solution

We want to prove the relationship:

1 $$u'^2 < c^2 .$$

We can rewrite u'^2 in components

2 $$u'^2 = u_x'^2 + u_y'^2 + u_z'^2 ,$$

[3] Max von Laue, "Die Mitführung des Lichtes durch bewegte Körper nach dem Relativitätsprinzip," (The Entrainment of Light by Moving Bodies according to Principle of Relativity) *Annalen der Physik* **328**, 989–990 (1907). Max von Laue (1879–1960), won the Nobel prize in 1914 "for his discovery of the diffraction of X-rays by crystals". A man with passion for truth in life as in physics.

where u'_x is the component of the velocity parallel to the x- and x'-axes, and the u'_y and u'_z are the two transverse components. Using Eq. (7.9a) and Eq. (7.9b) to express these components in terms of u and v we obtain, after some simple algebraic manipulations

3
$$u'^2 = \frac{(u_x - v)^2 + (u_y^2 + u_z^2)(1 - v^2/c^2)}{(1 - vu_x/c^2)^2} = \frac{u^2 + v^2 - 2u_x v - (u_y^2 + u_z^2)v^2/c^2}{(1 - vu_x/c^2)^2}$$
$$= \frac{(u^2 - c^2)(1 - v^2/c^2) + c^2(1 - v^2/c^2) - 2u_x v + v^2 + u_x^2 v^2/c^2}{(1 - vu_x/c^2)^2} \,,$$

where $u^2 = u_x^2 + u_y^2 + u_z^2 \equiv \vec{u}\,'^2$. The v^2 terms cancel and we obtain

4
$$\boxed{\vec{u}\,'^2 = c^2 \left(1 - \frac{(1 - \vec{u}^{\,2}/c^2)(1 - \vec{v}^{\,2}/c^2)}{(1 - \vec{v} \cdot \vec{u}/c^2)^2}\right)} \,,$$

where for $\vec{v} = v\hat{i}$ we used $v^2 = \vec{v}^{\,2}$ and $vu_x = \vec{v} \cdot \vec{u}$. Equation 4 presents the addition theorem of velocities for the speed, when the direction of motion is not aligned with the direction of LT.

Inspecting the large brackets in Eq. 4, we see that a manifestly positive term is always subtracted from unity. This term becomes arbitrarily small when either of the speeds, u, v approaches speed of light, yet always

5 $$u'^2 < c^2 \,.$$

Thus we confirm that irrespective of the direction of motion the addition theorem of velocities derived from the Lorentz coordinate transformation is consistent with c being the universal and highest speed which cannot be exceeded.

It is of interest to consider the special case of a light ray with $u = c$ observed in the moving frame S'. We find

6 $$u'^2 = c^2, \qquad \text{for} \quad u = c \,.$$

This result is of use in Sect. 7.2.

Another cross-check of interest is verification of the result Eq. (7.5), corresponding to the case $u_y = u_z = 0$; that is, $u^2 = u_x^2$. We form common denominator in Eq. 4 to obtain

7
$$u'^2 = c^2 \frac{1 - 2vu_x/c^2 + v^2u_x^2/c^4 - 1 - v^2u_x^2/c^4 + u_x^2/c^2 + v^2/c^2}{(1 - vu_x/c^2)^2}$$
$$= \frac{(u_x - v)^2}{(1 - vu_x/c^2)^2} \,,$$

which we recognize as the square of Eq. (7.5). The cross-check Eq. 7 confirms the principle results of this exercise, Eq. 4.

End III–11: Maximum speed for arbitrary $\vec{u}$

Exercise III–12: Relativistic speed – velocity addition theorem

State the magnitude of the speed seen in Eq. 4 in exercise III–11 in vector notation showing the nonrelativistic leading term explicitly.

Solution

We recall that in Eq. 4 in exercise III–11 the velocity v has a single component which we now denote $\vec{\beta}_2 = -(v/c, 0, 0)$. We call now $\vec{\beta}_1 = (u_x/c, u_y/c, u_z/c)$. In this notation Eq. 4 in exercise III–11 reads

$$\mathbf{1} \quad \beta^2 \equiv u'^2/c^2 = \left(1 - \frac{(1-\vec{\beta}_1^{\,2})(1-\vec{\beta}_2^{\,2})}{(1+\vec{\beta}_1\cdot\vec{\beta}_2)^2}\right).$$

We rewrite Eq. 1 with common denominator to obtain

$$\mathbf{2} \quad \beta^2 = \frac{1 + 2\vec{\beta}_1\cdot\vec{\beta}_2 + (\vec{\beta}_1\cdot\vec{\beta}_2)^2 - 1 + \vec{\beta}_1^{\,2} + \vec{\beta}_2^{\,2} - \vec{\beta}_1^{\,2}\vec{\beta}_2^{\,2}}{(1+\vec{\beta}_1\cdot\vec{\beta}_2)^2}.$$

Noting $(\vec{\beta}_1 \times \vec{\beta}_2)^2 = \vec{\beta}_1^2\vec{\beta}_2^2 - (\vec{\beta}_1\cdot\vec{\beta}_2)^2$ in the numerator we find

$$\mathbf{3} \quad \beta^2 = \frac{(\vec{\beta}_1+\vec{\beta}_2)^2 - (\vec{\beta}_1\times\vec{\beta}_2)^2}{(1+\vec{\beta}_1\cdot\vec{\beta}_2)^2}, \quad \text{or} \quad \boxed{\vec{v}^{\,2} = \frac{(\vec{v}_1+\vec{v}_2)^2 - (\vec{v}_1\times\vec{v}_2)^2/c^2}{(1+\vec{v}_1\cdot\vec{v}_2/c^2)^2}.}$$

Equation (3) displays a counterintuitive contribution, a term normal to both added velocity vectors which vanishes when $\vec{v}_1 \parallel \vec{v}_2$. As a cross-check we note for $\vec{v}_1 \times \vec{v}_2 \neq 0$ agreement with Eq. 7 in exercise III–11, of which Eq. 3 is a generalization. In the limit $c \to \infty$ only the first nonrelativistic term seen in the numerator remains.

End III–12: Relativistic speed – velocity addition theorem

7.2 Aberration of light

We consider a light ray originating in a distant star, see Fig. 7.4, where for the observer marked with $\vec{v}$ the light arrives more horizontally as discussed before in Sect. 1.3. Thus the inferred star location has moved by the aberration angle α from its position.

The following discussion is addressing the observation of well focused light-rays and not (spherical) plane wave light. This approach sidesteps arguments about modification of aberration effect by waves. This picture of the experimental situation is accurate since the light emitted by a star consists of an incoherent flux of photons produced in independent atomic processes. When we look at a star we see individual photons emitted across the entire source that make it at the same time, given the material such as telescope optics, into our eye. The accidental coherence of these photons is difficult to discern, but it is of use in evaluating the size of the random light source using the so called HBT[4] method.

[4]R. Hanbury Brown and R.Q. Twiss (1956). "A Test of a New Type of Stellar Interferometer on Sirius" *Nature* **178**, 1046 (1956). doi:10.1038/1781046a0.

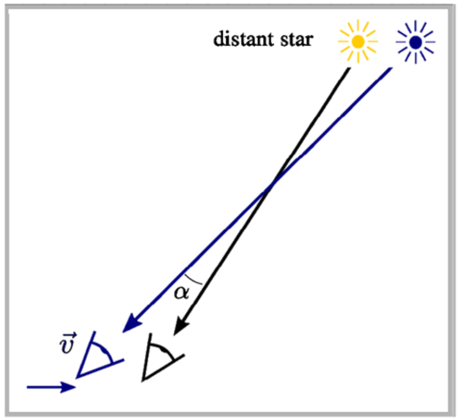

Fig. 7.4 Aberration angle α of light from a distant star: Comparison of the line of sight for an observer at rest and for another observer traveling at a constant velocity $\vec{v}$ relative to the line of sight

In a light-source rest-frame S we choose the coordinate axis pointing away from the Earth, hence the velocity components of the light ray are negative pointing 'down' at the Earth

$$u_x = -c\sin\theta\cos\phi, \qquad u_y = -c\sin\theta\sin\phi, \qquad u_z = -c\cos\theta\,. \tag{7.10}$$

The angle θ spans from the coordinate z-axis to the line connecting to the observer on Earth. The orientation of the Earthbound observer's (moving) S' coordinate system can be for convenience and without loss of generality chosen such that the relative motion velocity vector $\vec{v}$ points along one of the coordinate axes, and the natural choice is the z'-direction which we align with the z direction in S.

We apply the velocity addition theorem, Eq. (7.9a), where the boost is now along the z- and not the x-axis. The light ray velocity in the Earth's rest-frame of the observer S' becomes:

$$\begin{aligned}
u'_x &= \frac{u_x}{\gamma(1 - vu_z/c^2)} = -\frac{c\,\sin\theta\,\cos\phi}{\gamma(1 + (v/c)\cos\theta)} \equiv -c\sin\theta'\cos\phi'\,,\\
u'_y &= \frac{u_y}{\gamma(1 - vu_z/c^2)} = -\frac{c\,\sin\theta\,\sin\phi}{\gamma(1 + (v/c)\cos\theta)} \equiv -c\sin\theta'\sin\phi'\,,\\
u'_z &= \frac{u_z - v}{1 - vu_z/c^2} = -\frac{c\,\cos\theta + v}{1 + (v/c)\cos\theta} \equiv -c\cos\theta'\,,
\end{aligned} \tag{7.11}$$

where as usual $\gamma = 1/\sqrt{1 - v^2/c^2}$; it is easy to verify $u'^2_x/c^2 + u'^2_y/c^2 + u'^2_z/c^2 = 1$ as in exercise III–11. On the right-hand side in Eq. (7.11) we have introduced the observation angles θ', ϕ' that an observer on Earth reports for the location of the observer S. Thus Eq. (7.11) establishes a velocity dependent relation between the two sets of angles θ, ϕ and θ', ϕ' for the line of sight that connects the two systems: the source S and moving (by choice/definition) observer S'. We keep in mind that the relative velocity vector $\vec{v}$ defines the direction of both z and z' coordinate axis, hence

$$v \equiv |\vec{v}| \equiv v_{z'} \equiv -v_z\,. \tag{7.12}$$

Comparing the two first expressions in Eq. (7.11) we recognize that the azimuthal aberration vanishes,

$$\phi = \phi' \, . \tag{7.13}$$

However, according to the last expression in Eq. (7.11) the altitude aberration

$$\alpha \equiv \theta - \theta' \tag{7.14}$$

does not vanish:

$$\boxed{\cos\theta' \equiv \frac{\cos\theta + (v/c)}{1 + (v/c)\cos\theta} \simeq \cos\theta + \frac{v}{c}\sin^2\theta - \left(\frac{v}{c}\right)^2 \cos^2\theta(1 - \cos\theta) + \cdots \, .} \tag{7.15}$$

We note that since by definition $v > 0$, $\cos\theta' > \cos\theta$ and thus $\theta' < \theta$ for any altitude angle $\theta \in \{0, \pi/2\}$; see Fig. 7.4 on page 99. For $\theta \to 0$ the altitude aberration vanishes; this is the case of pure 'radial' motion; that is, motion along the line of sight. We next check the relativistic consistency of Eq. (7.15). Inverting Eq. (7.15) to obtain $\cos\theta$ we find

$$\cos\theta = \frac{\cos\theta' - (v/c)}{1 - (v/c)\cos\theta'} \, . \tag{7.16}$$

The altitude aberration expressions Eq. (7.14), Eq. (7.15), and Eq. (7.16) show the equivalence between the two observers S and S' with respect to

$$\alpha \leftrightarrow -\alpha, \quad v \leftrightarrow -v, \quad \theta \leftrightarrow \theta' \, . \tag{7.17}$$

The relativistically consistent expression Eq. (7.16) allows the computation in terms of the Earth observed altitude angle θ' the true altitude θ provided that we know the relative velocity vector $\vec{v}$, Eq. (7.12).

Equation (7.16) demonstrates that the effect of aberration for an observer S is analogous to that for the observer S'. The principles of special relativity are fully accounted for. We can go one step further: by the nature of the problem the speed of light must enter into the answer, and by dimensional considerations the effect has thus to involve the ratio of the relative speed of the light source with the Earth, v, with the speed of light c. Since the Galilean limit has to arise, we knew and expected that the aberration effect begins with the linear power v/c. In order to assure symmetry of the result between observers on Earth and on the Star, as described by Eq. (7.17), the functional dependence we found applying the Lorentz coordinate transformation, Eq. (7.15), is uniquely defined.

We now discuss the magnitude of the effects we expect. Considering the range of values $v/c < \mathcal{O}(10^{-3})$ we can neglect quadratic terms in the nonrelativistic expansion shown in Eq. (7.15). In the small aberration angle limit we use

$$\cos\theta' = \cos(\theta - \alpha) = \cos\alpha\cos\theta + \sin\alpha\sin\theta \simeq \cos\theta + \alpha\sin\theta \, . \tag{7.18}$$

The linear term in the nonrelativistic expansion shown in Eq. (7.15) provides

$$\alpha = \frac{v_\perp}{c}, \qquad v_\perp = v \sin\theta \, . \tag{7.19}$$

This is the final aberration result: the relative speed transverse to the line of sight fixes the aberration angle α. The radial (along line of sight) speed $v_r = v\cos\theta$ does not influence the magnitude of aberration. Naturally, the transverse speed can contain aside of the orbital motion of the Earth around the Sun also a transverse component inherent in the star motion.

For the Earthbound observer S', the velocity vector $\vec{v}$ includes the Sun's relative motion $\vec{v}_\odot$ with respect to the observed star, modulated by the annual rhythm of Earth's orbital velocity $\vec{v}_\oplus$ around the Sun, and daily rotation velocity $\vec{v}_{\rm rot}$:

$$\vec{v} = \vec{v}_\odot + \vec{v}_\oplus + \vec{v}_{\rm rot} \, . \tag{7.20}$$

We drop $\vec{v}_{\rm rot}$ ($v_{\rm rot} \simeq 0.46\,$km/s) from any further discussion; this is possible considering that Earth's orbital speed $\vec{v}_\oplus \simeq 30\,$km/s is 67 times greater. Relative star speeds $v_\odot$ range from essentially vanishing to a few times greater than the orbital speed. However, the nearby stars that are easily observed move along with the Earth around the galactic center and thus in general the dominant transverse component in the motion is the orbital velocity $\vec{v}_\oplus$.

Today we know that the London zenith star, Gamma Draconis, is approaching Earth with the radial speed $28.19 \pm 0.36\,$km/s, nearly the same as $v_\oplus = 30\,$km/s. However, the proper motion (angle expressed transverse speed) of Gamma Draconis is $-8.48\,$mas/yr (milli-arc-sec/year). Therefore the observed aberration motion $\pm 20\,$as/yr of Gamma Draconis, see Sect. 1.3, is an effect 2000 times larger, must be thus dominated by the Earth motion around the Sun.

We evaluate α according to Eq. (7.19) assuming that orbital speed is $v_\perp$, hence $v_\perp/c = \pm 10^{-4}$. While the Earth makes a full orbit around the Sun, with orbital speed $v_\oplus = 30\,$km/s, the Earth velocity vector $\vec{v}$ produces an image of the Earth's motion in the alteration angle[5] $\delta\alpha(t) \in (-10^{-4}, 10^{-4}) = (-20'', +20'')$, which is the apparent zenith star circular motion observed by Molyneux and Bradley and interpreted in this way by Bradley.

Discussion III-2 – **Light aberration and æther drag**

Topic: Aberration is a topic recurrent in the context of æther drag theories and the "relativity must be wrong" argument; *i.e.*, it is easy to argue that the aberration effect is a measurement of the absolute speed of the æther near the Earth. Is that really the case?

Simplicius: I saw a book where the author argues that since one measures absolute speed of Earth by observing fixed star periodic aberration, one actually observes the æther motion around the Sun.

[5] We recall that $1\,$arc min $= [(2\pi)/(60 \times 360)]\,$rad.

Student: Why should the measurement of Earth orbital motion create this æther drag problem? Plenty of good people measure the relative speed of the Earth to this and to that. The emphasis is on 'relative'; here, relative to the Sun.

Simplicius: But the point the author makes is that by looking at many fixed stars distributed uniformly in the sky we measure exclusively the Earth's orbital speed around the Sun. The author concludes that the measurement is that of the motion of the material æther that accompanies the Earth in the motion around the Sun.

Student: I read on Wikipedia that the aberration effect had a big influence on the development of the understanding leading to the invention of relativity. Wikipedia points out that the hypothesis of the motion of the material æther could not account for the effect of aberration – there is even a graphic simulation to explain why. But it was a winding road so it is likely that some of these nearly 200 year old ideas are rediscovered – without the arguments that led to their demise.

Simplicius: I also looked at Wikipedia; that is a very theoretical web page on "aberration of light". What I see in the book I was reading are lots of experimental results addressing the aberration of fixed stars placed in all directions in the sky. When the aberration data is analyzed, the author finds the orbital speed of the Earth in each case, so he attributes the effect to the motion of the dragged æther. I am more inclined to believe the experiment. A theory could be wrong.

Professor: The stars we can observe easily are in general located nearby in our Galaxy – at most a few 100 ly away. These nearby stars move along with us in the Milky Way, and do this in a manner that renders the aberration effect to be often mainly caused by the motion of the Earth around the Sun with the speed of 30 km/s. Stated in technical terms, for many nearby stars the motion transverse to the line of sight is dominated by the Earth's orbital motion around the Sun.

Simplicius: What you say also means that there should be some stars for which the analysis would not give the Earth speed as 30 km/s...

Professor: ...and you wonder why this book does not mention that? I see also in my own research domain that good people are inclined to reject measurements that contradict their view of the World.

Student: I see this all the time when grading labs: my students only retain results they know fit what they should find.

Professor: There are many people who look at experimental data selectively. Both by picking from the results they obtain, and by making the experimental data they do not like have a large error. I see where these habits come from: I wonder what would happen if a student in a lab report presented a result suggesting Newton or Hooke was wrong; can she expect a good grade? I imagine students 'learn' in their introductory labs to pick out 'good' data.

Simplicius: I would never do that, would I ...? So the table I saw was cherry-picked data? And the measurements that did not fit were ignored? I can see now how to begin to argue with my friend who loaned me that book. What other advice do you have?

Student: I would focus on experimental issues to avoid protracted argument and doubt. Argue one should consider a series of precise annual rhythm aberration measurements choosing stars randomly. Soon one finds cases that contradict æther drag. Today plenty of data is out there. So this can be done without any additional experimental effort.

Simplicius: The book shows so many data points ...

Student: ... you do not need to find an equal number of opposite examples. One case that strongly contradicts a hypothesis is enough.

Exercise III–13: Ultrarelativistic aberration of light

Calculate the ultrarelativistic aberration of the observation angle θ of a 'star' for an observer moving with velocity $\vec{v}$ with respect to the fixed star, see Sect. 7.2. Interpret the result for the case of a relativistic source emitting radiation observed in laboratory.

Solution

The aberration formula Eq. (7.15) is not suitable for evaluating the aberration when within measurement error we have $v/c \to 1$. The result suggests that all such objects appear in the zenith. This is clearly not right. The reason is that we lost 'relativistic' sensitivity when deriving Eq. (7.15).

An equivalent form of Eq. (7.15) is obtained using Eq. (7.13) in one of the first two expressions in Eq. (7.11)

$$\boxed{\sin\theta' = \frac{\sin\theta}{\gamma(1+(v/c)\cos\theta)}\,.} \tag{1}$$

Adding in squares of the left-hand sides of Eq. (7.15) and Eq. 1 we show $\sin^2\theta' + \cos^2\theta' = 1$. This verifies the consistency of Eq. (7.15) with Eq. 1. Taking the ratio of Eq. (7.15) with Eq. 1 we obtain

$$\cot\theta' = \frac{\gamma(\cos\theta + (v/c))}{\sin\theta} = \gamma\cot\theta + \frac{\gamma v/c}{\sin\theta}\,. \tag{2}$$

Both Eq. 1 and Eq. 2 are useful for the case $v/c \to 1$, since we now have the explicit value of $\gamma > 1$ available.

Even though we do not have a relativistic shooting star on our horizon, the results presented can be adapted to describe the case of light emitted by relativistic charged particles. However, it is the particle and not the observer (the laboratory in which the experiment is performed) that is moving. We thus need to change to the emitting frame of reference: we perform the transformation shown in Eq. (7.17): $\theta \leftrightarrow \theta'$ and $v \to -v$.

Now θ' is the angle between the emitted radiation and the direction of motion in the frame of reference of the charged particle moving with speed v with respect to the laboratory, and θ is the observation angle in the rest-frame of the laboratory. The two equivalent results are

$$\boxed{\sin\theta = \frac{\sin\theta'}{\gamma(1-(v/c)\cos\theta')}\,, \qquad \cot\theta = \gamma\cot\theta' - \frac{\gamma v/c}{\sin\theta'}\,.} \tag{3}$$

To see how this relation works, take $\theta' = 45^\circ$ such that $\sin\theta' = \cos\theta' = 1/\sqrt{2}$ and assume that the speed v is sufficiently ultrarelativistic to justify use of $v/c = 1$. Hence we find $\sin\theta \simeq \theta = 2.4/\gamma$, thus for $\gamma = 100$ the angle of emission in the lab is $\theta = 0.024 \times 360^\circ/(2\pi) = 1.4^\circ$. We recognize that the radiation is observed 'focused' into a forward cone (small θ) for large γ, even if in the emitter frame of reference it is emitted more uniformly in all directions; *i.e.*, uniformly as function of θ'.

End III–13: Ultrarelativistic aberration of light

7.3 Invariance of proper time

We compare

$$s^2 = c^2t^2 - x^2 - y^2 - z^2, \quad \text{with}$$
$$s'^{\,2} = c^2t'^2 - x'^2 - y'^2 - z'^2\,, \tag{7.21}$$

and equivalently

$$\Delta s^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2, \quad \text{with}$$
$$\Delta s'^{\,2} = c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2\,. \tag{7.22}$$

It suffices to consider only the LT in the x-direction. If other directions need to be transformed, we can always re-orient our coordinate system so that the x-axis points in the direction of the LT transformation. Therefore in what follows $y^2 + z^2 = y'^2 + z'^2$ in Eq. (7.21), and similarly for the difference between two events, Eq. (7.22).

We employ the LT Eq. (6.19) and Eq. (6.21) to find

$$\begin{aligned} s'^2 &= c^2t'^2 - x'^2 - y'^2 - z'^2 \\ &= \gamma^2(ct - \beta x)^2 - \gamma^2(x - \beta ct)^2 - y^2 - z^2 \\ &= \gamma^2(c^2t^2 + \beta^2x^2 - x^2 - \beta^2c^2t^2) - y^2 - z^2\,. \end{aligned} \tag{7.23}$$

Recognizing $\gamma^2 = (1 - \beta^2)^{-1}$ we have

$$s'^2 = \gamma^2(1 - \beta^2)(c^2t^2 - x^2) - y^2 - z^2 = s^2\,. \tag{7.24}$$

The same transformation property follows for the difference between two events

$$\begin{aligned} \Delta s'^2 &= c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2 \\ &= \gamma^2\left[c^2(t_2 - t_1)^2 + (\beta^2 - 1)(x_2 - x_1)^2 - \beta^2c^2(t_2 - t_1)^2\right] - (y_2 - y_1)^2 - (z_2 - z_1)^2 \\ &= c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = \Delta s^2\,. \end{aligned} \tag{7.25}$$

Thus for a small separation between events we have

$$\begin{aligned} ds'^2 &= c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 \\ &= c^2dt^2 - dx^2 - dy^2 - dz^2 \\ &= ds^2\,. \end{aligned} \tag{7.26}$$

One says that s^2 and, equivalently, the increment ds^2, are Lorentz invariant.

Dividing by c we see that $ds/c = d\tau$ and

$$\begin{aligned} d\tau'^{\,2} &= \frac{ds'^2}{c^2} \\ &= dt'^2 - \frac{dx'^2 + dy'^2 + dz'^2}{c^2} \end{aligned}$$

$$
\begin{aligned}
&= dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2} \\
&= \frac{ds^2}{c^2} = d\tau^2 \,. \qquad (7.27)
\end{aligned}
$$

The meaning of τ becomes clear when we consider an observer for whom there is no change of position $dx = 0$. We see that $dt = d\tau$. This observer's clock, which rests in the body frame of reference, measures the proper body time, τ.

We found by deriving Eq. (7.26) that the increment of proper time is an invariant. That means that all observers $S(v)$ having some speed v will agree to how fast a clock will tick inside a body,

$$
\begin{aligned}
d\tau &= \sqrt{dt^2 - dx^2/c^2} = dt\sqrt{1 - dx^2/(cdt)^2} \\
&= dt\sqrt{1 - v^2/c^2} \,, \qquad (7.28)
\end{aligned}
$$

where v is velocity of a body measured by the observer $S(v)$ and this velocity can depend on time t. We see that the proper time increment $d\tau$ is according to Eq. (7.28) shorter compared to the time of any observer $S(v)$.

Important: for all observers the proper time $d\tau$ of each and every body is the same; we say it is a Lorentz invariant; it is the observer's $S(v)$ clock that measures a different and longer time dt depending on how fast the observer measures the speed of the body. More generally we can say that each and every body has its proper time which is an invariant quantity.

Exercise III–14: Proper time of interstellar probe

The star Alpha Centauri is located 4.4 light years from our solar system. A probe leaves the solar system, traveling at a constant speed, and reaches Alpha Centauri six years later, according to observers on Earth. How much time passes on a clock traveling with the probe?

Solution

The use of the invariant τ leads us to an efficient solution of this seemingly complex problem. We evaluate the Lorentz invariant proper time given by:

1 $c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 \,.$

With $\Delta x = 4.4$ light years and $\Delta t = 6$ years we find

2 $c^2\Delta\tau^2 = (6c\mathrm{y})^2 - (4.4c\mathrm{y})^2 = c^2(4.1\,\mathrm{y})^2 \,.$

Thus $\Delta\tau = 4.1$ years. Note that with increasing (average) speed $\Delta x/\Delta t \equiv v \to c$ of the probe, we have $c\Delta t \to \Delta x$ and hence $\Delta\tau \to 0$. A probe moving at ultrarelativistic speed ages very little.

End III–14: Proper time of interstellar probe

Exercise III–15: Positronium annihilation

A metastable configuration of positronium (usually denoted by symbol Ps), the bound state of an electron and its antiparticle the positron, has a mean proper lifetime of $\tau = 142$ ns before it annihilates into three photons (this is the ortho-positronium[6] where the particle spins align forming spin 1 state). If the mean lifetime observed in the laboratory of the metastable Ps in a mono-energetic beam (that is all Ps at a constant and prescribed speed) in the lab frame is $\Delta t = 300$ ns, what is the mean lab frame distance traveled before annihilation by the Ps 'particle' in the beam?

Solution

Again, as in exercise III–14 we use the invariant,

1 $$c^2\tau^2 = c^2\Delta t^2 - \Delta x^2 \,.$$

If τ is the mean proper lifetime and Δt the mean laboratory lifetime, then the mean distance Ps traveled from the source is

2 $$\Delta x = c\sqrt{\Delta t^2 - \tau^2} \simeq 264\, c \cdot \mathrm{ns} = 260\,\mathrm{ft} = 79.2\,\mathrm{m}\,.$$

We used feet as a reminder that the order of magnitude estimate is always 1 ns at speed c $\simeq$ 1 ft.

We note that the reported mean observed lifespan $\Delta t = 300$ ns > 142 ns exceeds the proper lifespan due to the effect of time dilation. Without time dilation the maximum average travel distance must be less than 142 ft (corresponding to 142 ns mean lifespan and a speed which must be below light velocity). In this somewhat different context this exercise is a redo of the muon traveling from the upper atmosphere to the surface of the Earth, exercise II–2 on page 56, and conversation III-1 on page 83.

End III–15: Positronium annihilation

7.4 Two Lorentz coordinate transformations in sequence

We consider two subsequent Lorentz coordinate transformations: a) the event (t, x, y, z) in S transformed into (t', x', y', z') in S' using v_1, and b) (t', x', y', z') in S' transformed into (t'', x'', y'', z'') in S'' using v_2. We consider the case where both transformations are in the x-direction. We now show that the outcome is as if there was one transformation from S to S'' by v_x which arises from the relativistic addition of the two velocities.

We begin with

$$x'' = \gamma_2(x' - \beta_2 ct'), \quad ct'' = \gamma_2(ct' - \beta_2 x')\,, \tag{7.29}$$

[6] The other equally bound state of positronium, that annihilates into two photons, the 'para-positronium' with spin-0, has a lifespan of $\tau = 125$ picoseconds, that is more than 1000 times shorter than for the ortho-Ps considered in the exercise.

where we insert

$$x' = \gamma_1(x - \beta_1 ct), \quad ct' = \gamma_1(ct - \beta_1 x)\,, \tag{7.30}$$

which results in

$$x'' = \gamma_2\gamma_1(x - \beta_1 ct - \beta_2(ct - \beta_1 x))\,, \quad ct'' = \gamma_2\gamma_1(ct - \beta_1 x - \beta_2(x - \beta_1 ct))\,. \tag{7.31}$$

A reorganization of terms produces

$$x'' = \gamma_2\gamma_1(x(1 + \beta_2\beta_1) - (\beta_2 + \beta_1)ct)\,, \quad ct'' = \gamma_2\gamma_1(ct(1 + \beta_2\beta_1) - (\beta_2 + \beta_1)x)\,. \tag{7.32}$$

If this is to be a new Lorentz coordinate transformation again from $S \to S''$, then we must have

$$\boxed{\gamma = \gamma_2\gamma_1(1 + \beta_1\beta_2), \quad \gamma\beta = \gamma_2\gamma_1(\beta_1 + \beta_2)\,.} \tag{7.33}$$

Dividing these two equations by each other we obtain again the velocity addition theorem

$$\boxed{\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}, \quad \text{that is} \quad v_x = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}\,,} \tag{7.34}$$

where $v_x = c\beta$. We see that two Lorentz coordinate transformations in sequence in the x-direction are described by a transformation with the velocity obtained according to the relativistic addition theorem for the two velocities of these transformations.

Comparing to our earlier result Eq. (7.5) and Eq. (7.9b), we note a change in sign: in the prior result the velocity v was that of a moving body. In the present consideration β_2 describes a change of coordinate system. Thus we have shown the equivalence of both passive and active relativistic velocity addition theorems.

We next check if $\beta^2 < 1$. This is best done evaluating

$$1/\gamma^2 \equiv 1 - \beta^2 = \frac{(1 - \beta_1^2)(1 - \beta_2^2)}{(1 + \beta_1\beta_2)^2}\,, \tag{7.35}$$

which is positive and smaller than unity for any $\beta_1^2, \beta_2^2 < 1$.

For $\beta_1 = -\beta_2$ we find from Eq. (7.34) as well as from Eq. (7.35) that with the 2nd transformation we transformed back to the original system $S'' = S$, since $\beta = 0, \gamma = 1$.

Exercise III–16: Two LTs in different directions

Find the transformation describing the consequence of two Lorentz coordinate transformations carried out in sequence, the first one in x-direction with v_{1x} and the second one in y-direction with v_{2y}.

Solution

Since $z = z' = z''$, we will ignore this coordinate. The first transformation is from system S to S' along the x-axis

$$\underline{1} \quad x' = \gamma_{1x}(x - \beta_{1x}ct)\,, \qquad y' = y\,, \qquad ct' = \gamma_{1x}(ct - \beta_{1x}x)\,,$$

and the second from S' to S'' along the y-axis

$$\underline{2} \quad x'' = x'\,, \qquad y'' = \gamma_{2y}(y' - \beta_{2y}ct')\,, \qquad ct'' = \gamma_{2y}(ct' - \beta_{2y}y')\,.$$

By substitution we obtain the double primed coordinates expressed in terms of the not primed coordinates

$$\underline{3} \quad \begin{aligned} x'' &= \gamma_{1x}(x - \beta_{1x}ct)\,,\\ y'' &= \gamma_{2y}(y - \beta_{2y}\gamma_{1x}(ct - \beta_{1x}x)) = \gamma_{2y}(y + \gamma_{1x}\beta_{2y}\beta_{1x}x) - \gamma_{1x}\gamma_{2y}\beta_{2y}ct\,,\\ ct'' &= \gamma_{2y}(\gamma_{1x}(ct - \beta_{1x}x) - \beta_{2y}y) = \gamma_{2y}\gamma_{1x}ct - \gamma_{2y}\gamma_{1x}\beta_{1x}x - \gamma_{2y}\beta_{2y}y\,. \end{aligned}$$

We see that this format is very different from the usual Lorentz coordinate transformation.

An interested reader can now check that for two non-parallel Lorentz coordinate transformations the sequence in which they are carried out matters. One says that these transformations do not commute. This means is that if we were first to consider the transformation in the y-direction with v_{2y}, followed by a transformation in x-direction with v_{1x}, the result we obtain would be different from the result we presented. This is further developed in exercise VIII–2 on page 283.

End III–16: Two LTs in different directions

7.5 Rapidity

We now introduce a new way to characterize speed within the Lorentz coordinate transformation through a function[7] $y_r(\beta)$. We are motivated by the desire to find a new variable which, unlike v, is not bounded by c. Therefore, it can more accurately characterize motion at ultrarelativistic speeds. We also want that in nonrelativistic limit

$$y_r = \beta \qquad \text{for } \beta \ll 1\,. \tag{7.36}$$

This relation suggests that we call y_r something that relates to speed, and the name rapidity is in common use.

We further recall that in the nonrelativistic limit two velocities add vectorially. If these velocity vectors are parallel, they add as numbers in nonrelativistic limit; that is, the two speeds add. Among many functions $y_r(\beta)$ we choose a unique one such that for parallel relativistic motion the rapidities sum just like speeds do for nonrelativistic motion. How

[7] It is very inconvenient at this point to introduce the rapidity using the conventional symbol y which can be confounded in this book with a coordinate. We thus introduce a subscript, here y_r with 'r' for rapidity. Later we will see 'p' for particle and 's' for space ship or space rocket.

this could work is easily recognized by recalling the addition theorem

$$\tanh(y_{r\,1} + y_{r\,2}) = \frac{\tanh y_{r\,1} + \tanh y_{r\,2}}{1 + \tanh y_{r\,1} \tanh y_{r\,2}}\,, \tag{7.37}$$

which we compare with the addition theorem of relativistic parallel speeds

$$\beta_{12} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}\,. \tag{7.38}$$

Thus we explore a new variable y_r that satisfies

$$\boxed{\beta = \tanh y_r = \frac{e^{y_r} - e^{-y_r}}{e^{y_r} + e^{-y_r}} < 1\,.} \tag{7.39}$$

We find the useful relation

$$e^{y_r} = \sqrt{\frac{1+\beta}{1-\beta}} = \gamma(1+\beta)\,, \qquad e^{-y_r} = \sqrt{\frac{1-\beta}{1+\beta}} = \gamma(1-\beta)\,, \tag{7.40}$$

and hence

$$\boxed{y_r = \ln\sqrt{\frac{1+\beta}{1-\beta}} = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \equiv \tanh^{-1}(\beta)\,.} \tag{7.41}$$

Considering that hyperbolic exponential functions can be written in terms of the exponential function

$$\cosh y_r = \frac{e^{y_r} + e^{-y_r}}{2}\,, \qquad \sinh y_r = \frac{e^{y_r} - e^{-y_r}}{2}\,, \tag{7.42}$$

a short computation shows

$$\boxed{\cosh y_r = \gamma\,, \qquad \sinh y_r = \gamma\beta.} \tag{7.43}$$

We cross-check our computation

$$\cosh^2 y_r - \sinh^2 y_r = \gamma^2(1-\beta^2) = 1\,, \tag{7.44}$$

and furthermore

$$\tanh y_r = \frac{\sinh y_r}{\cosh y_r} = \frac{\gamma\beta}{\gamma} = \beta\,, \tag{7.45}$$

as expected, see Eq. (7.39).

Expanding Eq. (7.45) in a power series we see that

$$\beta = \tanh y_r = y_r - \frac{1}{3}y_r^3 + \frac{2}{15}y_r^5 - \cdots\,, \tag{7.46}$$

and from Eq. (7.41)

$$\begin{aligned} y_r &= \frac{1}{2}[\ln(1+\beta) - \ln(1-\beta)] \\ &= \frac{1}{2}\left[\left(\beta - \frac{\beta^2}{2} + \frac{\beta^3}{3} - \cdots\right) - \left(-\beta - \frac{\beta^2}{2} - \frac{\beta^3}{3} - \cdots\right)\right] \\ &= \beta + \frac{1}{3}\beta^3 + \frac{1}{5}\beta^5 + \cdots . \end{aligned} \tag{7.47}$$

On the other hand for $\beta \to 1$ we use a Taylor series for Eq. (7.39) to write

$$1 - \beta = 2\left(e^{-2y_r} - e^{-4y_r} + e^{-6y_r} - \cdots\right) . \tag{7.48}$$

It is interesting to note how rapidity describes the approach to the speed of light: $y_r = 10$ corresponds to a fractional deviation from the speed of light by $4 \cdot 10^{-9}$, corresponding to 1.2 m/s.

We now write the Lorentz coordinate transformation using rapidity in the form

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1-(v/c)^2}} = \gamma(x - \beta ct) = x \cosh y_r - ct \sinh y_r , \\ ct' &= \frac{ct - (v/c)x}{\sqrt{1-(v/c)^2}} = \gamma(ct - \beta x) = ct \cosh y_r - x \sinh y_r . \end{aligned} \tag{7.49}$$

The form of Eq. (7.49) is similar to a rotation around the z-axis:

$$\begin{aligned} x' &= x \cos\phi + y \sin\phi , \\ y' &= y \cos\phi - x \sin\phi , \end{aligned} \tag{7.50}$$

but with one sign being different and a hyperbolic angle of rotation. This takes into account the different properties of space-time: while the usual rotation *e.g.* around the z-axis leaves $\rho^2 = x^2 + y^2$ invariant, the Lorentz coordinate transformation leaves invariant $s^2 = (ct)^2 - \vec{x}^{\,2}$, compare Eq. (7.26).

With the help of rapidity we thus recognize the Lorentz coordinate transformation to be a new form of rotation in what one calls Minkowski space-time. LT is characterized by an angle which is hyperbolic and not trigonometric. This difference arises from the unbounded character of y_r compared to $0 \le \phi \le 2\pi$, where ϕ is a regular rotation angle.

We have constructed the rapidity variable to be additive, which means that when we consider two LT in sequence that the rapidities add just like two angles add in case of regular rotations

$$\boxed{y_r = y_{r\,1} + y_{r\,2} .} \tag{7.51}$$

This will be the content of the exercise III–17 below. It is important to remember that Eq. (7.51) is only true for two Lorentz coordinate transformations in the same spatial direction. Even so, Eq. (7.51) is a pivotal property of rapidity, which behaves just like nonrelativistic addition of velocity does. This makes rapidity an extraordinarily important tool in the study of many problems in physics.

For example, the rapidity formulation of LT Eq. (7.43) allows us to show the invariance of proper time just as we show that length of a vector remains the same under rotations:

$$\begin{aligned}
s'^{\,2} &= c^2 t'^{\,2} - x'^{\,2} \\
&= (ct\ \cosh y_r - x\ \sinh y_r)^2 - (x\ \cosh y_r - ct\ \sinh y_r)^2 \\
&= (\cosh^2 y_r - \sinh^2 y_r)c^2t^2 + (\sinh^2 y_r - \cosh^2 y_r)x^2 \\
&= c^2t^2 - x^2 = s^2\,, \qquad (7.52)
\end{aligned}$$

which is the same result as obtained in Sect. 7.3.

Exercise III–17: Addition of rapidity

Demonstrate that rapidities for two Lorentz coordinate transformations carried out in sequence add akin to the situation with rotation angles.

Solution

We now carry out two Lorentz coordinate transformations in sequence, see Sect. 7.4, employing rapidity format. We have

1 $$\begin{aligned} x' &= x\ \cosh y_{r\,1} - ct\ \sinh y_{r\,1}\,, & ct' &= ct\ \cosh y_{r\,1} - x\ \sinh y_{r\,1}\,,\\ x'' &= x'\ \cosh y_{r\,2} - ct'\ \sinh y_{r\,2}\,, & ct'' &= ct'\ \cosh y_{r\,2} - x'\ \sinh y_{r\,2}\,. \end{aligned}$$

We insert the first transformation into the second

2 $$\begin{aligned} x'' &= (x\ \cosh y_{r\,1} - ct\ \sinh y_{r\,1})\ \cosh y_{r\,2} - (ct\ \cosh y_{r\,1} - x\ \sinh y_{r\,1})\ \sinh y_{r\,2}\,,\\ ct'' &= (ct\ \cosh y_{r\,1} - x\ \sinh y_{r\,1})\ \cosh y_{r\,2} - (x\ \cosh y_{r\,1} - ct\ \sinh y_{r\,1})\ \sinh y_{r\,2}\,. \end{aligned}$$

Reordering terms we find

3 $$\begin{aligned} x'' &= x\,(\cosh y_{r\,1} \cosh y_{r\,2} + \sinh y_{r\,1} \sinh y_{r\,2})\\ &\quad - ct\,(\cosh y_{r\,1} \sinh y_{r\,2} + \sinh y_{r\,1} \cosh y_{r\,2})\,,\\ ct'' &= ct\,(\cosh y_{r\,1} \cosh y_{r\,2} + \sinh y_{r\,1} \sinh y_{r\,2})\\ &\quad - x\,(\cosh y_{r\,1} \sinh y_{r\,2} + \sinh y_{r\,1} \cosh y_{r\,2})\,. \end{aligned}$$

It is well known that

4 $$\begin{aligned} \cosh(y_{r\,1} + y_{r\,2}) &= \cosh y_{r\,1} \cosh y_{r\,2} + \sinh y_{r\,1} \sinh y_{r\,2}\,,\\ \sinh(y_{r\,1} + y_{r\,2}) &= \cosh y_{r\,1} \sinh y_{r\,2} + \sinh y_{r\,1} \cosh y_{r\,2}\,, \end{aligned}$$

which can be also checked using

5 $$\cosh y_r = \frac{e^{y_r} + e^{-y_r}}{2}\,, \qquad \sinh y_r = \frac{e^{y_r} - e^{-y_r}}{2}\,.$$

The combined LT transformations thus have the form

6 $$x'' = x\ \cosh y_r - ct\ \sinh y_r\,, \qquad ct'' = ct\ \cosh y_r - x\ \sinh y_r\,,$$

where

7 $y_r = y_{r\,1} + y_{r\,2}$.

The rapidities add under Lorentz coordinate transformations carried out in the same direction, just like is the case with speeds of two Galilean transformations carried out in sequence in the direction of two parallel velocities.

End III–17: Addition of rapidity

Exercise III–18: Shuttlecraft rescue: rapidity method

We return here to the exercise III–7 on page 91 employing instead of speeds/velocities the associated rapidities. We recall that in the Star Wars scene the insurrection base reports to its shuttle traveling with $u_{\text{sh}} = 0.3c$ that it is being chased by a rocket following it with $u_{\text{r}} = 0.6c$, and their rescue spaceship is approaching from the opposite direction traveling with the velocity $u_{\text{S}} = -0.6c$; all velocities are parallel. What relative speeds are registered in the shuttlecraft? Are these consistent with exercise III–6?

Solution

We first establish the appropriate rapidities using Eq. (7.41). We find:

1 $$y_{r\,\text{s}} = 0.5 \ln \frac{1.3}{0.7} = 0.310, \quad y_{r\,\text{r}} = 0.5 \ln \frac{1.6}{0.4} = 0.693, \quad y_{r\,\text{S}} = 0.5 \ln \frac{0.4}{1.6} = -0.693\,.$$

We boost to the shuttle frame. The spaceship approaches the shuttle with rapidity

2 $$y_{r\,\text{Ss}} = y_{r\,\text{S}} - y_{r\,\text{s}} = -1.003\,, \quad \to v_{\text{Ss}} = 0.763\,,$$

where we use Eq. (7.39) to compute velocity v. However, the rocket is gaining on the shuttle with rapidity

3 $$y_{r\,\text{rs}} = y_{r\,\text{r}} - y_{r\,\text{s}} = 0.383\,, \quad \to v_{\text{rs}} = 0.365\,,$$

so rescue is possible but not assured. We check if shuttlecraft crew computations are correct by boosting with $v = v_{rs} = -0.366c$ from the shuttle frame to the rocket frame of reference. This gives the rocket observed rapidity of the spaceship, independent of what the shuttle is doing:

4 $$y_{r\,\text{Sr}} = y_{r\,\text{Ss}} - y_{r\,\text{rs}} = -1.003 - 0.383 = -1.386\,, \quad \to v_{\text{Sr}} = 0.88\,.$$

This is the expected result

5 $$y_{r\,\text{Sr}} = y_{r\,\text{S}} - y_{r\,\text{r}} = -0.693 - 0.693 = -1.386$$

since trivially

6 $$y_{r\,\text{Sr}} = y_{r\,\text{Ss}} - y_{r\,\text{rs}} = y_{r\,\text{S}} - y_{r\,\text{s}} - (y_{r\,\text{r}} - y_{r\,\text{s}}) = y_{r\,\text{S}} - y_{r\,\text{r}}\,.$$

The advantage of the use of rapidity in the Star Wars context and for that matter all science fiction conforming to Einstein's relativity is now evident, but this simple variable has yet to be discovered by movie makers.

End III–18: Shuttlecraft rescue: rapidity method

Exercise III–19: Lorentz coordinate transformations in a sequence: factor γ_{12}

Obtain using rapidity the Lorentz factor γ_{12} for two Lorentz coordinate transformations carried out in sequence along the same axis.

Solution

We take advantage of the fact that rapidities of these two transformations add

1 $\gamma_{12} = \cosh(y_{r\,1} + y_{r\,2})\ .$

We further have in general

2 $\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b\ ,$

and thus

3 $\gamma_{12} = \cosh y_1 \cosh y_2 (1 + \tanh y_1 \tanh y_2)\ .$

We use Eq. (7.39) and Eq. (7.43) to obtain

4 $\gamma_{12} = \gamma_1 \gamma_2 (1 + \beta_1 \beta_2)\ .$

Naturally, this result follows in a more cumbersome computation using velocity addition and evaluating γ_{12} explicitly, as we have shown deriving Eq. (7.33). However, the use of rapidity simplified the evaluation considerably.

End III–19: Lorentz coordinate transformations in a sequence: factor γ_{12}