APPLICATION OF SIGNAL ANALYSIS TO THE EMBEDDING PROBLEM OF $\mathbb{Z}^k$-ACTIONS

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Abstract. We study the problem of embedding arbitrary $\mathbb{Z}^k$-actions into the shift action on the infinite dimensional cube $\left([0,1]^D\right)^{\mathbb{Z}^k}$. We prove that if a $\mathbb{Z}^k$-action X satisfies the marker property (in particular if X is a minimal system without periodic points) and if its mean dimension is smaller than $D/2$ then we can embed it in the shift on $\left([0,1]^D\right)^{\mathbb{Z}^k}$. The value $D/2$ here is optimal. The proof goes through signal analysis. We develop the theory of encoding $\mathbb{Z}^k$-actions into band-limited signals and apply it to proving the above statement. Main technical difficulties come from higher dimensional phenomena in signal analysis. We overcome them by exploring analytic techniques tailored to our dynamical settings. The most important new idea is to encode the information of a tiling of $\mathbb{R}^k$ into a band-limited function which is constructed from another tiling.

1. Introduction

1.1. **Background.** The main purpose of this paper is to deepen interactions between signal analysis and dynamical systems. (This subsection is just a motivation. So readers can skip unfamiliar/undefined terminologies.) If we think of signal analysis in a broad sense (like signal analysis $\approx$ Fourier analysis), then applications of signal analysis are ubiquitous in ergodic theory. For example, Fourier analysis proof of the unique ergodicity of the irrational rotation (due to Weyl) is a milestone of such applications. But here we would like to think of signal analysis in a narrower sense, say signal analysis is a part of communication theory. From this viewpoint, a masterpiece is Shannon's work on *communications over band-limited channels with Gaussian noise* ([Sha48], [CT06, Chapter 9]): Suppose that we try to send information by using signals (say, of telephone line) whose frequencies are limited in $[-W/2, W/2]$. If the averaged power of signal is P and if the noise is additive white Gaussian noise of spectral density N, then the capacity of the channel (i.e. the amount of information we can send by using this channel) is given by

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the formula

$$\text{Capacity} = \frac{W}{2}\log_2\left(1+\frac{P}{NW}\right) \quad \text{bits per second.}$$

The details of this formula are not important here. We just would like to emphasize that the **band-width** W is a crucial parameter in the communications over band-limited channels.

Recently the first and third named authors [GT] found an analogous theory in the context of topological dynamics. They started the theory of *encoding arbitrary dynamical systems into band-limited signals* and applied it to solving an open problem posed by Lindenstrauss [Lin99] in 1999. The purpose of the present paper is to expand this theory to *multi-dimensional case*.

1.2. **Encoding into discrete signals.** We start from a problem seemingly unrelated to signal analysis. Let k be a natural number. A triple $(X, \mathbb{Z}^k, T)$ (often abbreviated to X) is called a dynamical system if X is a compact metric space and

$$T : \mathbb{Z}^k \times X \to X, \quad (n, x) \mapsto T^n x$$

is a continuous action. A fundamental example of dynamical systems is the **shift action on the infinite dimensional cube**. Let D be a natural number. Consider the infinite product $\left([0,1]^D\right)^{\mathbb{Z}^k}$. Let $\sigma : \mathbb{Z}^k \times \left([0,1]^D\right)^{\mathbb{Z}^k} \to \left([0,1]^D\right)^{\mathbb{Z}^k}$ be the shift:

$$\sigma^m\left((x_n)_{n\in\mathbb{Z}^k}\right) = (x_{n+m})_{n\in\mathbb{Z}^k}.$$

The triple $\left(\left([0,1]^D\right)^{\mathbb{Z}^k}, \mathbb{Z}^k, \sigma\right)$ is a dynamical system and called the shift on $\left([0,1]^D\right)^{\mathbb{Z}^k}$.

We study the old problem of *embedding arbitrary dynamical systems in the shift on* $\left([0,1]^D\right)^{\mathbb{Z}^k}$. Let $(X, \mathbb{Z}^k, T)$ be a dynamical system. A map $f : X \to \left([0,1]^D\right)^{\mathbb{Z}^k}$ is called an embedding of a dynamical system if f is a $\mathbb{Z}^k$-equivariant continuous injection. We would like to understand when X can be embedded in the shift on $\left([0,1]^D\right)^{\mathbb{Z}^k}$. Notice that every compact metric space can be *topologically* embedded in $\left([0,1]^D\right)^{\mathbb{Z}^k}$. So the problem is a genuinely *dynamical* question.

It will be convenient later to see the problem from a slightly different viewpoint: A point $x = (x_n)_{n\in\mathbb{Z}^k}$ in $\left([0,1]^D\right)^{\mathbb{Z}^k}$ can be seen as a *discrete signal*[1] valued in $[0,1]^D$. Then the embedding problem asks *when we can encode a given dynamical system X into discrete signals.*

We review the history of the problem before explaining our main result (Main Theorem 1 below). The embedding problem was first studied by Jaworski [Jaw74] in 1974. For understanding his result, notice that periodic points form an obvious obstruction to the embedding. For example, the shift on $\left([0,1]^2\right)^{\mathbb{Z}^k}$ cannot be embedded in the shift on $[0,1]^{\mathbb{Z}^k}$ because the fixed points sets of $\left([0,1]^2\right)^{\mathbb{Z}^k}$ and $[0,1]^{\mathbb{Z}^k}$ are homeomorphic to the

[1]When $k = 2$, it might be better to call x a *discrete image*.

square $[0,1]^2$ and the line segment $[0,1]$ respectively and the square cannot be topologically embedded in the line segment.

A dynamical system $(X, \mathbb{Z}^k, T)$ is said to be **aperiodic** if $T^n x \neq x$ for all $x \in X$ and nonzero $n \in \mathbb{Z}^k$. Jaworski [Jaw74] proved that periodic points are the only obstruction if X is a finite dimensional system:

Theorem 1.1 (Jaworski, 1974)**.** *Let $(X, \mathbb{Z}, T)$ be an aperiodic finite dimensional dynamical system. Then we can embed it in the shift on $[0,1]^{\mathbb{Z}}$.*

Although this theorem is stated only for $\mathbb{Z}$-actions, it can be easily generalized to $\mathbb{Z}^k$-actions. (Indeed the $\mathbb{Z}^k$-case can be deduced from the $\mathbb{Z}$-case.)

After Jaworski, people were interested in whether the assumption of finite dimensionality is essential or not. Auslander [Aus88, p. 193] asked whether we can embed every minimal system $(X, \mathbb{Z}, T)$ in the shift on $[0,1]^{\mathbb{Z}}$. A system $(X, \mathbb{Z}^k, T)$ is said to be **minimal** if the orbit $\{T^n x\}_{n \in \mathbb{Z}^k}$ is dense in X for every $x \in X$. When $k = 1$, minimal systems X have no periodic points unless X is a finite set. (If X is finite, then it can be obviously embedded in the shift on $[0,1]^{\mathbb{Z}}$.) Therefore the question essentially asks whether there is another obstruction different from periodic points.

Lindenstrauss–Weiss [LW00] found that mean dimension provides a new obstruction to the embedding. Mean dimension (first introduced by Gromov [Gro99]) is a topological invariant of dynamical systems which counts the average number of parameters of a given system. The mean dimension of $(X, \mathbb{Z}^k, T)$ is denoted by $\mathrm{mdim}(X)$. We review its definition in Subsection 2.1. Finite dimensional systems and finite topological entropy systems are known to have zero mean dimension.

The mean dimension of the shift on $\left([0,1]^D\right)^{\mathbb{Z}^k}$ is equal to D, which means that $\left([0,1]^D\right)^{\mathbb{Z}^k}$ has D parameters in average. If $(X, \mathbb{Z}^k, T)$ can be embedded in the shift on $\left([0,1]^D\right)^{\mathbb{Z}^k}$ then $\mathrm{mdim}(X) \leq D$. Lindenstrauss–Weiss [LW00, Proposition 3.5] constructed a minimal system $(X, \mathbb{Z}, T)$ of mean dimension strictly greater than one. This system cannot be embedded in the shift on $[0,1]^{\mathbb{Z}}$ although it is minimal. Thus it solved Auslander's question.

Lindenstrauss went further; he proved a partial converse [Lin99, Theorem 5.1].

Theorem 1.2 (Lindenstrauss, 1999)**.** *If a minimal system $(X, \mathbb{Z}, T)$ satisfies $\mathrm{mdim}(X) < D/36$ then we can embed it in the shift on $\left([0,1]^D\right)^{\mathbb{Z}}$.*

Lindenstrauss [Lin99, p. 229] asked the problem of improving the condition $\mathrm{mdim}(X) < D/36$. This problem was solved by the first and third named authors [GT, Theorem 1.4]:

Theorem 1.3 (Gutman–Tsukamoto)**.** *If a minimal system $(X, \mathbb{Z}, T)$ satisfies $\mathrm{mdim}(X) < D/2$ then we can embed it in the shift on $\left([0,1]^D\right)^{\mathbb{Z}}$.*

The condition $\mathrm{mdim}(X) < D/2$ is optimal because there exists a minimal system $(X, \mathbb{Z}, T)$ of mean dimension $D/2$ which cannot be embedded in the shift on $\left([0,1]^D\right)^{\mathbb{Z}}$

([LT14]). Theorem 1.3 can be seen as a dynamical analogue of the classical theorem in dimension theory [HW41, Thm. V2]: A compact metric space X can be topologically embedded in $[0,1]^D$ if $\dim X < D/2$. We review the proof of this classical result in Subsection 2.3.

A motivation of the present paper is to generalize Theorem 1.3 to $\mathbb{Z}^k$-actions. It is convenient to introduce the following notion. A dynamical system $(X, \mathbb{Z}^k, T)$ is said to satisfy the **marker property** if for every natural number N there exists an open set $U \subset X$ satisfying

$$U \cap T^{-n}U = \emptyset \quad (0 < |n| < N), \quad X = \bigcup_{n \in \mathbb{Z}^k} T^{-n}U.$$

This property obviously implies the aperiodicity. The following dynamical systems are known to satisfy the marker property:

- Aperiodic minimal systems. (This is an immediate fact from the definition.)
- Aperiodic finite dimensional systems ([Gut12, Theorem 6.1])[2].
- If X satisfies the marker property and if $\pi : Y \to X$ is an extension (i.e. $\mathbb{Z}^k$-equivariant continuous surjection) then Y also satisfies the marker property.

The authors do not know an example of aperiodic actions which do not satisfy the marker property.

Lindenstrauss and the first and third named authors [GLT16, Theorem 1.5] proved:

Theorem 1.4 (Gutman–Lindenstrauss–Tsukamoto, 2016)**.** *If a dynamical system* $(X, \mathbb{Z}^k, T)$ *satisfies the marker property and*

$$\mathrm{mdim}(X) < \frac{D}{2^{k+1}}$$

then we can embed it in the shift on $\left([0,1]^{2D}\right)^{\mathbb{Z}^k}$.

After stating this theorem, the paper [GLT16] posed the problem of improving the condition $\mathrm{mdim}(X) < D/2^{k+1}$. Quoting [GLT16, p. 782]:

> Note also that in our condition $\mathrm{mdim}(X) < D/2^{k+1}$, the constant involved is likely far from optimal. Presumably for an aperiodic $\mathbb{Z}^k$-system if $\mathrm{mdim}(X) < \frac{D}{2}$ then X can be embedded in $([0,1]^D)^{\mathbb{Z}^k}$.

The first main theorem of the present paper confirms this conjecture under the assumption of the marker property:

[2] More generally, we can prove that if an aperiodic dynamical system satisfies the *small boundary property* then it satisfies the marker property. The small boundary property is a notion introduced by Lindenstrauss–Weiss [LW00, Definition 5.2], which is satisfied by every aperiodic finite dimensional system. We do not use these facts. So we omit the detailed explanation.

Main Theorem 1. *If a dynamical system $(X, \mathbb{Z}^k, T)$ satisfies the marker property and*

$$\mathrm{mdim}(X) < \frac{D}{2},$$

then we can embed it in the shift on $\left([0,1]^D\right)^{\mathbb{Z}^k}$.

As in the case of Theorem 1.3, the condition $\mathrm{mdim}(X) < D/2$ is optimal because the paper [LT14][3] provides an example of aperiodic minimal system $(X, \mathbb{Z}^k, T)$ of mean dimension $D/2$ which cannot be embedded in the shift on $\left([0,1]^D\right)^{\mathbb{Z}^k}$. Main Theorem 1 has some novelty even in the case of $k = 1$: Since both aperiodic finite dimensional systems and aperiodic minimal systems satisfy the marker property, Main Theorem 1 unifies Jaworski's theorem (Theorem 1.1) and Theorem 1.3 into a single statement.

1.3. **Encoding into band-limited signals.** A discovery of the paper [GT] is that we can approach to the embedding problem in Subsection 1.2 via signal analysis. We need to prepare some terminologies. For a rapidly decreasing function $\varphi : \mathbb{R}^k \to \mathbb{C}$ we define its Fourier transforms by

$$\mathcal{F}(\varphi)(\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}^k} \varphi(t)\, e^{-2\pi\sqrt{-1}t\cdot\xi}\, dt_1 \dots dt_k,$$
$$\overline{\mathcal{F}}(\varphi)(t) = \check{\varphi}(t) = \int_{\mathbb{R}^k} \varphi(\xi)\, e^{2\pi\sqrt{-1}t\cdot\xi} d\xi_1 \dots d\xi_k.$$

It follows that $\mathcal{F}\left(\overline{\mathcal{F}}(\varphi)\right) = \overline{\mathcal{F}}\left(\mathcal{F}(\varphi)\right) = \varphi$. We extend $\mathcal{F}$ and $\overline{\mathcal{F}}$ to tempered distributions ψ by the dualities $\langle \mathcal{F}(\psi), \varphi\rangle = \langle \psi, \overline{\mathcal{F}}(\varphi)\rangle$ and $\langle \overline{\mathcal{F}}(\psi), \varphi\rangle = \langle \psi, \mathcal{F}(\varphi)\rangle$ where φ are rapidly decreasing functions (Schwartz [Sch66, Chapter 7]). For example, if $\psi(t) = e^{2\pi\sqrt{-1}a\cdot t}$ $(a \in \mathbb{R}^k)$ then $\mathcal{F}(\psi) = \delta_a$ is the delta probability measure at a.

Let $a_1, \dots, a_k$ be positive numbers. A bounded continuous function $\varphi : \mathbb{R}^k \to \mathbb{R}$ is said to be band-limited in $[-a_1/2, a_1/2] \times \dots \times [-a_k/2, a_k/2]$ if the Fourier transform $\hat{\varphi}$ satisfies

$$\mathrm{supp}\, \hat{\varphi} \subset \left[-\frac{a_1}{2}, \frac{a_1}{2}\right] \times \dots \times \left[-\frac{a_k}{2}, \frac{a_k}{2}\right],$$

which means that $\langle \hat{\varphi}, \phi\rangle = \langle \varphi, \check{\phi}\rangle = 0$ for all rapidly decreasing functions ϕ satisfying

$$\mathrm{supp}\, \phi \cap \left[-\frac{a_1}{2}, \frac{a_1}{2}\right] \times \dots \times \left[-\frac{a_k}{2}, \frac{a_k}{2}\right] = \emptyset.$$

For example, when $k = 1$, the functions

$$\sin(\pi t_1), \quad \cos(\pi t_1), \quad \frac{\sin(\pi t_1)}{\pi t_1}$$

are band-limited in $[-1/2, 1/2]$.

We define $\mathcal{B}(a_1, \dots, a_k)$ as the Banach space of all bounded continuous functions $\varphi : \mathbb{R}^k \to \mathbb{R}$ band-limited in $[-a_1/2, a_1/2] \times \dots \times [-a_k/2, a_k/2]$. Its norm is the L^∞-norm.

[3]Strictly speaking, the paper [LT14] considered only the case of $k = 1$. But their construction can be generalized to $\mathbb{Z}^k$-actions without any changes.

We define $\mathcal{B}_1(a_1, \dots, a_k)$ as the set of $\varphi \in \mathcal{B}(a_1, \dots, a_k)$ satisfying $\|\varphi\|_{L^\infty(\mathbb{R}^k)} \le 1$. In this paper we promise that the space $\mathcal{B}_1(a_1, \dots, a_k)$ is always endowed with the **topology of uniform convergence over compact subsets**. Namely $\mathcal{B}_1(a_1, \dots, a_k)$ is endowed with the topology given by the distance

$$d(\varphi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \|\varphi - \psi\|_{L^\infty(B_n)}, \quad \left(B_n = \{t \in \mathbb{R}^k |\, |t| \le n\}\right).$$

$\mathcal{B}_1(a_1, \dots, a_k)$ is compact (see Corollary 2.3 below) and endowed with a natural continuous action of $\mathbb{Z}^k$ defined by

$$\sigma : \mathbb{Z}^k \times \mathcal{B}_1(a_1, \dots, a_k) \to \mathcal{B}_1(a_1, \dots, a_k), \quad \sigma^n(\varphi)(t) = \varphi(t+n).$$

We call the dynamical system $(\mathcal{B}_1(a_1, \dots, a_k), \mathbb{Z}^k, \sigma)$ the **shift on** $\mathcal{B}_1(a_1, \dots, a_k)$. Its mean dimension is equal to the product[4]

$$a_1 \dots a_k,$$

which is a multi-dimensional version of band-width W introduced in Subsection 1.1. As band-width W plays a crucial role in Shannon's theory, the quantity $a_1 \dots a_k$ becomes a key parameter below.

We consider the problem of embedding arbitrary dynamical systems in the shift on $\mathcal{B}_1(a_1, \dots, a_k)$. In other words we ask *when we can encode a given dynamical system into band-limited signals.* When $k = 1$, the first and third named authors [GT, Theorem 1.7] proved:

Theorem 1.5 (Gutman–Tsukamoto)**.** *If an aperiodic minimal system $(X, \mathbb{Z}, T)$ satisfies $\mathrm{mdim}(X) < a_1/2$ then we can embed it in the shift on $\mathcal{B}_1(a_1)$.*

The second main theorem of the present paper generalizes this theorem to the higher dimensional case:

Main Theorem 2. *If a dynamical system $(X, \mathbb{Z}^k, T)$ satisfies the marker property and*

$$\mathrm{mdim}(X) < \frac{a_1 \dots a_k}{2},$$

then we can embed it in the shift on $\mathcal{B}_1(a_1, \dots, a_k)$.

As in the case of Main Theorem 1, the condition $\mathrm{mdim}(X) < a_1 \dots a_k/2$ is optimal because a modification of [LT14] provides an aperiodic minimal system $(X, \mathbb{Z}^k, T)$ of mean dimension $a_1 \dots a_k/2$ which cannot be embedded in the shift on $\mathcal{B}_1(a_1, \dots, a_k)$.

We can prove Main Theorem 1 by using Main Theorem 2.

Proof: Main Theorem 2 implies Main Theorem 1. From $\mathrm{mdim}(X) < D/2$ we can choose positive numbers $a_1, \dots, a_k$ satisfying

$$a_1 < D,\ a_2 < 1,\ \dots, a_k < 1, \quad \mathrm{mdim}(X) < \frac{a_1 \dots a_k}{2}.$$

[4]This fact is not used in the paper. So we omit the proof.

Main Theorem 2 implies that we can embed X in the shift on $\mathcal{B}_1(a_1,\dots,a_k)$. We define a lattice $\Lambda\subset\mathbb{R}^k$ by

$$\Lambda=\left\{\left(\frac{n_1}{D},n_2,\dots,n_k\right)\middle|\, n_1,n_2\dots,n_k\in\mathbb{Z}\right\}.$$

It follows from a sampling theorem (see Lemma 2.4 below) that the map

$$\mathcal{B}_1(a_1,\dots,a_k)\to[-1,1]^\Lambda,\quad \varphi\mapsto\varphi|_\Lambda$$

is injective. Set $e=(1/D,0,\dots,0)\in\mathbb{R}^k$. The above injectivity means that the $\mathbb{Z}^k$-equivariant map

$$\mathcal{B}_1(a_1,\dots,a_k)\to\left([-1,1]^D\right)^{\mathbb{Z}^k},\quad \varphi\mapsto\left(\varphi(n),\varphi(n+e),\dots,\varphi\left(n+(D-1)e\right)\right)_{n\in\mathbb{Z}^k}$$

is an embedding. Since the system X can be embedded in the shift on $\mathcal{B}_1(a_1,\dots,a_k)$, it can be also embedded in the shift on $\left([-1,1]^D\right)^{\mathbb{Z}^k}$, which is obviously isomorphic to the shift on $\left([0,1]^D\right)^{\mathbb{Z}^k}$. □

1.4. **One dimension versus multi-dimension; what are the main difficulties?** This subsection explains what are the main difficulties in the proof of Main Theorem 2. When the authors started the research of this paper, they optimistically thought that the proof of Main Theorem 2 is probably more or less a direct generalization of the proof of Theorem 1.5 in [GT]. This expectation turned out to be wrong. Completely new difficulties arose in the context of signal analysis. The paper [GT] used the technique of *one dimensional* signal analysis. The proof of Main Theorem 2 uses *mutli-dimensional* signal analysis. The difference of one dimension/multi-dimension is fundamental and causes big changes of the approaches significantly.

We review some basic signal analysis in Subsection 2.2 below. The main results there are Paley–Wiener's theorem (Lemma 2.2) and a sampling theorem (Lemma 2.4) for functions in $\mathcal{B}(a_1,\dots,a_k)$, whose proofs are a simple generalization of (or a reduction to) the corresponding theorems in one-variable case. They might give readers an impression that multi-dimensional signal analysis is just a simple generalization of one-dimensional signal analysis. This is far from being true. It is well-known in classical analysis community (cf. [OU12]) that *multi-dimensional signal analysis is inherently more difficult than one-dimensional signal analysis.* Some important theorems in one-dimensional case *cannot* be generalized to higher dimension.

For explaining further, we need to recall the statement of Paley–Wiener's theorem (Lemma 2.2): A bounded continuous function $f:\mathbb{R}^k\to\mathbb{R}$ belongs to $\mathcal{B}(a_1,\dots,a_k)$ if and only if it can be extended to a holomorphic function in $\mathbb{C}^k$ satisfying

$$\left|f\left(x_1+y_1\sqrt{-1},\dots,x_k+y_k\sqrt{-1}\right)\right|\le \text{const}\cdot e^{\pi(a_1|y_1|+\cdots+a_k|y_k|)}.$$

Roughly speaking, band-limited functions are the same as *holomorphic functions of exponential type*. Therefore we can say that signal analysis is a part of complex analysis[5]. Then it is easy to see why multi-dimensional case is more difficult than one-dimensional case: Zero points of holomorphic functions in $\mathbb{C}$ are isolated, whereas zero points of holomorphic functions in $\mathbb{C}^k$ ($k \geq 2$) form positive dimensional complex varieties whose geometry are highly nontrivial in general. This causes a fundamental difference in the nature of one-dimensional/multi-dimensional signal analysis.

More concretely speaking, we face difficulties of higher dimension in the following two issues. (The second issue is more serious than the first one.)

(1) Interpolation: *Sampling and interpolation* are the most basic themes of signal analysis. We recall a sampling theorem stated in Lemma 2.4 with a bit simplification[6]: Let $f \in \mathcal{B}(1, \dots, 1)$ and $0 < c < 1$. If f vanishes over $\Lambda \overset{\text{def}}{=} c\mathbb{Z}^k$ then f is identically zero. This theorem is valid for all dimensions. The crucial point of the statement is that we consider only **regular sampling**, which means that the fundamental domain $[0, c]^k$ of the sampling set $\Lambda = c\mathbb{Z}^k$ is similar to the *frequency domain* $[-1/2, 1/2]^k$. It is also easy to prove the following *regular* interpolation theorem: If $c > 1$ then the map

$$\mathcal{B}(1, \dots, 1) \to \ell^\infty(c\mathbb{Z}^k), \quad f \mapsto (f(\lambda))_{\lambda \in c\mathbb{Z}^k}$$

is surjective.

When $k = 1$, Beurling [Beu89] proved that essentially the same results also hold for *irregular* sampling/interpolation: Let $\Lambda \subset \mathbb{R}$ be a uniformly discrete set, i.e. the infimum of $|\lambda_1 - \lambda_2|$ over distinct $\lambda_1, \lambda_2 \in \Lambda$ is positive. Suppose the **lower density**

$$\liminf_{R \to \infty} \inf_{t \in \mathbb{R}} \frac{\#(\Lambda \cap [t, t+R])}{R}$$

is larger than 1. Under this setting, Beurling proved that if $f \in \mathcal{B}(1)$ vanishes over Λ then f is identically zero. Similarly, he also proved that if $\Lambda \subset \mathbb{R}$ is a uniformly discrete set and its **upper density**

$$\limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{\#(\Lambda \cap [t, t+R])}{R}$$

is smaller than 1, then the map

$$\mathcal{B}(1) \to \ell^\infty(\Lambda), \quad f \mapsto (f(\lambda))_{\lambda \in \Lambda}$$

[5] Of course this is over-simplified. Signal analysis is a mutli-faceted discipline and cannot be classified as a sub-area of complex analysis. For example, we emphasized its communication theory aspect in Subsection 1.1. But the complex analysis viewpoint is convenient here.

[6] Here we state a (seemingly) simpler version of the theorem for clarifying the argument, but indeed this statement is equivalent to Lemma 2.4.

is surjective. Therefore only the lower/upper density of Λ determines its sampling/interpolation property.

When $k \geq 2$, the situation is much more messy. We have no clear theorems valid for irregular sampling and interpolation (see [OU12] for the details). For example, consider

$$\Lambda_1 = \left\{ \left(\varepsilon^2 t_1, \frac{t_2}{\varepsilon} \right) \middle| t_1, t_2 \in \mathbb{Z} \right\}, \quad \Lambda_2 = \left\{ \left(\varepsilon t_1, \frac{t_2}{\varepsilon^2} \right) \middle| t_1, t_2 \in \mathbb{Z} \right\} \quad (\varepsilon > 0).$$

Suppose ε is very small. The density of Λ_1 is equal to $1/\varepsilon$ (very large) and the density of Λ_2 is equal to ε (very small). The function $f(z_1, z_2) = \sin(\pi \varepsilon z_2) \in \mathcal{B}(1,1)$ vanishes over Λ_1 but is not identically zero. Thus Λ_1 is not a *sampling set* for functions in $\mathcal{B}(1,1)$ although its density is very large. Similarly, it is also easy to prove that Λ_2 is not an *interpolation set* for $\mathcal{B}(1,1)$, namely the map

$$\mathcal{B}(1,1) \to \ell^\infty(\Lambda_2), \quad f \mapsto (f(\lambda))_{\lambda \in \Lambda_2}$$

is not surjective, although its density is very small.

In the proof of Main Theorem 2 we need to construct interpolating functions for some *irregular* sets $\Lambda \subset \mathbb{R}^k$. The paper [GT] addressed the same problem (in dimension one) by employing Beurling's interpolation theorem mentioned above. But this theorem is not valid in higher dimension.

(2) Zero points of band-limited maps $f : \mathbb{C}^k \to \mathbb{C}^k$: One crucial ingredient of the proof of Main Theorem 2 is the study of (isolated) zero points[7] of some holomorphic maps $f = (f_1, \dots, f_k) : \mathbb{C}^k \to \mathbb{C}^k$ all of whose entries f_i are band-limited functions. We call such a map f a **band-limited map**. It becomes important to know the growth of

$$\{z \in \mathbb{C}^k |\, z \text{ is an isolated zero point of } f \text{ with } |z| < R\} \tag{1.1}$$

as R goes to infinity. If $k = 1$, then this is a very easy problem. It follows from *Jensen's formula* or *Nevanlinna's first main theorem* ([Hay64, Section 1.3], [NW14, Section 1.1]) that if $f \in B(a)$ then

$$\int_1^R \#\{z \in \mathbb{C} |\, f(x) = 0, |z| < r\} \frac{dr}{r} \leq 2aR + \mathrm{const}_f.$$

In particular (1.1) grows at most linearly in R. Moreover if $f \in \mathbb{B}(a)$ has no zero points, then it must be of the form

$$f(z) = e^{\alpha z + \beta}$$

for some constants α and β.

[7] A point $p \in \mathbb{C}^k$ is a zero point of $f : \mathbb{C}^k \to \mathbb{C}^k$ if $f(p) = 0$. Notice that the domain and target of f have the same dimension. Therefore *generically* zero points of f are expected to be isolated.

When $k \geq 2$, the situation is radically different. First, we have no simple description of $f : \mathbb{C}^k \to \mathbb{C}^k$ having no zero points. For example, the map of the form

$$f(z_1, z_2) = (e^{\alpha z_1 + \beta}, f_2(z_1, z_2)), \quad (f_2(z_1, z_2)\text{: arbitrary band-limited function}),$$

have no zero points. Second (and more importantly), the set (1.1) may have *arbitrarily fast growth*. This inconvenient phenomena was first observed by Cornalba–Shiffman [CS72] and known as *the failure of the transcendental Bezout theorem*. Here we briefly review their construction (with minor modification). Let $\{\alpha_n\}_{n=1}^{\infty}$ be an arbitrary sequence of positive integers (say, $\alpha_n = 2^{2^n}$). We choose a polynomial $p_n(w)$ (of one-variable) having α_n zeros in $|w| < 1$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a (nonzero) rapidly decreasing function satisfying $\operatorname{supp} \hat{\varphi} \subset [-1/2, 1/2]$. We choose $\beta_n > 0$ so that the function

$$g_n(w) \overset{\text{def}}{=} \beta_n \, \varphi(w) p_n(w)$$

is bounded by 2^{-n} over $\mathbb{R}$. We define $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ as follows:

$$f_1(z_1, z_2) = \sin(\pi z_1), \quad f_2(z_1, z_2) = \sum_{n=1}^{\infty} g_n(z_2) \frac{\sin \pi(z_1 - n)}{\pi(z_1 - n)}.$$

Both f_1 and f_2 belong to $\mathcal{B}(1,1)$. The map f has α_n zero points of the form (n, w) ($|w| < 1$). Therefore

$$\#\{z \in \mathbb{C}^2 \,|\, z \text{ is an isolated zero point of } f \text{ and } |z| < n+1\} \geq \alpha_n.$$

As a conclusion, it is hard (and sometimes impossible) to control the set (1.1) in higher dimension.

The above two issues show that general theory of multi-dimensional signal analysis is not strong enough for the proof of Main Theorem 2. Thus we have to develop *tailored methods* specific for the situation of the theorem. This becomes the main technical task of the paper. We address the issues (1) and (2) in Sections 4 and 5 respectively. The techniques developed there seem to have some independent interests, and possibly further applications will be found in a future.

The key ideas are as follows. (Here we ignore many details for simplicity. The real arguments are different.)

(1)' Almost regular interpolation: As we mentioned above, we need to construct interpolation for some irregular sets $\Lambda \subset \mathbb{R}^k$. But our sets Λ are not completely general uniformly discrete sets. They are "almost" regular sets in the sense that we have a decomposition

$$\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$$

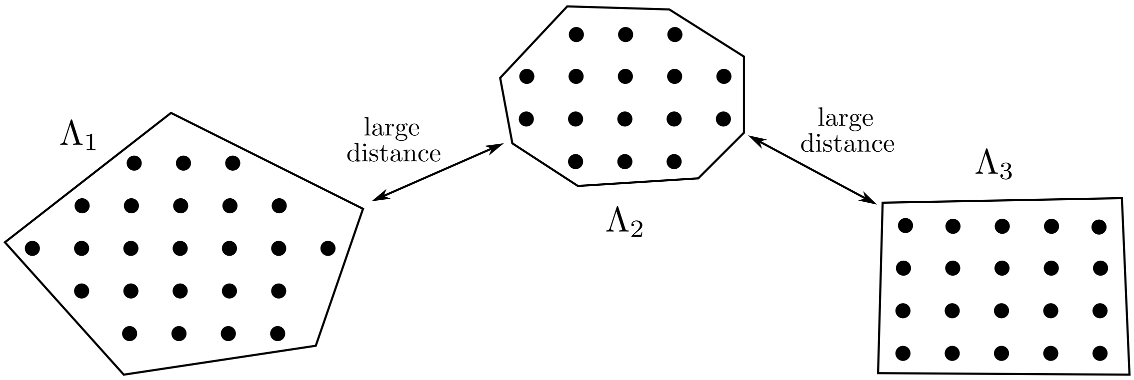

FIGURE 1.1. Almost regular interpolation set Λ. Different pieces Λ_m and Λ_n are sufficiently far from each other.

such that each Λ_n is a part of a regular interpolation set and that the distances between different pieces Λ_m and Λ_n are sufficiently large. See Figure 1.1. We develop a technique of constructing interpolation for such almost regular sets. As a conclusion, we can prove that they have (essentially) all the nice properties which regular interpolation sets possess.

(2)' Good/bad decomposition of the space: It is very difficult in general to control the zero points of holomorphic maps $f : \mathbb{C}^k \to \mathbb{C}^k$. The maps f used in the proof of Main Theorem 2 are very special, but still we cannot control all their (isolated) zero points. Therefore we simply give up to control *all* the zero points of f. Instead we decompose the space $\mathbb{C}^k$ into two disjoint regions:

$$\mathbb{C}^k = \mathbb{G} \cup \mathbb{B}.$$

In the "good region"$\mathbb{G}$, the zero points of f are very sparsely distributed and we can control them as we like. In the "bad region" $\mathbb{B}$, the zero points of f may have extremely high density, but the region $\mathbb{B}$ itself is very tiny. See Figure 1.2. We consider a *Voronoi diagram* with respect to the zero points of f. Then most of the space $\mathbb{C}^k$ is cover by very large (and hence well organized) tiles whose Voronoi centers are zero points in $\mathbb{G}$. We don't have any control of tiles whose centers are located in $\mathbb{B}$. They may have very complicated structure. But every messy phenomena is confined in the tiny region $\mathbb{B}$ and does not affect much the whole picture.

In Subsection 3.1 we state a key proposition (Proposition 3.1) and prove Main Theorem 2 assuming it. The rest of the paper is devoted to the proof of Proposition 3.1. The strategy of the proof is explained in Subsection 3.2. It might be better for some readers to go to Subsections 3.1 and 3.2 before reading Section 2.

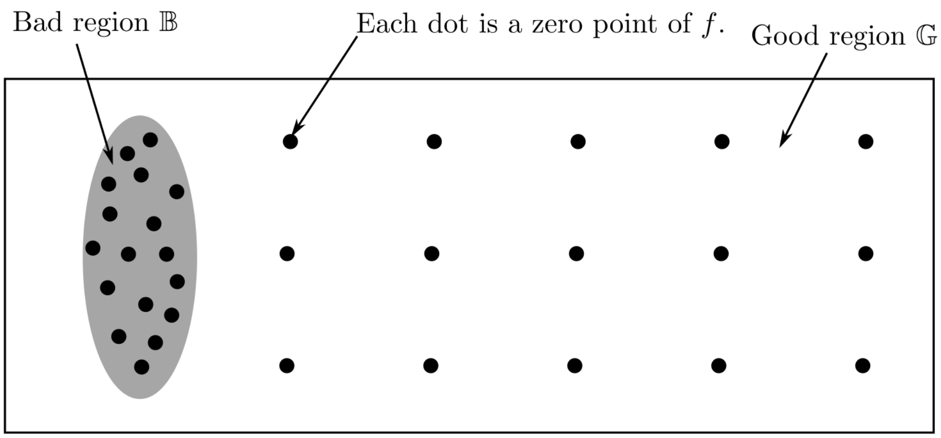

FIGURE 1.2. Schematic picture of the good/bad decomposition. Zero points of f are sparsely and almost regularly distributed inside the good region $\mathbb{G}$. The bad region $\mathbb{B}$ may contain plenty of zero points but it is tiny.

1.5. **Organization of the paper.** In Section 2 we review mean dimension, band-limited functions, simplicial complexes and convex sets. In Section 3 we state the main proposition and prove Main Theorem 2 assuming it. In particular, we give an overview of the proof of the main theorem. In Section 4 we construct interpolating functions. In Section 5 we construct certain "tiling-like maps" and study their properties. In Section 6 we prove the main proposition by using the results in Sections 2, 4 and 5. In Section 7 we explain some open problems.

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2. Preliminaries

2.1. **Review of mean dimension.** Here we review the definition of mean dimension [Gro99, LW00]. Throughout this paper we promise that every simplicial complex is finite (i.e. it has only finitely many simplicies) and that for a natural number N

$$[N] = \{0, 1, 2, \dots, N-1\}^k.$$

Let (X, d) be a compact metric space and Y a topological space. Let ε be a positive number and $f : X \to Y$ a continuous map. f is said to be an **ε-embedding** if $\mathrm{diam} f^{-1}(y) < \varepsilon$ for all $y \in Y$. We define the **ε-width dimension** $\mathrm{Widim}_\varepsilon(X, d)$ as the minimum integer n such that there exist an n-dimensional simplicial complex P and an ε-embedding $f : X \to P$. This is a *macroscopic dimension* of X in the scale of ε. The topological dimension $\dim X$ can be obtained by[8]

$$\dim X = \lim_{\varepsilon\to 0} \mathrm{Widim}_\varepsilon(X,d).$$

Let $(X,\mathbb{Z}^k,T)$ be a dynamical system, namely X is a compact metric space (with a distance d) and $T:\mathbb{Z}^k\times X\to X$ is a continuous action. For a finite subset $\Omega\subset\mathbb{Z}^k$ we define a distance d_Ω on X by

$$d_\Omega(x,y) = \max_{n\in\Omega} d(T^n x, T^n y).$$

It is easy to check the following *subadditivity* and *invariance*:

$$\mathrm{Widim}_\varepsilon(X,d_{\Omega_1\cup\Omega_2}) \le \mathrm{Widim}_\varepsilon(X,d_{\Omega_1}) + \mathrm{Widim}_\varepsilon(X,d_{\Omega_2}),$$
$$\mathrm{Widim}_\varepsilon(X,d_{a+\Omega}) = \mathrm{Widim}_\varepsilon(X,d_\Omega) \quad (a\in\mathbb{Z}^k, a+\Omega=\{a+x|\, x\in\Omega\}).$$

Then by the standard *division argument* we can show the existence of the limit

$$\lim_{N\to\infty}\frac{\mathrm{Widim}_\varepsilon\left(X,d_{[N]}\right)}{N^k}.$$

We define the **mean dimension** of $(X,\mathbb{Z}^k,T)$ by

$$\mathrm{mdim}(X,\mathbb{Z}^k,T) = \lim_{\varepsilon\to 0}\left(\lim_{N\to\infty}\frac{\mathrm{Widim}_\varepsilon\left(X,d_{[N]}\right)}{N^k}\right).$$

This is a topological invariant, namely it is independent of the choice of the distance d. We usually abbreviate this to $\mathrm{mdim}(X)$.

2.2. **Review of band-limited functions.** Here we review some basic theorems on band-limited functions. Throughout this subsection we assume that $a_1,\dots,a_k$ are positive numbers. We denote by $z = x+y\sqrt{-1}$ the standard coordinate of $\mathbb{C}^k$ with $x=(x_1,\dots,x_k), y=(y_1,\dots,y_k)\in\mathbb{R}^k$.

Lemma 2.1. *Let $f:\mathbb{C}^k\to\mathbb{C}$ be a holomorphic function such that there exists $C>0$ satisfying $\left|f\left(x+y\sqrt{-1}\right)\right| \le Ce^{\pi(a_1|y_1|+\cdots+a_k|y_k|)}$. Then it follows that*

$$\left|f\left(x+y\sqrt{-1}\right)\right| \le \|f\|_{L^\infty(\mathbb{R}^k)}\, e^{\pi(a_1|y_1|+\cdots+a_k|y_k|)}.$$

Proof. The paper [GT, Lemma 2.1] proves this for $f:\mathbb{C}\to\mathbb{C}$. So we consider the case of $k>1$. Fix $x+y\sqrt{-1}\in\mathbb{C}^k$. For $s+t\sqrt{-1}\in\mathbb{C}$ we set $g\left(s+t\sqrt{-1}\right) = f\left(x+y\left(s+t\sqrt{-1}\right)\right)$. g is a holomorphic function satisfying

$$\left|g\left(s+t\sqrt{-1}\right)\right| \le Ce^{\pi|t|(a_1|y_1|+\cdots+a_k|y_k|)}.$$

By the statement of the one variable case, $\left|g\left(s+t\sqrt{-1}\right)\right|$ is bounded by

$$\|g\|_{L^\infty(\mathbb{R})}\, e^{\pi|t|(a_1|y_1|+\cdots+a_k|y_k|)} \le \|f\|_{L^\infty(\mathbb{R}^k)}\, e^{\pi|t|(a_1|y_1|+\cdots+a_k|y_k|)}.$$

[8] If readers do not know the definition of topological dimension, then they may think that this is the *definition* of topological dimension. Only one nontrivial point is that $\dim X$ is a *topological invariant* whereas $\mathrm{Widim}_\varepsilon(X,d)$ depends on the distance d.

Letting $s=0$ and $t=1$, we get the statement. $\square$

Lemma 2.2 (Paley–Wiener's theorem)**.** *Let $f:\mathbb{R}^k\to\mathbb{C}$ be a bounded continuous function. Then the following two conditions are equivalent.*

(1) *The Fourier transform $\hat{f}$ is supported in $\prod_{i=1}^k[-a_i/2,a_i/2]$.*
(2) *f can be extended to a holomorphic function in $\mathbb{C}^k$ such that there exists $C>0$ satisfying $\left|f\left(x+y\sqrt{-1}\right)\right|\le Ce^{\pi(a_1|y_1|+\dots+a_k|y_k|)}$.*

Proof. This is a special case of a distribution version of Paley–Wiener's theorem ([Sch66, Chapter 7, Section 8]). Here we prove it directly only by using the standard version of Paley–Wiener's theorem ([DM72, Section 3.3]): For a (one-variable) L^2-function $f:\mathbb{R}\to\mathbb{C}$, the above two Conditions (1) and (2) are equivalent. Set $\Omega=\prod_{i=1}^k[-a_i/2,a_i/2]$.

First we assume that $f:\mathbb{R}^k\to\mathbb{C}$ is a L^2-function and prove (1) $\Leftrightarrow$ (2). Suppose f satisfies (1). Then

$$f(x)=\int_{\mathbb{R}^k}\hat{f}(\xi)e^{2\pi\sqrt{-1}x\cdot\xi}\,d\xi_1\dots d\xi_k=\int_\Omega\hat{f}(\xi)e^{2\pi\sqrt{-1}x\cdot\xi}\,d\xi_1\dots d\xi_k.$$

We extend f to a holomorphic function in $\mathbb{C}^k$ by

$$f\left(x+y\sqrt{-1}\right)=\int_\Omega\hat{f}(\xi)e^{2\pi\sqrt{-1}(x+y\sqrt{-1})\cdot\xi}\,d\xi_1\dots d\xi_k.$$

Then we can check Condition (2):

$$\left|f\left(x+y\sqrt{-1}\right)\right|\le\int_\Omega|\hat{f}(\xi)|e^{-2\pi y\cdot\xi}\,d\xi_1\dots d\xi_k\le\text{const}\cdot e^{\pi(a_1|y_1|+\dots+a_k|y_k|)}.$$

Next suppose f satisfies (2). Fix $\xi\in\mathbb{R}^k$ and suppose $|\xi_1|>a_1/2$. We will show $\hat{f}(\xi)=0$. We fix $x_2,\dots,x_k$ and consider $f(x_1,\dots,x_k)$ as a one-variable function of variable x_1. This is a L^2-function (of variable x_1) for almost every choice of $(x_2,\dots,x_k)$. It follows from $\left|f\left(x_1+y_1\sqrt{-1},x_2,\dots,x_k\right)\right|\le\text{const}\cdot e^{\pi a_1|y_1|}$ and the standard Paley–Wiener's theorem that

$$\int_{-\infty}^{\infty}f(x_1,x_2,\dots,x_k)e^{-2\pi\sqrt{-1}x_1\xi_1}\,dx_1=0\quad\text{for a.e. }(x_2,\dots,x_k).$$

Therefore

$$\hat{f}(\xi)=\int_{\mathbb{R}^{k-1}}\left(\int_{-\infty}^{\infty}f(x_1,x_2,\dots,x_k)e^{-2\pi\sqrt{-1}x_1\xi_1}\,dx_1\right)e^{-2\pi\sqrt{-1}(x_2\xi_2+\dots+x_k\xi_k)}\,dx_2\dots dx_k=0.$$

Thus the proof has been completed under the assumption $f\in L^2$.

Next we only assume that $f:\mathbb{R}^k\to\mathbb{C}$ is a bounded continuous function. We choose a rapidly decreasing function $\varphi:\mathbb{R}^k\to\mathbb{R}$ such that $\max_{x\in\mathbb{R}^k}|\varphi(x)|=\varphi(0)=1$ and $\operatorname{supp}\hat{\varphi}\subset(-1/2,1/2)^k$. We set $\varphi_\varepsilon(x)=\varphi(\varepsilon x)$ for $\varepsilon>0$. It satisfies $\hat{\varphi}_\varepsilon(\xi)=\varepsilon^{-k}\hat{\varphi}(\xi/\varepsilon)$ and hence $\operatorname{supp}\hat{\varphi}_\varepsilon\subset(-\varepsilon/2,\varepsilon/2)^k$. φ_ε converges to 1 uniformly over every compact subset of $\mathbb{R}^k$ as $\varepsilon \to 0$. Since φ_ε is rapidly decreasing (and hence in L^2), we can extend it to a holomorphic function satisfying

$$\left|\varphi_\varepsilon\left(x + y\sqrt{-1}\right)\right| \le e^{\pi\varepsilon(|y_1|+\cdots+|y_k|)} \quad \text{(here we used Lemma 2.1).} \tag{2.1}$$

We set $f_\varepsilon(x) = \varphi_\varepsilon(x) f(x)$. Then f_ε is a L^2-function and satisfies $|f_\varepsilon(x)| \le |f(x)|$.

Suppose f satisfies (1). Then $\hat{f}_\varepsilon$ is supported in $\prod_{i=1}^k [-(a_i+\varepsilon)/2, (a_i+\varepsilon)/2]$. Thus by the previous argument for L^2-functions and Lemma 2.1, we can extend f_ε to a holomorphic function satisfying

$$\begin{aligned} \left|f_\varepsilon\left(x + y\sqrt{-1}\right)\right| &\le \|f_\varepsilon\|_{L^\infty(\mathbb{R}^k)}\, e^{\pi\{(a_1+\varepsilon)|y_1|+\cdots+(a_k+\varepsilon)|y_k|\}} \\ &\le \|f\|_{L^\infty(\mathbb{R}^k)}\, e^{\pi\{(a_1+\varepsilon)|y_1|+\cdots+(a_k+\varepsilon)|y_k|\}}. \end{aligned} \tag{2.2}$$

Then we can extend f to a *meromorphic function* in $\mathbb{C}^k$ by

$$f\left(x + y\sqrt{-1}\right) = \frac{f_\varepsilon\left(x + y\sqrt{-1}\right)}{\varphi_\varepsilon\left(x + y\sqrt{-1}\right)}.$$

This is independent of $\varepsilon > 0$ because of the unique continuation. Since φ_ε converges to 1 uniformly over every compact subset, $f\left(x + y\sqrt{-1}\right)$ is actually a holomorphic function (namely it does not have neither poles nor indeterminacy points). By taking the limit in (2.2) we get

$$\left|f\left(x + y\sqrt{-1}\right)\right| \le \|f\|_{L^\infty(\mathbb{R}^k)}\, e^{\pi(a_1|y_1|+\cdots+a_k|y_k|)}.$$

Next suppose f satisfies (2). By (2.1) and Lemma 2.1, we get (2.2) again. Since $f_\varepsilon \in L^2$, the Fourier transform $\hat{f}_\varepsilon$ is supported in $\prod_{i=1}^k [-(a_i+\varepsilon)/2, (a_i+\varepsilon)/2]$. The functions $\hat{f}_\varepsilon$ converge to $\hat{f}$ in the topology of distribution. Thus $\hat{f}$ is supported in $\prod_{i=1}^k [-a_i/2, a_i/2]$. $\square$

Corollary 2.3. *The space $\mathcal{B}_1(a_1, \ldots, a_k)$ of continuous functions $f : \mathbb{R}^k \to [-1, 1]$ band-limited in $\prod_{i=1}^k [-a_i/2, a_i/2]$ is compact with respect to the topology of uniform convergence over compact subsets of $\mathbb{R}^k$.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathcal{B}_1(a_1, \ldots, a_k)$. By Lemmas 2.1 and 2.2 the functions f_n can be holomorphically extended over $\mathbb{C}^k$ satisfying $|f_n(z)| \le e^{\pi(a_1|y_1|+\cdots+a_k|y_k|)}$. In particular $\{f_n\}$ is bounded over every compact subset of $\mathbb{C}^k$. By the Cauchy integration formula, it becomes an equicontinuous family over every compact subset (i.e. normal family). So we can choose a converging subsequence by the Arzela–Ascoli theorem. $\square$

Lemma 2.4 (Sampling theorem)**.** *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function band-limited in $\prod_{i=1}^k [-a_i/2, a_i/2]$. Suppose that there exist positive numbers $b_1, \ldots, b_k$ such that $a_i b_i < 1$ for all $1 \le i \le k$ and that f vanishes over*

$$\{(b_1 n_1, \ldots, b_k n_k) \mid n_1, \ldots, n_k \in \mathbb{Z}\}.$$

Then f is identically zero.

Proof. As in the proof of Lemma 2.2, it is enough to prove the statement under the assumption that f is a L^2-function. Then the proof is standard as follows. Set $g(x_1, \dots, x_k) = f(b_1 x_1, \dots, b_k x_k)$. This vanishes on $\mathbb{Z}^k$.

$$\hat{g}(\xi) = \frac{1}{b_1 \dots b_k} \hat{f}\left(\frac{\xi_1}{b_1}, \dots, \frac{\xi_k}{b_k}\right).$$

Then $\hat{g}$ is supported in

$$\prod_{i=1}^{k} \left[-\frac{a_i b_i}{2}, \frac{a_i b_i}{2}\right] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^k.$$

Consider the Fourier series of $\hat{g}(\xi)$ for $\xi \in (-1/2, 1/2)^k$:

$$\hat{g}(\xi) = \sum_{n \in \mathbb{Z}^k} a_n e^{-2\pi\sqrt{-1} n \cdot \xi}.$$

The coefficients a_n are given by

$$a_n = \int_{[-1/2,1/2]^k} e^{2\pi\sqrt{-1} n \cdot \xi} \hat{g}(\xi) d\xi_1 \dots d\xi_k = \overline{\mathcal{F}}\left(\mathcal{F}(g)\right)(n) = g(n) = 0.$$

Thus $\hat{g} = 0$. This implies $g = 0$ and $f = 0$. $\square$

2.3. **Technical results on simplicial complexes.** The results of this subsection are used only in Section 6. So readers can postpone reading this subsection until they come to Section 6. Recall that we promised that every simplicial complex is finite. (This is mainly for the simplicity of the exposition.) Let P be a simplicial complex. A map $f : P \to \mathbb{R}^n$ is said to be **simplicial** if it has the form

$$f\left(\sum_{k=0}^{m} \lambda_k v_k\right) = \sum_{k=0}^{m} \lambda_k f(v_k), \quad \left(\lambda_k \geq 0, \sum_{k=0}^{m} \lambda_k = 1\right),$$

on every simplex $\Delta \subset P$, where $v_0, \dots, v_m$ are the vertices of Δ. We define $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Lemma 2.5 (Approximation lemma)**.** *Let ε and δ be positive numbers. Let (X, d) be a compact metric space and P a simplicial complex. Let $\pi : X \to P$ be an ε-embedding and $f : X \to \mathbb{R}^n$ a continuous map satisfying*

$$d(x, y) < \varepsilon \Longrightarrow \|f(x) - f(y)\|_\infty < \delta.$$

Then, after taking a sufficiently fine subdivision of P, we can find a simplicial map $g : P \to \mathbb{R}^n$ satisfying

$$\|f(x) - g(\pi(x))\|_\infty < \delta \quad (\forall x \in X).$$

Proof. We reproduce the proofs in [GLT16, GT] for the completeness. By subdividing P we can assume that for every vertex $v \in P$ the diameter of $\pi^{-1}(O(v))$ is smaller than ε, where $O(v)$ is the open star of v, i.e. the union of the relative interiors of all simplices containing v. For each vertex $v \in P$ we choose $g(v) \in f(\pi^{-1}(O(v)))$. (If this is empty, then we choose arbitrary $g(v)$.) We linearly extend g all over P. Take $x \in X$ and let $\Delta \subset P$ be the simplex containing $\pi(x)$ in its relative interior. Every vertex $v \in \Delta$ satisfies $\pi(x) \in O(v)$ and hence $\|f(x) - g(v)\|_\infty < \delta$ because the diameter of $f(\pi^{-1}(O(v)))$ (which contains both $f(x)$ and $g(v)$) is smaller than δ. $g(\pi(x))$ is a convex combination of such $g(v)$. Hence $\|f(x) - g(\pi(x))\|_\infty < \delta$. □

Let P be a simplicial complex. We denote by V the set of all vertices of P. The set $\mathrm{Hom}(P, \mathbb{R}^n)$ of simplicial maps $f : P \to \mathbb{R}^n$ can be identified with $(\mathbb{R}^n)^V$, which is endowed with the natural topology.

Lemma 2.6 (Embedding lemma). (1) *If* $\dim P < n/2$ *then the set of simplicial embeddings (i.e. injective simplicial maps)* $f : P \to \mathbb{R}^n$ *is an open dense subset of* $\mathrm{Hom}(P, \mathbb{R}^n)$.

(2) *Let* U *be the set of simplicial maps* $f : P \to \mathbb{R}^n$ *such that, for every subcomplex* $Q \subset P$ *and every subset* $A \subset \{1, 2, \dots, n\}$ *with* $\#A > 2\dim Q$*, the map*

$$Q \to \mathbb{R}^A, \quad x \mapsto f(x)|_A$$

is an embedding. Then U *is open and dense in* $\mathrm{Hom}(P, \mathbb{R}^n)$.

Proof. The statement (2) follows from (1) because the natural map (restriction)

$$\mathrm{Hom}(P, \mathbb{R}^n) \to \mathrm{Hom}(Q, \mathbb{R}^A)$$

is an open map and thus the preimage of an open and dense set under this map is also open and dense. (Notice that it corresponds to the natural projection $(\mathbb{R}^n)^{V(P)} \to (\mathbb{R}^A)^{V(Q)}$ where $V(P)$ and $V(Q)$ are the sets of vertices of P and Q.) Thus it is enough to prove (1). First we show that simplicial embeddings $f : P \to \mathbb{R}^n$ form an open set. Let $f : P \to \mathbb{R}^n$ be a simplicial embedding. We would like to show that "injectiveness" is preserved under a small perturbation of f.

- Let $\Delta \subset P$ be a simplex with the vertices $v_0, \dots, v_m$. Then $f|_\Delta : \Delta \to \mathbb{R}^n$ is an injection, which means that $f(v_0), \dots, f(v_m)$ are affinely independent. If $g : P \to \mathbb{R}^n$ is sufficiently close to f then $g(v_0), \dots, g(v_m)$ are also affinely independent.
- Let $\Delta_1, \Delta_2 \subset P$ be two disjoint simplices. Then the distance between $f(\Delta_1)$ and $f(\Delta_2)$ is positive. This condition is certainly preserved under a small perturbation.
- Let $\Delta_1, \Delta_2 \subset P$ be two simplices which share a nonempty simplex $\Delta_1 \cap \Delta_2$. We assume $\Delta_1 \neq \Delta_1 \cap \Delta_2$ and $\Delta_2 \neq \Delta_1 \cap \Delta_2$. Consider $V(\Delta_1) \setminus V(\Delta_1 \cap \Delta_2)$ and $V(\Delta_2) \setminus V(\Delta_1 \cap \Delta_2)$, and denote their convex hulls by Δ_1' and Δ_2' respectively. Fix $v \in V(\Delta_1 \cap \Delta_2)$. The map $f|_{\Delta_1 \cup \Delta_2} : \Delta_1 \cup \Delta_2 \to \mathbb{R}^n$ is injective, which means the positivity of

$$\min \left\{ \text{Angle between the vectors } f(\overrightarrow{vw_1}) \text{ and } f(\overrightarrow{vw_2}) \,|\, w_1 \in \Delta_1', w_2 \in \Delta_2' \right\}.$$

  This condition is preserved under a small perturbation.

The above proves that sufficiently small perturbations of f are also injective.

Next we prove that simplicial embeddings $f : P \to \mathbb{R}^n$ form a dense set. Let $f : P \to \mathbb{R}^n$ be an arbitrary simplicial map and $U \subset \mathrm{Hom}(P, \mathbb{R}^n)$ an open neighborhood of f. By simple linear algebra we can choose $g \in U$ such that for any subset $\{v_1, \dots, v_m\} \subset V(P)$ with $m \leq n+1$ the points $g(v_1), \dots, g(v_m)$ are affinely independent. As $2\dim P + 1 \leq n$, $g|_{\Delta_1 \cup \Delta_2}$ is an embedding for any two simplices $\Delta_1, \Delta_2 \subset P$ and thus g is an embedding. $\square$

How to use Lemmas 2.5 and 2.6: We would like to illustrate the above two lemmas by a simple application. (This is a prototype of the later argument but logically independent of the proof of the main theorems. So experienced readers can skip it.) Let (X, d) be a compact metric space. We prove a classical result in the dimension theory: If $\dim X < n/2$ then X can be topologically embedded into $\mathbb{R}^n$. The set of all embeddings $f : X \to \mathbb{R}^n$ is equal to

$$\bigcap_{m=1}^{\infty} \{f : X \to \mathbb{R}^n |\, (1/m)\text{-embedding}\}. \tag{2.3}$$

So it is a G_δ-subset of the space of all continuous maps $f : X \to \mathbb{R}^n$ since "$(1/m)$-embedding" is an open condition. Fix a natural number m and a positive number δ. Let $f : X \to \mathbb{R}^n$ be an arbitrary continuous map. Choose $0 < \varepsilon < 1/m$ so that

$$d(x, y) < \varepsilon \Longrightarrow \|f(x) - f(y)\|_\infty < \delta.$$

By $\dim X = \lim_{\varepsilon \to 0} \mathrm{Widim}_\varepsilon(X, d)$, we can find a simplicial complex P of $\dim P \leq \dim X < n/2$ and an ε-embedding $\pi : X \to P$. By Lemma 2.5 there exists a simplicial map $g : P \to \mathbb{R}^n$ satisfying $\|f(x) - g(\pi(x))\|_\infty < \delta$ for all $x \in X$. By $\dim P < n/2$ and Lemma 2.6 (1) we can find a simplicial *embedding* $h : P \to \mathbb{R}^n$ satisfying $\|g(p) - h(p)\|_\infty < \delta$ for all $p \in P$. Then the map $h \circ \pi : X \to \mathbb{R}^n$ is an ε-embedding (and hence $(1/m)$-embedding by $\varepsilon < 1/m$) and satisfies $\|f(x) - h \circ \pi(x)\|_\infty < 2\delta$ for all $x \in X$. Since δ is arbitrary, this shows that

$$\{f : X \to \mathbb{R}^n |\, (1/m)\text{-embedding}\}$$

is dense in the space of all continuous maps from X to $\mathbb{R}^n$. Thus, by the Baire Category Theorem, the set (2.3) is dense G_δ. In particular there exists a topological embedding of X into $\mathbb{R}^n$.

2.4. **A lemma on convex sets.** The result of this subsection is used only in Section 6. Let $r > 0$ and $W \subset \mathbb{R}^k$. We define $\partial_r W$ as the set of all $t \in \mathbb{R}^k$ such that the closed r-ball $B_r(t)$ around t have non-empty intersections both with W and $\mathbb{R}^k \setminus W$. We set $\mathrm{Int}_r W = W \setminus \partial_r W$.

Lemma 2.7. *For any $c > 1$ and $r > 0$ there exists $R > 0$ such that if a bounded closed convex subset $W \subset \mathbb{R}^k$ satisfies $\mathrm{Int}_R W \neq \emptyset$ then*

$$|W \cup \partial_r W| < c\,|\mathrm{Int}_r W|.$$

Here $|\cdot|$ *denotes the* k*-dimensional volume (Lebesgue measure).*

Proof. We can assume $0 \in \mathrm{Int}_R W$. For $a > 0$ we set $aW = \{at | \, t \in W\}$.

Claim 2.8. $\partial_r W \subset \left(1 + \frac{r}{R}\right) W$.

Proof. Take $u \in \mathbb{R}^k$ outside of $\left(1 + \frac{r}{R}\right) W$. There exists a hyperplane $H = \{a_1 x_1 + \cdots + a_k x_k = 1\} \subset \mathbb{R}^k$ *separating* u and $\left(1 + \frac{r}{R}\right) W$. Namely

$$u \in \{a_1 x_1 + \cdots + a_k x_k > 1\}, \quad \left(1 + \frac{r}{R}\right) W \subset \{a_1 x_1 + \cdots + a_k x_k < 1\}.$$

Then

$$W \subset \left\{a_1 x_1 + \cdots + a_k x_k < \left(1 + \frac{r}{R}\right)^{-1}\right\}.$$

It follows that the distance between u and W is greater than the distance between H and $\left(1 + \frac{r}{R}\right)^{-1} H$. Let h be the distance between $\left(1 + \frac{r}{R}\right)^{-1} H$ and the origin. $h > R$ since $0 \in \mathrm{Int}_R W$. Then the distance between H and $\left(1 + \frac{r}{R}\right)^{-1} H$ is

$$\left(1 + \frac{r}{R}\right) h - h = \frac{rh}{R} > r.$$

This implies that u does not belong to $\partial_r W$. $\square$

Claim 2.9. $\partial_r W \cap \left(1 - \frac{r}{R}\right) W = \emptyset$ *and hence* $\mathrm{Int}_r W \supset \left(1 - \frac{r}{R}\right) W$.

Proof. Almost the same argument shows that if we take a point u outside of W then the distance between u and $\left(1 - \frac{r}{R}\right) W$ is greater than r. This implies that every point $v \in \left(1 - \frac{r}{R}\right) W$ satisfies $B_r(v) \subset W$ and hence does not belong to $\partial_r W$. $\square$

It follows from these two claims that

$$|W \cup \partial_r W| \leq \left|\left(1 + \frac{r}{R}\right) W\right| = \left(1 + \frac{r}{R}\right)^k |W|,$$
$$|\mathrm{Int}_r W| \geq \left|\left(1 - \frac{r}{R}\right) W\right| = \left(1 - \frac{r}{R}\right)^k |W|.$$

Thus

$$|W \cup \partial_r W| \leq \left(\frac{1 + \frac{r}{R}}{1 - \frac{r}{R}}\right)^k |\mathrm{Int}_r W|.$$

Since $c > 1$, if r/R is sufficiently small then the right-hand side is smaller than $c \, |\mathrm{Int}_r W|$. $\square$

3. Main proposition

3.1. **Statement of the main proposition.** Main Theorem 2 in Subsection 1.3 follows from the next proposition whose proof occupies the rest of the paper.

Proposition 3.1 (Main Proposition). *Let $a_1, \dots, a_k$ and δ be positive numbers. Let $(X, \mathbb{Z}^k, T)$ be a dynamical system and d a distance on X. Let $f : X \to \mathcal{B}_1(a_1, \dots, a_k)$ be a $\mathbb{Z}^k$-equivariant continuous map. If X has the marker property and satisfies*

$$\mathrm{mdim}(X) < \frac{a_1 \dots a_k}{2}$$

then there exists a $\mathbb{Z}^k$-equivariant continuous map $g : X \to \mathcal{B}_1(a_1 + \delta, \dots, a_k + \delta)$ such that

- $\|f(x) - g(x)\|_{L^\infty(\mathbb{R}^k)} < \delta$ *for all* $x \in X$.
- g *is a* δ*-embedding with respect to the distance* d.

It is important to note that the map g takes values in $\mathcal{B}_1(a_1 + \delta, \dots, a_k + \delta)$ which is slightly larger than the target $\mathcal{B}_1(a_1, \dots, a_k)$ of the original map f. Here we prove that Proposition 3.1 implies Main Theorem 2.

Proof of Main Theorem 2, assuming Proposition 3.1. The proof is very close to the standard proof of the Baire Category Theorem. We assume $\mathrm{diam}(X, d) < 1$ by rescaling the distance (just for simplicity of the notation). For $1 \le i \le k$ we choose sequences $\{a_{in}\}_{n=1}^\infty$ such that

$$0 < a_{i1} < a_{i2} < a_{i3} < \dots < a_i, \quad \mathrm{mdim}(X) < \frac{a_{1n} a_{2n} \dots a_{kn}}{2} \quad (\forall n \ge 1).$$

For $n \ge 1$ we inductively define a positive number δ_n and a $\mathbb{Z}^k$-equivariant $(1/n)$-embedding (with respect to d) $f_n : X \to \mathcal{B}_1(a_{1n}, \dots, a_{kn})$. First we set

$$\delta_1 = 1, \quad f_1(x) = 0 \quad (\forall x \in X).$$

Notice that f_1 is a 1-embedding because $\mathrm{diam}(X, d) < 1$. Suppose we have defined δ_n and f_n. Since f_n is a $(1/n)$-embedding, we can choose $0 < \delta_{n+1} < \delta_n/2$ such that if a $\mathbb{Z}^k$-equivariant continuous map $g : X \to \mathcal{B}_1(a_1, \dots, a_k)$ satisfies

$$\sup_{x \in X} \|f_n(x) - g(x)\|_{L^\infty(\mathbb{R}^k)} \le \delta_{n+1}$$

then g is also a $(1/n)$-embedding. By applying Proposition 3.1 to f_n, we can find a $\mathbb{Z}^k$-equivariant continuous map $f_{n+1} : X \to \mathcal{B}_1(a_{1,n+1}, \dots, a_{k,n+1})$ such that

- $\sup_{x \in X} \|f_n(x) - f_{n+1}(x)\|_{L^\infty(\mathbb{R}^k)} < \delta_{n+1}/2$.
- f_{n+1} is a $\frac{1}{n+1}$-embedding with respect to d.

For $n > m \geq 1$

$$\begin{aligned}\sup_{x\in X} \|f_n(x) - f_m(x)\|_{L^\infty(\mathbb{R}^k)} &\leq \sum_{l=m}^{n-1} \sup_{x\in X} \|f_l(x) - f_{l+1}(x)\|_{L^\infty(\mathbb{R}^k)} \\ &< \sum_{l=m}^{n-1} \frac{\delta_{l+1}}{2} \\ &< \sum_{l=1}^{\infty} 2^{-l}\delta_{m+1} = \delta_{m+1} \to 0 \quad (m \to \infty).\end{aligned}$$

Here we used $\delta_{l+1} < \delta_l/2$. By taking a limit, we find a $\mathbb{Z}^k$-equivariant continuous map $f : X \to \mathcal{B}_1(a_1, \dots, a_k)$ satisfying

$$\sup_{x\in X} \|f(x) - f_n(x)\|_{L^\infty(\mathbb{R}^k)} \leq \delta_{n+1}$$

for all $n \geq 1$. It follows from the definition of δ_{n+1} that f is a $(1/n)$-embedding. Since n is arbitrary, f is an embedding. □

The above proof used a flexibility exhibited by band-limited signals and not by discrete signals. At each step of the induction, the target $\mathcal{B}_1(a_{1,n+1}, \dots, a_{k,n+1})$ of the map f_{n+1} is slightly bigger than the previous target $\mathcal{B}_1(a_{1n}, \dots, a_{kn})$. The differences $a_{i,n+1} - a_{in}$ converge to zero as n goes to infinity. Such an argument is possible because $a_1, \dots, a_k$ are *continuous* parameters. We cannot apply the same argument to $\left([0,1]^D\right)^{\mathbb{Z}^k}$ because it depends on a *discrete* parameter D.

3.2. **Strategy of the proof.** Most known theorems about embedding (e.g. the Whitney embedding theorem for manifolds) are proved by *perturbation*. The proof of Proposition 3.1 also uses this idea *but with a twist*. The map g in the statement of Proposition 3.1 will have the form

$$g(x) = g_1(x) + g_2(x), \quad (x \in X).$$

The first term $g_1(x)$ is a band-limited function in $\mathcal{B}_1(a_1, \dots, a_k)$, which is constructed by perturbing the function $f(x) \in \mathcal{B}_1(a_1, \dots, a_k)$. The second term $g_2(x)$ is also a band-limited function whose frequencies are restricted in a compact subset of

$$\prod_{i=1}^{k}\left[-\frac{a_i+\delta}{2}, \frac{a_i+\delta}{2}\right] \setminus \prod_{i=1}^{k}\left[-\frac{a_i}{2}, \frac{a_i}{2}\right].$$

The construction of $g_2(x)$ is (essentially) independent of $f(x)$. The function $g_2(x)$ *encodes how to perturb* $f(x)$. In other words, the function $g_1(x)$ is constructed by perturbing the function $f(x)$, and the perturbation *method* is determined by the function $g_2(x)$. Probably this explanation is not very clear. So let us try a more naive explanation. Consider *cooking*. Each point $x \in X$ is a cook. The function $f(x)$ is an ingredient of cooking (e.g. a raw salmon) that the cook x chooses. The function $g_2(x)$ is a cookware (e.g. kitchen knife and oven) that the cook x possesses. The function $g_1(x)$ is the result of cooking (e.g. grilled salmon). The grilled salmon $g_1(x)$ is made from the raw salmon $f(x)$ by using the knife and oven $g_2(x)$. The knife and oven $g_2(x)$ are independent of the raw salmon $f(x)$.

The Fourier transforms $\mathcal{F}(g_1(x))$ and $\mathcal{F}(g_2(x))$ have disjoint supports. Hence if we have the equation $g(x) = g(y)$ for some x and y in X, then it follows that

$$g_1(x) = g_1(y), \quad g_2(x) = g_2(y).$$

We would like to deduce $d(x, y) < \delta$ from these equations. In other words we try to determine who is the cook (up to some small error) by knowing the result of cooking $(g_1(x) = g_1(y))$ and what cookware was used $(g_2(x) = g_2(y))$.

More precisely, the proof goes as follows. Take $x \in X$. The function $f(x)$ is defined over $\mathbb{R}^k$. It is difficult to control functions over unbounded domains. So we introduce a *tiling* (indexed by $\mathbb{Z}^k$)

$$\mathbb{R}^k = \bigcup_{n \in \mathbb{Z}^k} W(x, n),$$

such that

(1) Each tile $W(x, n)$ is a bounded convex set, and different tiles $W(x, n)$ and $W(x, m)$ are disjoint except for their boundaries.
(2) $W(x, n)$ depends continuously on $x \in X$ in the Hausdorff topology.
(3) The tiles are equivariant in the sense that

$$W(T^n x, m) = -n + W(x, n + m) = \{-n + t | \, t \in W(x, n + m)\}.$$

(4) "Most" part of the space $\mathbb{R}^k$ is covered by "sufficiently large" tiles.

We try to perturb the function $f(x)$ over each tile $W(x, n)$. We can construct a good perturbation over sufficiently large tiles. So if every tile is sufficiently large, then we can construct a good perturbation of $f(x)$ over the whole space $\mathbb{R}^k$. Unfortunately some tiles may be small in general. We cannot construct a good perturbation over such tiny tiles. Condition (4) helps us here. We will construct a "social welfare system" of the tiles $W(x, n)$. (This idea was first introduced in [GT].) Large tiles help small tiles and bear "additional" perturbations which are originally "duties" of small tiles. By using this social welfare system we construct the perturbation $g_1(x)$ of the function $f(x)$.

The map g_1 will have the following property: If x and y in X satisfy $g_1(x) = g_1(y)$ and

$$\forall n \in \mathbb{Z}^k : \quad W(x, n) = W(y, n) \tag{3.1}$$

then it follows that $d(x, y) < \delta$. Namely if we know $g_1(x)$ and the tiling $\{W(x, n)\}$ then we can recover the point x up to error δ. So the next problem is how to encode the information of the tiles. The function $g_2(x)$ is introduced for solving this problem. We encode all the information of the tiles $W(x, n)$ to a band-limited function $g_2(x)$. (Indeed the real argument below goes in a reverse way. First we construct the function $g_2(x)$, and then the tiles $W(x, n)$ are constructed from $g_2(x)$. So $g_2(x)$ knows everything about the tiles $W(x,n)$.) In particular the equation $g_2(x) = g_2(y)$ implies (3.1). As a conclusion, the function $g(x) = g_1(x) + g_2(x)$ satisfies the requirements of Proposition 3.1. This is the outline of the proof.

We would like to emphasize that the most important new idea introduced in this paper is the technique to encode the information of the tiling $\{W(x,n)\}$ into the band-limited function $g_2(x)$. This idea is new even in the one dimensional case. This is the main reason why Main Theorems 1 and 2 have some novelty even in the case of $k = 1$.

Section 4 introduces interpolating functions which will be used in the perturbation process. In Section 5 we construct the tiling $W(x,n)$ and their social welfare system and explain how to encode them into a band-limited function. The proof of Proposition 3.1 is finished in Section 6.

3.3. **Fixing some notations.** The proof of Proposition 3.1 is notationally messy. So here we gather some of the notations for the convenience of readers. Throughout the rest of the paper (except for Section 7 where we discuss open problems) we fix the following.

- $a_1, \dots, a_k$ and δ are positive numbers. We additionally assume

$$\delta < \min(1, a_1, \dots, a_k). \tag{3.2}$$

- $(X, \mathbb{Z}^k, T)$ is a dynamical system satisfying the marker property and

$$\mathrm{mdim}(X) < \frac{a_1 \dots a_k}{2}.$$

- We fix positive *rational numbers* $\rho_1, \dots, \rho_k$ satisfying

$$\rho_i < a_i \quad (\forall 1 \le i \le k), \quad \mathrm{mdim}(X) < \frac{\rho_1 \dots \rho_k}{2}. \tag{3.3}$$

- We define a lattice $\Gamma \subset \mathbb{R}^k$ by

$$\Gamma = \left\{ \left(\frac{t_1}{\rho_1}, \dots, \frac{t_k}{\rho_k} \right) \middle| \, t_1, \dots, t_k \in \mathbb{Z} \right\}. \tag{3.4}$$

  Moreover we define $\Gamma_1 \subset \mathbb{R}^k$ by

$$\Gamma_1 = \bigcup_{n \in \mathbb{Z}^k} (n + \Gamma), \quad (n + \Gamma = \{n + \gamma \,|\, \gamma \in \Gamma\}). \tag{3.5}$$

  Since ρ_i are rational numbers, Γ_1 is also a lattice.

4. Interpolating functions

Recall that we fixed positive numbers $a_1, \dots, a_k$, positive rational numbers $\rho_1, \dots, \rho_k$ with $\rho_i < a_i$ and lattices $\Gamma \subset \Gamma_1 \subset \mathbb{R}^k$ in Subsection 3.3. We fix a positive number τ satisfying

$$\forall 1 \le i \le k : \quad \rho_i + \tau < a_i. \tag{4.1}$$

For $r > 0$ and $a \in \mathbb{R}^k$ we denote by $B_r(a)$ the closed r-ball around a, and we set $B_r = B_r(0)$. We choose a rapidly decreasing function $\chi_0 : \mathbb{R}^k \to \mathbb{R}$ satisfying $\chi_0(0) = 1$ and $\operatorname{supp} \mathcal{F}(\chi_0) \subset B_{\tau/2}$.

Notation 4.1. (1) We set

$$K_0 = \sup_{t \in \mathbb{R}^k} \sum_{\lambda \in \Gamma_1} |\chi_0(t - \lambda)|. \tag{4.2}$$

This is finite because χ_0 is rapidly decreasing and Γ_1 is a lattice.

(2) We fix $r_0 > 0$ satisfying

$$\sum_{\lambda \in \Gamma_1 \setminus B_{r_0}} |\chi_0(\lambda)| < \frac{1}{2}. \tag{4.3}$$

Definition 4.2. (1) A subset $\Lambda \subset \Gamma_1$ is called an **admissible set** if any two points $\lambda_1, \lambda_2 \in \Lambda$ satisfy at least one of the following two conditions:

- $\lambda_1 - \lambda_2 \in \Gamma$.
- $|\lambda_1 - \lambda_2| > r_0$.

This notion is an elaborated formulation of "almost regular interpolation set" introduced in Subsection 1.4. We denote by Ads the set of all admissible sets $\Lambda \subset \Gamma_1$.

(2) A function $p : \Gamma_1 \to [0, 1]$ is called an **admissible function** if the set

$$\{\lambda \in \Gamma_1 | \, p(\lambda) > 0\}$$

is an admissible set. We denote by Adf the set of all admissible functions.

We define the function $\operatorname{sinc} : \mathbb{R} \to \mathbb{R}$ by

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}, \quad \operatorname{sinc}(0) = 1.$$

This is one of the most famous functions in signal analysis. The function $\operatorname{sinc}(t)$ vanishes on $\mathbb{Z} \setminus \{0\}$ and its Fourier transform is supported in $[-1/2, 1/2]$. It satisfies $|\operatorname{sinc}(t)| \leq 1$ for all $t \in \mathbb{R}$.

Let $\Lambda \subset \Gamma_1$ be an admissible set. We denote by $\ell^\infty(\Lambda)$ the Banach space of all bounded functions $u : \Lambda \to \mathbb{R}$ endowed with the supremum norm $\|\cdot\|_\infty$. For $u \in \ell^\infty(\Lambda)$ we define a band-limited function $\varphi_\Lambda(u) \in \mathcal{B}(\rho_1 + \tau, \ldots, \rho_k + \tau)$ by

$$\varphi_\Lambda(u)(t) = \sum_{\lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda} u(\lambda) \chi_0(t - \lambda) \prod_{i=1}^k \operatorname{sinc}\left(\rho_i(t_i - \lambda_i)\right), \quad (t = (t_1, \ldots, t_k) \in \mathbb{R}^k).$$

Here ρ_i are rational numbers introduced in Subsection 3.3. It follows from the definition of K_0 that

$$\|\varphi_\Lambda(u)\|_{L^\infty(\mathbb{R}^k)} \leq K_0 \|u\|_\infty .$$

Since we promised $\rho_i + \tau < a_i$ in (4.1), the function $\varphi_\Lambda(u)$ belongs to $\mathcal{B}(a_1, \dots, a_k)$. We define a bounded linear operator $S_\Lambda : \ell^\infty(\Lambda) \to \ell^\infty(\Lambda)$ by

$$S_\Lambda(u) = \varphi_\Lambda(u)|_\Lambda \left(= (\varphi_\Lambda(u)(\lambda))_{\lambda\in\Lambda}\right).$$

Remark 4.3. When Λ is the empty set, we define $\ell^\infty(\Lambda)$ to be the trivial vector space consisting only of zero and $\varphi_\Lambda(0) = 0$, the operator S_Λ the identity operator of the trivial vector space.

Lemma 4.4. $\|S_\Lambda - \mathrm{id}\| \le 1/2$, *where the left-hand side is the operator norm of* $S_\Lambda - \mathrm{id}$.

Proof. For $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda$ the function

$$\chi_0(t-\lambda) \prod_{i=1}^k \operatorname{sinc}\left(\rho_i(t_i - \lambda_i)\right)$$

vanishes on $(\lambda + \Gamma) \setminus \{\lambda\}$ and its value at $t = \lambda$ is 1. Then by the admissibility of Λ

$$\varphi_\Lambda(u)(\lambda) - u(\lambda) = \sum_{\lambda'\in\Lambda\setminus B_{r_0}(\lambda)} u(\lambda')\chi_0(\lambda'-\lambda) \prod_{i=1}^k \operatorname{sinc}\left(\rho_i(\lambda'_i - \lambda_i)\right).$$

By the choice of r_0 in (4.3),

$$|\varphi_\Lambda(u)(\lambda) - u(\lambda)| \le \left(\sum_{\lambda'\in\Lambda\setminus B_{r_0}(\lambda)} |\chi_0(\lambda'-\lambda)|\right) \|u\|_\infty \le \frac{1}{2}\|u\|_\infty.$$

$\square$

Therefore the operator S_Λ has the inverse

$$S_\Lambda^{-1} = \sum_{n=0}^\infty (\mathrm{id} - S_\Lambda)^n \text{ with } \left\|S_\Lambda^{-1}\right\| \le 2.$$

For $u \in \ell^\infty(\Lambda)$ we define a band-limited function $\psi_\Lambda(u) \in \mathcal{B}(a_1, \dots, a_k)$ by

$$\psi_\Lambda(u) = \varphi_\Lambda(S_\Lambda^{-1}(u)).$$

This function satisfies

$$\psi_\Lambda(u)(\lambda) = u(\lambda) \quad (\forall \lambda \in \Lambda), \quad \|\psi_\Lambda(u)\|_{L^\infty(\mathbb{R}^k)} \le 2K_0 \|u\|_\infty.$$

The former property means that $\psi_\Lambda(u)$ interpolates the sequence u. The construction of $\psi_\Lambda(u)$ is "$\mathbb{Z}^k$-equivariant": For $n \in \mathbb{Z}^k$ and $u \in \ell^\infty(\Lambda)$ we define $v \in \ell^\infty(n+\Lambda)$ by $v(n+\lambda) = u(\lambda)$. Then

$$\psi_{n+\Lambda}(v)(t+n) = \psi_\Lambda(u)(t).$$

If Λ is the empty set, then $\psi_\Lambda(0) = 0$.

We denote by $\mathbb{B}_1(\ell^\infty(\Lambda))$ the closed unit ball of $\ell^\infty(\Lambda)$ around the origin with respect to the supremum norm. The function $\psi_\Lambda(u)$ depends continuously on Λ and u as follows:

Lemma 4.5 (Continuity of the construction of $\psi_\Lambda(u)$)**.** *For any $r>0$ and $\varepsilon>0$ there exist $r'>0$ and $\varepsilon'>0$ so that if $\Lambda_1,\Lambda_2\in \mathrm{Ads}$, $u_1\in \mathbb{B}_1(\ell^\infty(\Lambda_1))$ and $u_2\in\mathbb{B}_1(\ell^\infty(\Lambda_2))$ satisfy*

$$\Lambda_1\cap B_{r'}=\Lambda_2\cap B_{r'},\quad |u_1(\lambda)-u_2(\lambda)|<\varepsilon'\quad (\forall \lambda\in\Lambda_1\cap B_{r'}) \tag{4.4}$$

then for all $t\in B_r$

$$|\psi_{\Lambda_1}(u_1)(t)-\psi_{\Lambda_2}(u_2)(t)|<\varepsilon.$$

Proof. By the linearity of φ_Λ,

$$\psi_\Lambda(u)=\varphi_\Lambda(S_\Lambda^{-1}(u))=\sum_{n=0}^{\infty}\varphi_\Lambda\left((\mathrm{id}-S_\Lambda)^n u\right).$$

By Lemma 4.4

$$\left\|\varphi_\Lambda\left((\mathrm{id}-S_\Lambda)^n u\right)\right\|_{L^\infty(\mathbb{R}^k)}\leq K_0\left\|(\mathrm{id}-S_\Lambda)^n u\right\|_\infty\leq \frac{K_0}{2^n}\left\|u\right\|_\infty.$$

Therefore the statement follows from the next claim.

Claim 4.6. (1) *For any $r>0$ and $\varepsilon>0$ there exist $r'>0$ and $\varepsilon'>0$ so that if $\Lambda_1,\Lambda_2\in\mathrm{Ads}$, $u_1\in\mathbb{B}_1(\ell^\infty(\Lambda_1))$ and $u_2\in\mathbb{B}_1(\ell^\infty(\Lambda_2))$ satisfy (4.4) then for all $t\in B_r$*

$$|\varphi_{\Lambda_1}(u_1)(t)-\varphi_{\Lambda_2}(u_2)(t)|<\varepsilon.$$

(2) *For any $r>0$, $\varepsilon>0$ and a natural number n there exist $r'>r$ and $\varepsilon'>0$ so that if $\Lambda_1,\Lambda_2\in\mathrm{Ads}$, $u_1\in\mathbb{B}_1(\ell^\infty(\Lambda_1))$ and $u_2\in\mathbb{B}_1(\ell^\infty(\Lambda_2))$ satisfy (4.4) then for all $\lambda\in\Lambda_1\cap B_r$*

$$|(\mathrm{id}-S_{\Lambda_1})^n(u_1)(\lambda)-(\mathrm{id}-S_{\Lambda_2})^n(u_2)(\lambda)|<\varepsilon.$$

Proof. (1) We choose $r'>r$ so large that for all $t\in B_r$

$$\sum_{\lambda\in\Gamma_1\setminus B_{r'}}|\chi_0(t-\lambda)|<\frac{\varepsilon}{4}.$$

Then Condition (4.4) implies that for $t\in B_r$

$$\begin{aligned}|\varphi_{\Lambda_1}(u_1)(t)-\varphi_{\Lambda_2}(u_2)(t)|&<\frac{\varepsilon}{2}+\sum_{\lambda\in\Lambda_1\cap B_{r'}}|u_1(\lambda)-u_2(\lambda)|\cdot|\chi_0(t-\lambda)|\\&\leq\frac{\varepsilon}{2}+\varepsilon'\sum_{\lambda\in\Gamma_1\cap B_{r'}}|\chi_0(t-\lambda)|.\end{aligned}$$

We can choose $\varepsilon'>0$ so small that the right-hand side is smaller than ε.

(2) The statement can be reduced to the case of $n=1$ by induction. The case of $n=1$ immediately follows from the statement (1). □

□

$\psi_\Lambda(u)$ is a nice interpolating function. But we need one more twist. We naturally identify the set $\{0,1\}^{\Gamma_1}$ with the set of all subsets of Γ_1. Namely, $\Lambda \subset \Gamma_1$ is identified with the characteristic function $1_\Lambda \in \{0,1\}^{\Gamma_1}$. Then in particular Ads (the set of admissible sets) is a Borel subset of $\{0,1\}^{\Gamma_1}$. Let $p : \Gamma_1 \to [0,1]$ be an admissible *function*, namely $\{\lambda \in \Gamma_1 | \, p(\lambda) > 0\}$ belongs to Ads. We define a probability measure μ_p on $\{0,1\}^{\Gamma_1}$ by

$$\mu_p = \prod_{\lambda \in \Gamma_1} \left((1-p(\lambda))\,\boldsymbol{\delta}_0 + p(\lambda)\boldsymbol{\delta}_1\right),$$

where $\boldsymbol{\delta}_0$ and $\boldsymbol{\delta}_1$ are the delta probability measures at 0 and 1 on $\{0,1\}$ respectively. Since p is admissible, the set Ads has full measure: $\mu_p(\mathrm{Ads}) = 1$.

Let $\ell^\infty(\Gamma_1)$ be the Banach space of bound functions $u : \Gamma_1 \to \mathbb{R}$. For $u \in \ell^\infty(\Gamma_1)$ we define a band-limited function $\Psi(p,u) \in \mathcal{B}(a_1,\dots,a_k)$ by

$$\Psi(p,u) = \int_{\Lambda \in \mathrm{Ads}} \psi_\Lambda(u|_\Lambda)\, d\mu_p(\Lambda). \tag{4.5}$$

This function is the main product of this section.

Lemma 4.7 (Basic properties of Ψ)**.** Ψ *satisfies:*

1. **Interpolation:** *If* $\lambda \in \Gamma_1$ *satisfies* $p(\lambda) = 1$ *then* $\Psi(p,u)(\lambda) = u(\lambda)$.
2. **Boundedness:** $\|\Psi(p,u)\|_{L^\infty(\mathbb{R}^k)} \le 2K_0 \|u\|_\infty$.
3. **Equivariance:** *For* $n \in \mathbb{Z}^k$ *we define* $q \in \mathrm{Adf}$ *and* $v \in \ell^\infty(\Gamma_1)$ *by* $q(\lambda+n) = p(\lambda)$ *and* $v(\lambda+n) = u(\lambda)$ *for* $\lambda \in \Gamma_1$*. Then* $\Psi(q,v)(t+n) = \Psi(p,u)(t)$.

Proof. These three properties immediately follow from the corresponding properties of $\psi_\Lambda(u)$. □

We would like to show that the map Ψ depends continuously on p and u (Proposition 4.9 below). But we need a preparation before that. For $r > 0$ and $p \in \mathrm{Adf}$ we define probability measures $\mu_{p,r}$ and $\nu_{p,r}$ on $\{0,1\}^{\Gamma_1 \cap B_r}$ and $\{0,1\}^{\Gamma_1 \setminus B_r}$ respectively by

$$\mu_{p,r} = \prod_{\lambda \in \Gamma_1 \cap B_r} \left((1-p(\lambda))\,\boldsymbol{\delta}_0 + p(\lambda)\boldsymbol{\delta}_1\right), \quad \nu_{p,r} = \prod_{\lambda \in \Gamma_1 \setminus B_r} \left((1-p(\lambda))\,\boldsymbol{\delta}_0 + p(\lambda)\boldsymbol{\delta}_1\right).$$

It follows $\mu_p = \mu_{p,r} \otimes \nu_{p,r}$.

Lemma 4.8 (Truncation of the integral)**.** *For any* $r > 0$ *and* $\varepsilon > 0$ *if we choose* $r' > 0$ *sufficiently large then for any* $p \in \mathrm{Adf}$*,* $u \in \mathbb{B}_1(\ell^\infty(\Gamma_1))$ *and* $t \in B_r$

$$\left| \Psi(p,u)(t) - \int_{\Lambda \in \mathrm{Ads} \cap \{0,1\}^{\Gamma_1 \cap B_{r'}}} \psi_\Lambda(u|_\Lambda)(t)\, d\mu_{p,r'}(\Lambda) \right| < \varepsilon. \tag{4.6}$$

Proof. For an admissible set $\Lambda_1 \subset \Gamma_1 \cap B_{r'}$ we define $\mathrm{Ads}(\Lambda_1, r') \subset \mathrm{Ads}$ as the set of $\Lambda_2 \subset \Gamma_1 \setminus B_{r'}$ satisfying $\Lambda_1 \cup \Lambda_2 \in \mathrm{Ads}$. If $\mu_{p,r'}(\{\Lambda_1\}) > 0$ then $\mathrm{Ads}(\Lambda_1, r')$ has full measure with respect to $\nu_{p,r'}$ because $\mu_p(\mathrm{Ads}) = 1$.

By $\mu_p = \mu_{p,r'} \otimes \nu_{p,r'}$

$$\Psi(p,u) = \int_{\Lambda_1 \in \mathrm{Ads} \cap \{0,1\}^{\Gamma_1 \cap B_{r'}}} \left(\int_{\Lambda_2 \in \mathrm{Ads}(\Lambda_1, r')} \psi_{\Lambda_1 \cup \Lambda_2}(u|_{\Lambda_1 \cup \Lambda_2})\, d\nu_{p,r'}(\Lambda_2) \right) d\mu_{p,r'}(\Lambda_1).$$

If r' is sufficiently large, then it follows from Lemma 4.5 that

$$|\psi_{\Lambda_1 \cup \Lambda_2}(u|_{\Lambda_1 \cup \Lambda_2}) - \psi_{\Lambda_1}(u|_{\Lambda_1})| < \varepsilon \text{ on } B_r.$$

Then (4.6) follows. ☐

Proposition 4.9 (Continuity of Ψ)**.** *For any $r > 0$ and $\varepsilon > 0$ there exist $r' > 0$ and $\varepsilon' > 0$ so that if $p, q \in \mathrm{Adf}$ and $u, v \in \mathbb{B}_1(\ell^\infty(\Gamma_1))$ satisfy*

$$\forall \lambda \in \Gamma_1 \cap B_{r'}: \quad |p(\lambda) - q(\lambda)| < \varepsilon', \quad |u(\lambda) - v(\lambda)| < \varepsilon' \tag{4.7}$$

then for all $t \in B_r$

$$|\Psi(p,u)(t) - \Psi(q,v)(t)| < \varepsilon.$$

Proof. From Lemma 4.8 it is enough to show that if Condition (4.7) holds for sufficiently large $r' > 0$ and sufficiently small $\varepsilon' > 0$ then for all $t \in B_r$

$$\left| \int_{\Lambda \in \mathrm{Ads} \cap \{0,1\}^{\Gamma_1 \cap B_{r'}}} \psi_\Lambda(u|_\Lambda)(t)\, d\mu_{p,r'}(\Lambda) - \int_{\Lambda \in \mathrm{Ads} \cap \{0,1\}^{\Gamma_1 \cap B_{r'}}} \psi_\Lambda(v|_\Lambda)(t)\, d\mu_{q,r'}(\Lambda) \right| < \varepsilon.$$

The left-hand side is bounded by

$$\int_{\Lambda \in \mathrm{Ads} \cap \{0,1\}^{\Gamma_1 \cap B_{r'}}} |\psi_\Lambda(u|_\Lambda)(t) - \psi_\Lambda(v|_\Lambda)(t)|\, d\mu_{p,r'}(\Lambda) + \int_{\Lambda \in \mathrm{Ads} \cap \{0,1\}^{\Gamma_1 \cap B_{r'}}} |\psi_\Lambda(v|_\Lambda)(t)|\, d|\mu_{p,r'} - \mu_{q,r'}|(\Lambda).$$

The first term can be made arbitrarily small by Lemma 4.5. The second term is bounded by

$$2K_0 \int_{\{0,1\}^{\Gamma_1 \cap B_{r'}}} d|\mu_{p,r'} - \mu_{q,r'}|.$$

The integral here is the total variation of the signed measure $\mu_{p,r'} - \mu_{q,r'}$, which is equal to the sum

$$\sum_{\Lambda \in \{0,1\}^{\Gamma_1 \cap B_{r'}}} |\mu_{p,r'}(\{\Lambda\}) - \mu_{q,r'}(\{\Lambda\})|.$$

This can be made arbitrarily small by choosing $\varepsilon' > 0$ in (4.7) sufficiently small. ☐

5. Dynamical construction of tiling-like band-limited maps

This section is the longest section of the paper. As we described in Subsection 3.2, the proof of Proposition 3.1 uses a *tiling* $\mathbb{R}^k = \bigcup_{n\in\mathbb{Z}^k} W(x,n)$ and a *social welfare system among tiles*, which should be *encoded into a certain band-limited function* $g_2(x)$. The purpose of this section is to construct these ingredients. The following diagram outlines the construction.

$$\begin{array}{c}
\text{A point } x \text{ in a dynamical system } (X,\mathbb{Z}^k,T) \text{ satisfying the marker property.}\\
\downarrow\\
\text{A tiling } \mathbb{R}^k = \bigcup_{n\in\mathbb{Z}^k} W_0(x,n).\\
\downarrow\\
\text{A tiling-like band-limited map } \Phi(x):\mathbb{C}^k\to\mathbb{C}^k,\\
\text{which is equivalent to a band-limited function } g_2(x).\\
\downarrow\\
\text{A tiling } \mathbb{R}^k = \bigcup_{n\in\mathbb{Z}^k} W(x,n).\\
\downarrow\\
\text{Weight functions } w(x,n)\in[0,1]^{\mathbb{Z}^k} \text{ for } n\in\mathbb{Z}^k,\\
\text{which form a social welfare system among tiles } W(x,n).
\end{array}$$

Take $x\in X$. First we construct a tiling $\mathbb{R}^k = \bigcup_{n\in\mathbb{Z}^k} W_0(x,n)$ from the point x. We use the marker property assumption only here. This tiling is *not* used directly in the proof of Proposition 3.1. Instead we construct a certain holomorphic map $\Phi(x):\mathbb{C}^k\to\mathbb{C}^k$ called a "tiling-like band-limited map" from the tiles $W_0(x,n)$. Write $\Phi(x) = (\Phi(x)_1,\dots,\Phi(x)_k)$. The Fourier transforms of $\Phi(x)_1|_{\mathbb{R}^k},\dots,\Phi(x)_k|_{\mathbb{R}^k}$ and their complex conjugates are adjusted to have disjoint supports. Then the band-limited function

$$g_2(x) \overset{\text{def}}{=} \sum_{i=1}^k \mathrm{Re}\,(\Phi(x)_i)$$

becomes "equivalent" to $\Phi(x)$. Namely if $g_2(x)=g_2(y)$ for some $x,y\in X$ then $\Phi(x)=\Phi(y)$. (Indeed we will define $g_2(x)$ in Section 6 by multiplying a small constant to the above function so that the norm of $g_2(x)$ becomes sufficiently small.)

Let us give some more heuristics concerning the construction of $\Phi(x)$. Consider the function

$$\tilde{\Theta}_L = (\sin\frac{\pi z_1}{L},\cdots,\sin\frac{\pi z_k}{L}):\mathbb{R}^k\to\mathbb{R}^k$$

for some positive integer L thought to be very big. Every entry $\sin(\pi z_i/L)$ of the map $\tilde{\Theta}_L$ vanishes on $L\mathbb{Z}^k\subset\mathbb{R}^k$ and belongs to $B(\frac{1}{L},\cdots,\frac{1}{L})$. Suppose we are given a family of tilings $W_0(x) = \{W_0(x,n)\}_{n\in\mathbb{Z}^k}$ of $\mathbb{R}^k$ depending continuously[9] and $\mathbb{Z}^k$-equivariantly (i.e. $W_0(T^n x, m) = -n + W_0(x, n+m)$) on $x \in X$. We would like to replace this family of tilings by a family of Voronoi tilings[10] generated by the zeros of a continuous function $F : X \times \mathbb{R}^k \to \mathbb{R}^k$, where F has the following properties:

- **(1) Band-limited:** Every entry of $F(x,\cdot)$ belongs to $B(\frac{1}{L},\cdots,\frac{1}{L})$.
- **(2) Equivariance:** $F(T^n x, z) = F(x, z+n)$ for any $x \in X$, $z \in \mathbb{R}^k$ and $n \in \mathbb{Z}^k$.

A candidate for F is the following function

$$X \times \mathbb{R}^k \to \mathbb{R}^k, \quad (x,t) \mapsto \sum_{n\in\mathbb{Z}^k} \widetilde{\Theta}_L(t-n)\, 1_{W_0(x,n)}(t).$$

This function is equivariant but unfortunately not continuous. However a small change (for details see Subsection 5.2) makes it band-limited (and continuous).

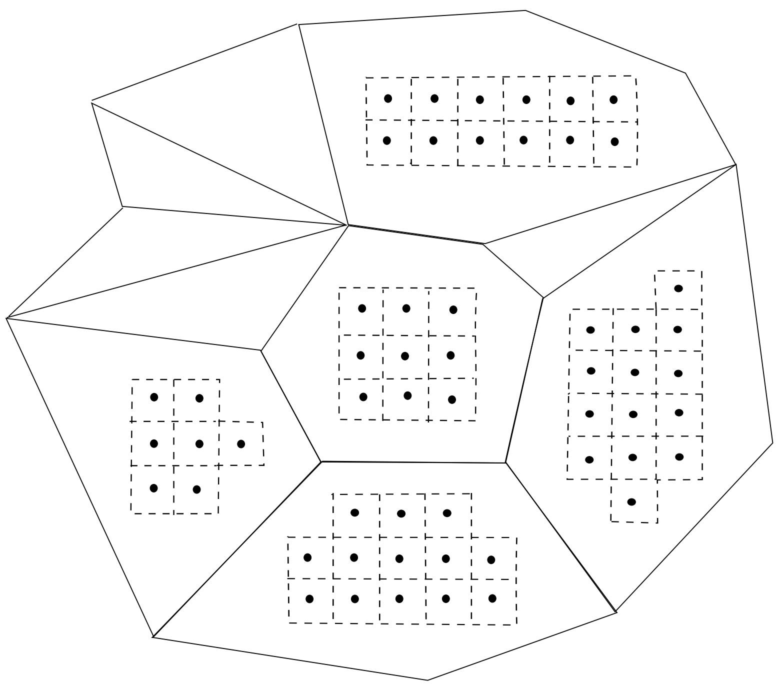

FIGURE 5.1. The original tiling $W_0(x,n)$: ——; The zeros of F in the "deep" interior of "big" tiles: •; "Regular" part of the new tiling $W(x,n)$: - - - - -.

The most simple way of generating Voronoi tilings would be to use the zeros of $F(x,\cdot)$ as generating sets for the Voronoi tiling, i.e. $W(x) = \mathrm{Voronoi}\big(\text{zeros of } F(x,\cdot)\big)$. However, the zeros of band-limited maps behave rather wildly in general, e.g. they are not isolated and form a positive dimensional variety in general. We will thus instead adopt a more

[9]More precisely, for $\varepsilon > 0$ and for each $x \in X$ and $n \in \mathbb{Z}^k$ with $\mathrm{Int}\, W_0(x,n) \neq \emptyset$, if $y \in X$ is sufficiently close to x, then the Hausdorff distance between $W_0(x,n)$ and $W_0(y,n)$ is smaller than ε.

[10]A Voronoi tiling generated by a discrete set $A \subset \mathbb{R}^k$ is a partitioning of $\mathbb{R}^k$ into regions based on the closest distance to points in A. For more details, see Subsection 5.3.

elaborate scheme. We give a weight $v(x,n) \in [0,1]$ to every $x \in X$ and $n \in \mathbb{Z}^k$, which is constructed from (some appropriate) zeros of $F(x,\cdot)$ in a neighborhood of n. We use these weights to generate a family of indexed Voronoi tilings $\{W(x,n)\}_{n\in\mathbb{Z}^k}$. The exact procedure is described in Subsection 5.4. For big tiles and big $L>0$ the picture schematically looks like Figure 5.1.

The new tiling is used in the proof of Proposition 3.1 in Section 6. We construct a certain "weight function" (not to be confused with $v(x,n)$ of the previous paragraph)

$$w(x) = (w(x,n))_{n\in\mathbb{Z}^k} \in \left([0,1]^{\mathbb{Z}^k}\right)^{\mathbb{Z}^k}$$

from the tiles $W(x,n)$. This will play a role of a social welfare system among the tiles $W(x,n)$. The tiles $W(x,n)$ and functions $w(x,n)$ are constructed from $\Phi(x)$. So all their informations are encoded into $\Phi(x)$, which is equivalent to the band-limited function $g_2(x)$.

Subsection 5.1 is a function-theoretic preparation for the construction of tiling-like band-limited maps. Subsection 5.2 constructs a tiling-like band-limited map from a tiling of $\mathbb{R}^k$ (not necessarily coming from a dynamical system). Subsection 5.3 describes a construction of the first tiling $\mathbb{R}^k = \bigcup_{n\in\mathbb{Z}^k} W_0(x,n)$. Subsection 5.4 uses the preparations in Subsections 5.1, 5.2 and 5.3 and constructs $\Phi(x)$, $W(x,n)$ and $w(x,n)$.

5.1. **Quantitative persistence of zero points.** We start from the following elementary observation: Let $f:\mathbb{C}^k\to\mathbb{C}^k$ be a holomorphic map such that $f(0)=0$ and the derivative df_0 at the origin is an invertible matrix. For any neighborhood U of the origin in $\mathbb{C}^k$, if $g:\mathbb{C}^k\to\mathbb{C}^k$ is a holomorphic map such that $|f(z)-g(z)|$ is sufficiently small over the unit ball B_1, then there exists $z\in U$ such that $g(z)=0$ and the derivative dg_z at z is also invertible. The purpose of this subsection is to develop a *quantitative* version of this observation for some very special f (i.e. the function Θ_L introduced below). Here "quantitative" means that we would like to specify *how small* $|g(x)-f(x)|$ *should be* and *how much non-degenerate* dg_z *is.*

Let $r>0$. For $u\in\mathbb{C}$ we define $D_r(u)$ as the closed disk of radius r around u in $\mathbb{C}$. We set $D_r = D_r(0)$. For $u=(u_1,\dots,u_k)\in\mathbb{C}^k$ we define the polydisc $D_r^k(u)$ in $\mathbb{C}^k$ by $D_r^k(u) = D_r(u_1)\times\cdots\times D_r(u_k)$. We set $D_r^k = D_r^k(0)$.

First we study the sine function $\sin z$ in the complex domain. Consider the image $\sin(\partial D_{\pi/2})$ of the circle $\partial D_{\pi/2}$ of radius $\pi/2$ under the map $\sin:\mathbb{C}\to\mathbb{C}$. This curve does not contain the origin and its rotation number around the origin is one:

$$\frac{1}{2\pi\sqrt{-1}}\int_{\sin(\partial D_{\pi/2})}\frac{dz}{z} = \frac{1}{2\pi\sqrt{-1}}\int_{\partial D_{\pi/2}}\frac{\cos z}{\sin z}\,dz = 1.$$

This implies that the map

$$\sin_*: H_1\left(D_{\pi/2}\setminus\{0\}\right)\to H_1\left(\mathbb{C}\setminus\{0\}\right)$$

is an isomorphism. Here $H_1(\cdot)$ is the standard homology group of $\mathbb{Z}$-coefficients. (Both sides of the above are isomorphic to $\mathbb{Z}$.) The map $\sin_*$ is the homomorphism induced by the map $\sin : D_{\pi/2} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$. The natural maps

$$H_2\left(D_{\pi/2}, D_{\pi/2} \setminus \{0\}\right) \to H_1\left(D_{\pi/2} \setminus \{0\}\right), \quad H_2\left(\mathbb{C}, \mathbb{C} \setminus \{0\}\right) \to H_1\left(\mathbb{C} \setminus \{0\}\right)$$

are isomorphic by the canonical long exact sequence ([Hat02, Theorem 2.16]). It follows by composition that

$$\sin_* : H_2\left(D_{\pi/2}, D_{\pi/2} \setminus \{0\}\right) \to H_2\left(\mathbb{C}, \mathbb{C} \setminus \{0\}\right)$$

is also an isomorphism. In the same way, for any $n \in \mathbb{Z}$, the map

$$\sin_* : H_2\left(D_{\pi/2}(\pi n), D_{\pi/2}(\pi n) \setminus \{\pi n\}\right) \to H_2\left(\mathbb{C}, \mathbb{C} \setminus \{0\}\right)$$

is an isomorphism.

Definition 5.1. (1) Recall that we introduced positive numbers $a_1, \dots, a_k$ and δ in Subsection 3.3. We set $b_i = a_i + \delta/2$. For $L > 1$ we define $\Theta_L : \mathbb{C}^k \to \mathbb{C}^k$ by

$$\Theta_L(z_1, \dots, z_k) = \left(e^{\pi\sqrt{-1}\, b_1 z_1} \sin(\pi z_1/L), \dots, e^{\pi\sqrt{-1}\, b_k z_k} \sin(\pi z_k/L)\right).$$

This map vanishes on $L\mathbb{Z}^k$. The Fourier transform of the i-th entry

$$e^{\pi\sqrt{-1}\, b_i t_i} \sin(\pi t_i/L), \quad (t = (t_1, \dots, t_k) \in \mathbb{R}^k)$$

of $\Theta_L|_{\mathbb{R}^k}$ (the restriction of Θ_L to $\mathbb{R}^k$) is supported in

$$\{0\}^{i-1} \times \left[\frac{b_i}{2} - \frac{1}{2L}, \frac{b_i}{2} + \frac{1}{2L}\right] \times \{0\}^{k-i}, \quad (1 \le i \le k).$$

We will choose a very large L later. So this line segment will be very small.

(2) Let A be a $k \times k$ matrix. We set

$$\nu(A) = \min_{z \in \mathbb{C}^k, |z|=1} |Az|.$$

The matrix A is invertible if and only if $\nu(A) > 0$. So $\nu(A)$ quantifies the amount of non-degeneracy of A. For another $k \times k$ matrix B we have

$$|\nu(A) - \nu(B)| \le |A - B| \quad (\text{the operator norm of } A - B). \tag{5.1}$$

In particular $\nu(A)$ depends continuously on A.

Lemma 5.2. *For every $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$ the homomorphism*

$$(\Theta_L)_* : H_{2k}\left(D^k_{L/2}(Ln), D^k_{L/2}(Ln) \setminus \{Ln\}\right) \to H_{2k}\left(\mathbb{C}^k, \mathbb{C}^k \setminus \{0\}\right)$$

is isomorphic. (Notice that the both sides are isomorphic to $\mathbb{Z}$.)

Proof. We define a homotopy $f_t : \left(D^k_{L/2}(Ln), D^k_{L/2}(Ln)\setminus\{Ln\}\right) \to \left(\mathbb{C}^k, \mathbb{C}^k\setminus\{0\}\right)$ for $0\le t\le 1$ by

$$f_t(z_1,\dots,z_k) = \left(e^{\pi\sqrt{-1}\,t\,b_1 z_1}\sin(\pi z_1/L),\dots,e^{\pi\sqrt{-1}\,t\,b_k z_k}\sin(\pi z_k/L)\right).$$

$f_1=\Theta_L$. So it is enough to show that f_0 induces an isomorphism on the $2k$-th homology groups (see [Hat02, Theorem 2.10]). The map f_0 is given by

$$f_0(z_1,\dots,z_k) = (\sin(\pi z_1/L),\dots,\sin(\pi z_k/L)).$$

Set $\varphi(z)=\sin(\pi z/L)$ for $z\in\mathbb{C}$. We have natural isomorphisms by the Künneth theorem (Spanier [Spa66, p. 235, 10 Theorem]):

$$H_{2k}\left(D^k_{L/2}(Ln), D^k_{L/2}(Ln)\setminus\{Ln\}\right) \cong \bigotimes_{i=1}^k H_2\left(D_{L/2}(Ln_i), D_{L/2}(Ln_i)\setminus\{Ln_i\}\right)$$

$$H_{2k}\left(\mathbb{C}^k,\mathbb{C}^k\setminus\{0\}\right) \cong \bigotimes_{i=1}^k H_2\left(\mathbb{C},\mathbb{C}\setminus\{0\}\right).$$

Under these isomorphisms, the homomorphism $(f_0)_*$ is identified with

$$\otimes_{i=1}^k\varphi_* : \bigotimes_{i=1}^k H_2\left(D_{L/2}(Ln_i), D_{L/2}(Ln_i)\setminus\{Ln_i\}\right) \to \bigotimes_{i=1}^k H_2\left(\mathbb{C},\mathbb{C}\setminus\{0\}\right).$$

By the argument in the beginning of this subsection, each factor

$$\varphi_* : H_2\left(D_{L/2}(Ln_i), D_{L/2}(Ln_i)\setminus\{Ln_i\}\right) \to H_2\left(\mathbb{C},\mathbb{C}\setminus\{0\}\right)$$

is isomorphic. Thus $(f_0)_*$ is an isomorphism. $\square$

The derivative $(d\Theta_L)_z$ at $z=(z_1,\dots,z_k)$ is a diagonal matrix whose (i,i)-th entry is given by

$$\pi e^{\pi\sqrt{-1}\,b_i z_i}\left(\sqrt{-1}\,b_i\sin(\pi z_i/L)+\frac{\cos(\pi z_i/L)}{L}\right),\quad (1\le i\le k).$$

Notation 5.3. We choose $0<r_1<1/4$ so small that for all $1\le i\le k$ and $z\in D_{r_1}$

$$\pi\left|e^{\pi\sqrt{-1}\,b_i z}\left(\sqrt{-1}\,b_i\sin(\pi z/L)+\frac{\cos(\pi z/L)}{L}\right)\right| > \frac{3}{L}.$$

Since for small z

$$\sin(\pi z/L)\approx\frac{\pi z}{L},\quad \frac{\cos(\pi z/L)}{L}\approx\frac{1}{L},$$

we can choose r_1 independent of $L>1$. (This independence is a plausible fact but it is not really needed in the argument below.)

It immediately follows from the definition that for all $n\in\mathbb{Z}^k$ and $z\in D^k_{r_1}(Ln)$

$$\nu\left((d\Theta_L)_z\right) > \frac{3}{L}. \tag{5.2}$$

Notation 5.4 (The threshold θ_L)**.** We define $\Omega \subset \mathbb{C}^k$ as the set of $z = (z_1, \dots, z_k)$ satisfying $|\mathrm{Im}(z_i)| \le 1$ for all $1 \le i \le k$. We define $\theta_L > 0$ as the minimum of the following two quantities:

$$k^{-1/2}(3/4)^{k+1}\frac{1}{L}, \quad \inf\left\{ |\Theta_L(z)| \,\middle|\, z \in \Omega \setminus \bigcup_{n\in\mathbb{Z}^k} D^k_{r_1}(Ln) \right\}.$$

The next lemma is the main result of this subsection.

Lemma 5.5 (Persistence of zero points of Θ_L)**.** *Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a holomorphic map. Let $W \subset \mathbb{C}^k$ be a subset and suppose $|f(z) - \Theta_L(z)| < \theta_L$ for all $z \in W$. Then the following (1) and (2) hold.*

(1)
$$\{z \in W \cap \Omega |\, f(z) = 0\} \subset \bigcup_{n\in\mathbb{Z}^k} D^k_{r_1}(Ln).$$

(2) *For every $n \in \mathbb{Z}^k$ with $D^k_1(Ln) \subset W$ there exists $z \in D^k_{r_1}(Ln)$ satisfying*
$$f(z) = 0, \quad \nu(df_z) > \frac{2}{L}.$$

Proof. (1) is a direct consequence of the definition of θ_L. So we consider (2). For the notational simplicity, assume $n = 0$. It follows from the definition of θ_L that

$$\max_{z\in\partial D^k_{r_1}} |f(z) - \Theta_L(z)| < \min_{z\in\partial D^k_{r_1}} |\Theta_L(z)|,$$

where $\partial D^k_{r_1}$ is the boundary of $D^k_{r_1}$. (So in particular its $(2k-1)$-th homology group is isomorphic to $\mathbb{Z}$.) Then we can define a homotopy $f_t : \partial D^k_{r_1} \to \mathbb{C}^k \setminus \{0\}$ between Θ_L and f by

$$f_t(z) = (1-t)\Theta_L(z) + tf(z), \quad (0 \le t \le 1).$$

Here the point is that f_t has no zero on $\partial D^k_{r_1}$. It follows from Lemma 5.2 that $f_* = (\Theta_L)_* : H_{2k-1}(\partial D^k_{r_1}) \to H_{2k-1}\left(\mathbb{C}^k \setminus \{0\}\right)$ is isomorphic. Then f must attain zero at some (interior) point of $D^k_{r_1}$. Otherwise the map $f_* : H_{2k-1}\left(\partial D^k_{r_1}\right) \to H_{2k-1}\left(\mathbb{C}^k \setminus \{0\}\right)$ is equal to the composition of the maps

$$H_{2k-1}\left(\partial D^k_{r_1}\right) \to H_{2k-1}\left(D^k_{r_1}\right) \xrightarrow{f_*} H_{2k-1}\left(\mathbb{C}^k \setminus \{0\}\right).$$

But $H_{2k-1}\left(D^k_{r_1}\right) = 0$ implies that the map $f_* : H_{2k-1}\left(\partial D^k_{r_1}\right) \to H_{2k-1}\left(\mathbb{C}^k \setminus \{0\}\right)$ is zero. This is a contradiction.

The rest of the work is to prove the statement about the derivative of f at zero points. By (5.1) and (5.2) it is enough to prove that for all $z \in D^k_{r_1}$ the operator norm of $df_z - (d\Theta_L)_z$ is smaller than $1/L$. Let $z = (z_1, \dots, z_k) \in D^k_{r_1}$. By the Cauchy integration formula,

$$f(z) - \Theta_L(z) = \frac{1}{(2\pi\sqrt{-1})^k} \int_{\partial D_1} \cdots \int_{\partial D_1} \frac{f(w) - \Theta_L(w)}{(w_1 - z_1)\dots(w_k - z_k)}\, dw_1 \dots dw_k.$$

For $u = (u_1, \dots, u_k) \in \mathbb{C}^k$ with $|u| = 1$, the vector $df_z(u) - (d\Theta_L)_z(u)$ is given by

$$\frac{1}{(2\pi\sqrt{-1})^k} \sum_{i=1}^{k} u_i \int_{\partial D_1} \cdots \int_{\partial D_1} \frac{f(w) - \Theta_L(w)}{(w_1 - z_1)\dots(w_i - z_i)^2 \ \dots(w_k - z_k)} \, dw_1 \dots dw_k.$$

For $z_i \in D_{r_1}$ and $w_i \in \partial D_1$ we have $|w_i - z_i| > 3/4$ because $r_1 < 1/4$. Hence

$$\begin{aligned} |df_z(u) - (d\Theta_L)_z(u)| &< \sum_{i=1}^{k} |u_i| (4/3)^{k+1} \theta_L \\ &\le \sqrt{k}(4/3)^{k+1}\theta_L \le \frac{1}{L}, \quad \left(\text{by } \theta_L \le k^{-1/2}(3/4)^{k+1}\frac{1}{L}\right). \end{aligned}$$

This proves $|df_z - (d\Theta_L)| < 1/L$. □

5.2. **Tiling-like band-limited maps.** We choose a rapidly decreasing function $\chi_1 : \mathbb{R}^k \to \mathbb{R}$ satisfying $\operatorname{supp}\mathcal{F}(\chi_1) \subset B_{\delta/8}$ and

$$\mathcal{F}(\chi_1)(0) = \int_{\mathbb{R}^k} \chi_1(t) \, dt_1 \dots dt_k = 1. \tag{5.3}$$

Notation 5.6. We set

$$K_1 = \int_{\mathbb{R}^k} |\chi_1(t)| \, dt_1 \dots dt_k.$$

The function χ_1 can be extended holomorphically over $\mathbb{C}^k$ (see Lemma 2.2) by

$$\chi_1(z) = \int_{B_{\delta/8}} \mathcal{F}(\chi_1)(\xi) \, e^{2\pi\sqrt{-1}\xi \cdot z} \, d\xi_1 \dots d\xi_k, \quad (z = (z_1, \dots, z_k) \in \mathbb{C}^k).$$

For any polynomial $P(z_1, \dots, z_k)$, by the integration by parts,

$$P(z_1, \dots, z_k)\chi_1(z) = \int_{B_{\delta/8}} \left(P\left(\frac{\sqrt{-1}}{2\pi}\frac{\partial}{\partial \xi_1}, \dots, \frac{\sqrt{-1}}{2\pi}\frac{\partial}{\partial \xi_k} \right) \mathcal{F}(\chi_1)(\xi) \right) e^{2\pi\sqrt{-1}\xi \cdot z} \, d\xi_1 \dots d\xi_k.$$

Hence there exists a positive constant const_P depending on P and satisfying

$$|P(z_1, \dots, z_k)\chi_1(z)| \le \mathrm{const}_P \cdot e^{(\pi\delta/4)|\mathrm{Im}(z)|}, \quad \left(|\mathrm{Im}(z)| = \sqrt{\sum_{i=1}^{k} |\mathrm{Im}(z_i)|^2} \right).$$

In particular for each $l \ge 0$ there exists a constant const_l satisfying

$$|\chi_1(z)| \le \mathrm{const}_l \cdot (1 + |z|)^{-l} \cdot e^{(\pi\delta/4)|\mathrm{Im}(z)|}. \tag{5.4}$$

It follows that the integral

$$\int_{\mathbb{R}^k} \chi_1(z - t) \, dt_1 \dots dt_k$$

converges and becomes a holomorphic function in $z \in \mathbb{C}^k$. It is constantly equal to one for $z \in \mathbb{R}^k$ by (5.3). Therefore indeed it is identically equal to one:

$$\int_{\mathbb{R}^k} \chi_1(z - t) \, dt_1 \dots dt_k = 1, \quad (z \in \mathbb{C}^k). \tag{5.5}$$

Let $\mathcal{W} = \{W_n\}_{n\in\mathbb{Z}^k}$ be a set of bounded convex sets of $\mathbb{R}^k$ (indexed by $\mathbb{Z}^k$) such that it **tiles the whole space** $\mathbb{R}^k$, namely

- $\mathbb{R}^k = \bigcup_{n\in\mathbb{Z}^k} W_n$.
- For any two distinct m and n in $\mathbb{Z}^k$ the sets W_m and W_n intersect at most on their boundaries ∂W_m and ∂W_n: $W_m \cap W_n = \partial W_m \cap \partial W_n$.

Notice that some W_n may be empty. For $L > 1$ we define a **tiling-like band-limited map** $\Phi_L(\mathcal{W}) : \mathbb{C}^k \to \mathbb{C}^k$ by

$$\Phi_L(\mathcal{W})(z) = \sum_{n\in\mathbb{Z}^k} \Theta_L(z-n) \int_{W_n} \chi_1(z-t)\, dt_1 \dots dt_k.$$

As was already explained before, intuitively we would like to consider

$$\sum_{n\in\mathbb{Z}^k} \Theta_L(t-n)\, 1_{W_n}(t), \quad (t\in\mathbb{R}^k).$$

Namely we would like to "paint" the map $\Theta_L(t-n)$ over each tile W_n. But this is not even continuous in t. So instead we defined $\Phi_L(\mathcal{W})$ as in the above.

It follows from $\|\Theta_L\|_{L^\infty(\mathbb{R}^k)} = \sqrt{k}$ and Notation 5.6 that the map $\Phi_L(\mathcal{W})(t)$ is bounded over $t \in \mathbb{R}^k$:

$$\begin{aligned}
|\Phi_L(\mathcal{W})(t)| &\le \sum_{n\in\mathbb{Z}^k} \|\Theta_L\|_{L^\infty(\mathbb{R}^k)} \int_{W_n} |\chi_1(t-x)|\, dx_1 \dots dx_k \\
&= \|\Theta_L\|_{L^\infty(\mathbb{R}^k)} \int_{\mathbb{R}^k} |\chi_1(t-x)|\, dx_1 \dots dx_k \\
&= \sqrt{k} K_1.
\end{aligned}$$

The Fourier transform of the i-th entry of $\Phi_L(\mathcal{W})|_{\mathbb{R}^k}$ is supported in

$$\left[-\frac{\delta}{8}, \frac{\delta}{8}\right]^{i-1} \times \left[\frac{b_i}{2} - \frac{1}{2L} - \frac{\delta}{8}, \frac{b_i}{2} + \frac{1}{2L} + \frac{\delta}{8}\right] \times \left[-\frac{\delta}{8}, \frac{\delta}{8}\right]^{k-i}, \quad (1 \le i \le k).$$

Since $b_i = a_i + \delta/2$, if $L > 4/\delta$ then this is contained in

$$\left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^{i-1} \times \left(\frac{a_i}{2}, \frac{a_i}{2} + \frac{\delta}{2}\right) \times \left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^{k-i}$$

Since we assumed $\delta < \min(a_1, \dots, a_k)$ in (3.2) in Subsection 3.3, these k sets are disjoint with each other.

Notation 5.7. Recall that $\Omega \subset \mathbb{C}^k$ is the set of $z = (z_1, \dots, z_k)$ satisfying $|\mathrm{Im}(z_i)| \le 1$ for all $1 \le i \le k$. By (5.4) we can choose $E = E(L) > 0$ such that for all $z \in \Omega$

$$\|\Theta_L\|_{L^\infty(\Omega)} \int_{\mathbb{R}^k \setminus B_E(\mathrm{Re}(z))} |\chi_1(z-t)|\, dt_1 \dots dt_k < \frac{\theta_L}{2},$$

where θ_L is the constant introduced in Notation 5.4.

We recall the notations introduced in Subsection 2.4: For $r > 0$ and $A \subset \mathbb{R}^k$, we define $\partial_r A$ as the set of all $t \in \mathbb{R}^k$ satisfying $B_r(t) \cap A \neq \emptyset$ and $B_r(t) \cap \left(\mathbb{R}^k \setminus A\right) \neq \emptyset$. We set $\mathrm{Int}_r A = A \setminus \partial_r A$, namely $\mathrm{Int}_r A$ is the set of $t \in A$ satisfying $B_r(t) \subset A$.

Lemma 5.8. *Let $n \in \mathbb{Z}^k$ and $z \in \Omega$. If $\mathrm{Re}(z) \in \mathrm{Int}_E W_n$ then*

$$|\Phi_L(\mathcal{W})(z) - \Theta_L(z-n)| < \theta_L.$$

Proof. For simplicity of the notation we assume $n = 0$. By (5.5)

$$\begin{aligned}\Phi_L(\mathcal{W})(z) - \Theta_L(z) = &\sum_{m \neq 0} \Theta_L(z-m) \int_{W_m} \chi_1(z-t)\, dt_1 \dots dt_k \\ &- \Theta_L(z) \int_{\mathbb{R}^k \setminus W_0} \chi_1(z-t)\, dt_1 \dots dt_k.\end{aligned}$$

Hence

$$|\Phi_L(\mathcal{W})(z) - \Theta_L(z)| \leq 2 \left\|\Theta_L\right\|_{L^\infty(\Omega)} \int_{\mathbb{R}^k \setminus W_0} |\chi_1(z-t)|\, dt_1 \dots dt_k.$$

This is smaller than θ_L by the definition of E. $\square$

The next lemma is the summary of this subsection.

Lemma 5.9 (Main properties of $\Phi_L(\mathcal{W})$)**.** (1) *The restriction of $\Phi_L(\mathcal{W})$ to $\mathbb{R}^k$ is a bounded continuous map such that $\left\|\Phi_L(\mathcal{W})\right\|_{L^\infty(\mathbb{R}^k)} \leq \sqrt{k} K_1$ and that if $L > 4/\delta$ then the Fourier transform of the i-th entry of $\Phi_L(\mathcal{W})|_{\mathbb{R}^k}$ is supported in*

$$\left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^{i-1} \times \left(\frac{a_i}{2}, \frac{a_i}{2} + \frac{\delta}{2}\right) \times \left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^{k-i}, \quad (1 \leq i \leq k),$$

which are disjoint with each other.

(2) *Let $n \in \mathbb{Z}^k$ and $z \in \Omega$. If $\mathrm{Re}(z) \in \mathrm{Int}_E W_n$ and $\Phi_L(\mathcal{W})(z) = 0$ then there exists $m \in \mathbb{Z}^k$ satisfying $z \in D^k_{r_1}(n + Lm)$.*

(3) *Let $n, m \in \mathbb{Z}^k$ with $n + Lm \in \mathrm{Int}_{E+\sqrt{k}} W_n$. Then there exists $z \in D^k_{r_1}(n + Lm)$ satisfying*

$$\Phi_L(\mathcal{W})(z) = 0, \quad \nu\left(\left(d\Phi_L(\mathcal{W})\right)_z\right) > \frac{2}{L}.$$

Proof. (1) was already proved. (2) follows from Lemmas 5.5 (1) and 5.8. For (3), notice that $n + Lm \in \mathrm{Int}_{E+\sqrt{k}} W_n$ implies $B_{\sqrt{k}}(n + Lm) \subset \mathrm{Int}_E(W_n)$ and that every point $z \in D^k_1(n + Lm)$ satisfies $\mathrm{Re}(z) \in \mathrm{Int}_E(W_n)$. Then (3) follows from Lemmas 5.5 (2) and 5.8. $\square$

Conditions (2) and (3) in Lemma 5.9 are the rigorous formulation of “good/bad decomposition” explained in Subsection 1.4. The domain Ω is decomposed into two regions:

$$\text{Good region} = \left\{ z \in \Omega \,\middle|\, \mathrm{Re} z \in \bigcup_{n\in\mathbb{Z}^k} \mathrm{Int}_{E+\sqrt{k}} W_n \right\},$$

$$\text{Bad region} = \left\{ z \in \Omega \,\middle|\, \mathrm{Re} z \in \bigcup_{n\in\mathbb{Z}^k} \partial_{E+\sqrt{k}} W_n \right\}.$$

The above (2) and (3) mean that we have a good control (for our purpose here) of zero points of $\Phi_L(\mathcal{W})$ over the good region. As we explained in Subsection 1.4, the bad region should be “tiny”. This means that we need to introduce a tiling $\mathcal{W}$ such that the “boundary region”

$$\bigcup_{n\in\mathbb{Z}^k} \partial_{E+\sqrt{k}} W_n$$

is sufficiently small. (Indeed we will need a bit more involved condition later; see (5.9) below). The purpose of the next subsection is to construct such tilings.

5.3. **Dynamical Voronoi diagram.** This subsection is largely a reproduction of [GLT16, Section 4]. Here we introduce a *dynamically generated Voronoi diagram*. Our use of Voronoi diagram is conceptually influenced by the works of Lightwood [Lig03, Lig04].

Let $(X, \mathbb{Z}^k, T)$ be a dynamical system having the marker property as we promised in Subsection 3.3. Let M be a natural number. It follows from the marker property that there exists an open set $U \subset X$ satisfying

$$U \cap T^n U = \emptyset \quad (0 < |n| < M), \quad X = \bigcup_{n\in\mathbb{Z}^k} T^n U.$$

We can find a natural number $M_1 \geq M$ and a compact set $F \subset U$ satisfying $X = \bigcup_{|n|<M_1} T^n F$. Choose a continuous function $h : X \to [0,1]$ satisfying $\operatorname{supp} h \subset U$ and $h = 1$ on F. Then it satisfies

$$(\operatorname{supp} h) \cap T^n (\operatorname{supp} h) = \emptyset \quad (0 < |n| < M), \quad X = \bigcup_{|n|<M_1} T^n \left(\{x \in X \,|\, h(x) = 1\}\right). \tag{5.6}$$

For $x \in X$ consider the following discrete set in $\mathbb{R}^{k+1}$:

$$\left\{ \left(n, \frac{1}{h(T^n x)}\right) \,\middle|\, n \in \mathbb{Z}^k, h(T^n x) > 0 \right\}.$$

We consider the **Voronoi diagram** determined by this set. Namely we define a convex set $V_0(x,n)$ for $n \in \mathbb{Z}^k$ as follows: If $h(T^n x) = 0$ then we set $V_0(x,n) = \emptyset$. If $h(T^n x) > 0$ then we define it as the set of all $u \in \mathbb{R}^{k+1}$ satisfying

$$\forall m \in \mathbb{Z}^k \text{ with } h(T^m x) > 0 : \quad \left| u - \left(n, \frac{1}{h(T^n x)}\right) \right| \leq \left| u - \left(m, \frac{1}{h(T^m x)}\right) \right|.$$

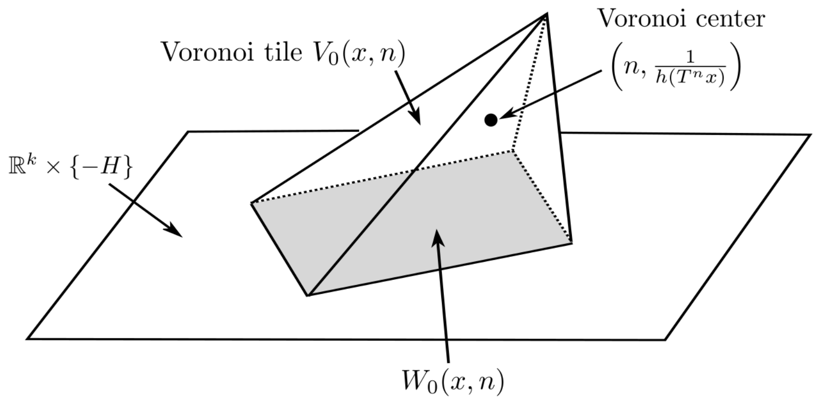

FIGURE 5.2. Voronoi tiling. The Voronoi center $(n, 1/h(T^n x))$ is located inside $V_0(x,n)$. The shadowed region is $W_0(x,n)$.

The sets $V_0(x,n)$ form a tiling of $\mathbb{R}^{k+1}$:

$$\mathbb{R}^{k+1} = \bigcup_{n\in\mathbb{Z}^k} V_0(x,n).$$

Let $\pi_{\mathbb{R}^k} : \mathbb{R}^{k+1} \to \mathbb{R}^k$ be the projection to the first k coordinates (i.e. forgetting the last coordinate). Set $H = \left(M_1 + \sqrt{k}\right)^2$ and

$$W_0(x,n) = \pi_{\mathbb{R}^k}\left(V_0(x,n) \cap \left(\mathbb{R}^k \times \{-H\}\right)\right).$$

See Figure 5.2. These form a tiling of $\mathbb{R}^k$:

$$\mathbb{R}^k = \bigcup_{n\in\mathbb{Z}^k} W_0(x,n).$$

This tiling is $\mathbb{Z}^k$-equivariant in the sense that

$$W_0(T^n x, m) = -n + W_0(x, n+m) = \{-n + u \mid u \in W_0(x, n+m)\}.$$

The tiles $W_0(x,n)$ depend continuously on $x \in X$ in the Hausdorff topology. More precisely, for $\varepsilon > 0$ and for each $x \in X$ and $n \in \mathbb{Z}^k$ with $\operatorname{Int} W_0(x,n) \neq \emptyset$, if $y \in X$ is sufficiently close to x, then the Hausdorff distance between $W_0(x,n)$ and $W_0(y,n)$ is smaller than ε. Here $\operatorname{Int} W_0(x,n)$ denotes the interior of $W_0(x,n)$. (When the tile $W_0(x,n)$ has no interior point, it may vanish after x moves slightly. But this does not cause any problem.)

The set $W_0(x,n)$ depends on the choice of the parameter M. So we sometimes write $W_0^M(x,n) = W_0(x,n)$ for clarifying that M is the parameter[11]. In the next subsection we consider the tiling-like band-limited map for this tiling. The following property of

[11] It might look that M_1 and h are also parameters. But we can choose them to be *functions of* M. Namely for each natural number M we fix a natural number $M_1 = M_1(M)$ and a continuous function $h = h_M : X \to [0,1]$ satisfying (5.6). Then only M remains to be a parameter.

$W_0^M(x,n)$ becomes crucial there, which means that the "bad region" is negligible as we explained at the end of the last section.

Lemma 5.10. *For any $r>0$*

$$\lim_{M\to\infty}\left\{\limsup_{R\to\infty}\left(\sup_{x\in X}\frac{\left|B_R\cap\bigcup_{n\in\mathbb{Z}^k}\partial_r W_0^M(x,n)\right|}{|B_R|}\right)\right\}=0.$$

Here $|\cdot|$ is the k-dimensional volume (Lebesgue measure).

Proof. We need some auxiliary claims.

Claim 5.11. *Let $x\in X$ and $n\in\mathbb{Z}^k$ with $h(T^n x)>0$.*

1. *$V_0(x,n)$ contains the ball $B_{M/2}\left(n,1/h(T^n x)\right)$.*
2. *$W_0(x,n)\subset B_{M_1+\sqrt{k}}(n)$.*
3. *If $W_0(x,n)\neq\emptyset$ then $h(T^n x)>1/2$.*

Proof. It follows from the first property of h in (5.6) that if $m\in\mathbb{Z}^k$ also satisfies $h(T^m x)>0$ then $|n-m|\geq M$. Property (1) easily follows from this. For the proof of (2) and (3), take $u\in W_0(x,n)$. Let $v\in\mathbb{Z}^k$ be the integer point closest to u. By the second property of h in (5.6) there exists $m\in\mathbb{Z}^k$ satisfying $|m|<M_1$ and $h(T^{v+m}x)=1$. It follows from the definition of the Voronoi tiling that

$$\left|(u,-H)-\left(n,\frac{1}{h(T^n x)}\right)\right|\leq\left|(u,-H)-(v+m,1)\right|. \tag{5.7}$$

Since $|H+1/h(T^n x)|\geq H+1$,

$$|u-n|\leq|u-v-m|<|u-v|+M_1<\sqrt{k}+M_1.$$

This proves (2). It follows from (5.7) that

$$\left(H+\frac{1}{h\left(T^n x\right)}\right)^2\leq|u-v-m|^2+(H+1)^2<\left(\sqrt{k}+M_1\right)^2+(H+1)^2.$$

Then Property (3) follows:

$$\frac{1}{h(T^n x)}<\frac{\left(\sqrt{k}+M_1\right)^2}{2H}+1+\frac{1}{2H}<\frac{1}{2}+1+\frac{1}{2}=2,$$

where we used $H=\left(M_1+\sqrt{k}\right)^2>1$. $\square$

Let $\varepsilon>0$. We take $c>1$ satisfying

$$c^{-k}>1-\varepsilon.$$

We choose M so large that

$$\frac{(c-1)M}{2(c+1)}>r,\quad\left(c+\frac{2}{M}\right)^{-k}>1-\varepsilon. \tag{5.8}$$

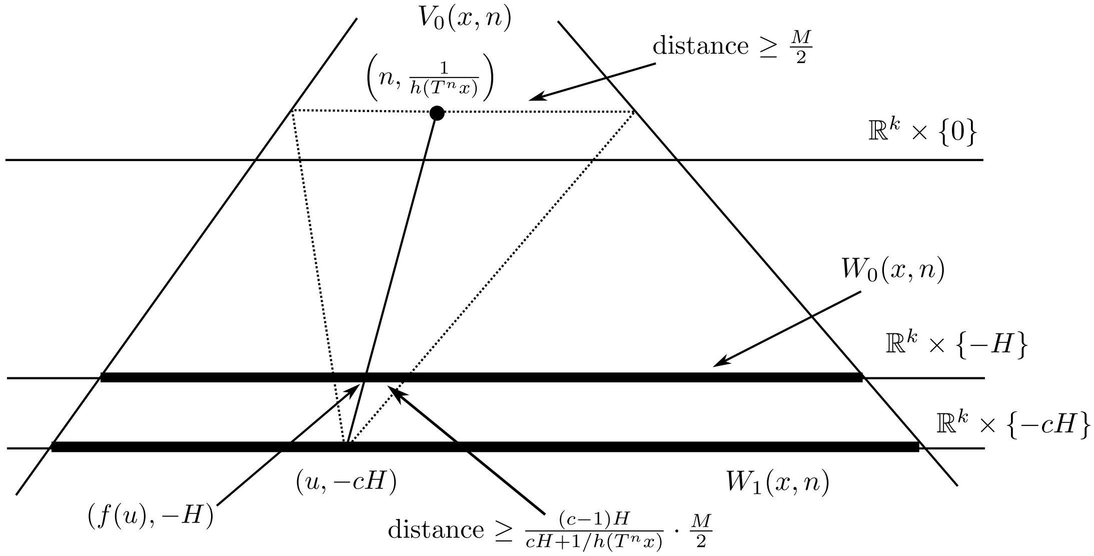

FIGURE 5.3. Configurations of $V_0(x,n)$, $W_0(x,n)$ and $W_1(x,n)$.

For $x \in X$ and $n \in \mathbb{Z}^k$ we set $W_1^M(x,n) = W_1(x,n) = \pi_{\mathbb{R}^k}\left(V(x,n) \cap \mathbb{R}^k \times \{-cH\}\right)$, where $\pi_{\mathbb{R}^k} : \mathbb{R}^{k+1} \to \mathbb{R}^k$ is the projection to the first k coordinates. It follows from the same proof of Claim 5.11 (2) that

$$W_1(x,n) \subset B_{M_1+\sqrt{k}}(n).$$

For $u \in W_1(x,n)$ we denote by $(f(u), -H)$ the intersection point of the hyperplane $\mathbb{R}^k \times \{-H\}$ and the line segment between $(u, -cH)$ and the Voronoi center $(n, 1/h(T^n x))$. It follows from the convexity of $V_0(x,n)$ that $f(u) \in W_0(x,n)$. See Figure 5.3.

Claim 5.12. (1) *For any $u \in W_1(x,n)$ the point $f(u)$ belongs to* $\mathrm{Int}_r W_0(x,n)$.

(2) $|\mathrm{Int}_r W_0(x,n)| \geq (1-\varepsilon)\,|W_1(x,n)|$.

Proof. (1) From $B_{M/2}\left(n, 1/h(T^n x)\right) \subset V_0(x,n)$, the set $W_0(x,n) \times \{-H\}$ contains the intersection point of the hyperplane $\mathbb{R}^k \times \{-H\}$ and the line segment between $(u, -cH)$ and $(v, 1/h(T^n x))$ for $v \in B_{M/2}(n)$. Then $W_0(x,n)$ contains the ball of radius

$$\frac{cH - H}{cH + \frac{1}{h(T^n x)}} \cdot \frac{M}{2}$$

around the point $f(u)$. See Figure 5.3 again. From $h(T^n x) > 1/2$ (Claim 5.11 (3)),

$$\frac{cH - H}{cH + \frac{1}{h(T^n x)}} \cdot \frac{M}{2} > \frac{(c-1)H}{cH + 2} \cdot \frac{M}{2} = \frac{(c-1)M}{2\left(c + \frac{2}{H}\right)}.$$

By $H = \left(M_1 + \sqrt{k}\right)^2 > 2$, this is larger than

$$\frac{(c-1)M}{2(c+1)} > r \quad \text{(by the first condition of (5.8))}.$$

This shows $f(u) \in \mathrm{Int}_r W_0(x,n)$.

(2) We can assume $W_1(x,n) \neq \emptyset$. (Otherwise the statement is trivial.) From (1)

$$|\mathrm{Int}_r W_0(x,n)| \geq |\{f(u) \mid u \in W_1(x,n)\}| = \left(\frac{H + \frac{1}{h(T^n x)}}{cH + \frac{1}{h(T^n x)}}\right)^k |W_1(x,n)|.$$

From $h(T^n x) > 1/2$ (Claim 5.11 (3)),

$$|\mathrm{Int}_r W_0(x,n)| \geq \left(\frac{H}{cH+2}\right)^k |W_1(x,n)| = \left(\frac{1}{c + \frac{2}{H}}\right)^k |W_1(x,n)|.$$

Since $H = \left(M_1 + \sqrt{k}\right)^2 > M$

$$|\mathrm{Int}_r W_0(x,n)| \geq \left(\frac{1}{c + \frac{2}{M}}\right)^k |W_1(x,n)|.$$

By the second condition of (5.8) this proves (2). $\square$

Let $x \in X$ and $R > 2M_1 + 2\sqrt{k}$.

$$\begin{aligned}\left|B_R \cap \bigcup_{n\in\mathbb{Z}^k} \partial_r W_0(x,n)\right| &\leq |B_R| - \left|\bigcup_{|n|\leq R - M_1 - \sqrt{k}} \mathrm{Int}_r W_0(x,n)\right| \quad (\text{by } W_0(x,n) \subset B_{M_1+\sqrt{k}}(n)) \\ &\leq |B_R| - (1-\varepsilon)\left|\bigcup_{|n|\leq R-M_1-\sqrt{k}} W_1(x,n)\right| \quad (\text{by Claim 5.12 (2)}).\end{aligned}$$

Since $W_1(x,n) \subset B_{M_1+\sqrt{k}}(n)$ and $\mathbb{R}^k = \bigcup_{n\in\mathbb{Z}^k} W_1(x,n)$,

$$\bigcup_{|n|\leq R-M_1-\sqrt{k}} W_1(x,n) \supset B_{R-2M_1-2\sqrt{k}}.$$

Hence

$$\left|B_R \cap \bigcup_{n\in\mathbb{Z}^k} \partial_r W_0(x,n)\right| \leq |B_R| - (1-\varepsilon)\left|B_{R-2M_1-2\sqrt{k}}\right|.$$

Thus

$$\limsup_{R\to\infty}\left(\sup_{x\in X} \frac{\left|B_R \cap \bigcup_{n\in\mathbb{Z}^k} \partial_r W_0(x,n)\right|}{|B_R|}\right) \leq \varepsilon.$$

This proves the statement. $\square$

5.4. **Tiling and weight functions from tiling-like band-limited maps.** Let A, L_0 and L be positive numbers. These three parameters control the construction of this subsection. We assume:

Condition 5.13 (L is much larger than AL_0)**.** L is an *integer* satisfying

$$L > 1000^k (A+1)\left(L_0 + 1 + \sqrt{k}\right).$$

Let $E = E(L)$ be the positive constant introduced in Notation 5.7. By Lemma 5.10 we can choose $M = M(A, L_0, L) > 0$ so that the sets $W_0(x, n) = W_0^M(x, n)$ in Subsection 5.3 satisfy

$$\limsup_{R\to\infty} \left(\sup_{x\in X} \frac{\left| B_{3R} \cap \bigcup_{n\in\mathbb{Z}^k} \partial_{E+2(L+1)\sqrt{k}+L_0+1} W_0(x,n) \right|}{\left| B_{R/2} \right|} \right) < \frac{1}{6A+2}. \tag{5.9}$$

Let $x \in X$. We define $\Phi_{A,L_0,L}(x) = \Phi_L\left(\{W_0(x,n)\}_{n\in\mathbb{Z}^k}\right) : \mathbb{C}^k \to \mathbb{C}^k$ as the tiling-like band-limited map with respect to the tiling $\{W_0(x,n)\}_{n\in\mathbb{Z}^k}$, namely for $z \in \mathbb{C}^k$

$$\Phi_{A,L_0,L}(x)(z) = \sum_{n\in\mathbb{Z}^k} \Theta_L(z-n) \int_{W_0(x,n)} \chi_1(z-t)\, dt_1 \dots dt_k.$$

We often abbreviate $\Phi_{A,L_0,L}(x)$ to $\Phi(x)$.

Lemma 5.14 (Basic properties of $\Phi(x)$)**.** Φ *satisfies*

- **(1) Boundedness:** $\|\Phi(x)\|_{L^\infty(\mathbb{R}^k)} \le \sqrt{k}K_1$.
- **(2) Frequencies:** *If $L > 4/\delta$ then the Fourier transform of the i-th entry of $\Phi(x)|_{\mathbb{R}^k}$ $(1 \le i \le k)$ is supported in*
$$\left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^{i-1} \times \left(\frac{a_i}{2}, \frac{a_i}{2} + \frac{\delta}{2}\right) \times \left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^{k-i},$$
which are disjoint with each other.
- **(3) Equivariance:** $\Phi(T^n x)(z) = \Phi(x)(z+n)$ *for all $x \in X$, $n \in \mathbb{Z}^k$ and $z \in \mathbb{C}^k$.*
- **(4) Continuity:** *If a sequence x_n in X converges to x then $\Phi(x_n)$ converges to $\Phi(x)$ uniformly over every compact subset of $\mathbb{C}^k$.*

Proof. (1) and (2) immediately follow from Lemma 5.9 (1). (3) and (4) follow from the equivariance and continuity of the tiles $W_0(x, n)$. □

Let $0 < r_1 < 1/4$ be the positive number introduced in Notation 5.3. Let $\alpha_1 : \mathbb{C}^k \to [0, 1]$ and $\alpha_2 : \mathbb{R} \to [0, 1]$ be continuous functions satisfying

$$\alpha_1(z) = 1 \quad \left(z \in D^k_{r_1}\right), \quad \alpha_1(z) = 0 \quad \left(z \notin D^k_{2r_1}\right),$$
$$\alpha_2(t) = 0 \quad \left(t \le \frac{1}{L}\right), \quad \alpha_2(t) = 1 \quad \left(t \ge \frac{2}{L}\right).$$

For each $x \in X$ and $n \in \mathbb{Z}^k$ we define a non-negative number $\nu_{A,L_0,L}(x, n) = \nu(x, n)$ by

$$\nu(x,n) = \min\left\{1, \sum_{\Phi(x)(z)=0} \alpha_1(z-n)\, \alpha_2\left(\nu\left(d\Phi(x)_z\right)\right)\right\}.$$

Here the sum is taken over all $z \in \mathbb{C}^k$ satisfying $\Phi(x)(z) = 0$, $z \in D^k_{2r_1}(n)$ and $\nu\left(d\Phi(x)_z\right) \ge 1/L$. The set of such z is a finite set because non-degenerate zero points are isolated. So this is a finite sum. (Notice that the polydisk $D^k_{2r_1}(n)$ may contain infinitely many zeros of $\Phi(x)$. The above definition of $\nu(x, n)$ addresses this problem. The number $1/L$ is chosen so that the zero points found in Lemma 5.9 (3) contribute to the above sum.) For each $n \in \mathbb{Z}^k$ the number $\nu(x,n)$ depends continuously on $x \in X$.

We again consider a Voronoi tiling. Let $x \in X$. Consider the following discrete set in $\mathbb{R}^{k+1}$:

$$\left\{ \left(n, \frac{1}{\nu(x,n)}\right) \middle| n \in \mathbb{Z}^k,\ \nu(x,n) > 0 \right\}.$$

This set is not empty. (See the proof of Lemma 5.15 (1) below.) So this determines a Voronoi diagram. Namely for $n \in \mathbb{Z}^k$ we define $V(x,n)$ as the set of $u \in \mathbb{R}^{k+1}$ satisfying

$$\forall m \in \mathbb{Z}^k \text{ with } \nu(x,m) > 0: \quad \left| u - \left(n, \frac{1}{\nu(x,n)}\right) \right| \le \left| u - \left(m, \frac{1}{\nu(x,m)}\right) \right|.$$

If $\nu(x,n) = 0$ then $V(x,n) = \emptyset$. These form a tiling of $\mathbb{R}^{k+1}$:

$$\mathbb{R}^{k+1} = \bigcup_{n \in \mathbb{Z}^k} V(x,n).$$

We set $W(x,n) = \pi_{\mathbb{R}^k}\left(V(x,n) \cap \left(\mathbb{R}^k \times \{0\}\right)\right)$, where $\pi_{\mathbb{R}^k} : \mathbb{R}^{k+1} \to \mathbb{R}^k$ is the projection to the first k coordinates as before. The sets $W(x,n)$ form a tiling of $\mathbb{R}^k$:

$$\mathbb{R}^k = \bigcup_{n \in \mathbb{Z}^k} W(x,n).$$

This is also $\mathbb{Z}^k$-equivariant, namely $W(T^n x, m) = -n + W(x, n+m)$. Each $W(x,n)$ is a bounded convex set of $\mathbb{R}^k$ (see Lemma 5.15 (1) below) and depends continuously on $x \in X$ in the Hausdorff topology as in the case of $W_0(x,n)$. We sometimes denote $W(x,n)$ by $W^{A,L_0,L}(x,n)$ for clarifying the dependence on the parameters A, L_0, L. For $x \in X$ we define $\partial(x) = \partial^{A,L_0,L}(x)$ as $\bigcup_{n \in \mathbb{Z}^k} \partial W(x,n)$.

Lemma 5.15 (Basic properties of $W(x,n)$)**.** (1) *The diameter of $W(x,n)$ is bounded uniformly in $x \in X$ and $n \in \mathbb{Z}^k$. Moreover*

$$\sup_{x \in X} \sup_{n \in \mathbb{Z}^k} \sup_{u \in W(x,n)} |n - u| < \infty.$$

(2) *Let $\alpha_3 : \mathbb{R} \to [0,3]$ be a continuous function satisfying*

$$\alpha_3(t) = 3 \quad (t \le L_0), \quad \alpha_3(t) = 0 \quad (t \ge L_0 + 1).$$

There exists $R_0 = R_0(A, L_0, L) > 0$ such that for all $x \in X$ and all $u \in \mathbb{Z}^k$

$$A \sum_{|n-u| \le 2R_0} \alpha_3\left(d\left(n, \partial(x)\right)\right) < \sum_{|n-u| \le R_0} \left|\mathrm{Int}_{L_0} W(x,n)\right|,$$

where $d(n, \partial(x)) = \inf_{t \in \partial(x)} |n - t|$ and the sums are taken over all $n \in \mathbb{Z}^k$ satisfying $|n-u| \le 2R_0$ and $|n-u| \le R_0$ in the left-hand side and right-hand side respectively.

Proof. Let $x \in X$ and $n \in \mathbb{Z}^k$. It follows from Lemma 5.9 (2) and (3) that

$$\begin{aligned} &\forall \text{ integer point } m \in \mathrm{Int}_{E+2r_1\sqrt{k}} W_0(x,n) \setminus \left(n + L\mathbb{Z}^k\right) : \ \nu(x,m) = 0, \\ &\forall\, m \in \left(n + L\mathbb{Z}^k\right) \cap \mathrm{Int}_{E+\sqrt{k}} W_0(x,n) : \ \nu(x,m) = 1. \end{aligned} \tag{5.10}$$

(1) By (5.9) if we choose R sufficiently large then for all $x \in X$

$$B_R \cap \bigcup_{n \in \mathbb{Z}^k} \partial_{E+(L+1)\sqrt{k}} W_0(x,n) \neq B_R.$$

Then there exists $n \in \mathbb{Z}^k$ satisfying $B_R \cap \mathrm{Int}_{E+(L+1)\sqrt{k}} W_0(x,n) \neq \emptyset$. Then we can find a point $m \in \left(n + L\mathbb{Z}^k\right) \cap B_{R+L\sqrt{k}} \cap \mathrm{Int}_{E+\sqrt{k}} W_0(x,n)$. It follows from the second line of (5.10) that $\nu(x,m) = 1$. By the $\mathbb{Z}^k$-equivariance we conclude that for any $p \in \mathbb{Z}^k$ there exists an integer point $q \in B_{R+L\sqrt{k}}(p)$ satisfying $\nu(x,q) = 1$.

Take $u \in W(x,n)$. Let p be the integer point closest to u. Then we can find an integer point $q \in B_{R+L\sqrt{k}}(p)$ satisfying $\nu(x,q) = 1$. It follows form the definition of the Voronoi tiling that

$$\left|(u,0) - \left(n, \frac{1}{\nu(x,n)}\right)\right| \leq |(u,0) - (q,1)|\,.$$

From $\nu(x,n) \leq 1$,

$$|u-n| \leq |u-q| \leq |u-p| + |p-q| \leq \sqrt{k} + R + L\sqrt{k}.$$

This proves

$$\sup_{x \in X} \sup_{n \in \mathbb{Z}^k} \sup_{u \in W(x,n)} |n-u| \leq R + (L+1)\sqrt{k}.$$

(2) The above (5.10) implies

$$\forall m \in \left(n + L\mathbb{Z}^k\right) \cap \mathrm{Int}_{E+(L+1)\sqrt{k}} W_0(x,n) :\ W(x,m) = m + \left[-\frac{L}{2}, \frac{L}{2}\right]^k,$$

because for such m

$$\begin{aligned} &\forall \text{ integer point } l \in m + L\,(-1,1)^k \text{ with } l \neq m : \quad \nu(x,l) = 0, \\ &\forall\, l \in m + L\,\{-1,0,1\}^k : \quad \nu(x,l) = 1. \end{aligned}$$

We introduce a dichotomy: For $x \in X$ we define a "good set" $\mathbb{G}_x$ as the set of $m \in \mathbb{Z}^k$ satisfying $W(x,m) = m + [-L/2, L/2]^k$. We define a "bad set" $\mathbb{B}_x$ as the complement of $\mathbb{G}_x$ in $\mathbb{Z}^k$. Then it follows that

$$\begin{aligned} &\mathrm{Int}_{E+(2L+1)\sqrt{k}} W_0(x,n) \subset \bigcup_{m \in \mathbb{G}_x} W(x,m), \\ &\bigcup_{m \in \mathbb{B}_x} W(x,m) \subset \bigcup_{n \in \mathbb{Z}^k} \partial_{E+(2L+1)\sqrt{k}} W_0(x,n). \end{aligned} \tag{5.11}$$

For simplicity of the notation, we set $E' = E + (2L+1)\sqrt{k}$. We define D as the supremum of $|n-u|$ over all $x \in X$, $n \in \mathbb{Z}^k$ and $u \in W(x,n)$. This is finite by (1).

Let r and R be positive numbers and $u \in \mathbb{Z}^k$. We estimate $\sum_{n \in B_R(u)} |\partial_r W(x,n)|$. Noting

$$B_{R-D}(u) \subset \bigcup_{n \in B_R(u)} W(x,n) \subset B_{R+D}(u),$$

we can estimate

(5.12)
$$\begin{aligned}\sum_{n\in B_R(u)} |\partial_r W(x,n)| &= \sum_{n\in B_R(u)\cap \mathbb{G}_x} |\partial_r W(x,n)| + \sum_{n\in B_R(u)\cap \mathbb{B}_x} |\partial_r W(x,n)| \\ &\le 2^{k+1} r(L+2r)^{k-1}\frac{|B_{R+D}|}{L^k} + \left| B_{R+D+r}(u)\cap \bigcup_{n\in\mathbb{Z}^k} \partial_{E'+r} W_0(x,n)\right| \\ &= \frac{2^{k+1} r(L+2r)^{k-1}}{L^k}|B_{R+D}| + \left| B_{R+D+r}\cap \bigcup_{n\in\mathbb{Z}^k} \partial_{E'+r} W_0(T^u x,n)\right|.\end{aligned}$$

We would like to prove that for sufficiently large R (uniformly in $x\in X$ and $u\in\mathbb{Z}^k$)

$$A\sum_{n\in B_{2R}(u)} \alpha_3\left(d\left(n,\partial(x)\right)\right) < \sum_{n\in B_R(u)} |\mathrm{Int}_{L_0} W(x,n)|. \tag{5.13}$$

It follows from (5.12) that

$$\begin{aligned}\sum_{n\in B_R(u)} |\mathrm{Int}_{L_0} W(x,n)| &\ge \sum_{n\in B_R(u)} \left(|W(x,n)| - |\partial_{L_0} W(x,n)|\right) \\ &\ge |B_{R-D}| - \frac{2^{k+1} L_0 (L+2L_0)^{k-1}}{L^k}|B_{R+D}| \\ &\quad - \left| B_{R+D+L_0}\cap \bigcup_{n\in\mathbb{Z}^k} \partial_{E'+L_0} W_0(T^u x,n)\right|.\end{aligned}$$

If R is sufficiently larger than $D+L_0$ then (noting $L>2L_0$ by Condition 5.13)

(5.14)
$$\begin{aligned}\sum_{n\in B_R(u)} |\mathrm{Int}_{L_0} W(x,n)| &\ge \left|B_{R/2}\right| - \frac{4^k L_0}{L}|B_{2R}| - \left| B_{2R}\cap \bigcup_{n\in\mathbb{Z}^k} \partial_{E'+L_0} W_0(T^u x,n)\right| \\ &= \left(1-\frac{16^k L_0}{L}\right)\left|B_{R/2}\right| - \left| B_{2R}\cap \bigcup_{n\in\mathbb{Z}^k} \partial_{E'+L_0} W_0(T^u x,n)\right|.\end{aligned}$$

The left-hand side of (5.13) is bounded by

$$\begin{aligned}&3A\#\left(\mathbb{Z}^k\cap B_{2R}(u)\cap \bigcup_{n\in\mathbb{Z}^k} \partial_{L_0+1} W(x,n)\right) \\ &\le 3A\left| B_{2R+\sqrt{k}}(u)\cap \bigcup_{n\in\mathbb{Z}^k} \partial_{L_0+1+\sqrt{k}} W(x,n)\right| \\ &\le 3A \sum_{n\in B_{2R+2\sqrt{k}+D+L_0+1}(u)} \left|\partial_{L_0+1+\sqrt{k}} W(x,n)\right|.\end{aligned}$$

As in the case of (5.14) if R is sufficiently large then this is bounded by (noting $L > 2L_0 + 2 + 2\sqrt{k}$ by Condition 5.13)

$$\begin{aligned} &3A\left(\frac{4^k\left(L_0+1+\sqrt{k}\right)}{L}\left|B_{3R}\right| + \left|B_{3R}\cap \bigcup_{n\in\mathbb{Z}^k}\partial_{E'+L_0+1+\sqrt{k}}W_0(T^u x, n)\right|\right) \\ (5.15)\quad &= 3A\left(\frac{24^k\left(L_0+1+\sqrt{k}\right)}{L}\left|B_{R/2}\right| + \left|B_{3R}\cap \bigcup_{n\in\mathbb{Z}^k}\partial_{E'+L_0+1+\sqrt{k}}W_0(T^u x, n)\right|\right). \end{aligned}$$

By combining (5.14), (5.15) with Condition 5.13 and (5.9), we can conclude that (5.13) holds for sufficiently large R. $\square$

As we explained in Subsection 3.2 we need a social welfare system among the tiles $W(x,n)$. The next proposition provides it.

Proposition 5.16 (Construction of weight functions)**.** *We can construct a continuous map*

$$(5.16)\quad X \to \left([0,1]^{\mathbb{Z}^k}\right)^{\mathbb{Z}^k},\ x \mapsto (w(x,n))_{n\in\mathbb{Z}^k},\ \text{where } w(x,n) = (w_m(x,n))_{m\in\mathbb{Z}^k} \in [0,1]^{\mathbb{Z}^k}$$

satisfying the following.

1. *The map is equivariant in the sense that for all $m, n \in \mathbb{Z}^k$ and $x \in X$*
$$w(T^m x, n) = w(x, n+m).$$
2. *If $x, y \in X$ satisfy $\Phi(x) = \Phi(y)$ then $w(x,n) = w(y,n)$ for all $n \in \mathbb{Z}^k$.*
3. *For all $x \in X$ and $n \in \mathbb{Z}^k$*
$$\#\{m \in \mathbb{Z}^k \mid w_m(x,n) > 0\} < 1 + \frac{1}{A}\left|\mathrm{Int}_{L_0} W(x,n)\right|.$$
In particular if $\mathrm{Int}_{L_0} W(x,n) = \emptyset$ then $w_m(x,n) = 0$ for all $m \in \mathbb{Z}^k$.
4. *Let $x \in X$ and $p \in \mathbb{Z}^k$ with $d(p, \partial(x)) \le L_0$. There exists $n \in \mathbb{Z}^k \cap B_{R_0}(p)$ satisfying $w_{p-n}(x,n) = 1$. Here $R_0 = R_0(A, L_0, L)$ is the positive number introduced in Lemma 5.15 (2).*

We call $w(x,n)$ ***weight functions*** *and sometimes write $w^{A,L_0,L}(x,n) = w(x,n)$ for clarifying the dependence on parameters.*

Probably the intuitive meaning of the statement is not clear at all. So we explain it before the proof. (The idea of the social welfare system was first introduced in [GT]. The explanation below is more or less a reproduction of the argument around [GT, Lemma 6.4].) In the proof of Proposition 3.1 we construct a perturbation $g_1(x)$ of a given band-limited function $f(x)$ for each $x \in X$. It is difficult to construct a perturbation over the whole space $\mathbb{R}^k$ at once. So we will perturb $f(x)$ over each tile $W(x,n)$ separately. The difficulty arises near the boundary $\partial W(x,n)$. The parameter L_0 will be chosen so that we can construct a good perturbation in $\mathrm{Int}_{L_0} W(x,n)$. But we cannot control the perturbation over $\partial_{L_0} W(x,n)$. (Note that if the tile $W(x,n)$ is tiny, it may be contained in $\partial_{L_0} W(x,n)$. So the problem is approximately equivalent to "how to help small tiles".)

The function $g_1(x)$ should encode the information of the orbit of $x \in X$. The above argument means that it becomes difficult to encode the information of $T^p x$ for $p \in \mathbb{Z}^k$ with $d(p, \partial(x)) \le L_0$. The weight functions $w_m(x,n)$ help with this situation. Roughly (and very inaccurately) speaking, we will encode the $(100 \times w_m(x,n))\,\%$ of the information of $T^{n+m}x$ to $g_1(x)|_{W(x,n)}$. So $w_m(x,n)$ is the "amount of help" that the tile $W(x,n)$ gives to the point $n+m$. In particular, if $w_m(x,n) = 1$ then the information of $T^{n+m}x$ is perfectly encoded into $g_1(x)|_{W(x,n)}$.

Now we can explain the intuitive meaning of the above (1)–(4). Condition (1) is an obvious requirement. Condition (2)[12] means that the weight function $w(x,n)$ is determined by the tiling-like band-limited map $\Phi(x)$. Condition (3) means that the amount of "additional encoding" which $W(x,n)$ has to bear is controlled by the volume of $\mathrm{Int}_{L_0} W(x,n)$. Condition (4) means that every point near the boundary $\partial(x)$ is "successfully rescued" by some large tile. Namely we can solve all the difficulties coming from the boundary effect.

Proof of Proposition 5.16. We fix a bijection between $\mathbb{Z}^k$ and $\mathbb{N} = \{1,2,3,\dots\}$

$$\mathbb{N} \ni l \mapsto m_l \in \mathbb{Z}^k$$

such that if $l_1 < l_2$ then $|m_{l_1}| \le |m_{l_2}|$. (Then in particular $m_1 = 0$.)

Fix $x \in X$. For $n \in \mathbb{Z}^k$ we set

$$u_0(n) = \frac{|\mathrm{Int}_{L_0} W(x,n)|}{A}, \quad v_0(n) = \alpha_3\left(d\left(n, \partial(x)\right)\right).$$

Here $\alpha_3 : \mathbb{R} \to [0,3]$ is the continuous function introduced in Lemma 5.15 (2). We define $\omega_l(n), u_l(n), v_l(n)$ inductively with respect to $l \ge 1$ by

$$\begin{aligned}
\omega_l(n) &= \min\left(u_{l-1}(n), v_{l-1}(n+m_l)\right),\\
u_l(n) &= u_{l-1}(n) - \omega_l(n),\\
v_l(n) &= v_{l-1}(n) - \omega_l(n-m_l).
\end{aligned}$$

This process can be explained in "social welfare" terms:

- The tiles $W(x,n)$ should pay tax. Tax is used for helping integer points $p \in \mathbb{Z}^k$. The tile $W(x,n)$ can pay at most $u_0(n)$ and the point p needs $v_0(p)$.
- At the l-th step of the induction, the tile $W(x,n)$ gives the money $\omega_l(n)$ to the point $n+m_l$. After the step, $W(x,n)$ still can pay $u_l(n)$ and points p still need $v_l(p)$.

[12]The above explanation does not feature Condition (2). But indeed this is a very important condition. It did not appear in the paper [GT]. A substantial amount of the argument in this section has been developed for establishing this condition.

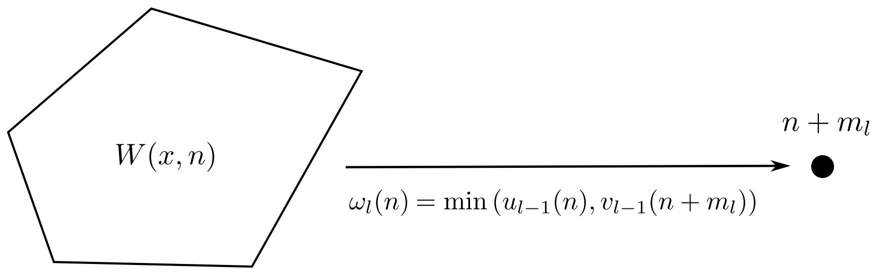

FIGURE 5.4. At the l-th step of the induction, the tile $W(x,n)$ gives the amount of money $\omega_l(n)$ to the point $n+m_l$. Before this step, $W(x,n)$ has the amount of money $u_{l-1}(n)$ and the point $n+m_l$ needs $v_{l-1}(n+m_l)$. After this step, either $W(x,n)$ loses all its money (i.e. $u_l(n)=0$) or $n+m_l$ becomes satisfied (i.e. $v_l(n+m_l)=0$). Namely, $\min\left(u_l(n), v_l(n+m_l)\right)=0$.

- At each step, $W(x,n)$ pays as much as possible. Namely, after the l-th step, at least one of $u_l(n)$ and $v_l(n+m_l)$ is zero. This is a key property of the process. Figure 5.4 schematically explains the process.

Set $l_0 = \#\left(\mathbb{Z}^k \cap B_{R_0}\right)$. Then $\mathbb{Z}^k \cap B_{R_0} = \{m_1, \dots, m_{l_0}\}$. It follows from Lemma 5.15 (2) that this process terminates: For all $l \geq l_0$ and $n \in \mathbb{Z}^k$

$$v_l(n) = 0, \quad \omega_{l+1}(n) = 0,$$

because if $v_{l_0}(n_0) > 0$ for some n_0 then $u_{l_0}(n) = 0$ for all $n \in B_{R_0}(n_0)$ and hence

$$\sum_{n \in B_{R_0}(n_0)} u_0(n) = \sum_{n \in B_{R_0}(n_0)} \sum_{l=1}^{l_0} \omega_l(n) < \sum_{n \in B_{2R_0}(n_0)} v_0(n),$$

which contradicts Lemma 5.15 (2).

It follows from the construction that for $n, p \in \mathbb{Z}^k$

$$\sum_{l=1}^{l_0} \omega_l(n) \leq u_0(n) = \frac{|\mathrm{Int}_{L_0} W(x,n)|}{A}, \tag{5.17}$$

$$\sum_{l=1}^{l_0} \omega_l(p - m_l) = v_0(p) = \alpha_3\left(d\left(n, \partial(x)\right)\right). \tag{5.18}$$

We define continuous functions $\alpha_4 : \mathbb{R} \to [1, 2l_0]$ and $\alpha_5 : \mathbb{R} \to [0,1]$ such that

$$\alpha_4(0) = 2l_0, \quad \alpha_4(t) = 1 \quad (t \geq 1),$$
$$\alpha_5(t) = 0 \quad (t \leq 1), \quad \alpha_5(t) = 1 \quad (t \geq 2).$$

We define $F : \mathbb{R}^{l_0} \to \mathbb{R}^{l_0}$ by $F(x_1, \dots, x_{l_0}) = (y_1, \dots, y_{l_0})$ where

$$y_{l_0} = 2l_0 x_{l_0}, \quad y_{l_0-1} = \alpha_4(y_{l_0}) x_{l_0-1}, \quad y_{l_0-2} = \alpha_4\left(\max(y_{l_0}, y_{l_0-1})\right) x_{l_0-2}, \quad \dots$$
$$y_1 = \alpha_4\left(\max(y_{l_0}, \dots, y_2)\right) x_1.$$

This satisfies

$$\#\{1 \leq l \leq l_0 |\, y_l > 1\} \leq 1 + \#\{1 \leq l \leq l_0 |\, x_l > 1\}. \tag{5.19}$$

We define $G : \mathbb{R}^{l_0} \to [0,1]^{l_0}$ by $G(y_1, \dots, y_{l_0}) = (\alpha_5(y_1), \dots, \alpha_5(y_{l_0}))$. Then it follows from the above (5.19) that if we set $GF(x_1, \dots, x_{l_0}) = (z_1, \dots, z_{l_0})$ then

$$\#\{1 \leq l \leq l_0 |\, z_l > 0\} \leq 1 + \#\{1 \leq l \leq l_0 |\, x_l > 1\}. \tag{5.20}$$

We define $w_m(x,n)$ for $|m| \leq R_0$ and $n \in \mathbb{Z}^k$ by

$$\left(w_{m_1}(x,n), w_{m_2}(x,n), \dots, w_{m_{l_0}}(x,n)\right) = GF\left(\omega_1(n), \dots, \omega_{l_0}(n)\right).$$

We set $w_m(x,n) = 0$ for $|m| > R_0$.

Now we have defined the map (5.16) in the statement. We need to check its properties. It is continuous and equivariant (i.e. Property (1)) because the tiles $W(x,n)$ depend continuously on $x \in X$ and $W(T^m x, n) = -m + W(x, n+m)$. Property (2) is obvious because the tiles $W(x,n)$ are constructed from the function $\Phi(x)$.

Fix $x \in X$ again. From (5.17)

$$\#\{1 \leq l \leq l_0 |\, \omega_l(n) > 1\} < \frac{|\mathrm{Int}_{L_0} W(x,n)|}{A}.$$

Then Property (3) follows from (5.20):

$$\#\{m |\, w_m(x,n) > 0\} \leq 1 + \#\{1 \leq l \leq l_0 |\, \omega_l(n) > 1\} < 1 + \frac{|\mathrm{Int}_{L_0} W(x,n)|}{A}.$$

Finally we check Property (4). Let $p \in \mathbb{Z}^k$ with $d\left(p, \partial(x)\right) \leq L_0$. Then $v_0(p) = 3$. From (5.18)

$$\sum_{l=1}^{l_0} \omega_l(p - m_l) = 3.$$

We define l_1 as the maximum of $1 \leq l \leq l_0$ satisfying $\omega_l(p - m_l) > 0$. If $\omega_{l_1}(p - m_{l_1}) \geq 2$ then $w_{m_{l_1}}(p - m_{l_1}) = 1$. Otherwise

$$\sum_{l=1}^{l_1 - 1} \omega_l(p - m_l) > 1.$$

Then there exists $l_2 < l_1$ satisfying

$$\omega_{l_2}(p - m_{l_2}) > \frac{1}{l_1} \geq \frac{1}{l_0}.$$

The condition $l_2 < l_1$ implies $u_{l_2}(p - m_{l_2}) = 0$ (otherwise $v_{l_2}(p) = 0$ and $\omega_{l_1}(p - m_{l_1}) = 0$) and hence $\omega_l(p - m_{l_2}) = 0$ for all $l > l_2$. It follows from the definition of the maps F and G that

$$w_{l_2}(p - m_{l_2}) = 1.$$

This proves Property (4). $\square$

6. Proof of Main Proposition

We prove Proposition 3.1 in this section. We recommend readers to review the notations introduced in Subsection 3.3. We repeat the statement of the proposition (assuming the notations in Subsection 3.3):

Proposition 6.1 (= Proposition 3.1). *Let d be a distance on X and $f : X \to \mathcal{B}_1(a_1, \dots, a_k)$ a $\mathbb{Z}^k$-equivariant continuous map. Then there exists a $\mathbb{Z}^k$-equivariant continuous map $g : X \to \mathcal{B}_1(a_1 + \delta, \dots, a_k + \delta)$ such that*

- *$\|g(x) - f(x)\|_{L^\infty(\mathbb{R}^k)} < \delta$ for all $x \in X$.*
- *g is a δ-embedding with respect to the distance d.*

We can assume $\|f(x)\|_{L^\infty(\mathbb{R}^k)} \le 1 - \delta$ for all $x \in X$ by replacing f with $(1-\delta)f$ if necessary. We fix $0 < \delta' < \delta$ satisfying

$$4K_0\delta' < \delta \tag{6.1}$$

where K_0 is the positive number introduced in Notation 4.1 (1). We choose $0 < \varepsilon < \delta$ such that for any $x, y \in X$

$$d(x, y) < \varepsilon \Longrightarrow \|f(x) - f(y)\|_{L^\infty\left([0,1]^k\right)} < \delta'. \tag{6.2}$$

From $\mathrm{mdim}(X) < \rho_1 \dots \rho_k/2$ we can find $c_0 > 1$ and a natural number N such that

- $\rho_i N$ are integers for all $1 \le i \le k$. (Recall $\rho_i \in \mathbb{Q}$.)
- $$c_0 \cdot \frac{\mathrm{Widim}_\varepsilon\left(X, d_{[N]}\right) + 1}{N^k} < \frac{\rho_1 \dots \rho_k}{2}.$$

From the second condition, we can find a simplicial complex P and an ε-embedding $\Pi : \left(X, d_{[N]}\right) \to P$ satisfying

$$c_0 \frac{\dim P + 1}{N^k} < \frac{\rho_1 \dots \rho_k}{2}. \tag{6.3}$$

We take a simplicial complex Q and an ε-embedding $\pi : (X, d) \to Q$.

We define the **cone** CP as $[0,1] \times P/\sim$, where $(0, p) \sim (0, q)$ for all $p, q \in P$. We denote by tp the equivalence class of $(t, p) \in [0,1] \times P$. We set $* = 0p$ and call it the **vertex** of the cone. We define CQ in the same way. It follows from (6.3) that

$$c_0 \frac{\dim CP}{N^k} < \frac{\rho_1 \dots \rho_k}{2}. \tag{6.4}$$

We set

$$c_1 = \frac{\rho_1 \dots \rho_k}{2} - c_0 \frac{\dim CP}{N^k} > 0. \tag{6.5}$$

Recall that we introduced the positive number r_0 in Notation 4.1 (2). We set

$$r_2 = r_0 + \sqrt{\sum_{i=1}^k (1/\rho_i)^2}. \tag{6.6}$$

Notation 6.2 (Fixing the parameters A, L_0, L)**.** We take positive numbers A, L_0 and L satisfying the following conditions.

(1) $c_1 A > 2\dim CQ$.

(2) L_0 is sufficiently larger than $r_2 + N\sqrt{k}$ such that if a bounded closed convex set $W \subset \mathbb{R}^k$ satisfies $\mathrm{Int}_{L_0} W \neq \emptyset$ then

$$2\dim CQ < c_1 \left|\mathrm{Int}_{r_2+N\sqrt{k}} W\right|, \tag{6.7}$$

$$\left|W \cup \partial_{N\sqrt{k}} W\right| < c_0 \left|\mathrm{Int}_{r_2+N\sqrt{k}} W\right|. \tag{6.8}$$

The second condition is satisfied for sufficiently large L_0 by Lemma 2.7.

(3) $L > 4/\delta$ (see Lemma 5.14 (2)) and Condition 5.13 holds.

For these A, L_0 and L we construct the tiles $W(x,n) = W^{A,L_0,L}(x,n)$ and the weight functions $w(x,n) = w^{A,L_0,L}(x,n)$ as in Subsection 5.4.

Lemma 6.3. *Let $x \in X$ and $n \in \mathbb{Z}^k$. We define*

$$\#(x,n) = \#\{m \in \mathbb{Z}^k \mid w_m(x,n) > 0\}.$$

If $\mathrm{Int}_{L_0} W(x,n) \neq \emptyset$ *then*

$$\frac{\left|W(x,n) \cup \partial_{N\sqrt{k}} W(x,n)\right|}{N^k} \dim CP + \#(x,n)\cdot \dim CQ < \frac{\rho_1 \dots \rho_k}{2} \left|\mathrm{Int}_{r_2+N\sqrt{k}} W(x,n)\right|. \tag{6.9}$$

Proof. By (6.8) and Proposition 5.16 (3), the left-hand side of (6.9) is smaller than

$$c_0 \frac{\left|\mathrm{Int}_{r_2+N\sqrt{k}} W(x,n)\right|}{N^k} \dim CP + \left(1 + \frac{|\mathrm{Int}_{L_0} W(x,n)|}{A}\right) \dim CQ. \tag{6.10}$$

By Notation 6.2 (1) and (2), the second term is smaller than

$$\frac{1}{2} c_1 \left|\mathrm{Int}_{r_2+N\sqrt{k}} W(x,n)\right| + \frac{1}{2} c_1 \left|\mathrm{Int}_{L_0} W(x,n)\right| \le c_1 \left|\mathrm{Int}_{r_2+N\sqrt{k}} W(x,n)\right| \quad (\text{by } L_0 > r_2 + N\sqrt{k}).$$

Then it follows from the definition of c_1 in (6.5) that the above (6.10) is smaller than

$$\left(c_0 \frac{\dim CP}{N^k} + c_1\right) \left|\mathrm{Int}_{r_2+N\sqrt{k}} W(x,n)\right| = \frac{\rho_1 \dots \rho_k}{2} \left|\mathrm{Int}_{r_2+N\sqrt{k}} W(x,n)\right|.$$

□

Notation 6.4 (Choice of R)**.** We take a positive number R such that

- $R \in N\mathbb{Z}$.
- $R > R_0(A, L_0, L)$ where $R_0(A, L_0, L)$ is the positive number introduced in Lemma 5.15 (2).

- For all $x \in X$ and $n \in \mathbb{R}^k$, the tile $W(x,n)$ satisfies $-n + W(x,n) \subset [-R,R)^k$, where $-n+W(x,n) = \{-n+t | \, t \in W(x,n)\}$. This condition is satisfied for sufficiently large R by Lemma 5.15 (1).

We set

$$\boldsymbol{\mathcal{P}} = (CP)^{[-R,R)^k \cap N\mathbb{Z}^k}, \quad \boldsymbol{\mathcal{Q}} = (CQ)^{[-R,R)^k \cap \mathbb{Z}^k}.$$

For $i \geq 0$ we define $\boldsymbol{\mathcal{Q}}(i) \subset \boldsymbol{\mathcal{Q}}$ as the set of $(q_n)_{n\in[-R,R)^k\cap\mathbb{Z}^k}$ satisfying $q_n = *$ except for at most i entries. In particular $\boldsymbol{\mathcal{Q}}(0) = \{(*, \dots, *)\}$ and $\dim \boldsymbol{\mathcal{Q}}(i) \leq i \dim CQ$. Let $W \subset [-R,R)^k$ be a convex set. We define $\boldsymbol{\mathcal{P}}(W) \subset \boldsymbol{\mathcal{P}}$ as the set of $(p_n)_{n\in[-R,R)^k\cap N\mathbb{Z}^k}$ such that if $W \cap \left(n+[0,N)^k\right) = \emptyset$ then $p_n = *$. Its dimension is estimated by

$$\dim \boldsymbol{\mathcal{P}}(W) \leq \frac{\left|W \cup \partial_{N\sqrt{k}} W\right|}{N^k} \dim CP.$$

We define a map $\Pi_W : X \to \boldsymbol{\mathcal{P}}(W)$ by

$$\Pi_W(x) = \left(\frac{\left|W \cap \left(n+[0,N)^k\right)\right|}{N^k} \cdot \Pi(T^n x)\right)_{n\in[-R,R)^k\cap N\mathbb{Z}^k}.$$

Π_W is an ε-embedding with respect to the distance $d_{\mathrm{Int}W\cap\mathbb{Z}^k}$, where $\mathrm{Int}W$ is the interior of W. It follows from Lemma 6.3 that for any $x \in X$ and $n \in \mathbb{Z}^k$ with $\mathrm{Int}_{L_0} W(x,n) \neq \emptyset$

$$\dim\left(\boldsymbol{\mathcal{P}}\left(-n+W(x,n)\right) \times \boldsymbol{\mathcal{Q}}\left(\#(x,n)\right)\right) < \frac{\rho_1 \cdots \rho_k}{2}\left|\mathrm{Int}_{r_2+N\sqrt{k}} W(x,n)\right|. \tag{6.11}$$

Recall that we introduced lattices Γ and Γ_1 of $\mathbb{R}^k$ in Subsection 3.3. Applying Lemma 2.5 to the ε-embedding $\Pi : (X, d_{[N]}) \to P$ and a map

$$X \to \mathbb{R}^{\Gamma\cap[0,N)^k}, \quad x \mapsto \left(f(x)(\lambda)\right)_{\lambda\in\Gamma\cap[0,N)^k},$$

we can find a simplicial map $F : P \to \mathbb{R}^{\Gamma\cap[0,N)^k}$ satisfying

$$\forall x \in X, \lambda \in \Gamma \cap [0,N)^k : \quad \left|F\left(\Pi(x)\right)(\lambda) - f(x)(\lambda)\right| < \delta'.$$

We extend F over CP by $F(tp) = tF(p)$. We define a simplicial map $G : \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{Q}} \to \mathbb{R}^{\Gamma\cap[-R,R)^k}$ as follows: For $p = (p_n)_{n\in[-R,R)^k\cap N\mathbb{Z}^k} \in \boldsymbol{\mathcal{P}}$, $q \in \boldsymbol{\mathcal{Q}}$, $n \in [-R,R)^k \cap N\mathbb{Z}^k$ and $\lambda \in \Gamma \cap [0,N)^k$

$$G(p,q)(n+\lambda) = F(p_n)(\lambda).$$

Using Lemma 2.6 (2), we perturb G and construct a simplicial map $\mathbb{G} : \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{Q}} \to \mathbb{R}^{\Gamma\cap[-R,R)^k}$ satisfying the following.

Property 6.5 (Properties on $\mathbb{G}$)**.** (1) $\mathbb{G}$ *is sufficiently close to* G *in the sense that for all* $(p,q) \in \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{Q}}$ *and* $\lambda \in \Gamma \cap [-R,R)^k$

$$\left|\mathbb{G}(p,q)(\lambda) - G(p,q)(\lambda)\right| < \delta' - \sup_{x\in X, \lambda\in\Gamma\cap[0,N)^k} \left|F\left(\Pi(x)\right)(\lambda) - f(x)(\lambda)\right|.$$

This implies that for all convex sets $W \subset [-R,R)^k$, $x \in X$ *and* $q \in \boldsymbol{\mathcal{Q}}$

$$\forall \lambda \in \Gamma \cap \mathrm{Int}_{N\sqrt{k}} W : \quad \left|\mathbb{G}\left(\Pi_W(x), q\right)(\lambda) - f(x)(\lambda)\right| < \delta'.$$

(2) *If a convex set* $W \subset [-R, R)^k$, $i \geq 0$ *and a subset* $\Lambda \subset \Gamma \cap [-R, R)^k$ *satisfy*

$$\dim\left(\boldsymbol{\mathcal{P}}(W) \times \boldsymbol{\mathcal{Q}}(i)\right) < \frac{|\Lambda|}{2},$$

then the map

$$\boldsymbol{\mathcal{P}}(W) \times \boldsymbol{\mathcal{Q}}(i) \to \mathbb{R}^{\Lambda}, \quad (p, q) \mapsto \mathbb{G}(p, q)|_{\Lambda}$$

is an embedding.

Now we are ready to prove the main proposition.

Proof of Proposition 3.1. We choose continuous functions $\beta_1, \beta_2 : \mathbb{R} \to [0, 1]$ satisfying

$$\beta_1(t) = 0 \quad (t \leq r_0), \quad \beta_1(t) = 1 \quad (t \geq r_0 + 1),$$
$$\beta_2(t) = 0 \quad \left(t \leq N\sqrt{k}\right), \quad \beta_2(t) = 1 \quad \left(t \geq r_0 + N\sqrt{k}\right).$$

Let $x \in X$. Recall that we introduced $\partial(x) = \bigcup_{n \in \mathbb{Z}^k} \partial W(x, n)$. We define $p_x : \Gamma_1 \to [0, 1]$ and $u_x = (u_x(\lambda))_{\lambda \in \Gamma_1} \in \ell^\infty(\Gamma_1)$ as follows: For $\lambda \in \Gamma_1$

- If there exists $n \in \mathbb{Z}^k$ satisfying $\lambda \in (n + \Gamma) \cap W(x, n)$ then

$$\begin{aligned} p_x(\lambda) =& \beta_1\left(d(\lambda, \partial(x))\right), \\ u_x(\lambda) =& \beta_2\left(d(\lambda, \partial(x))\right)\{-f(x)(\lambda) \\ & + \mathbb{G}\left(\Pi_{-n+W(x,n)}(T^n x), \left(w_m(x, n) \cdot \pi(T^{n+m}x)\right)_{m \in [-R,R)^k \cap \mathbb{Z}^k}\right)(\lambda - n)\}. \end{aligned}$$

- Otherwise we set $p_x(\lambda) = 0$ and $u_x(\lambda) = 0$.

p_x is an *admissible* function in the sense of Definition 4.2 (2). It follows from Property 6.5 (1) with $f(x)(\lambda) = f(T^n x)(\lambda - n)$ that

$$\|u_x\|_\infty < \delta'. \tag{6.12}$$

Since the tiles $W(x, n)$ and the weight functions $w(x, n)$ depend continuously on $x \in X$, the numbers $p_x(\lambda)$ and $u_x(\lambda)$ depend continuously on x for each fixed $\lambda \in \Gamma_1$. We set

$$g_1(x) = f(x) + \Psi(p_x, u_x) \in \mathcal{B}(a_1, \dots, a_k)$$

where Ψ is the interpolating function introduced in Section 4. By the continuity of Ψ (Proposition 4.9), $g_1(x)$ depends continuously on $x \in X$. Since the tiles $W(x, n)$ and the weight functions $w(x, n)$ satisfy the natural $\mathbb{Z}^k$-equivariance, $p_x(\lambda)$ and $u_x(\lambda)$ also satisfy the $\mathbb{Z}^k$-equivariance (i.e. $p_{T^n x}(\lambda) = p_x(\lambda + n)$ and $u_{T^n x}(\lambda) = u_x(\lambda + n)$). The interpolating function Ψ also satisfies it (Lemma 4.7 (3)). Therefore the map $X \ni x \mapsto g_1(x) \in \mathcal{B}_1(a_1, \dots, a_k)$ is $\mathbb{Z}^k$-equivariant. By Lemma 4.7 (2) and (6.12)

$$\begin{aligned} \|g_1(x) - f(x)\|_{L^\infty(\mathbb{R}^k)} &= \|\Psi(p_x, u_x)\|_{L^\infty(\mathbb{R}^k)} \\ &< 2K_0\delta' < \frac{\delta}{2} \quad \left(\text{we assumed } 4K_0\delta' < \delta \text{ in } (6.1)\right). \end{aligned} \tag{6.13}$$

Let $\Phi_{A,L_0,L}(x) = (\Phi(x)_1, \dots, \Phi(x)_k) : \mathbb{C}^k \to \mathbb{C}^k$ be the tiling-like band-limited map introduced in Subsection 5.4 for the parameters A, L_0, L in Notation 6.2. We define $g_2(x) : \mathbb{R}^k \to \mathbb{R}$ by

$$g_2(x) = \frac{\delta}{2kK_1} \mathrm{Re}\left(\sum_{i=1}^{k} \Phi(x)_i\right).$$

It follows from $\|\Phi(x)\|_{L^\infty(\mathbb{R}^k)} \le \sqrt{k}K_1$ (Lemma 5.14 (1)) that

$$\|g_2(x)\|_{L^\infty(\mathbb{R}^k)} \le \frac{\delta}{2}. \tag{6.14}$$

By Lemma 5.14 (2) with $L > 4/\delta$, the Fourier transform of each $\Phi(x)_i|_{\mathbb{R}^k}$ is supported in

$$\Delta_i \overset{\mathrm{def}}{=} \left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^{i-1} \times \left(\frac{a_i}{2}, \frac{a_i}{2} + \frac{\delta}{2}\right) \times \left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^{k-i}, \quad (1 \le i \le k).$$

Then the Fourier transform of $g_2(x)$ is supported in

$$\bigcup_{i=1}^{k} \Delta_i \cup (-\Delta_i), \quad (-\Delta_i = \{-\xi \mid \xi \in \Delta_i\}). \tag{6.15}$$

This is contained in $\prod_{i=1}^{k}[-(a_i+\delta)/2, (a_i+\delta)/2]$. So $g_2(x) \in \mathcal{B}(a_1+\delta, \dots, a_k+\delta)$. The map $X \ni x \mapsto g_2(x) \in \mathcal{B}(a_1+\delta, \dots, a_k+\delta)$ is continuous and $\mathbb{Z}^k$-equivariant by the corresponding properties of $\Phi(x)$ in Lemma 5.14 (3) and (4).

Finally we set $g(x) = g_1(x) + g_2(x)$. By (6.13) and (6.14)

$$\|f(x) - g(x)\|_{L^\infty(\mathbb{R}^k)} < \delta.$$

In particular $\|g(x)\|_{L^\infty(\mathbb{R}^k)} < 1$ by the assumption $\|f(x)\|_{L^\infty(\mathbb{R}^k)} \le 1 - \delta$ in the beginning of the section. The map

$$X \ni x \mapsto g(x) \in \mathcal{B}_1(a_1+\delta, \dots, a_k+\delta)$$

is continuous and $\mathbb{Z}^k$-equivariant.

We need to prove that $g : X \to \mathcal{B}_1(a_1+\delta, \dots, a_k+\delta)$ is a δ-embedding with respect to d. Suppose $g(x) = g(y)$ for some $x, y \in X$. We would like to show $d(x,y) < \delta$. It follows from $g(x) = g(y)$ that $g_1(x) = g_1(y)$ and $g_2(x) = g_2(y)$ because the Fourier transforms of $g_1(x)$ and $g_1(y)$ are supported in $\prod_{i=1}^{k}[-a_i/2, a_i/2]$, which is disjoint with (6.15). The equation $g_2(x) = g_2(y)$ implies $\Phi(x)_i = \Phi(y)_i$ for all $1 \le i \le k$ because $2k$ sets Δ_i and $-\Delta_i$ are disjoint with each other. So $\Phi(x) = \Phi(y)$. Then it follows that $W(x,n) = W(y,n)$ and $w(x,n) = w(y,n)$ for all $n \in \mathbb{Z}^k$ because the tiles $W(x,n)$ and the weight functions $w(x,n)$ are constructed from the tiling-like band-limited map $\Phi(x)$.

Case 1: Suppose $d(0, \partial(x)) > L_0$. Then there exists $n \in \mathbb{Z}^k$ satisfying $0 \in \mathrm{Int}_{L_0} W(x,n)$. For all $\lambda \in (n+\Gamma) \cap \mathrm{Int}_{r_0+N\sqrt{k}} W(x,n)$

$$p_x(\lambda) = 1,$$

$$u_x(\lambda) = -f(x)(\lambda) + \mathbb{G}\left(\Pi_{-n+W(x,n)}(T^n x), \left(w_m(x,n)\cdot\pi(T^{n+m}x)\right)_{m\in[-R,R)^k\cap\mathbb{Z}^k}\right)(\lambda - n).$$

It follows from the interpolation property of Ψ in Lemma 4.7 (1) that for all $\lambda \in (n + \Gamma) \cap \mathrm{Int}_{r_0+N\sqrt{k}} W(x,n)$

$$
\begin{aligned}
g_1(x)(\lambda) &= \mathbb{G}\left(\Pi_{-n+W(x,n)}(T^n x), \left(w_m(x,n)\cdot \pi(T^{n+m}x)\right)_{m\in[-R,R)^k\cap\mathbb{Z}^k}\right)(\lambda - n),\\
g_1(y)(\lambda) &= \mathbb{G}\left(\Pi_{-n+W(y,n)}(T^n y), \left(w_m(y,n)\cdot \pi(T^{n+m}y)\right)_{m\in[-R,R)^k\cap\mathbb{Z}^k}\right)(\lambda - n).
\end{aligned} \tag{6.16}
$$

From $\mathrm{Int}_{L_0} W(x,n) \neq \emptyset$ (because $0 \in \mathrm{Int}_{L_0} W(x,n)$) and (6.11)

$$
\begin{aligned}
\dim\left(\boldsymbol{\mathcal{P}}\left(-n + W(x,n)\right) \times \boldsymbol{\mathcal{Q}}\left(\#(x,n)\right)\right) &< \frac{\rho_1\dots\rho_k}{2}\left|\mathrm{Int}_{r_2+N\sqrt{k}} W(x,n)\right| \\
&\le \frac{1}{2}\#\left(\Gamma \cap \left(-n + \mathrm{Int}_{r_0+N\sqrt{k}} W(x,n)\right)\right), \\
&\quad \left(\text{by } r_2 = r_0 + \sqrt{\sum_i (1/\rho_i)^2} \text{ in (6.6)}\right).
\end{aligned}
$$

Therefore by Property 6.5 (2) the map

$$
\begin{aligned}
\boldsymbol{\mathcal{P}}\left(-n + W(x,n)\right) \times \boldsymbol{\mathcal{Q}}\left(\#(x,n)\right) &\to \mathbb{R}^{\Gamma\cap\left(-n+\mathrm{Int}_{r_0+N\sqrt{k}} W(x,n)\right)} \\
(p,q) &\mapsto \mathbb{G}(p,q)|_{\Gamma\cap\left(-n+\mathrm{Int}_{r_0+N\sqrt{k}} W(x,n)\right)}
\end{aligned} \tag{6.17}
$$

is an embedding. Thus $g_1(x) = g_1(y)$ implies that $\Pi_{-n+W(x,n)}(T^n x) = \Pi_{-n+W(y,n)}(T^n y)$. Recall that the map $\Pi_{-n+W(x,n)}$ is an ε-embedding with respect to the distance $d_{-n+\mathrm{Int}W(x,n)}$. It follows from $0 \in \mathrm{Int}_{L_0} W(x,n)$ that $-n \in -n + \mathrm{Int}W(x,n)$. Thus we get

$$
d(x,y) \le d_{-n+\mathrm{Int}W(x,n)}(T^n x, T^n y) < \varepsilon < \delta.
$$

Case 2: Suppose $d\left(0, \partial(x)\right) \le L_0$. It follows from Proposition 5.16 (3) and (4) that there exists $n \in B_{R_0} \cap \mathbb{Z}^k$ satisfying $\mathrm{Int}_{L_0} W(x,n) \neq \emptyset$ and $w_{-n}(x,n) = w_{-n}(y,n) = 1$. As in Case 1, we have the same formula (6.16) for all $\lambda \in (n+\Gamma) \cap \mathrm{Int}_{r_0+N\sqrt{k}} W(x,n)$. The map (6.17) is also an embedding. Then the equation $g_1(x) = g_1(y)$ implies that

$$
\left(w_m(x,n)\cdot \pi(T^{n+m}x)\right)_{m\in[-R,R)^k\cap\mathbb{Z}^k} = \left(w_m(y,n)\cdot \pi(T^{n+m}y)\right)_{m\in[-R,R)^k\cap\mathbb{Z}^k}.
$$

Since $w_{-n}(x,n) = w_{-n}(y,n) = 1$, we get $\pi(x) = \pi(y)$. Thus $d(x,y) < \varepsilon < \delta$ because $\pi : X \to Q$ is an ε-embedding with respect to d. □

Now we have completed the proof of Main Theorem 2 (and hence the proof of Main Theorem 1).

7. Open problems

Here we explain major open problems in the direction of the paper.

(1) From Main Theorem 1, we have a good understanding of when dynamical systems can be embedded in $\left([0,1]^D\right)^{\mathbb{Z}^k}$ under the assumption of the marker property. So the next step is how to remove the assumption. We would like to propose

Conjecture 7.1. *Let $(X, \mathbb{Z}^k, T)$ be a dynamical system. For a subgroup $A \subset \mathbb{Z}^k$ we define $X_A \subset X$ as the space of $x \in X$ satisfying $T^a x = x$ for all $a \in A$. The quotient group $\mathbb{Z}^k/A$ naturally acts on X_A. Then X can be embedded in the shift on $\left([0,1]^D\right)^{\mathbb{Z}^k}$ if for every subgroup $A \subset \mathbb{Z}^k$ the mean dimension of X_A with respect to the $(\mathbb{Z}^k/A)$-action is smaller than $D/2$.*

If $\mathbb{Z}^k/A$ is a finite group, then the mean dimension of X_A is just

$$\frac{\dim X_A}{\#(\mathbb{Z}^k/A)}.$$

When $k = 1$, Conjecture 7.1 is equivalent to [LT14, Conjecture 1.2]. Notice that if we set $Y = \left([0,1]^D\right)^{\mathbb{Z}^k}$ then for any subgroup $A \subset \mathbb{Z}^k$ the system Y_A is naturally identified with $\left([0,1]^D\right)^{\mathbb{Z}^k/A}$, whose mean dimension is D. So Conjecture 7.1 can be rephrased; if $(X, \mathbb{Z}^k, T)$ satisfies

$$\forall \text{ subgroup } A \subset \mathbb{Z}^k : \quad \mathrm{mdim}(X_A) < \frac{\mathrm{mdim}(Y_A)}{2},$$

then it can be embedded in $Y = \left([0,1]^D\right)^{\mathbb{Z}^k}$.

(2) Let $\Omega \subset \mathbb{R}^k$ be a compact subset. We define $\mathcal{B}_1(\Omega)$ as the space of continuous functions $f : \mathbb{R}^k \to \mathbb{C}$ satisfying $\mathrm{supp}\hat{f} \subset \Omega$ and $\|f\|_{L^\infty(\mathbb{R}^k)} \leq 1$. The mean dimension of the natural $\mathbb{Z}^k$-action on $\mathcal{B}_1(\Omega)$ is equal to $2|\Omega|$. (Here notice that f are complex-valued functions. The factor 2 comes from $\dim \mathbb{C} = 2$.) We would like to ask when a dynamical system $(X, \mathbb{Z}^k, T)$ can be embedded in the shift on $\mathcal{B}_1(\Omega)$. By almost the same argument as the proof of Main Theorem 2, we can prove

Theorem 7.2. *Suppose Ω is a rectangle (i.e. congruent to a set $[a_1, b_1] \times \cdots \times [a_k, b_k]$). If a dynamical system $(X, \mathbb{Z}^k, T)$ satisfies the marker property and*

$$\mathrm{mdim}(X) < |\Omega| = (b_1 - a_1) \times \cdots \times (b_k - a_k),$$

then we can embed it in the shift on $\mathcal{B}_1(\Omega)$.

The proof of this theorem is notationally more messy than the proof of Main Theorem 2. So we concentrate on Main Theorem 2 in this paper.

If Ω is not a rectangle, then the method of this paper does not work directly. Nevertheless, it seems reasonable to conjecture that if Ω is a "nice" set (e.g. a semi-algebraic set) and if a dynamical system $(X, \mathbb{Z}^k, T)$ satisfies the marker property and

$$\mathrm{mdim}(X) < |\Omega|$$

then X can be embedded in the shift on $\mathcal{B}_1(\Omega)$. We hope to return to this problem in a future.

(3) An ambitious question is how to generalize Main Theorems 1 and 2 to the actions of noncommutative groups. In particular it is very interesting to see what kind of signal analysis should be involved. We don't have an answer even for nilpotent groups.

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