

Implementing Anti-Unification Modulo Equational Theory

Jochen Burghardt¹ and Birgit Heinz²

¹ GMD Berlin, jochen@first.gmd.de, <http://www.first.gmd.de/persons/Burghardt.Jochen.html>

² TU Berlin, heinz@cs.tu-berlin.de, <http://www.cs.tu-berlin.de/~heinz>

arXiv:1404.0953v2 [cs.LO] 4 Apr 2014

Technical Report
Arbeitspapiere der GMD 1006
June 1996
ISSN 0723-0508

GMD – Forschungszentrum
Informationstechnik GmbH
D-53754 Sankt Augustin
Tel. *49-2241-14-0
Fax *49-2241-14-2618
Telex 889469 gmd d
<http://www.gmd.de>

Abstract. We present an implementation of E-anti-unification as defined in [Hei95], where tree-grammar descriptions of equivalence classes of terms are used to compute generalizations modulo equational theories. We discuss several improvements, including an efficient implementation of variable-restricted E-anti-unification from [Hei95], and give some runtime figures about them.

We present applications in various areas, including lemma generation in equational inductive proofs, intelligence tests, diverging Knuth-Bendix completion, strengthening of induction hypotheses, and theory formation about finite algebras.

1 Introduction

An important task in the field of artificial intelligence is generalization. To the extent that a generalization approach allows us to incorporate certain background knowledge, it opens up applications in various fields of computer science. Inductive logic programming, for example, is concerned with generalization wrt. Horn-logic theories, with potential applications in automated scientific discovery, knowledge discovery in databases, automatic programming, and other areas.

[Hei95] gives several algorithms for generalizing terms wrt. equational background theories; they are applied to lemma generation in equational induction proofs. In this paper, we describe the implementation of these algorithms and give some further applications.

As a motivating example, consider the sequence 0, 1, 4, 9, represented by terms 0, $suc(0)$, $suc^4(0)$, $suc^9(0)$, respectively. Trained people with a knowledge of addition and multiplication of natural numbers will easily recognize that square numbers computable by the scheme $x * x$ are the desired solution for finding the continuation of the numbers. However, syntactical anti-unification merely leads to the term y , which is too general to be used for computing a continuation of the above sequence.

A useful mechanism for computing schemes is anti-unification extended to take account of equational theories. Addition and multiplication of natural numbers can be specified by the equations of Theory (1) in Fig. 3. In Fig. 1, we demonstrate that $x * x$ is one of the generalizations resulting from anti-unification of the above terms 0, $suc(0)$, $suc^4(0)$, $suc^9(0)$ modulo equational Theory (1), since each of them equals an instance of $x * x$.

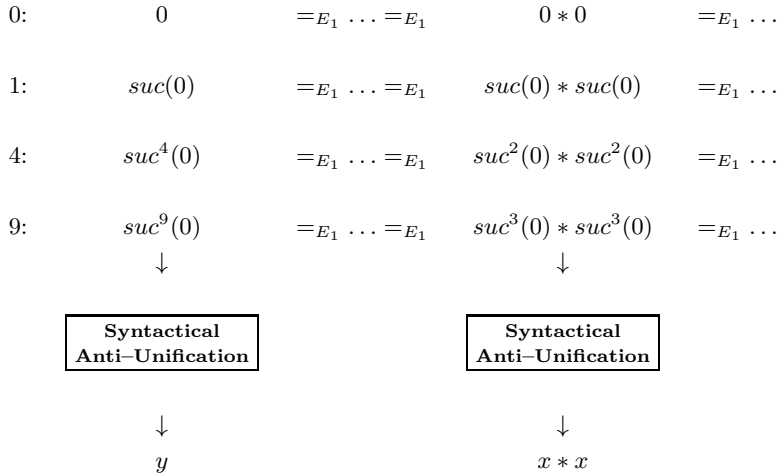


Fig. 1. Anti-Unification Modulo Equational Theory (1) from Fig. 3

Some problems arise if generalizations of terms wrt. equational theories have to be computed:

- There may exist many generalizations allowing the elements of a sequence of terms to be computed.
- The set of generalizations wrt. equational theory is usually infinite (see, however, Fig. 42 in Sect. 7.2).
- Hence, only approaches enumerating its elements can be provided.
- Depending on the application area, the set of possible generalizations may contain useless or undesired computation schemes.

In [Hei95], an approach to generalization modulo canonical equational theories, called anti-narrowing, has been developed. This approach simply allows all generalizations of terms modulo canonical theories to be enumerated; useless generalizations can only be eliminated *after* their computation using corresponding criteria.

Instead of enumerating all generalizations of terms modulo equational theory, a compact, finite representation of the set of all generalizations is desirable for several reasons, e.g. to enable all useless generalizations to be eliminated at one go.

[Hei95] therefore investigates a second approach to E-anti-unification based on regular tree grammars¹. At the heart of the approach is an algorithm that takes two sorts or regular tree languages s, s' , and computes a sort or regular tree language θ of all syntactical generalizations of terms from s and s' :

$$L(\theta) = \{t \sqcap t' \mid t \in L(s), t' \in L(s')\}.$$

In order to E-anti-unify two terms modulo a given background equational theory E , their equivalence classes modulo E are represented by sorts s, s' ; the corresponding θ then contains all E-generalizations of both terms.

In this paper, we describe the implementation of the sort approach to E-anti-unification including technical optimizations and run-time measurements. We present several applications:

- generation of lemma candidates in blocked situations of an inductive proof (Sect. 6)
- construction of series-formation laws (Sect. 7)
- some other potential applications, which are only sketched (Sect. 8).

Section 2 introduces some necessary formal definitions and notations. Section 3 presents the implementation of E-Anti-Unification. Section 4 discusses how to enumerate the terms of a computed sort. Section 5 gives an elaborate example of E-anti-unification. Sections 6 to 8 present the above-mentioned applications. In Sect. 11, we show that for each finite algebra we can always generate a closed representation of all its quantifier-free and variable-bounded theorems. The PROLOG source code of our implementation is listed in Appendix A.

¹ Called “sorts” in this paper, following [Com90]

2 Definitions and Notations

Definition 1. Let \mathcal{V} be an enumerable set of variables, \mathcal{F} a finite set of function symbols, each with fixed arity; $ar(f)$ denotes the arity of $f \in \mathcal{F}$. Terms are built from $\mathcal{V} \cup \mathcal{F}$; \mathcal{T} denotes the set of terms.

We assume familiarity with the classic definitions and notations of terms. We say that a term t starts with a function symbol f if $t = f(t_1, \dots, t_n)$ for some terms t_1, \dots, t_n . $vars(t)$ describes the set of variables occurring in a term $t \in \mathcal{T}$. A term is called linear if no variable occurs in it twice. A term t is called a ground term if t does not contain any variables.

Definition 2. Lists are built from $[]$ and $cons$, written as $(.)$ as in PROLOG; $length(L)$ denotes the number of elements of the list L ; $head(L)$ denotes its first element. The list-comprehension notation $[t \mid p(t)]$ denotes a list of all t , such that $p(t)$ holds, in some arbitrary order.

$\langle t_1, \dots, t_n \rangle$ denotes an n -tuple; $\pi_i^n(\langle t_1, \dots, t_n \rangle) := t_i$ denotes its i -th projection.

$:=$ means equal by definition. $A \subset B$ means that A is a subset of B , or A is equal to B .

$\#A$ denotes the number of elements of a finite set A . $\varphi[A] := \{\varphi(a) \mid a \in A\}$ denotes the image of the set A under the mapping φ ; $\varphi^{-1}[A] := \{a \mid \varphi(a) \in A\}$ denotes the inverse image of A under φ . For a set A , we denote its n -fold cartesian product by A^n .

Definition 3. Substitutions are defined as usual; $\{x_1 \leftarrow t_1, \dots, x_n \leftarrow t_n\}$ denotes a substitution that maps each variable x_i to the term t_i . Sometimes we also use a set-comprehension-like notation: $\{x \leftarrow t \mid p(x, t)\}$ denotes a substitution that maps each x to t such that $p(x, t)$ holds. $dom(\sigma) := \{x \in \mathcal{V} \mid \sigma x \neq x\}$ denotes the domain of σ .

σ is called a linear substitution if $\sigma\langle x_1, \dots, x_n \rangle$ is a linear term, where $dom(\sigma) = \{x_1, \dots, x_n\}$. σ is called a flat substitution if $\sigma x \in \mathcal{V}$ for all $x \in \mathcal{V}$. σ is called a renaming substitution if it is a bijective mapping from variables to variables, and hence is both linear and flat.

t is called an instance of t' , and t' an anti-instance of t , if a substitution σ exists such that $t = \sigma t'$. If σ is a renaming, t is also called a variant of t' .

Definition 4. An equational theory E is a finite set of equations $t_l = t_r$. The relation $(=_E)$ is defined as usual as the smallest reflexive, symmetric, and transitive rewrite relation that contains E .

The equivalence class of a term t mod. $(=_E)$ is denoted by $\underline{t} := \{t' \in \mathcal{T} \mid t' =_E t\}$. $\mathcal{T}/_E$ denotes the algebra of equivalence classes. We assume that in each equivalence class \underline{t} , some term $nf(t)$ is distinguished, which we call the normal form of t . Let $\mathcal{N} := nf[\mathcal{T}]$ be the set of all normal forms. \underline{t} , $nf(t)$, and \mathcal{N} depend on E .

Note that we do not require $(=_E)$ to be computable by a confluent and noetherian term-rewriting system. However, we require that each equivalence class \underline{t} be a regular tree language.

Definition 5. Let \mathcal{S}_{NAME} be an enumerable set of sort names, and let the sets \mathcal{V} , \mathcal{F} , and \mathcal{S}_{NAME} be pairwise disjoint. Let \mathcal{S} denote the set of all sort expressions; a sort expression is a sort name or has one of the forms

$$\begin{aligned} & s_1 \mid s_2, \text{ where } s_1, s_2 \in \mathcal{S}; \\ & f(s_1, \dots, s_n), \text{ where } f \in \mathcal{F}, s_1, \dots, s_n \in \mathcal{S}; \text{ or} \end{aligned}$$

x for $x \in \mathcal{V}$.

A sort definition is of the form $s_n \doteq s_e$, where s_n is a sort name and s_e is a sort expression. We say that s_n is defined by s_e .

Given a system of sort definitions where each occurring sort name has exactly one definition, we define their semantics as their least fixed point. We denote the semantics of a sort expression s by $L(s) \subset \mathcal{T}$, which has the following properties:

$$\begin{aligned} L(s_1 \mid s_2) &= L(s_1) \cup L(s_2) \\ L(x) &= \{x\} && \text{for } x \in \mathcal{V} \\ L(f(s_1, \dots, s_n)) &= \{f(t_1, \dots, t_n) \mid t_i \in L(s_i)\} && \text{for } f \in \mathcal{F} \\ L(s_n) &= L(s_e) && \text{for } s_n \doteq s_e \end{aligned}$$

Observe that $\mathcal{T} \subset \mathcal{S}$, and $L(t) = \{t\}$ for each $t \in \mathcal{T}$. The empty sort is denoted by \perp . A sort definition of the form $s \doteq f_1(s_{11}, \dots, s_{1n_1}) \mid \dots \mid f_m(s_{m1}, \dots, s_{mn_m})$ is said to be in head normal form, where $s, s_{ij} \in \mathcal{S}_{NAME}$, $f_i \in \mathcal{F} \cup \mathcal{V}$, and $n_i = ar(f_i)$. For proof-technical reasons, we define a system of sort definitions as being in normal form if each sort definition has either the form

- $s \doteq s_1 \mid \dots \mid s_n$ with $s, s_1, \dots, s_n \in \mathcal{S}_{NAME}$, or
- $s \doteq f(s_1, \dots, s_n)$ with $s, s_1, \dots, s_n \in \mathcal{S}_{NAME}$,

and if no cycles

$$\begin{aligned} s_1 &\doteq \dots \mid s_2 \mid \dots \\ s_2 &\doteq \dots \mid s_3 \mid \dots \\ &\dots \\ s_{n-1} &\doteq \dots \mid s_n \mid \dots \\ s_n &\doteq \dots \mid s_1 \mid \dots \end{aligned}$$

occur. Each system of sort definitions can be transformed into head normal form, and into normal form, maintaining its semantics. Any system of sort definitions in head normal form, or in normal form, has exactly one fixed point.

Theorem 6. Assume a system of sort definitions in normal form. Let p be a family of unary predicates, indexed over the set of all defined sort names. Show for each defined sort name s :

$$\begin{aligned} \forall t \in \mathcal{T} \quad p_s(t) &\leftrightarrow p_{s_1}(t) \vee \dots \vee p_{s_n}(t) && \text{if } s \doteq s_1 \mid \dots \mid s_n \\ \forall t \in \mathcal{T} \quad p_s(t) &\leftrightarrow \exists t_1, \dots, t_n \in \mathcal{T} \quad t = f(t_1, \dots, t_n) \wedge p_{s_1}(t_1) \wedge \dots \wedge p_{s_n}(t_n) && \text{if } s \doteq f(s_1, \dots, s_n) \\ \forall t \in \mathcal{T} \quad p_s(t) &\leftrightarrow t = x && \text{if } s \doteq x \end{aligned}$$

Then, $\forall t \in \mathcal{T} \quad t \in L(s) \leftrightarrow p_s(t)$ holds for each defined sort name s .

Definition 7. A term t is called a generalization of t_1 and t_2 iff there exist substitutions σ_1 and σ_2 such that $\sigma_1 t = t_1$ and $\sigma_2 t = t_2$. t is called the most specific generalization iff each generalization t' of t_1 and t_2 is an anti-instance of t . The most specific generalization t of two terms t_1, t_2 always exists and is unique up to renaming. We sometimes use the notation $t_1 \sqcap t_2 := t$.

The above definitions can be extended to generalization of n terms; we write $t_1 \sqcap \dots \sqcap t_n$ to denote the most specific generalization of t_1, \dots, t_n .

Definition 8. A term t is called an E -generalization of terms t_1, t_2 iff there exist substitutions σ_1 and σ_2 such that $\sigma_1 t =_E t_1$ and $\sigma_2 t =_E t_2$.

As in to unification, a most specific E -generalization of arbitrary terms does not usually exist. A set $G \subset \mathcal{T}$ is called a correct set of E -generalizations of t_1, t_2 iff each member is an E -generalization of t_1, t_2 . G is called complete if, for each E -generalization t of t_1, t_2 , G contains an instance of t . G is called complete wrt. linear generalizations if for each linear term t , which is an E -generalization of t_1, t_2 , G contains an instance of t .

The following algorithm can be traced back to the early seventies [Plo70,Plo71,Rey70]. It takes two terms t, t' and computes the syntactical generalization $t \sqcap t'$.

Algorithm 9. Let V be an infinite set of new variables and $\varphi : \mathcal{T} \times \mathcal{T} \rightarrow V$ an injective mapping.

1. Define $sg(f(t_1, \dots, t_n), f(t'_1, \dots, t'_n)) := f(sg(t_1, t'_1), \dots, sg(t_n, t'_n))$.
2. Define $sg(f(t_1, \dots, t_n), f'(t'_1, \dots, t'_m)) := \varphi(f(t_1, \dots, t_n), f'(t'_1, \dots, t'_m))$, if $f \neq f'$.

Since syntactical anti-unification is unique only up to renaming, the mapping φ is used to fix one concrete variable naming that is the same in all subterms. In the example in Fig. 2, the use of φ ensures that both occurrences of $sg(3, 4)$ yield the same result, viz. the variable v_{34} .

$$\begin{aligned}
& sg(3 + y * 3, 4 + x * 4) \\
&= sg(3, 4) + sg(y * 3, x * 4) \\
&= sg(3, 4) + sg(y, x) * sg(3, 4) \\
&= v_{34} + v_{yx} * v_{34}
\end{aligned}$$

Fig. 2. Example: Computation of Syntactical Anti-Unification

Lemma 10. Algorithm 9 can be extended to anti-unify N terms simultaneously, requiring $\varphi : \mathcal{T}^N \rightarrow \mathcal{V}$. For any such φ and any finite $V \subset \mathcal{V}$, we may define the substitutions σ_i for $i = 1, \dots, N$ by $\sigma_i \varphi(t_1, \dots, t_N) := t_i$. We have $dom(\sigma_i) = V$ for all i .

The result of Alg. 9 then satisfies the following correctness property:

$\sigma_i sg([t_1, \dots, t_N]) = t_i$ for $i = 1, \dots, N$ provided we choose $V \supset vars(sg([t_1, \dots, t_N]))$.

3 E–Anti–Unification

In this section, we describe the implementation of E–anti–unification based on sorts.

Section 3.2 restates the algorithm *rsg* from [Hei95] for computing a sort containing all linear generalizations; Sect. 3.3 discusses how to subsequently modify this sort in order to get all nonlinear generalizations as well. To obtain the sort of all generalizations, Alg. 12 from Sect. 3.2 is run as a first phase, then the algorithm from Sect. 3.3 is run as the second phase; cf. also Sect. 5.

In Sect. 3.4, the algorithm *rsg_V* for computing the sort of all generalization terms that contain only variables from a given set V is restated from [Hei95]. Note that all linear *and* nonlinear generalizations are computed in one phase only, thus saving a large amount of computation time; cf. Sect. 3.6, Fig. 15.

Section 3.5 discusses a technical optimization that applies to both *rsg* and *rsg_V* and helps to avoid many useless recursive calls. Sect. 3.6 presents some figures for runtime measurements and the improvement factors of technical optimizations.

3.1 Modeling Equivalence Classes as Sorts

The algorithms from Sect. 3.2/3.3 and from Sect. 3.4 both require the representation of the equivalence classes of the input terms as sorts. In this paper, we do not treat this issue in detail; rather, we drew up the respective sort definitions manually.

Figure 3 shows the equational theories used in this paper; we will refer to them as background theories. For example, Theory (1) consists of all equations that have a “1” in the leftmost column. Figure 23 refers to Theory (1) and shows the sort definition of s_0 and s_1 , representing $\underline{0}$ and $\underline{suc(0)}$, respectively. Figure 4 shows the sort representation of $\underline{0}$ to $\underline{suc^5(0)}$ wrt. Theory (5); s_p , s_e , s_o , and s_n denote the sets of positive (i.e. > 0), even, odd, and arbitrary natural numbers, respectively.

In Theory (6), we have included the $(-)$ operator, such that equivalence classes of terms are no longer regular tree languages, and hence cannot be described by our sorts; for example, the sort definition $s_0 \doteq 0 \mid \dots \mid s_0 - s_0 \mid s_1 - s_1 \mid \dots \mid s_i - s_i \mid \dots$ would become infinite. We can, however, approximate the equivalence classes from below by cutting off the sort definitions at a certain n , i.e., omitting all greater numbers. Figure 5 shows the approximations of $\underline{0}$ to $\underline{suc^5(0)}$, where we set the cut-off point to $n = 5$, i.e. $s_6 - s_6 \mid \dots \mid s_i - s_i \mid \dots$ is missing; hence, the equivalence classes do not contain terms that yield an intermediate result greater than 5 when evaluated to normal form.

In Sect. 9, we discuss the use of sort-definition schemes that allow several sort definitions to be abbreviated by one scheme. Figure 51 shows a sort-definition scheme for background Theory (0). Figure 50, shows sort-definition schemes of equivalence classes mod. *append* and *reverse*.

[Emm94] describes an algorithm for building the sort definitions automatically from a given confluent and noetherian term-rewriting system. However, there is no implementation that fits our data structures. See also Thm. 15 and Cor. 16 in Sect. 11, which provide sufficient criteria for an equational theory in order that the equivalence classes are regular tree languages.

3.2 Linear Generalizations

The following algorithm from [Hei95] takes two sorts s, s' and computes a sort θ containing all linear syntactical generalizations $t \sqcap t'$ of terms $t \in \mathbf{L}(s)$, $t' \in \mathbf{L}(s')$, i.e.

$$\theta \supset \{t \sqcap t' \mid t \in \mathbf{L}(s), t' \in \mathbf{L}(s'), t \sqcap t' \text{ linear} \}.$$

012	567	9	$x + 0$	$= x$	
012	567	9	$x + \text{suc}(y)$	$= \text{suc}(x + y)$	
	67		$x - 0$	$= x$	
	67		$\text{suc}(x) - \text{suc}(y)$	$= x - y$	
12	5	7	9	$x * 0$	$= 0$
12	5	7	9	$x * \text{suc}(y)$	$= x * y + x$
	9			$0/y$	$= 0$
	9			$\text{suc}(x)/\text{suc}(0)$	$= \text{suc}(x)$
	9			$\text{suc}(x)/\text{suc}(\text{suc}(y)) = \text{suc}((x - \text{suc}(\text{suc}(y))) / \text{suc}(\text{suc}(y)))$	
2				$\text{dup}(0)$	$= 0$
2				$\text{dup}(\text{suc}(x))$	$= \text{suc}(\text{suc}(\text{dup}(x)))$
	5			$\text{if}(0, x, y)$	$= y$
	5			$\text{if}(\text{suc}(z), x, y)$	$= x$
	5			$\text{ev}(0)$	$= \text{suc}(0)$
	5			$\text{ev}(\text{suc}(0))$	$= 0$
	5			$\text{ev}(\text{suc}(\text{suc}(x)))$	$= \text{ev}(x)$
4				$\text{len}([])$	$= 0$
4				$\text{len}(x.y)$	$= \text{suc}(\text{len}(y))$
34	8			$\text{app}([], x)$	$= x$
34	8			$\text{app}(x.y, z)$	$= x.\text{app}(y, z)$
34	8			$\text{rev}([])$	$= []$
34	8			$\text{rev}(x.y)$	$= \text{app}(\text{rev}(y), [x])$
	8			$\text{rot}([])$	$= []$
	8			$\text{rot}(x.y)$	$= \text{app}(y, [x])$
	8			$\text{il}(x, [])$	$= x$
	8			$\text{il}([], x)$	$= x$
	8			$\text{il}(x.y, z.w)$	$= x.z.\text{il}(y, w)$

Fig. 3. Background Equational Theories Used

$$\begin{aligned}
s_0 &\doteq 0 && | \text{if}(s_p, s_0, s_n) & | \text{if}(s_0, s_n, s_0) & | \text{ev}(s_o) & | s_0 + s_0 & | s_0 * s_n & | s_n * s_0 \\
s_1 &\doteq \text{suc}(s_0) && | \text{if}(s_p, s_1, s_n) & | \text{if}(s_0, s_n, s_1) & | \text{ev}(s_e) & | s_1 + s_0 & | s_0 + s_1 & | s_1 * s_1 \\
s_2 &\doteq \text{suc}(s_1) && | \text{if}(s_p, s_2, s_n) & | \text{if}(s_0, s_n, s_2) & | s_2 + s_0 & | s_1 + s_1 & | s_0 + s_2 & | s_1 * s_2 & | s_2 * s_1 \\
s_3 &\doteq \text{suc}(s_2) && | \text{if}(s_p, s_3, s_n) & | \text{if}(s_0, s_n, s_3) & | s_3 + s_0 & | s_2 + s_1 & | s_1 + s_2 & | s_0 + s_3 & | s_1 * s_3 & | s_3 * s_1 \\
s_4 &\doteq \text{suc}(s_3) && | \text{if}(s_p, s_4, s_n) & | \text{if}(s_0, s_n, s_4) & | s_4 + s_0 & | s_3 + s_1 & | s_2 + s_2 & | s_1 + s_3 & | s_0 + s_4 & | s_1 * s_4 & | s_2 * s_2 & | s_4 * s_1 \\
s_5 &\doteq \text{suc}(s_4) && | \text{if}(s_p, s_5, s_n) & | \text{if}(s_0, s_n, s_5) & | s_5 + s_0 & | s_4 + s_1 & | s_3 + s_2 & | s_2 + s_3 & | s_1 + s_4 & | s_0 + s_5 & | s_1 * s_5 & | s_5 * s_1 \\
\\
s_p &\doteq \text{suc}(s_n) && | \text{if}(s_p, s_p, s_n) & | \text{if}(s_0, s_n, s_p) & | \text{ev}(s_e) & | s_p + s_n & | s_n + s_p & | s_p * s_p \\
s_e &\doteq 0 && | \text{if}(s_p, s_e, s_n) & | \text{if}(s_0, s_n, s_e) & | \text{ev}(s_o) & | \text{suc}(s_o) & | s_e + s_e & | s_o + s_o & | s_e * s_n & | s_n * s_e \\
s_o &\doteq \text{suc}(s_e) && | \text{if}(s_p, s_o, s_n) & | \text{if}(s_0, s_n, s_o) & | \text{ev}(s_e) & | s_o + s_e & | s_e + s_o & | s_o * s_o \\
s_n &\doteq 0 && | \text{suc}(s_n) & | \text{if}(s_n, s_n, s_n) & | \text{ev}(s_n) & | s_n + s_n & | s_n * s_n
\end{aligned}$$

Fig. 4. Equivalence Classes for Background Theory (5) from Fig. 3

$s_0 \doteq 0$	$ s_0 + s_0$	$ s_0 - s_0$	$ s_1 - s_1$	$ s_2 - s_2$	$ s_3 - s_3$	$ s_4 - s_4$	$ s_5 - s_5$
$s_1 \doteq \text{suc}(s_0)$	$ s_1 + s_0$	$ s_0 + s_1$	$ s_1 - s_0$	$ s_2 - s_1$	$ s_3 - s_2$	$ s_4 - s_3$	$ s_5 - s_4$
$s_2 \doteq \text{suc}(s_1)$	$ s_2 + s_0$	$ s_1 + s_1$	$ s_0 + s_2$	$ s_2 - s_0$	$ s_3 - s_1$	$ s_4 - s_2$	$ s_5 - s_3$
$s_3 \doteq \text{suc}(s_2)$	$ s_3 + s_0$	$ s_2 + s_1$	$ s_1 + s_2$	$ s_0 + s_3$	$ s_3 - s_0$	$ s_4 - s_1$	$ s_5 - s_2$
$s_4 \doteq \text{suc}(s_3)$	$ s_4 + s_0$	$ s_3 + s_1$	$ s_2 + s_2$	$ s_1 + s_3$	$ s_0 + s_4$	$ s_4 - s_0$	$ s_5 - s_1$
$s_5 \doteq \text{suc}(s_4)$	$ s_5 + s_0$	$ s_4 + s_1$	$ s_3 + s_2$	$ s_2 + s_3$	$ s_1 + s_4$	$ s_0 + s_5$	$ s_5 - s_0$
$s_n \doteq 0$	$ \text{suc}(s_n)$	$ s_n + s_n$	$ s_n - s_n$				

Fig. 5. Equivalence Classes for Background Theory (6) from Fig. 3

Algorithm 11. Let s and s' be sort names, and θ be a new sort name. Let V be an infinite set of new variables, and $\varphi : \mathcal{S} \times \mathcal{S} \rightarrow V$ an injective mapping. Define $\text{rsg}(s, s') := \theta$, where a new sort definition is introduced for θ :

1. If $\text{rsg}(s, s')$ has been called before, then θ is already defined.
2. If $s \doteq s_1 \mid \dots \mid s_n$, then define $\theta \doteq \text{rsg}(s_1, s') \mid \dots \mid \text{rsg}(s_n, s')$.
3. If $s' \doteq s'_1 \mid \dots \mid s'_n$, then define $\theta \doteq \text{rsg}(s, s'_1) \mid \dots \mid \text{rsg}(s, s'_n)$.
4. If $s \doteq f(s_1, \dots, s_n)$ and $s' \doteq f(s'_1, \dots, s'_n)$, then define $\theta \doteq f(\text{rsg}(s_1, s'_1), \dots, \text{rsg}(s_n, s'_n))$.
5. If $s \doteq f(s_1, \dots, s_n)$ and $s' \doteq f'(s'_1, \dots, s'_n)$ with $f \neq f'$, then define $\theta \doteq \varphi(s, s')$.

As shown in [Hei95], $\text{L}(\text{rsg}(s, s'))$ is a correct set of generalizations which is complete for the linear ones, i.e., $t \in \text{L}(\text{rsg}(s_1, s_2)) \Rightarrow \exists t_1 \in \text{L}(s_1), t_2 \in \text{L}(s_2) \ t = t_1 \sqcap t_2$, and $t_1 \in \text{L}(s_1) \wedge t_2 \in \text{L}(s_2) \wedge t := t_1 \sqcap t_2 \text{ linear} \Rightarrow t \in \text{L}(\text{rsg}(s_1, s_2))$.

Note that Case 1. requires us to maintain a set of argument pairs for which rsg has already been called, and to check the argument pair of each new call against this set. This set is called `Occ` in the implementation; it was initially implemented by a PROLOG list, then by a binary search tree, and now by a balanced binary search tree. A membership test in a balanced tree with 50 entries takes about 4 msec user time. See also Sect. 3.6 for the impact of balancing on runtime.

Assuming all sort definitions in head normal form, we can slightly improve the above algorithm in order to produce less variables. For example, given the sort definitions of s_0, s_1 from Fig. 23, the sort s_{01} computed in Fig. 24 would comprise twelve different variables if computed by rsg , while it actually – computed by hsg below – has only one variable.

Since $\text{L}(\text{hsg}(s_1, s_2)) \subset \text{L}(\text{rsg}(s_1, s_2))$, Alg. 12 is still correct; since for each generalization $t \in \text{L}(\text{rsg}(s_1, s_2))$ we have $\sigma t \in \text{L}(\text{hsg}(s_1, s_2))$ for some flat substitution σ , Alg. 12 is also complete for the linear generalizations.

Algorithm 12. Let s, s', V , and φ be as in Alg. 11. Define $\text{hsg}(s, s')$ as follows:

1. If $\text{hsg}(s, s')$ has been called earlier, its result is already defined.
2. If $s \doteq \bigsqcup_{i=1}^m f_i(s_{i1}, \dots, s_{in_i})$, and $s' \doteq \bigsqcup_{j=1}^{m'} f'_j(s'_{j1}, \dots, s'_{jn'_j})$,
then let θ be a new sort name,
define $\theta \doteq \varphi(s, s') \mid \bigsqcup_{i=1}^m \bigsqcup_{j=1}^{m'} \text{hsg}(f_i(s_{i1}, \dots, s_{in_i}), f'_j(s'_{j1}, \dots, s'_{jn'_j}))$;
let $\text{hsg}(s, s') := \theta$.
3. Define $\text{hsg}(f(s_1, \dots, s_n), f(s'_1, \dots, s'_n)) := f(\text{hsg}(s_1, s'_1), \dots, \text{hsg}(s_n, s'_n))$.
4. Define $\text{hsg}(f(s_1, \dots, s_n), f'(s'_1, \dots, s'_m)) := \perp$, if $f \neq f'$.

Usually, most of the disjuncts in Case 2. start with different function symbols, $f_i \neq f'_j$, and hence evaluate to \perp by Case 4. In Sect. 3.5, we will discuss an optimization that avoids these recursive calls of hsg .

Algorithms 11 and 12 can both be extended to compute the linear generalizations of N sorts simultaneously, requiring $\varphi : \mathcal{S}^N \rightarrow V$. Our implementation is capable of that, and we will tacitly use $hsg([s', s'', \dots, s^{(N)}])$ with N arguments where appropriate in this paper. Note that it makes sense to have multiple occurrences of the same sort among the input arguments. For example, using the sort definitions from Fig. 23, we have $0 * 0 \sqcap (0 + 0) * (0 + 0) = v * v \in L(hsg(s_0, s_0)) \setminus L(s_0)$. Moreover, it is important to maintain the order of argument sorts during computation, since otherwise, for example

$$\begin{aligned} & hsg(s_1, s_1) \\ &= \dots \mid hsg(s_0, s_1) + hsg(s_1, s_0) \mid \dots \\ &= \dots \mid hsg(s_0, s_1) + hsg(s_0, s_1) \mid \dots \\ &= \dots \mid v_{01} + v_{01} \mid \dots \end{aligned}$$

although 1 is not equal to any instance of $v_{01} + v_{01}$. For these reasons, we treat the argument of the extended hsg as a list rather than as a set of sorts.

3.3 Nonlinear Generalizations

The result sorts of the above algorithm contains all linear generalizations, but only some nonlinear ones. That is not generally sufficient, as the example in Sect. 5 shows (see in particular the difference between s_{01} and s'_{01} in Fig. 30).

In order to obtain as well all nonlinear generalizations, which are more specific, we need a second phase which introduces common variables to related sorts. We use the abbreviation $inf s(V) := \{\bigcap_{v \in V} \pi_i^N(\varphi^{-1}(v)) \mid 1 \leq i \leq N\}$; note that φ^{-1} exists since φ is injective. Below, we will assume $N = 2$ for the sake of simplicity.

[Hei95] proceeds as follows: for each set of variables $\{v_1, \dots, v_n\}$ from the result sorts, whenever $inf s(\{v_1, \dots, v_n\})$ does not contain the empty sort, a new variable v' is introduced which can be thought of intuitively as generalizing the sorts in $inf s(V)$, and each occurrence of each v_i is replaced by $v_i \mid v'$. For example, using the definitions from Sect. 5, we have $v_{01} = \varphi(s_0, s_1)$, $v_{0n} = \varphi(s_0, s_n)$, $v_{n1} = \varphi(s_n, s_1)$, and $v_{nn} = \varphi(s_n, s_n)$; hence $inf s(\{v_{0n}, v_{n1}\}) = \{s_0 \cap s_n, s_n \cap s_1\} = \{s_0, s_1\}$ since $s_0 \cup s_1 \subset s_n$. Thus, v_{0n}, v_{0n} are replaced by $v_{0n} \mid v'$, and $v_{0n} \mid v'$, respectively. Similarly, for any subset of $\{v_{01}, v_{0n}, v_{n1}, v_{nn}\}$ that contains more than one element, a new variable would be added.

We follow this approach, except that we only consider *maximal* sets $\{v_1, \dots, v_n\}$ of variables, i.e. where $\{v_i\} \in inf s(\{v_1, \dots, v_n, v_{n+1}\})$ for any v_{n+1} . For each generalization term t obtained by the [Hei95] approach, we get a more specific generalization term σt for some flat substitution σ . By analogy with the remark in Sect. 3.2, this shows the algorithm is still complete. In the above example, we would add only one variable for the set $\{v_{01}, v_{0n}, v_{n1}, v_{nn}\}$, which is maximal.

In the example in Sect. 5 / Fig. 30, the second phase leads to the modification of sort s_{01} to s'_{01} , etc. Unlike the former sort, the latter, for example, contains the term $v'_{01} * v'_{01}$, meaning that 0 and 1 are both quadratic numbers.

We explain below how the n -tuples v_1, \dots, v_n of variables with non-empty sort intersections $\{v_i\} \notin inf s(\{v_1, \dots, v_n\})$ are found. The replacement of each v_i by $v_i \mid v'$ is then straightforward. As the computation of sort intersections is very time-consuming, we try to minimize the number of such computations by applying some — correct — heuristics explained below. Note, however, that when each defined sort name corresponds to an equivalence class (unlike s_n), computing their intersections is trivial, since different equivalence classes are always disjoint.

Let V be the set of variables newly generated by the first phase. Each variable $v \in V$ is assigned a pair $\langle s_v, s'_v \rangle := \varphi^{-1}(v)$ of sorts such that v represents all generalizations of any term from s_v with

any term from s'_v . Let $S := \{s_v \mid v \in V\} \cup \{s'_v \mid v \in V\}$ be the set of all relevant sorts. In the first step, we compute the set

$$I2 := \{\langle s_1, s_2, s_{12} \rangle \mid s_1, s_2 \in S, \{\} \neq s_{12} = s_1 \cap s_2, s_1 \prec s_2\}$$

of all non-empty intersections of sorts from S . This is accomplished by the PROLOG predicate `calc_sort_infs`. The relation (\prec) is an arbitrary irreflexive total ordering on S . Without the condition $s_1 \prec s_2$, we had $\langle s_2, s_1, s_{12} \rangle \in I2$ whenever $\langle s_1, s_2, s_{12} \rangle \in I2$; the additional ordering condition helps to keep the memory requirements small. $I2$ is represented by a balanced binary tree. Then, the set

$$L2 := \{\langle v_1, v_2 \rangle \mid v_1, v_2 \in V, \{\} \notin \text{infs}(\{v_1, v_2\}), v_1 \prec v_2\}$$

of all pairs of variables with non-empty intersections is computed by `calc_inf2s`. The remarks on $s_1 \prec s_2$ apply similarly to the ordering condition $v_1 \prec v_2$. Each variable in the set V is equipped with its number of occurrences in $L2$, i.e. for each $v \in V$. Let

$$i_v := \#(\{\langle v, v' \rangle \mid \langle v, v' \rangle \in L2\} \cup \{\langle v', v \rangle \mid \langle v', v \rangle \in L2\});$$

then the set $V2 := \{\langle v, i_v \rangle \mid v \in V\}$ is computed using the predicates `calc_varcnt` and `collect_varcnt`. The predicate `calc_infs_list` then takes $L2$ and $V2$ as input: it calls `calc_infs` for each $\langle v_1, v_2 \rangle \in L2$, which in turn computes the set of all maximal V_{12} such that $v_1, v_2 \in V_{12}$, and $\{\} \notin \text{infs}(V_{12})$. The rules for `calc_infs` are given in Fig. 6; the start configuration is

$$\overline{\{v_1, v_2\} \quad L2 \quad []}$$

for each pair $\langle v_1, v_2 \rangle \in L2$. All the rules in this paper are prioritized, i.e. the n -th rule is tried only if the 1-st, \dots , $n - 1$ -th did not succeed. *Out* denotes the output so far. The set *Susp* contains “suspended” variable sets V that are subsumed by some earlier output, see rule (*Suplist*). The suspension is released as soon as a new variable can be joined to the actual set V , see rule (*Inf Inh*). The actual set V is reported as a maximal one only if no suspended variable sets exist, see rules (*Output Max*), (*Max Subs Susp*). Rule (*Inf Inh*) has two successors, the output of both of which has to be joined; $\#$ denotes the appending of two lists. Rule (*Count*) is justified since v' can have non-empty intersections with at most $i_{v'}$ different variables (“intersection of variables” v_1, \dots, v_n meaning $\text{infs}(\{v_1, \dots, v_n\})$). In order to allow fast lookup of $i_{v'}$ in rule (*Count*), and fast testing of $\langle v', v'', \dots \rangle \notin L2$ in rule (*Sublist2*), both $L2$ and $V2$ are given additionally in balanced-tree form to the algorithm.

Figure 7 shows an example intersection hierarchy for the variables a, b, c, d ; a common upper bound of v_1, \dots, v_n indicates that $\{\} \notin \text{infs}(v_1, \dots, v_n)$. While setting up $I2$, the six intersection sets $\text{infs}(\{a, b\})$, $\text{infs}(\{a, c\})$, $\text{infs}(\{a, d\})$, $\text{infs}(\{b, c\})$, $\text{infs}(\{b, d\})$, $\text{infs}(\{c, d\})$ are computed and tested to determine whether they contain the empty set. Note that these computations are unavoidable, since any two sorts could intersect independently of the others. During the run of `calc_infs_list`, only two more intersection sets need to be computed and checked; they are flagged right to the rule (*Inf Inh*) in Figs. 8 and 9 which show the corresponding example run. See also Sect. 5 for a second example.

$\frac{V \quad [v' \mid List] \quad Susp}{V \quad List \quad Susp}$	$v' \in V$	(Member)
$\frac{V \quad [v' \mid List] \quad Susp}{V \quad List \quad Susp}$	$i_{v'} < \#V$	(Count)
$\frac{V \quad [v' \mid List] \quad Susp}{V \quad List \quad Susp}$	$\exists v'' \in V \langle v', v'', \dots \rangle \notin L2$ $\wedge \langle v'', v', \dots \rangle \notin L2$	(Sublist2)
$\frac{V \quad [v' \mid List] \quad Susp}{V \quad List \quad [v' \mid Susp]}$	$\exists V' \in Out \quad V \cup \{v'\} \subset V'$	(Suplist)
$\frac{V \quad [v' \mid List] \quad Susp}{V \cup \{v'\} \quad List \uplus Susp \quad []}$ $V \quad List \uplus Susp \quad []$	$\{\} \notin \text{infs}(V \cup \{v'\})$	(Inf Inh)
$\frac{V \quad [v' \mid List] \quad Susp}{V \quad List \quad Susp}$		(Inf Empty)
$\frac{V \quad [] \quad []}{}$	$\exists V' \in Out \quad V \subset V'$	(Max Subsumed)
$\frac{V \quad [] \quad []}{}$		(Output Max)
$\frac{V \quad [] \quad [v' \mid Susp]}{}$		(Max Subs Susp)

Fig. 6. Rules for `calc_infs`

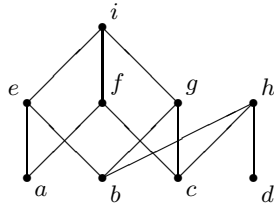


Fig. 7. Example: Intersection Hierarchy

$$L2 = \{\langle a, b \rangle, \langle a, c \rangle, \langle b, c \rangle, \langle b, d \rangle, \langle c, d \rangle\}$$

$$V2 = \{\langle a, 2 \rangle, \langle b, 3 \rangle, \langle c, 3 \rangle, \langle d, 2 \rangle\}$$

```

[a,b]  [a,b,c,d]  []
(Member)
[a,b]  [b,c,d]  []
(Member)
[a,b]  [c,d]  []
(Inf Inh)          infs({a,b,c})
[a,b,c]  [d]  []
(Sublist2)
[a,b,c]  []  []
(Output Max [a,b,c])
[a,b]  [d]  []
(Sublist2)
[a,b]  []  []
(Max Subsumed)
[a,c]  [a,b,c,d]  []
(Member)
[a,c]  [b,c,d]  []
(Suplist)
[a,c]  [c,d]  [b]
(Member)
[a,c]  [d]  [b]
(Sublist2)
[a,c]  []  [b]
(Max Subs Susp)

```

(Contd. in Fig. 9)

Fig. 8. Example: Intersection Computation Run

```

[b,c] [a,b,c,d] []
(Suplist)
[b,c] [b,c,d] [a]
(Member)
[b,c] [c,d] [a]
(Member)
[b,c] [d] [a]
(Inf Inh) infs({b,c,d})
[b,c,d] [a] []
(Sublist2)
[b,c,d] [] []
(Output Max [b,c,d])
[b,c] [a] []
(Suplist)
[b,c] [] [a]
(Max Subs Susp)

[b,d] [a,b,c,d] []
(Sublist2)
[b,d] [b,c,d] []
(Member)
[b,d] [c,d] []
(Suplist)
[b,d] [d] [c]
(Member)
[b,d] [] [c]
(Max Subs Susp)

[c,d] [a,b,c,d] []
(Sublist2)
[c,d] [b,c,d] []
(Suplist)
[c,d] [c,d] [b]
(Member)
[c,d] [d] [b]
(Member)
[c,d] [] [b]
(Max subs susp)

```

Fig. 9. Example: Intersection Computation Run (contd.)

3.4 Variable–Restricted E–Anti–Unification

The following algorithm from [Hei95] takes two input sorts s, s' and a finite set V of variables and computes the set of all generalizations of terms from s and s' that contain only variables from the set V . The call $rsg_V(\{\}, s, s')$ computes precisely the intersection of the sorts s and s' . Note that not only linear generalizations are computed; hence, the computationally expensive second phase described in Sect. 3.3 can be omitted. See Sect. 3.6 for figures on the runtime improvement thus achieved. The remarks from Sect. 3.2 about extending the algorithm to N arguments also apply to rsg_V .

Algorithm 13. Let s and s' be sorts and θ a new sort name. Let $\varphi : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{V}$ be an injective mapping. Let V be a finite set of variables, such that $TP := \varphi^{-1}[V] \subset \mathcal{T} \times \mathcal{T}$ is a finite set of term pairs. (Note that each term is also a sort expression.) Define $rsg_V(V, s, s') := \theta$, where a new sort definition for θ is introduced by the following cases.

1. If $rsg_V(V, s, s')$ has been called before, then θ is already defined.
2. If $s \doteq s_1 \mid \dots \mid s_n$, then define $\theta \doteq rsg_V(V, s_1, s') \mid \dots \mid rsg_V(V, s_n, s')$.
3. If $s' \doteq s'_1 \mid \dots \mid s'_n$, then define $\theta \doteq rsg_V(V, s, s'_1) \mid \dots \mid rsg_V(V, s, s'_n)$.
4. If $s \doteq f(s_1, \dots, s_n)$, and $s' \doteq f(s'_1, \dots, s'_n)$,
then define $\theta \doteq f(rsg_V(V, s_1, s'_1), \dots, rsg_V(V, s_n, s'_n))$.
5. If $s \doteq f(s_1, \dots, s_n)$, and $s' \doteq f'(s'_1, \dots, s'_n)$, then define $\theta \doteq \bigsqcup_{\langle t, t' \rangle \in TP \cap (\mathbf{L}(s) \times \mathbf{L}(s'))} \varphi(t, t')$.

Lemma 14. Given $\varphi : \mathcal{T}^N \rightarrow \mathcal{V}$ and V , we may define $\sigma_i \varphi(t_1, \dots, t_N) := t_i$ for all $\langle t_1, \dots, t_N \rangle \in \varphi^{-1}[V]$ and $i = 1, \dots, N$. We have $\text{dom}(\sigma_i) = V$ for all i . We assume that for each $x \in V$, $x = \varphi(t_1, \dots, t_n)$ such that not all t_1, \dots, t_n start with the same function symbol.

The result of Alg. 13 then satisfies the following correctness property:

$$t \in \mathbf{L}(rsg_V(V, [s_1, \dots, s_N])) \Leftrightarrow \text{vars}(t) \subset V \wedge \bigwedge_{i=1}^N \sigma_i t \in \mathbf{L}(s_i)$$

Hence, when the argument sorts of rsg_V are equivalence classes of terms, we have

$$t \in \mathbf{L}(rsg_V(V, [\underline{t}_1, \dots, \underline{t}_N])) \Leftrightarrow \text{vars}(t) \subset V \wedge \bigwedge_{i=1}^N \sigma_i t =_E t_i$$

Proof. We show the lemma only for $N = 2$, using Thm. 6 with

$$\begin{aligned} p_\theta(t) &\Leftrightarrow \sigma_1 t \in \mathbf{L}(s) \wedge \sigma_2 t \in \mathbf{L}(s') \wedge \text{vars}(t) \subset V && \text{if } \theta \doteq rsg_V(V, s, s') \\ p_\theta(t) &\Leftrightarrow t = \varphi(t', t'') && \text{if } \theta \doteq \varphi(t', t'') \\ p_\theta(t) &\Leftrightarrow t \in \mathbf{L}(\theta) && \text{else} \end{aligned}$$

In the latter case, we have nothing to show. In the two former cases, we make a case distinction according to the rule from Alg. 13 that defined θ . The second case has been introduced to put the result sort definitions into normal form.

1. In this case, no new sort definition is introduced.

2. We have $p_\theta(t)$
- $$\begin{aligned} &\Leftrightarrow \sigma_1 t \in \mathbf{L}(s) \wedge \sigma_2 t \in \mathbf{L}(s') \wedge \text{vars}(t) \subset V && \text{by Def. of } p_\theta \\ &\Leftrightarrow \exists i \ \sigma_1 t \in \mathbf{L}(s_i) \wedge \sigma_2 t \in \mathbf{L}(s') \wedge \text{vars}(t) \subset V && \text{by Def. of } \mathbf{L}(s) \\ &\Leftrightarrow \exists i \ p_{\theta_i}(t) && \text{by Def. of } p_{\theta_i} \end{aligned}$$
3. Similar to 2.
4. $p_\theta(t)$
- $$\begin{aligned} &\Leftrightarrow \sigma_1 t \in \mathbf{L}(s) \wedge \sigma_2 t \in \mathbf{L}(s') \wedge \text{vars}(t) \subset V && \text{by Def. of } p_\theta \\ &\Leftrightarrow \exists t_{11} \in \mathbf{L}(s_1), \dots, t_{1n} \in \mathbf{L}(s_n) \ \exists t_{21} \in \mathbf{L}(s'_1), \dots, t_{2n} \in \mathbf{L}(s'_n) \\ &\quad \sigma_1 t = f(t_{11}, \dots, t_{1n}) \wedge \sigma_2 t = f(t_{21}, \dots, t_{2n}) \wedge \text{vars}(t) \subset V && \text{by Def. of } \mathbf{L}(s), \mathbf{L}(s') \\ &\Leftrightarrow \exists t_1, \dots, t_n \ t = f(t_1, \dots, t_n) \\ &\quad \wedge \bigwedge_{i=1}^n \sigma_1 t_i \in \mathbf{L}(s_i) \wedge \sigma_2 t_i \in \mathbf{L}(s'_i) \wedge \text{vars}(t_i) \subset V && (*) \\ &\Leftrightarrow \exists t_1, \dots, t_n \ t = f(t_1, \dots, t_n) \wedge \bigwedge_{i=1}^n p_{\theta_i}(t_i) && \text{by Def. of } p_{\theta_i} \end{aligned}$$
- (*) : “ \Rightarrow ”: since $\sigma_1 x$ and $\sigma_2 x$ never start with the same function symbol f ,
we must have $t = f(t_1, \dots, t_n)$
with $\sigma_1 t_i = t_{1i} \in \mathbf{L}(s_i)$,
the case is similar for σ_2
“ \Leftarrow ”: choose $t_{1i} := \sigma_1 t_i$, and $t_{2i} := \sigma_2 t_i$
5. $p_\theta(t)$
- $$\begin{aligned} &\Leftrightarrow \sigma_1 t \in \mathbf{L}(f(s_1, \dots, s_n)) \text{ and } \sigma_2 t \in \mathbf{L}(f'(s'_1, \dots, s'_n)) \text{ and } \text{vars}(t) \subset V && \text{by Def. of } p_\theta \\ &\Rightarrow \sigma_1 t = f(\dots) \text{ and } \sigma_2 t = f'(\dots) \text{ and } \text{vars}(t) \subset V \\ &\Rightarrow t \in \mathcal{V} \text{ and } \text{vars}(t) \subset V \\ &\Rightarrow t \in V && \text{by Def. of } \text{vars} \\ &\Rightarrow t = \varphi(t', t'') \text{ for some } \langle t', t'' \rangle \in TP && \text{by Def. of } TP \\ &\Rightarrow \bigvee_{\langle t', t'' \rangle \in TP \cap (\mathbf{L}(s) \times \mathbf{L}(s'))} p_{\varphi(t', t'')}(t) && \text{by Def. of } p_{\varphi(t', t'')} \end{aligned}$$
- conversely:
- $$\begin{aligned} &\bigvee_{\langle t', t'' \rangle \in TP \cap (\mathbf{L}(s) \times \mathbf{L}(s'))} p_{\varphi(t', t'')}(t) \\ &\Rightarrow t = \varphi(t', t'') \text{ for some } \langle t', t'' \rangle \in TP \cap (\mathbf{L}(s) \times \mathbf{L}(s')) && \text{by Def. of } p_{\varphi(t', t'')} \\ &\Rightarrow \sigma_1 t = t' \in \mathbf{L}(s) \text{ and } \sigma_2 t = t'' \in \mathbf{L}(s') \text{ and } t \in V && \text{by Def. of } \sigma_1, \sigma_2 \\ &\Rightarrow p_\theta(t) && \text{by Def. of } p_\theta \end{aligned}$$

In Case 5., all pairs in $\langle t, t' \rangle \in TP$ are checked for whether $t \in \mathbf{L}(s)$ and $t' \in \mathbf{L}(s')$. For checking $t \in \mathbf{L}(s)$, a straightforward algorithm as shown in Fig. 10 is used. The notation $t : s$ means that term t is a member of the sort s ; intermediate results are cached if s is a sort name.

3.5 Optimized Argument Selection

We now discuss how to avoid recursive calls of rsg_V , and analogously of rsg , that yield \perp .

Consider the sort definitions of s_0 and s_1 in Fig. 23. In the naive implementation, $hsg(s_0, s_1)$ amounts to sixteen recursive calls, shown in lines 3 to 6 of Fig. 24. However, by comparing the leading function symbols in each call, we can see immediately that twelve of them will return the empty sort, and can hence be ignored, while only four of them start with identical function symbols and thus need to be further evaluated:

$$hsg(s_0 + s_0, s_0 + s_1) \mid hsg(s_0 + s_0, s_1 + s_0) \mid hsg(s_0 * s_n, s_1 * s_1) \mid hsg(s_n * s_0, s_1 * s_1).$$

The situation is slightly more complicated when computing rsg_V : certain combinations of different leading function symbols may lead to a variable in V and thus cannot be ignored; e.g. if $v_1 =$

$$\begin{array}{c}
\frac{t : s}{t : s'} \quad s \doteq s' \quad (\text{Sort Def}) \\
\\
\frac{t : s_1 \mid s_2}{t : s_1} \quad (\text{Disjunction L}) \\
\\
\frac{t : s_1 \mid s_2}{t : s_2} \quad (\text{Disjunction R}) \\
\\
\frac{f(t_1, \dots, t_n) : f(s_1, \dots, s_n)}{t_1 : s_1 \quad \dots \quad t_n : s_n} \quad (\text{Function})
\end{array}$$

Fig. 10. Simple Algorithm for Testing Sort Membership

$\varphi(0, \text{succ}(0))$ and $v_2 = \varphi(0, 0 + 0)$, the calls $\text{rsg}_V(\{v_1, v_2\}, 0, \text{succ}(0))$ and $\text{rsg}_V(\{v_1, v_2\}, 0, s_0 + s_0)$ will result in v_1 , and v_2 , respectively.

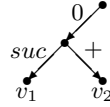


Fig. 11. Example: Search Tree

We present below an algorithm for selecting the argument pairs for recursive rsg_V calls. In a first step, we use the set V to build a search tree VT^* of pairs² of different function symbols which need to be considered by rsg_V . Building the search tree is a straightforward matter. In the above example, we get the tree shown in Fig. 11. In the general case, each path in the search tree has a length of N . The sort definitions are expected to be in head normal form, i.e., of the general form

$$s \doteq f_1(s_{11}, \dots, s_{1n_1}) \mid \dots \mid f_m(s_{m1}, \dots, s_{mn_m}).$$

We sort the disjuncts $f_1(s_{11}, \dots, s_{1n_1}), \dots, f_m(s_{m1}, \dots, s_{mn_m})$ ascendingly by their leading function symbols, assuming an arbitrary irreflexive total ordering \prec , and group together all disjuncts with the same function symbol (called “pre-grouping” to distinguish it from the kernel grouping algorithm). For example, from the definitions of s_0 and s_1 in Fig. 23, we get after sorting

$$\begin{array}{l}
[0, \quad s_0 + s_0, \quad s_0 * s_n, s_n * s_0] \\
[\text{succ}(s_0), s_0 + s_1, s_1 + s_0, s_1 * s_1],
\end{array}$$

and after pre-grouping

$$\begin{array}{l}
[[0], \quad [s_0 + s_0], \quad [s_0 * s_n, s_n * s_0]] =: Ps_1 \\
[[\text{succ}(s_0)], [s_0 + s_1, s_1 + s_0], [s_1 * s_1]] =: Ps_2.
\end{array}$$

In the general case, we get N such lists of pre-groups. These lists of pre-groups, together with the search tree, are given to the kernel grouping algorithm shown in Fig. 12, which produces a list of N -tuples, each serving as arguments for a subsequent recursive rsg_V call.

² N -tuples in the general case, where N sorts are simultaneously anti-unified.

The kernel grouping algorithm takes four input arguments:

- a flag E which can take the values eq or ne , the latter indicating that different function symbols have occurred in the current group
- the search tree VT^*
- the current group (a list of pre-groups)
- the current list of lists of pre-groups.

A pre-group, like $[s_0 + s_1, s_1 + s_0]$, is denoted by P , or P' ; a list of pre-groups by Ps , Ps' , or $Group$. $\uparrow P$ denotes the common head symbol of all terms in the pre-group P . For a search (sub)tree VT , we denote by VT_f the subtree of VT at the branch labeled f ; if a branch labeled f does not exist within VT , we get $VT_f = nil$, which denotes the empty search tree.

The output accumulator is not shown in Fig. 12; instead, in rule (*Output*), the statement “output $reverse(Group)$ ” occurs. If this statement is reached, $Group = [P_N, \dots, P_1]$ is a list of length N , and a list $[\langle t_1, \dots, t_N \rangle \mid t_i \in P_i, i = 1, \dots, N]$ of all N -tuples, such that the i -th component is a member of the i -th pre-group of $reverse(Group)$, is added to the output; the order of tuples within the list does not matter. The rule (*Output*) and the rules named (*Abort ...*) do not have a successor, whereas rules (*Join*) and (*Follow*) have two. The start configuration is

$$\overline{eq \quad VT^* \quad [] \quad [Ps_1, \dots, Ps_N]}$$

where VT^* is the initial search tree and Ps_1, \dots, Ps_N are the N lists of pre-groups obtained after pre-grouping. Figure 13 shows a computation example. For the sake of readability, pre-groups are written in braces, and the Ps_i in the fourth argument are separated by semicolons.

The same algorithm can also be used for rsg and hsg instead of rsg_V by providing an empty search tree.

We give below an estimation of the time complexity of the grouping algorithm. Assume that we are given N sorts whose definitions are in head normal form and have each at most m disjuncts on their right-hand side; moreover, assume that the initial search tree VT^* was built from n variables. Sorting the disjuncts by leading function symbols takes $N \cdot m \cdot \log m$ time; pre-grouping takes $N \cdot m$ time. We obtain N lists, each of which contains at most m groups. For the sake of brevity, we denote the third argument of the rules in Fig. 12 by G , and the fourth by H . Any 4-tuple $\langle E, VT, G, H \rangle$ that occurs during the computation is called a state; a state is called final if a rule without successors is applied to it. If VT of a state is a subtree of VT^* , we call the list of function symbols along the way from the root of VT^* to the root of VT the path of that state. Proceeding from VT to VT_f means appending f to the actual path. Observe the following properties of the rules:

1. To any state with $E = ne$ whose actual VT is not a subtree of VT^* , the rule (*Abort Ne*) will be applied, i.e. this state is final. Hence, each non-final state with $E = ne$ corresponds to a node in VT^* , viz. the root of the actual VT .
2. A non-final state with $E = ne$ and a path $[f_1, \dots, f_k]$ has $H = [Ps_{k+1}, \dots, Ps_N]$. Moreover, we have $f_i = \uparrow P_i$ for some P_i in Ps_i for all $i = 1, \dots, k$.
3. Each successor of a state with $E = ne$ corresponds to a different subtree VT_f of VT^* (possibly $VT_f = nil$), or has $head(H)$ shortened by one element. The latter makes it impossible to reach VT_f again, since $f = \uparrow P' \prec P''$ for all later P'' in Ps' , using the terminology common to rules (*Join*), (*Follow*), (*Skip*), and (*Init*).
4. At most m different non-final states with $E = ne$ may correspond to the same node of VT^* . This follows from 1 and 3.
5. E has the value eq iff there is a function symbol f such that $\uparrow P = f$ for all P in G ; in this case, we write $\uparrow G := f$.

eq	VT	$[P \mid Group]$	$[[P' \mid Ps'] \mid Rest]$	$f = \uparrow P = \uparrow P'$ (Join)
eq	VT_f	$[P', P \mid Group]$	$Rest$	
eq	VT	$[P \mid Group]$	$[Ps' \mid Rest]$	
<hr/>				
E	VT	$Group$	$[]$	(Output)
$output\ reverse(Group)$				
<hr/>				
ne	nil	$Group$	Any	(Abort Ne)
<hr/>				
eq	nil	$[P \mid Group]$	$[[P' \mid Ps'] \mid Rest]$	$\uparrow P \prec \uparrow P'$ (Abort Eq)
<hr/>				
<hr/>				
E	VT	$Group$	$[[P' \mid Ps'] \mid Rest]$	$f = \uparrow P'$ $Group \neq []$ (Follow)
ne	VT_f	$[P' \mid Group]$	$Rest$	
E	VT	$Group$	$[Ps' \mid Rest]$	
<hr/>				
E	VT	$Group$	$[[P' \mid Ps'] \mid Rest]$	$Group \neq []$ (Skip)
E	VT	$Group$	$[Ps' \mid Rest]$	
<hr/>				
E	VT	$[]$	$[[P' \mid Ps'] \mid Rest]$	$f = \uparrow P'$ (Init)
eq	VT_f	$[P']$	$Rest$	
eq	VT	$[]$	$[Ps' \mid Rest]$	
<hr/>				
E	VT	$Group$	$[[\] \mid Rest]$	(Abort $[\]$)
<hr/>				

Fig. 12. Grouping Rules

Precedence: $(0) \prec (s) \prec (*) \prec (+)$
 Input: $[[0, s0*sn, sn*s0, s0+s0], [suc(s0), s1*s1, s0+s1, s1+s0]]$
 Pre-Grouped: $[[[0], [s0*sn, sn*s0], [s0+s0]], [[suc(s0)], [s1*s1], [s0+s1, s1+s0]]]$
 Initial Search Tree VT^* : $\xrightarrow{0} \xrightarrow{s}$ (nil denotes empty subtree)
 Output: $[[0, suc(s0)], [s0*sn, s1*s1], [sn*s0, s1*s1], [s0+s0, s0+s1], [s0+s0, s1+s0]]$

```

eq 0 [] [ [{0},{S0*Sn,Sn*S0},{S0+S0}] ; [{suc(S0)},{S1*S1},{S0+S1,S1+S0}] ]
(Init)
eq s [{0}] [ [{suc(S0)},{S1*S1},{S0+S1,S1+S0}] ]
(Follow s)
ne nil [{suc(S0)},{0}] [ ]
(Output [0,suc(S0)])
eq s [{0}] [ [{S1*S1},{S0+S1,S1+S0}] ]
(Skip)
eq s [{0}] [ [{S0+S1,S1+S0}] ]
(Skip)
eq s [{0}] [ [ ] ]
(Abort [ ])
eq 0 [] [ [{S0*Sn,Sn*S0},{S0+S0}] ; [{suc(S0)},{S1*S1},{S0+S1,S1+S0}] ]
(Init)
eq nil [{S0*Sn,Sn*S0}] [ [{suc(S0)},{S1*S1},{S0+S1,S1+S0}] ]
(Skip)
eq nil [{S0*Sn,Sn*S0}] [ [{S1*S1},{S0+S1,S1+S0}] ]
(Join *)
eq nil [{S1*S1},{S0*Sn,Sn*S0}] [ ]
(Output [S0*Sn,S1*S1],[Sn*S0,S1*S1])
eq nil [{S0*Sn,Sn*S0}] [ [{S0+S1,S1+S0}] ]
(Abort Eq)
eq 0 [] [ [{S0+S0}] ; [{suc(S0)},{S1*S1},{S0+S1,S1+S0}] ]
(Init)
eq nil [{S0+S0}] [ [{suc(S0)},{S1*S1},{S0+S1,S1+S0}] ]
(Skip)
eq nil [{S0+S0}] [ [{S1*S1},{S0+S1,S1+S0}] ]
(Skip)
eq nil [{S0+S0}] [ [{S0+S1,S1+S0}] ]
(Join +)
eq nil [{S0+S1,S1+S0},{S0+S0}] [ ]
(Output [S0+S0,S0+S1],[S0+S0,S1+S0])
eq nil [{S0+S0}] [ [ ] ]
(Abort [ ])
eq 0 [] [ [ ] ; [{suc(S0)},{S1*S1},{S0+S1,S1+S0}] ]
(Abort [ ])

```

Fig. 13. Example: Argument Selection Run

6. At most m different states with $E = eq$ and $G \neq []$ may have the same $\uparrow G$ and the same $length(G)$, since each rule either increases $length(G)$ or shortens $head(H)$, the latter being possible at most m times. No rule decreases $length(G)$; and it always holds that $length(G) \leq N$.
7. Similar to 6, at most m different states with $E = eq$ and $G = []$ can exist.
8. Each non-final state has at most two final states as successors.
9. The initial search tree has $N \cdot n + 1$ nodes, if no two variables share a part of their path except for the root node. In all other cases, the number of nodes is decreased by path sharing.

From 9, there are at most $N \cdot n + 1$ nodes in VT^* ; from 1 and 4, there are at most $m \cdot (N \cdot n + 1)$ non-final states with $E = ne$; from 6 and 7, there are at most $m \cdot m \cdot N + m$ non-final states with $E = eq$. Hence, there are at most $m \cdot (N \cdot n + 1) + m \cdot m \cdot N + m = m \cdot (N \cdot n + N \cdot m + 2) \leq m \cdot N \cdot (m + n + 1)$ non-final states, and, by 8, at most $3 \cdot m \cdot N \cdot (m + n + 1)$ states in all. A final state may result in an output of at most g^N N -tuples, where g is the maximum number of disjuncts of a sort-definition's right-hand side that start with the same function symbol. Including sorting and pre-grouping, our approach takes $\mathcal{O}(N \cdot m \cdot (m + n) \cdot g^N)$ time. Since the naive approach — without grouping — takes $\mathcal{O}(m^N)$ time, grouping makes sense especially for $N > 2$, i.e. when more than two input sorts are to be anti-unified. Moreover, using the grouping algorithm makes the runtime more insensitive to the number of different function symbols in the background theory. Figure 14 shows some runtime measurements for grouping including pre-grouping. See Sect. 3.6 for more measurements.

Sort Defs of	User Time in msec	No. of Output Groups
s_0, s_1	13	4
s_0, s_1, s_2	23	10
s_0, s_1, s_2, s_3	39	32
s_0, s_1, s_2, s_3, s_4	102	144

Fig. 14. Runtimes for Grouping

3.6 Runtimes

In this section, we discuss some figures for runtime measurements. All runtimes given in this paper refer to user time of a SICStus PROLOG 2.1 #8 implementation on a SUN 4/50GX SPARC with 40 MHz clock and 64 MBytes main storage.

During development of the implementation prototype, we experimented with several technical optimizations. The impact of the successful ones (see Fig. 15) on runtimes are discussed below. Since runtime measurements have not been conducted systematically at all stages of development, we tried to estimate their impact from the available data shown in Figs. 38 and 40. These data are categorized and summarized in Fig. 16, yielding estimations of the improvement factors obtained by applying several optimizations simultaneously. Figure 17 gives a graphical representation of these factors, the x-axis being scaled logarithmically. In Fig. 18, we have computed the impact of each single technical improvement as far as possible, based on the data from Fig. 17.

Several measurements have been made in order to explore the practical runtime behavior of the final version (with optimizations *bdgsv*) of the algorithm wrt. number and size of input sorts. Figure 19(a) shows the E-anti-unification runtime vs. size of sort definitions; figures denote runtime of $rsq_V(\{\}, Sorts)$ in seconds. s_i denotes the equivalence class of i wrt. the respective background theory, cf. Fig. 4, 5. “ $> n$ ” means running out of memory after n seconds.

In the third column, we have included the $(-)$ operator in the background theory, such that equivalence classes of terms can no longer be described by our sorts. We therefore cut off the sort definitions

Abbr.	Technical Optimization	Described in	Improvement Factor
b	use of balanced binary trees to implement <code>Occ</code> sets	Sect. 3.2	1.5
d	use of <code>calc_sort_depths</code>	Sect. 4 below	6.3
g	use of <code>group_cr_sorts</code>	Sect. 3.5	8.3
s	restricting the sort simplifier to <code>simp1 ... simp4</code>	(obsolete)	
t	without user trace		2.1
v	use of <code>rsgv</code>	Sect. 3.4	21.4

Fig. 15. Technical Optimizations

From	To	Factor	Line	vs. Line	in Figure
g	v	3.0			
		1.9	5	6	38
		2.0	1	2	38
		2.0	3	4	38
		3.7	14	15	38
		3.7	16	17	38
		4.6	12	13	38
v	gstv	15.0			
		12.0	17	18	38
		29.0	19	22	38
gstv	dgsv	3.0			
		–	1	2	40
		0.3	3	4	40
		3.5	5	6	40
		7.8	7	8	40
dgsv	bdgsv	1.5			
		1.4	10	11	40
		2.1	14	15	40
stv	gstv	8.3	20	22	38
gsv	gstv	2.1	21	22	38

Fig. 16. Measured Improvements of Technical Optimization Groups

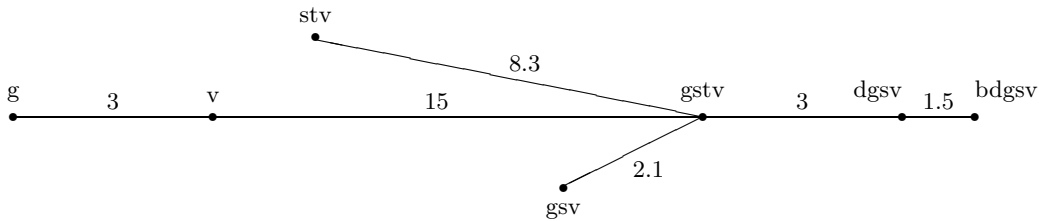


Fig. 17. Graphical Representation of Improvement Factors

at 20 — cf. Sect. 3.1. Since adding $(-)$ to the background theory led to an erratic increase in runtime, we additionally measured the dependency of runtime on the “cut-off point”; the result is shown in Fig. 19 (b), indicating the runtimes in seconds for $rsg_V(\{\}, s_0, s_2)$ with cut-off points n from 2 to 20.

Figure 20 shows the runtime for $rsg_V(\{\}, s_X^i)$ wrt. the background theory of $\{+\}$, $\{+, *\}$, and $\{+, *, /\}$, respectively, where s_X^i denotes the list that contains i times the sort s_X .

Optimization	From To	Factor
b	dgsv bdgsv	1.5
d	gsv dgsv	$6.3 = 2.1 * 3.0$
g	stv gstv	8.3
t	gsv gstv	2.1
sv	g gsv	$21.4 = 3.0 * 15.0 / 2.1$

Fig. 18. Runtime Impact of Single Technical Optimizations

Sorts	+	+, *	+, -, *
$[s_0, s_1, s_2]$	0	0	> 266
$[s_0, s_1, s_3]$	1	2	
$[s_0, s_1, s_4]$	0	3	
$[s_0, s_1, s_5]$	1	4	
$[s_0, s_1, s_6]$	1	5	
$[s_0, s_1, s_7]$	1	6	
$[s_0, s_1, s_8]$	1	9	
$[s_0, s_1, s_9]$	2	8	
$[s_0, s_1, s_{10}]$	2	10	
$[s_0, s_1, s_{11}]$	2	13	
$[s_0, s_1, s_{12}]$	2	15	
$[s_0, s_1, s_{13}]$	3	15	
$[s_0, s_1, s_{14}]$	3	17	
$[s_0, s_1, s_{15}]$	4	19	
$[s_0, s_1, s_{16}]$	6	21	
$[s_0, s_1, s_{17}]$	5	23	
$[s_0, s_1, s_{18}]$	7	24	
$[s_0, s_1, s_{19}]$	5	26	
$[s_0, s_1, s_{20}]$	7	29	

(a)

n	+, -	+, -, *
2	0	
3	1	
4	3	
5	7	
6	15	
7	24	
8	41	
9	61	
10	96	
11	135	
12	194	
13	272	
14	363	
15	485	
16	640	
17	847	
18	1066	
19	1349	
20	1804	1912

(b)

Fig. 19. E–Anti–Unification Runtime vs. Size of Sort Definitions

+	$i = 2$	3	4	5	6	7	8	9
s_0	0	1	1	1	1	2	1	0
s_1	0	0	1	6	11	27	87	279
s_2	1	2	16	84	579 > 3034			
s_3	1	13	106	1267 > 5313				
s_4	2	31	576 > 5009					
s_5	4	92	2546					
s_6	7	221 > 4518						
s_7	10	492						
s_8	19	1080						
s_9	27	2092						
s_{10}	39 > 1874							
s_{11}	53							
s_{12}	77							
s_{13}	105							
s_{14}	138							
s_{15}	178							
s_{16}	232							
s_{17}	290							
s_{18}	379							
s_{19}	466							
s_{20}	577							

+, *	$i = 2$	3	4	5	6	7	8	9
s_0	1	1	1	4	15	38	113	344
s_1	1	2	8	33	138	577	2400 > 7464	
s_2	2	9	49	318	2399 > 6466			
s_3	2	21	217	2632 > 5775				
s_4	3	57	962 > 5316					
s_5	7	133 > 2511						
s_6	10	309						
s_7	17	643						
s_8	24	1310						
s_9	34	2507						
s_{10}	49 > 2202							
s_{11}	67							
s_{12}	96							
s_{13}	123							
s_{14}	160							
s_{15}	209							
s_{16}	261							
s_{17}	335							
s_{18}	423							
s_{19}	525							
s_{20}	632							

+, *, /	$i = 2$	3	4	5	6	7	8	9
s_0	1	1	3	8	17	44		
s_1	794							
s_2								

Fig. 20. E–Anti–Unification Runtime vs. Size and Number of Sorts

4 Sort Enumeration

The algorithms rsg and rsg_V both return a sort as result. In this section, we describe how the terms belonging to a given sort are enumerated.

Enumeration of all terms of a given sort is done depth first. It is checked whether the sort is finite; in this case, a simple depth-first traversal is sufficient. For an infinite sort, the depth is limited to a fixed bound which is iteratively deepened. The depth of a term is currently measured as the maximum number of replacements of sort names by their definitions of all paths from the term root to its leaves. However, several other depth measures are possible; see the remarks on the 0, 1, 2, 1, 4, 1, 6 series example in Sect. 7.2.

Cutting the depth at a fixed bound n introduces many dead ends in sort enumeration when disjunctions in sort definitions are tried that can lead only to terms of a depth greater than n . As this usually happens at each of the n levels of recursive descent, the number of dead ends is in the order of magnitude m^n , where m is the maximum number of disjuncts occurring in a sort definition. For an example, see Figs. 35 and 36 in Sect. 5. An important optimization therefore consists in recording for each sort s the minimum depth $mindepth(s)$ of any of its terms; rule (*Sort Def*) is applicable only if this minimum depth is smaller than the remaining depth bound.

The rules of the optimized enumeration algorithm are given in Fig. 21. The notation $s : dp : t$ means that t is a term of a depth $\leq dp$ in $L(s)$, where s is an arbitrary sort expression. The resulting speedup of 6.3 compared to standard enumeration can be seen from line “d” in Fig. 18. The values of the sort’s minimum depths are computed using a bottom-up algorithm acting on all (new) sort definitions simultaneously, similar to the inhabitation algorithm from [AM91], which is described in Sect. 4.1.

Originally, our implementation had the additional capability of enumerating only terms in normal form wrt. a given set of (linear) redices. However, line 9 in Fig. 40 shows that sort enumeration may still take unacceptably long if only terms in normal form are to be enumerated. The restriction to terms in normal form is therefore achieved by intersecting the solution sort with the sort of all terms in normal form (provided all redices are linear terms, cf. Sect. 6.2), rather than by restricting the enumeration algorithm to terms in normal form. See line 10 in Fig. 40.

$\frac{\perp : dp : t}{\text{fail}}$		(Bottom)
$\frac{s : dp : t}{s' : dp-1 : t}$	$s \doteq s',$ $dp \geq mindepth(s)$	(Sort Def)
$\frac{s_1 s_2 : dp : t}{s_1 : dp : t}$		(Disjunction L)
$\frac{s_1 s_2 : dp : t}{s_2 : dp : t}$		(Disjunction R)
$\frac{f(s_1, \dots, s_n) : dp : f(t_1, \dots, t_n)}{s_1 : dp : t_1 \quad \dots \quad s_n : dp : t_n}$		(Function)

Fig. 21. Sort Enumeration Rules

4.1 Sort Depth Computation

Assume that all sort definitions are in head normal form. To each sort name s , associate a PROLOG variable S that will hold the minimal depth of s when the algorithm is finished. For each sort definition $s \doteq f_1(s_{11}, \dots, s_{1n_1}) \mid \dots \mid f_m(s_{m1}, \dots, s_{mn_m})$, introduce an equation

$$S = \min(\max(S_{11}, \dots, S_{1n_1}), \dots, \max(S_{m1}, \dots, S_{mn_m})) + 1$$

into the set of depth equations. Since \max is idempotent, multiple occurrences among its arguments may be replaced by one. Since \max is monotonic, $\max(S_{i1}, \dots, S_{in_i})$ can be omitted if S itself occurs among S_{i1}, \dots, S_{in_i} . If any nullary symbol is among f_1, \dots, f_m , the above depth equation simply reads $S = 1$. Building the set of depth equations is achieved by the PROLOG predicate `calc_sort_depths1_list`, which additionally distinguishes solved equations (i.e. ones containing no uninstantiated PROLOG variables) from unsolved ones.

In a second phase, repeat the following solving algorithm until no further progress is obtained, which will be the case after at least k cycles if there are k sort definitions. The solving algorithm checks each unsolved equation as to whether one or more of its right-hand side $\max(\dots)$ expressions are ground terms, meaning that all necessary data is available to compute its value. These expressions are evaluated and substituted by their result. If one of the results is minimal, i.e. equals $i - 1$ in the i -th cycle, the whole equation can be solved: $S = i$. Fig. 22 gives an example of a computation of the depths of sorts from Fig. 4.

All equations that remain unsolved after the final cycle correspond to empty sorts, cf. [AM91]. Their sort definitions' right-hand sides are simply replaced by \perp , and any occurrence of them in other sort definitions is also replaced by \perp , which may give rise to some obvious further simplifications.

In order to supply a different depth measure, \max has to be replaced accordingly, and the optimization based on its idempotence property may no longer be applicable.

$i = 1$	2	3
$S_0 = 1$		
$S_1 = \min(S_0, S_e) + 1$	$= 2$	
$S_2 = \min(S_1) + 1$	$= \min(2) + 1$	$= 3$
$S_3 = \min(S_2, \max(S_1, S_2)) + 1$	$= \min(S_2, \max(2, S_2)) + 1$	$= 4$
$S_4 = \min(S_3, \max(S_1, S_3), S_2) + 1$	$= \min(S_3, \max(2, S_3), S_2) + 1$	$= 4$
$S_5 = \min(S_4, \max(S_1, S_4), \max(S_2, S_3)) + 1$	$= \min(S_4, \max(2, S_4), \max(3, S_3)) + 1$	$= 5$
$S_p = \min(S_n, S_e) + 1$	$= 2$	
$S_e = 1$		
$S_o = \min(S_e) + 1$	$= 2$	
$S_n = 1$		

Fig. 22. Example: Sort Depth Computation

5 An E-Anti-Unification Example

In this section, we give an elaborate example of how the E-anti-unification algorithms work. Consider the background equational Theory (1) from Fig. 3, defining $(+)$ and $(*)$ over the 0-*suc* algebra. The equivalence class of 0 and *suc*(0) can be defined as sorts s_0 and s_1 , respectively, as shown in Fig. 23. s_n is not an equivalence class, but the sort of all terms. For example, the sort definition of s_0 can be read as follows: there are four possibilities for forming a term of the value 0: first, the constant 0 itself; second, adding two terms of value 0; third, multiplying a term of value 0 by any other term; and fourth, multiplying them conversely.

The computation of $hsg(s_0, s_1)$, according to Alg. 12, is shown in Fig. 24. The necessary auxiliary computations are shown in Figs. 25 to 29. The resulting sort definitions are summarized in the upper part of Fig. 30; They contain all the linear generalizations but only some of the nonlinear ones. Figures 31 to 34 show the computation of variables with non-empty sort intersections using the algorithm from Sect. 3.3, resulting in the new variable v'_{00} for $v_{00}, v_{0n}, v_{n0}, v_{nn}$, and v'_{01} for $v_{01}, v_{0n}, v_{n1}, v_{nn}$. The lower part of Fig. 30 shows the appropriately modified sort definitions. $L(s'_{01})$ is the set of all generalizations of 0 and *suc*(0) modulo $+, *$.

Figures 35 and 36 show the enumeration of sort s'_{01} with a depth bound of 2. The actual depth bound is shown in the second column, the i -th number referring to the i -th identifier (variable or sort name) in the term. Terms flagged “<*****” do not contain any sort names and are output. In Fig. 36, most terms that do not yield output are omitted.

$$\begin{array}{lcl} s_0 \doteq & 0 & | s_0 + s_0 | s_0 * s_n | s_n * s_0 \\ s_1 \doteq & suc(s_0) & | s_0 + s_1 | s_1 + s_0 | s_1 * s_1 \\ s_n \doteq & 0 & | suc(s_n) | s_n + s_n | s_n * s_n \end{array}$$

Fig. 23. Sort Definition of s_0, s_1

$$\begin{array}{l} s_{01} \\ \doteq hsg(s_0, s_1) \\ = v_{01} | \begin{array}{l} hsg(0, suc(s_0)) \\ hsg(s_0 + s_0, suc(s_0)) \\ hsg(s_0 * s_n, suc(s_0)) \\ hsg(s_n * s_0, suc(s_0)) \end{array} | \begin{array}{l} hsg(0, s_0 + s_1) \\ hsg(s_0 + s_0, s_0 + s_1) \\ hsg(s_0 * s_n, s_0 + s_1) \\ hsg(s_n * s_0, s_0 + s_1) \end{array} | \begin{array}{l} hsg(0, s_1 + s_0) \\ hsg(s_0 + s_0, s_1 + s_0) \\ hsg(s_0 * s_n, s_1 + s_0) \\ hsg(s_n * s_0, s_1 + s_0) \end{array} | \begin{array}{l} hsg(0, s_1 * s_1) \\ hsg(s_0 + s_0, s_1 * s_1) \\ hsg(s_0 * s_n, s_1 * s_1) \\ hsg(s_n * s_0, s_1 * s_1) \end{array} | \\ = v_{01} | \perp | \perp | \perp | \perp | \\ \perp | hsg(s_0 + s_0, s_0 + s_1) | hsg(s_0 + s_0, s_1 + s_0) | \perp | \\ \perp | \perp | \perp | hsg(s_0 * s_n, s_1 * s_1) | \\ \perp | \perp | \perp | hsg(s_n * s_0, s_1 * s_1) | \\ = v_{01} | hsg(s_0, s_0) + hsg(s_0, s_1) | hsg(s_0, s_1) + hsg(s_0, s_0) | hsg(s_0, s_1) * hsg(s_n, s_1) | hsg(s_n, s_1) * hsg(s_0 * s_1) \\ = v_{01} | s_{00} + s_{01} | s_{01} + s_{00} | s_{01} * s_{n1} | s_{n1} * s_{01} \end{array}$$

Fig. 24. Computation of $hsg(s_0, s_1)$

$$\begin{aligned}
& s_{00} \\
& \doteq hsg(s_0, s_0) \\
& = v_{00} \mid \begin{array}{l} hsg(0, 0) \\ hsg(s_0 + s_0, 0) \\ hsg(s_0 * s_n, 0) \\ hsg(s_n * s_0, 0) \end{array} \mid \begin{array}{l} hsg(0, s_0 + s_0) \\ hsg(s_0 + s_0, s_0 + s_0) \\ hsg(s_0 * s_n, s_0 + s_0) \\ hsg(s_n * s_0, s_0 + s_0) \end{array} \mid \begin{array}{l} hsg(0, s_0 * s_n) \\ hsg(s_0 + s_0, s_0 * s_n) \\ hsg(s_0 * s_n, s_0 * s_n) \\ hsg(s_n * s_0, s_0 * s_n) \end{array} \mid \begin{array}{l} hsg(0, s_n * s_0) \\ hsg(s_0 + s_0, s_n * s_0) \\ hsg(s_0 * s_n, s_n * s_0) \\ hsg(s_n * s_0, s_n * s_0) \end{array} \mid \\
& = v_{00} \mid 0 \mid hsg(s_0, s_0) + hsg(s_0, s_0) \mid hsg(s_0, s_0) * hsg(s_n, s_n) \mid \\
& \quad hsg(s_0, s_n) * hsg(s_n, s_0) \mid hsg(s_n, s_0) * hsg(s_0, s_n) \mid hsg(s_n, s_n) * hsg(s_0, s_0) \\
& = v_{00} \mid 0 \mid s_{00} + s_{00} \mid s_{00} * s_{nn} \mid \\
& \quad s_{0n} * s_{n0} \mid s_{n0} * s_{0n} \mid s_{nn} * s_{00}
\end{aligned}$$

Fig. 25. Computation of $hsg(s_0, s_0)$

$$\begin{aligned}
& s_{0n} \\
& \doteq hsg(s_0, s_n) \\
& = v_{0n} \mid \begin{array}{l} hsg(0, 0) \\ hsg(s_0 + s_0, 0) \\ hsg(s_0 * s_n, 0) \\ hsg(s_n * s_0, 0) \end{array} \mid \begin{array}{l} hsg(0, suc(s_n)) \\ hsg(s_0 + s_0, suc(s_n)) \\ hsg(s_0 * s_n, suc(s_n)) \\ hsg(s_n * s_0, suc(s_n)) \end{array} \mid \begin{array}{l} hsg(0, s_n + s_n) \\ hsg(s_0 + s_0, s_n + s_n) \\ hsg(s_0 * s_n, s_n + s_n) \\ hsg(s_n * s_0, s_n + s_n) \end{array} \mid \begin{array}{l} hsg(0, s_n * s_n) \\ hsg(s_0 + s_0, s_n * s_n) \\ hsg(s_0 * s_n, s_n * s_n) \\ hsg(s_n * s_0, s_n * s_n) \end{array} \mid \\
& = v_{0n} \mid 0 \mid hsg(s_0, s_n) + hsg(s_0, s_n) \mid hsg(s_0, s_n) * hsg(s_n, s_n) \mid hsg(s_n, s_n) * hsg(s_0, s_n) \\
& = v_{0n} \mid 0 \mid s_{0n} + s_{0n} \mid s_{0n} * s_{nn} \mid s_{nn} * s_{0n}
\end{aligned}$$

Fig. 26. Computation of $hsg(s_0, s_n)$

$$\begin{aligned}
& s_{n0} \\
& \doteq hsg(s_n, s_n) \\
& = v_{n0} \mid \begin{array}{l} hsg(0, 0) \\ hsg(suc(s_n), 0) \\ hsg(s_n + s_n, 0) \\ hsg(s_n * s_n, 0) \end{array} \mid \begin{array}{l} hsg(0, s_0 + s_0) \\ hsg(suc(s_n), s_0 + s_0) \\ hsg(s_n + s_n, s_0 + s_0) \\ hsg(s_n * s_n, s_0 + s_0) \end{array} \mid \begin{array}{l} hsg(0, s_0 * s_n) \\ hsg(suc(s_n), s_0 * s_n) \\ hsg(s_n + s_n, s_0 * s_n) \\ hsg(s_n * s_n, s_0 * s_n) \end{array} \mid \begin{array}{l} hsg(0, s_n * s_0) \\ hsg(suc(s_n), s_n * s_0) \\ hsg(s_n + s_n, s_n * s_0) \\ hsg(s_n * s_n, s_n * s_0) \end{array} \mid \\
& = v_{n0} \mid 0 \mid hsg(s_n, s_0) + hsg(s_n, s_0) \mid hsg(s_n, s_0) * hsg(s_n, s_n) \mid hsg(s_n, s_n) * hsg(s_n, s_0) \\
& = v_{n0} \mid 0 \mid s_{n0} + s_{n0} \mid s_{n0} * s_{nn} \mid s_{nn} * s_{n0}
\end{aligned}$$

Fig. 27. Computation of $hsg(s_n, s_0)$

$$\begin{aligned}
& s_{n1} \\
& \doteq hsg(s_n, s_1) \\
& = v_{n1} \mid hsg(0, suc(s_0)) \mid hsg(0, s_0 + s_1) \mid hsg(0, s_1 + s_0) \mid hsg(0, s_1 * s_1) \mid \\
& \quad hsg(suc(s_n), suc(s_0)) \mid hsg(suc(s_n), s_0 + s_1) \mid hsg(suc(s_n), s_1 + s_0) \mid hsg(suc(s_n), s_1 * s_1) \mid \\
& \quad hsg(s_n + s_n, suc(s_0)) \mid hsg(s_n + s_n, s_0 + s_1) \mid hsg(s_n + s_n, s_1 + s_0) \mid hsg(s_n + s_n, s_1 * s_1) \mid \\
& \quad hsg(s_n * s_n, suc(s_0)) \mid hsg(s_n * s_n, s_0 + s_1) \mid hsg(s_n * s_n, s_1 + s_0) \mid hsg(s_n * s_n, s_1 * s_1) \\
& = v_{n1} \mid suc(hsg(s_n, s_0)) \mid hsg(s_n, s_0) + hsg(s_n, s_1) \mid hsg(s_n, s_1) + hsg(s_n, s_0) \mid hsg(s_n, s_1) * hsg(s_n, s_1) \\
& = v_{n1} \mid suc(s_{n0}) \mid s_{n0} + s_{n1} \mid s_{n1} + s_{n0} \mid s_{n1} * s_{n1}
\end{aligned}$$

Fig. 28. Computation of $hsg(s_n, s_1)$

$$\begin{aligned}
& s_{nn} \\
& \doteq hsg(s_n, s_n) \\
& = v_{nn} \mid hsg(0, 0) \mid hsg(0, suc(s_n)) \mid hsg(0, s_n + s_n) \mid hsg(0, s_n * s_n) \mid \\
& \quad hsg(suc(s_n), 0) \mid hsg(suc(s_n), suc(s_n)) \mid hsg(suc(s_n), s_n + s_n) \mid hsg(suc(s_n), s_n * s_n) \mid \\
& \quad hsg(s_n + s_n, 0) \mid hsg(s_n + s_n, suc(s_n)) \mid hsg(s_n + s_n, s_n + s_n) \mid hsg(s_n + s_n, s_n * s_n) \mid \\
& \quad hsg(s_n * s_n, 0) \mid hsg(s_n * s_n, suc(s_n)) \mid hsg(s_n * s_n, s_n + s_n) \mid hsg(s_n * s_n, s_n * s_n) \\
& = v_{nn} \mid 0 \mid suc(hsg(s_n, s_n)) \mid hsg(s_n, s_n) + hsg(s_n, s_n) \mid hsg(s_n, s_n) * hsg(s_n, s_n) \\
& = v_{nn} \mid 0 \mid suc(s_{nn}) \mid s_{nn} + s_{nn} \mid s_{nn} * s_{nn}
\end{aligned}$$

Fig. 29. Computation of $hsg(s_n, s_n)$

Linear Generalization:

$$\begin{aligned}
s_{00} & \doteq v_{00} \mid 0 \mid s_{00} + s_{00} \mid s_{00} * s_{nn} \mid s_{0n} * s_{n0} \mid s_{n0} * s_{0n} \mid s_{nn} * s_{00} \\
s_{01} & \doteq v_{01} \mid s_{00} + s_{01} \mid s_{01} + s_{00} \mid s_{01} * s_{n1} \mid s_{n1} * s_{01} \\
s_{0n} & \doteq v_{0n} \mid 0 \mid s_{0n} + s_{0n} \mid s_{0n} * s_{nn} \mid s_{nn} * s_{0n} \\
s_{n0} & \doteq v_{n0} \mid 0 \mid s_{n0} + s_{n0} \mid s_{n0} * s_{nn} \mid s_{nn} * s_{n0} \\
s_{n1} & \doteq v_{n1} \mid suc(s_{n0}) \mid s_{n0} + s_{n1} \mid s_{n1} + s_{n0} \mid s_{n1} * s_{n1} \\
s_{nn} & \doteq v_{nn} \mid 0 \mid suc(s_{nn}) \mid s_{nn} + s_{nn} \mid s_{nn} * s_{nn}
\end{aligned}$$

Complete Generalization:

$$\begin{aligned}
s'_{00} & \doteq v_{00} \mid v'_{00} \mid 0 \mid s'_{00} + s'_{00} \mid s'_{00} * s'_{nn} \mid s'_{0n} * s'_{n0} \mid s'_{n0} * s'_{0n} \mid s'_{nn} * s'_{00} \\
s'_{01} & \doteq v_{01} \mid v'_{01} \mid s'_{00} + s'_{01} \mid s'_{01} + s'_{00} \mid s'_{01} * s'_{n1} \mid s'_{n1} * s'_{01} \\
s'_{0n} & \doteq v_{0n} \mid v'_{00} \mid v'_{01} \mid 0 \mid s'_{0n} + s'_{0n} \mid s'_{0n} * s'_{nn} \mid s'_{nn} * s'_{0n} \\
s'_{n0} & \doteq v_{n0} \mid v'_{00} \mid 0 \mid s'_{n0} + s'_{n0} \mid s'_{n0} * s'_{nn} \mid s'_{nn} * s'_{n0} \\
s'_{n1} & \doteq v_{n1} \mid v'_{01} \mid suc(s'_{n0}) \mid s'_{n0} + s'_{n1} \mid s'_{n1} + s'_{n0} \mid s'_{n1} * s'_{n1} \\
s'_{nn} & \doteq v_{nn} \mid v'_{00} \mid v'_{01} \mid 0 \mid suc(s'_{nn}) \mid s'_{nn} + s'_{nn} \mid s'_{nn} * s'_{nn}
\end{aligned}$$

Fig. 30. Generalization of s_0 and s_1

$$\begin{aligned}
S &= \{ s_0, s_1, s_n \} \\
I2 &= \{ \langle s_0, s_n, s_0 \rangle, \langle s_1, s_n, s_1 \rangle \} \\
L2 &= \{ \langle v_{00}, v_{0n} \rangle, \langle v_{00}, v_{n0} \rangle, \langle v_{00}, v_{nn} \rangle, \langle v_{01}, v_{0n} \rangle, \langle v_{01}, v_{n1} \rangle, \langle v_{01}, v_{nn} \rangle, \langle v_{0n}, v_{n0} \rangle, \langle v_{0n}, v_{n1} \rangle, \langle v_{0n}, v_{nn} \rangle, \\
&\quad \langle v_{n0}, v_{nn} \rangle, \langle v_{n1}, v_{nn} \rangle \} \\
V2 &= \{ \langle v_{00}, 3 \rangle, \langle v_{01}, 3 \rangle, \langle v_{0n}, 5 \rangle, \langle v_{n0}, 3 \rangle, \langle v_{n1}, 3 \rangle, \langle v_{nn}, 5 \rangle \}
\end{aligned}$$

```

[V00,V0n]      [V00,V01,V0n,Vn0,Vn1,Vnn]      []
(Member)
[V00,V0n]      [V01,V0n,Vn0,Vn1,Vnn]      []
(Sublist2)
[V00,V0n]      [V0n,Vn0,Vn1,Vnn]      []
(Member)
[V00,V0n]      [Vn0,Vn1,Vnn]      []
(Inf Inh)      infs({V00,V0n,Vn0})
[V00,V0n,Vn0]  [Vn1,Vnn]      []
(Sublist2)
[V00,V0n,Vn0]  [Vnn]      []
(Inf Inh)      infs({V00,V0n,Vn0,Vnn})
[V00,V0n,Vn0,Vnn]  []      []
(Output Max [V00,V0n,Vn0,Vnn])
[V00,V0n,Vn0]  []      []
(Max Subsumed)
[V00,V0n]      [Vn1,Vnn]      []
(Sublist2)
[V00,V0n]      [Vnn]      []
(Suplist)
[V00,V0n]      []      [Vnn]
(Max Subs Susp)

[V00,Vn0]      [V00,V01,V0n,Vn0,Vn1,Vnn]      []
(Member)
[V00,Vn0]      [V01,V0n,Vn0,Vn1,Vnn]      []
(Sublist2)
[V00,Vn0]      [V0n,Vn0,Vn1,Vnn]      []
(Suplist)
[V00,Vn0]      [Vn0,Vn1,Vnn]      [V0n]
(Member)
[V00,Vn0]      [Vn1,Vnn]      [V0n]
(Sublist2)
[V00,Vn0]      [Vnn]      [V0n]
(Suplist)
[V00,Vn0]      []      [Vnn,V0n]
(Max Subs Susp)

```

(Contd. in Fig. 32)

Fig. 31. Intersection Computation for s_0, s_1

```

[V00,Vnn]    [V00,V01,V0n,Vn0,Vn1,Vnn]    []
(Member)
[V00,Vnn]    [V01,V0n,Vn0,Vn1,Vnn]    []
(Sublist2)
[V00,Vnn]    [V0n,Vn0,Vn1,Vnn]    []
(Suplist)
[V00,Vnn]    [Vn0,Vn1,Vnn]    [V0n]
(Suplist)
[V00,Vnn]    [Vn1,Vnn]    [Vn0,V0n]
(Sublist2)
[V00,Vnn]    [Vnn]    [Vn0,V0n]
(Member)
[V00,Vnn]    []    [Vn0,V0n]
(Max Subs Susp)

[V01,V0n]    [V00,V01,V0n,Vn0,Vn1,Vnn]    []
(Sublist2)
[V01,V0n]    [V01,V0n,Vn0,Vn1,Vnn]    []
(Member)
[V01,V0n]    [V0n,Vn0,Vn1,Vnn]    []
(Member)
[V01,V0n]    [Vn0,Vn1,Vnn]    []
(Sublist2)
[V01,V0n]    [Vn1,Vnn]    []
(Inf Inh)                                infs({V01,V0n,Vn1})
[V01,V0n,Vn1]    [Vnn]    []
(Inf Inh)                                infs({V01,V0n,Vn1,Vnn})
[V01,V0n,Vn1,Vnn]    []    []
(Output Max [V01,V0n,Vn1,Vnn])
[V01,V0n,Vn1]    []    []
(Max Subsumed)
[V01,V0n]    [Vnn]    []
(Suplist)
[V01,V0n]    []    [Vnn]
(Max Subs Susp)

[V01,Vn1]    [V00,V01,V0n,Vn0,Vn1,Vnn]    []
(Sublist2)
[V01,Vn1]    [V01,V0n,Vn0,Vn1,Vnn]    []
(Member)
[V01,Vn1]    [V0n,Vn0,Vn1,Vnn]    []
(Suplist)
[V01,Vn1]    [Vn0,Vn1,Vnn]    [V0n]
(Sublist2)
[V01,Vn1]    [Vn1,Vnn]    [V0n]
(Member)
[V01,Vn1]    [Vnn]    [V0n]
(Suplist)
[V01,Vn1]    []    [Vnn,V0n]
(Max Subs Susp)

```

(Contd. in Fig. 33)

Fig. 32. Intersection Computation for s_0, s_1 (1. Contin.)


```

[V01,Vnn]    [V00,V01,V0n,Vn0,Vn1,Vnn]    []
(Sublist2)
[V01,Vnn]    [V01,V0n,Vn0,Vn1,Vnn]    []
(Member)
[V01,Vnn]    [V0n,Vn0,Vn1,Vnn]    []
(Suplist)
[V01,Vnn]    [Vn0,Vn1,Vnn]    [V0n]
(Sublist2)
[V01,Vnn]    [Vn1,Vnn]    [V0n]
(Suplist)
[V01,Vnn]    [Vnn]    [Vn1,V0n]
(Member)
[V01,Vnn]    []    [Vn1,V0n]
(Max Subs Susp)

[V0n,Vn0]    [V00,V01,V0n,Vn0,Vn1,Vnn]    []
(Suplist)
[V0n,Vn0]    [V01,V0n,Vn0,Vn1,Vnn]    [V00]
(Sublist2)
[V0n,Vn0]    [V0n,Vn0,Vn1,Vnn]    [V00]
(Member)
[V0n,Vn0]    [Vn0,Vn1,Vnn]    [V00]
(Member)
[V0n,Vn0]    [Vn1,Vnn]    [V00]
(Sublist2)
[V0n,Vn0]    [Vnn]    [V00]
(Suplist)
[V0n,Vn0]    []    [Vnn,V00]
(Max Subs Susp)

[V0n,Vn1]    [V00,V01,V0n,Vn0,Vn1,Vnn]    []
(Sublist2)
[V0n,Vn1]    [V01,V0n,Vn0,Vn1,Vnn]    []
(Suplist)
[V0n,Vn1]    [V0n,Vn0,Vn1,Vnn]    [V01]
(Member)
[V0n,Vn1]    [Vn0,Vn1,Vnn]    [V01]
(Sublist2)
[V0n,Vn1]    [Vn1,Vnn]    [V01]
(Member)
[V0n,Vn1]    [Vnn]    [V01]
(Suplist)
[V0n,Vn1]    []    [Vnn,V01]
(Max Subs Susp)

```

(Contd. in Fig. 34)

Fig. 33. Intersection Computation for s_0, s_1 (2. Contin.)

```

[V0n,Vnn]    [V00,V01,V0n,Vn0,Vn1,Vnn]    []
(Suplist)
[V0n,Vnn]    [V01,V0n,Vn0,Vn1,Vnn]    [V00]
(Suplist)
[V0n,Vnn]    [V0n,Vn0,Vn1,Vnn]    [V01,V00]
(Member)
[V0n,Vnn]    [Vn0,Vn1,Vnn]    [V01,V00]
(Suplist)
[V0n,Vnn]    [Vn1,Vnn]    [Vn0,V01,V00]
(Suplist)
[V0n,Vnn]    [Vnn]    [Vn1,Vn0,V01,V00]
(Member)
[V0n,Vnn]    []    [Vn1,Vn0,V01,V00]
(Max Subs Susp)

[Vn0,Vnn]    [V00,V01,V0n,Vn0,Vn1,Vnn]    []
(Suplist)
[Vn0,Vnn]    [V01,V0n,Vn0,Vn1,Vnn]    [V00]
(Sublist2)
[Vn0,Vnn]    [V0n,Vn0,Vn1,Vnn]    [V00]
(Suplist)
[Vn0,Vnn]    [Vn0,Vn1,Vnn]    [V0n,V00]
(Member)
[Vn0,Vnn]    [Vn1,Vnn]    [V0n,V00]
(Sublist2)
[Vn0,Vnn]    [Vnn]    [V0n,V00]
(Member)
[Vn0,Vnn]    []    [V0n,V00]
(Max Subs Susp)

[Vn1,Vnn]    [V00,V01,V0n,Vn0,Vn1,Vnn]    []
(Sublist2)
[Vn1,Vnn]    [V01,V0n,Vn0,Vn1,Vnn]    []
(Suplist)
[Vn1,Vnn]    [V0n,Vn0,Vn1,Vnn]    [V01]
(Suplist)
[Vn1,Vnn]    [Vn0,Vn1,Vnn]    [V0n,V01]
(Sublist2)
[Vn1,Vnn]    [Vn1,Vnn]    [V0n,V01]
(Member)
[Vn1,Vnn]    [Vnn]    [V0n,V01]
(Member)
[Vn1,Vnn]    []    [V0n,V01]
(Max Subs Susp)

```

Fig. 34. Intersection Computation for s_0, s_1 (3. Contin.)

S'01	2	
V01	1	<*****
V'01	1	<*****
S'00+S'01	1,1	
V00+V01	0,0	<*****
V00+V'01	0,0	<*****
V00+(S'00+S'01)	0,0	
V00+(S'01+S'00)	0,0	
V00+(S'01*S'n1)	0,0	
V00+(S'n1*S'01)	0,0	
V'00+V01	0,0	<*****
V'00+V'01	0,0	<*****
V'00+(S'00+S'01)	0,0	
V'00+(S'01+S'00)	0,0	
V'00+(S'01*S'n1)	0,0	
V'00+(S'n1*S'01)	0,0	
0+V01	0,0	<*****
0+V'01	0,0	<*****
0+(S'00+S'01)	0,0	
0+(S'01+S'00)	0,0	
0+(S'01*S'n1)	0,0	
0+(S'n1*S'01)	0,0	
(S'00+S'00)+V01	0,0	
(S'00+S'00)+V'01	0,0	
(S'00+S'00)+(S'00+S'01)	0,0	
(S'00+S'00)+(S'01+S'00)	0,0	
(S'00+S'00)+(S'01*S'n1)	0,0	
(S'00+S'00)+(S'n1*S'01)	0,0	
(S'00*S'nn)+V01	0,0	
(S'00*S'nn)+V'01	0,0	
(S'00*S'nn)+(S'00+S'01)	0,0	
(S'00*S'nn)+(S'01+S'00)	0,0	
(S'00*S'nn)+(S'01*S'n1)	0,0	
(S'00*S'nn)+(S'n1*S'01)	0,0	
(S'0n*S'n0)+V01	0,0	
(S'0n*S'n0)+V'01	0,0	
(S'0n*S'n0)+(S'00+S'01)	0,0	
(S'0n*S'n0)+(S'01+S'00)	0,0	
(S'0n*S'n0)+(S'01*S'n1)	0,0	
(S'0n*S'n0)+(S'n1*S'01)	0,0	
(S'n0*S'0n)+V01	0,0	
(S'n0*S'0n)+V'01	0,0	
(S'n0*S'0n)+(S'00+S'01)	0,0	
(S'n0*S'0n)+(S'01+S'00)	0,0	
(S'n0*S'0n)+(S'01*S'n1)	0,0	
(S'n0*S'0n)+(S'n1*S'01)	0,0	
(S'nn*S'00)+V01	0,0	
(S'nn*S'00)+V'01	0,0	
(S'nn*S'00)+(S'00+S'01)	0,0	
(S'nn*S'00)+(S'01+S'00)	0,0	
(S'nn*S'00)+(S'01*S'n1)	0,0	
(S'nn*S'00)+(S'n1*S'01)	0,0	

(Contd. in Fig. 36)

Fig. 35. Enumeration of s_{01}

$S'01+S'00$	1,1	
$V01+V00$	0,0	<*****
$V01+V'00$	0,0	<*****
$V01+0$	0,0	<*****
...		
$V'01+V00$	0,0	<*****
$V'01+V'00$	0,0	<*****
$V'01+0$	0,0	<*****
...		
$S'01*S'n1$	1,1	
$V01*Vn1$	0,0	<*****
$V01*V'01$	0,0	<*****
...		
$V'01*Vn1$	0,0	<*****
$V'01*V'01$	0,0	<*****
...		
$S'n1*S'01$	1,1	
$Vn1*V01$	0,0	<*****
$Vn1*V'01$	0,0	<*****
...		
$V'01*V01$	0,0	<*****
$V'01*V'01$	0,0	<*****
...		
$(S'n1*S'n1)*(S'n1*S'01)$	0,0	

Fig. 36. Enumeration of s_{01} (contd.)

6 Lemma Generation

6.1 Implementation

One application of E-anti-unification is for generating lemma candidates in equational induction proofs. In this section, we follow [Hei95] in showing how to generate candidates that are applicable in a given blocked proof situation. The algorithm presented here takes as input a term t and a list $[\sigma_1, \dots, \sigma_n]$ of ground substitutions, and generates as output the sort s^r of all terms t' such that $\sigma_i t =_E \sigma_i t'$ for $i = 1, \dots, n$. In other words, each generated t' equals the given t at least on all given example instances. The appropriate selection of the σ_i is discussed in Sect. 6.3 below.

Note that for each universally valid equation $t =_E t'$, the right-hand side t' will be a member of $L(s^r)$, and that for each non-universally valid equation $t =_E t'$, t' can be excluded from s^r by adding a “counter-example instance” σ' with $\sigma' t \neq_E \sigma' t'$ to $[\sigma_1, \dots, \sigma_n]$.

The algorithm first computes the simultaneous syntactical anti-unification of all example instances

$$t^l := \sigma_1 t \sqcap \dots \sqcap \sigma_n t.$$

Usually, t^l is a variant of t , cf. Sect. 6.3. Next, we compute the sort of all E-anti-unifiers of example instances

$$s^r := rsg_V(\text{vars}(t^l), [\underline{\sigma_1 t}, \dots, \underline{\sigma_n t}]).$$

For each $t^r \in L(s^r)$, the equation $t^l = t^r$ is then a lemma candidate. Note that using rsg_V ensures that $\text{vars}(t^r) \subset \text{vars}(t^l)$, thus excluding nonsense candidates like $(x_1 + x_2) + x_3 = x_4 + x_5$. In order to additionally restrict s^r to terms in normal form, we may define the sort s_{NF} of all terms in normal form (see Sect. 6.2 below) and intersect s^r with s_{NF} . This will exclude candidates such as $(x_1 + x_2) + x_3 = x_1 + (x_2 + (x_3 + 0))$.

6.2 Filter Sorts

In order to restrict the output of lemma generation or other applications to make it satisfy certain additional criteria, we may intersect the result sort of E-anti-unification with appropriate filter sorts. The definition and construction of these filter sorts is described below.

- The top sort s_{TOP} of all valid terms at all depends on the variables that have been introduced during E-anti-unification. Each such variable must be treated as a nullary function symbol. Let V be the set of all variables occurring in any sort definition; then

$$s_{TOP} \doteq \prod_{x \in V} x \mid \prod_{f \in \mathcal{F}} f(s_{TOP}, \dots, s_{TOP}).$$

- Provided all term-rewrite rules of the given background equational theory have linear left-hand sides, we can compute the sort s_{RED} of all reducible terms. Let l_1, \dots, l_n be the left-hand sides; let

$$L_i := \{x \leftarrow s_{TOP} \mid x \in \text{vars}(l_i)\} (l_i)$$

be the sorts obtained by replacing every variable by s_{TOP} . L_i contains all terms where l_i matches at the root. Define the set of all reducible terms by

$$s_{RED} \doteq \prod_{i=1}^n L_i \mid \prod_{f \in \mathcal{F}} f(s_{RED}, s_{TOP}, \dots, s_{TOP}) \mid \dots \mid f(s_{TOP}, \dots, s_{TOP}, s_{RED}).$$

- Since our sorts are closed wrt. relative complements, the sort of irreducible terms can simply be computed by $s_{NF} := s_{TOP} \setminus s_{RED}$.
- If there are nonlinear redices, the set \mathcal{T}_{RED} of reducible terms as well as the set \mathcal{T}_{NF} of irreducible terms may no longer be a regular tree language and may thus not be representable by our sorts, cf. e.g. [HH94, Com90]. However, by omitting all nonlinear left-hand sides, we can achieve $L(s_{RED}) \subset \mathcal{T}_{RED}$, and $\mathcal{T}_{NF} \subset L(s_{NF})$. Conversely, by replacing each nonlinear redex with a linear anti-instance, we get $\mathcal{T}_{RED} \subset L(s_{RED})$ and $L(s_{NF}) \subset \mathcal{T}_{NF}$.
- For every finite set of variables $V \subset W$, we can compute the sort S_V^W of all terms t such that $V \subset vars(t) \subset W$. Let A be the set of all occurring arities; for any $a \in A$, let

$$parts(V, a) := \{\langle P_1, \dots, P_a \rangle \mid P_1 \cup \dots \cup P_a = V, \forall i, j \ i \neq j \rightarrow P_i \cap P_j = \{\}\}$$

denote the set of all partitions of the set V into a disjoint subsets; let $parts(V, 0) = \{\}$. Define

$$\begin{aligned} s_{\{\}}^W &\doteq \mid_{x \in W} x \mid \mid_{f \in \mathcal{F}} f(s_{\{\}}^W, \dots, s_{\{\}}^W), \\ s_{\{v\}}^W &\doteq v \mid \mid_{f \in \mathcal{F}, ar(f) \geq 1} f(s_{\{v\}}^W, s_{\{\}}^W, \dots, s_{\{\}}^W) \mid \dots \mid f(s_{\{v\}}^W, \dots, s_{\{\}}^W, s_{\{v\}}^W), \\ s_V^W &\doteq \mid_{a \in A} \mid_{f \in \mathcal{F}, ar(f)=a} \mid_{\langle P_1, \dots, P_a \rangle \in parts(V, a)} f(s_{P_1}^W, \dots, s_{P_a}^W). \end{aligned}$$

Fig. 37 shows an example for $\mathcal{F} = \{0, s, +\}$, $V = \{v_1, v_2\}$ and $W = \{v_1, v_2, v_3, v_4\}$, using the abbreviations $s_{12}^{1234} := s_{\{v_1, v_2, v_3, v_4\}}^{\{v_1, v_2, v_3, v_4\}}$, $s_{\varepsilon}^{1234} := s_{\{\}}^{\{v_1, v_2, v_3, v_4\}}$, and so on.

Note that filtering with s_V^W can be avoided by computing with $rs g_V$ instead of $rs g$, which is also much faster. However, for $V \neq \{\}$ the filter sorts s_V^W provide a means of ensuring that each solution term contains at *least* all variables from a given set V .

$$\begin{aligned} s_{12}^{1234} &\doteq suc(s_{12}^{1234}) \mid s_{12}^{1234} + s_{\varepsilon}^{1234} \mid s_{\varepsilon}^{1234} + s_{12}^{1234} \mid s_1^{1234} + s_2^{1234} \mid s_2^{1234} + s_1^{1234} \\ s_1^{1234} &\doteq v_1 \mid suc(s_1^{1234}) \mid s_{\varepsilon}^{1234} + s_1^{1234} \mid s_1^{1234} + s_{\varepsilon}^{1234} \\ s_2^{1234} &\doteq v_2 \mid suc(s_2^{1234}) \mid s_{\varepsilon}^{1234} + s_2^{1234} \mid s_2^{1234} + s_{\varepsilon}^{1234} \\ s_{\varepsilon}^{1234} &\doteq 0 \mid v_1 \mid v_2 \mid v_3 \mid v_4 \mid suc(s_{\varepsilon}^{1234}) \mid s_{\varepsilon}^{1234} + s_{\varepsilon}^{1234} \end{aligned}$$

Fig. 37. Example: Variable Filter Sort

6.3 Selection of Ground Instances

As shown in Sect. 6.1, our lemma-generation algorithm needs some ground instances as input. Regarding their selection, we have the following requirements: Let $\langle v_1, \dots, v_n \rangle$ be the variables occurring in the given left-hand side term t . A ground instance is defined by an n -tuple $\langle t_1, \dots, t_n \rangle$, viz. $\{v_1 \leftarrow t_1, \dots, v_n \leftarrow t_n\}(t)$. Similarly, a matrix

$$G = \begin{pmatrix} t_{11} & \dots & t_{1n} \\ \vdots & & \vdots \\ t_{m1} & \dots & t_{mn} \end{pmatrix}$$

defines m ground instances of t , viz. one for each row. We have the following requirements with respect to G :

1. In each column there occur two terms starting with different function symbols.
2. No two columns are identical.

3. In each column at least one “non-trivial” value occurs (e.g. $\notin \{0, 1\}$, $\notin \{[], [a]\}$).
4. There are not too many rows.
5. No two rows are identical.
6. The t_{ij} are not too large values.

Requirements 1. and 2. ensure that the syntactical anti-unification of all m ground instances yields a variant of t ; 3. enhances the quality of results; 4. to 6. are designed to obtain small runtimes. Fig. 20 gives an impression of the size and number of ground instances that can currently be handled by our implementation. Note that by reimplementing in C and using up-to-date hardware, the runtimes could be improved by about one order of magnitude.

By “quality of results” we mean – informally – the ratio of “desired” terms to overall enumerated terms, weighted somehow by order of appearance. In other words, the earlier a desired term appears, the higher the quality; and the more desired terms appear, the higher the quality. Which terms are considered to be “desired” depends on the application; e.g. in lemma generation, t^r is desired if the equation $t^l = t^r$ is universally valid. Since, in general, infinitely many terms are contained in the result sort, a formal measure of quality can be defined e.g. by the expression $q := \sum_{i=1}^{\infty} d_i \cdot \delta^i$, where $d_i = 1$ if the i -th enumerated term belongs to the desired terms, $d_i = 0$ else, and $0 < \delta < 1$ is some real number. We then always have $0 \leq q \leq \frac{\delta}{1-\delta}$.

Currently, the example ground instances still have to be provided manually. When automating this process, Requirements 1., 2., and 5. are easy to satisfy as they are precisely defined. Requirement 3. would call for special knowledge about the equational theory which is not generally available. We propose dropping this requirement which is vague anyway, and enhancing the results’ quality by adding more instances (i.e. rows), maintaining, however, a balance wrt. the contradicting Requirement 4. On a similarly informal level, Requirement 3. usually contradicts Requirement 6., since, for example, the smallest terms are the nullary function symbols which mostly exhibit special behavior wrt. equationally defined functions like $+$, $*$, app and others.

The more the algorithms speeds can be improved, the less critical the selection of appropriate ground examples becomes.

6.4 Results and Runtimes

Figure 38 shows some generated lemmas together with the required runtimes — cf. also Sect. 3.6. The column “T” shows the background equational theory — cf. Fig. 3. The left-hand side of the equation in the column “Law” was input to the algorithm; the right-hand side was generated; note, for example, the difference between lines 23 and 24. Column “Rhs” indicates the normal forms of the left-hand side’s ground instances. This is considered a measure of the size of the input sorts to be anti-unified. In lines 12 to 24, ε stands for the empty list $[]$, while ab , for example, stands for the list $[a, b]$. Column “Nr” indicates the number of terms from the result sort that were enumerated before the desired right-hand side appeared. Columns “A”, “I”, and “E” show the runtime for anti-unification, intersecting with left-hand side’s variables sort (left empty if rsg_V is used), and enumeration, respectively. Column “ Σ ” shows the total runtime. Since the time resolution was 5 seconds, shorter runtimes appear as “0”. The rightmost column indicates which technical optimizations have been switched on — cf. Fig. 15.

As an example, in line 1, the algorithm was provided with the following instances of the left-hand-side term $v1 + (v2 + v3)$:

$$\begin{aligned} 1 + (0 + 0) &= 1, \\ 0 + (1 + 0) &= 1, \\ 2 + (0 + 1) &= 3. \end{aligned}$$

It thus had to anti-unify the equivalence classes of $suc(0)$, $suc(0)$, and $suc(suc(suc(0)))$ wrt. the given background theory consisting of the four equations defining $+$ and $*$. Anti-Unification took 35 seconds; intersecting the sort of all E-generalizations with the sort of all terms in variables $v1, v2, v3$ took another 55 seconds, and enumerating the result sort took 10 seconds until the desired right hand side $v1 + (v2 + v3)$ appeared.

As can be seen from a comparison of lines 7 and 8, for example, there is a trade-off effect between runtime and the quality of the result which can be controlled by the number and size of input sorts.

	T	Law	Rhs	Nr	A	I	E	Σ	
1	1	$v1 + v2 + v3 = v1 + (v2 + v3)$	1,1,3	6.	35	55	10	100	g
2	1	$v1 + v2 + v3 = v1 + (v2 + v3)$	1,1,3	6.	40		10	50	v
3	1	$v1 * (v2 + v3) = v1 * v2 + v1 * v3$	0,2,2	10.	20	30	30	80	g
4	1	$v1 * (v2 + v3) = v1 * v2 + v1 * v3$	0,2,2	10.	30		10	40	v
5	1	$v1 * v1 + v1 * v2 + v1 * v2 + v2 * v2 = (v1 + v2) * (v1 + v2)$	4,1,4	4.	285	25	70	380	g
6	1	$v1 * v1 + v1 * v2 + v1 * v2 + v2 * v2 = (v1 + v2) * (v1 + v2)$	4,1,4	4.	180		20	200	v
7	1	$v1 * v2 = v2 * v1$	0,0	3.	0	0	0	0	g
8	1	$v1 * v2 * v3 = v1 * (v2 * v3)$	0,0,2	31.	10	5	20	35	g
9	2	$dup(v1) + dup(v2) = dup(v1 + v2)$	2,4	2.	5	20	5	30	g
10	2	$dup(v1) = v1 + v1$	0,4	4.	0	5	0	5	g
11	2	$v1 * dup(v2) = dup(v1 * v2)$	0,0	13.	0	0	5	5	g
12	3	$app(rev(v1), rev(v2)) = rev(app(v2, v1))$	ba,cd,cb	1.	185	5	230	420	g
13	3	$app(rev(v1), rev(v2)) = rev(app(v2, v1))$	ba,cd,cb	1.	90		0	90	v
14	3	$app(rev(v1), rev(v2)) = rev(app(v2, v1))$	ε ,bac,deb	1.	110	5	150	265	g
15	3	$app(rev(v1), rev(v2)) = rev(app(v2, v1))$	ε ,bac,deb	1.	70		0	70	v
16	3	$app(v1, app(v2, v3)) = app(app(v1, v2), v3)$	ab,cd,bd	3.	185	5	260	450	g
17	3	$app(v1, app(v2, v3)) = app(app(v1, v2), v3)$	ab,cd,bd	3.	115		5	120	v
18	3	$app(v1, app(v2, v3)) = app(app(v1, v2), v3)$	ab,cd,bd	3.	?		?	10	gstv
19	3	$app(v1, app(v2, v3)) = app(app(v1, v2), v3)$	abc,bde,cb	3.	855		15	870	v
20	3	$app(v1, app(v2, v3)) = app(app(v1, v2), v3)$	abc,bde,cb	3.	240		10	250	stv
21	3	$app(v1, app(v2, v3)) = app(app(v1, v2), v3)$	abc,bde,cb	1.	50		15	65	gs v
22	3	$app(v1, app(v2, v3)) = app(app(v1, v2), v3)$	abc,bde,cb	1.	20		10	30	gstv
23	3	$v1 = rev(rev(v1))$	ε ,ab	4.	0	0	5	5	g
24	3	$rev(rev(v1)) = v1$	ε ,ab	1.	0	0	5	5	g
25	4	$len(app(v1, v2)) = len(v1) + len(v2)$	1,2	4.	5	10	5	20	g
26	4	$len(v1.app(v2, v3)) = suc(len(v2) + len(v3))$	2,3	10.	40	5	55	100	g

Fig. 38. Lemma-Generation Runtimes

7 Series Guessing

7.1 Implementation

A second application of E-anti-unification is the computation of possible continuations of term sequences, wellknown from intelligence tests. In this section, we discuss a corresponding algorithm. Given a list of terms $[t_n, \dots, t_1]$ and an example count $k \leq n$, the algorithm described below generates the sort s^r of all its possible formation laws wrt. a given equational background theory. Each formation-law term $t' \in s^r$ computes each list member beyond $n - k$ from its predecessors, using only functions of the equational background theory. Note that for technical reasons we reverse the usual list order, i.e. to compute formation laws of the quadratic numbers $0, 1, 4, 9, \dots$, we start with the list $[suc^9(0), suc^4(0), suc(0), 0]$.

We proceed in a similar way to lemma generation (cf. Sect. 6.1). First, we select the k suffixes of length $n - k, \dots, n - 1$ and annotate each one with its length plus one, i.e. with the rank of the respective successor term within the series. We then syntactically anti-unify these length-annotated suffixes:

$$t^l = \frac{\begin{array}{c} [\quad suc^{n-k+1}(0), \quad t_{n-k}, \quad \dots, \quad t_1 \quad] \\ \quad \quad \quad \square \dots \square \\ [\quad suc^n(0), \quad t_{n-1}, \quad \dots, \quad t_k, \quad t_{k-1}, \dots, t_1 \quad] \end{array}}{[\quad suc^{n-k+1}(v_{0\dots k-1}), \quad t'_{n-k\dots n-1}, \quad \dots, \quad t'_{1\dots k}, \quad t'' \quad]}$$

where $t'_{i\dots j} = t_i \square \dots \square t_j$, and $t'' = [] \square [t_1] \square \dots \square [t_{k-1}, \dots, t_1]$. We compute the sort of all E-anti-unifiers $s^r := rsg_V(vars(t^l), [t_{n-k+1}, \dots, t_n])$. Each $t^r \in L(s^r)$ can be interpreted as a series-formation law: applying the term-rewriting rule $t^l \rightsquigarrow t^r$ to a length-annotated suffix $[suc^i(0), t_i, \dots, t_1]$ will result in the next series element t_{i+1} , for $n - k \leq i < n$. As in Sect. 6.1, using rsg_V ensures that $vars(t^r) \subset vars(t^l)$. By intersecting s^r with s_{NF} , we can additionally restrict it to terms in normal form. Fig. 39 gives an example for $n = 4$ and $k = 3$.

Note that t^l is not usually a list of variables, but it contains function symbols. Consequently, the terms from s^r are more complicated than is intuitively expected; e.g. in the example in Fig. 39, the result term considered is $suc(v_1) * suc(v_1)$, whereas the usual way would be to give the series-construction law as $v * v$. Alternatively, one might provide the rsg_V call directly with $V = \{\varphi(suc^{n-k+1}(0), \dots, suc^n(0)), \varphi(t_{n-k}, \dots, t_{n-1}), \dots, \varphi(t_1, \dots, t_k), \varphi([], \dots, [t_{k-1}, \dots, t_1])\}$, thus intentionally over-generalizing. This might require some changes in the grouping algorithm from Sect. 3.5. Moreover, Lemma 14 is no longer valid.

7.2 Results and Runtimes

Figure 40 shows some series laws that have been guessed together with the required runtime. Columns “T”, “Nr”, “A”, “E”, “ Σ ” and the rightmost one are as in Fig. 38. The column “Series” shows the series given to the algorithm; the column “L” shows the number k of example suffixes, i.e. the number of anti-unified input sorts. The column “Law” shows the law that is expected to be generated; v_1 denotes the previous member of series, v_2 the pre-previous, and so on; m denotes the rank within the series. Using the notions from Sect. 7.2, we have $v_1 = t'_{n-k\dots n-1}$, $v_2 = t'_{n-k-1\dots n-2}$, \dots , and $m = suc^{n-k+1}(v_{0\dots k-1})$. Columns “D” and “N” show the runtimes for computing sort depths (cf. Sect. 4), and for intersecting with the normal forms’ sort (cf. Sect. 6.2), respectively. In lines 1 to 9, the enumeration algorithm was restricted to output only terms in normal form. However, this contradicted the technical optimization obtained by using sort depths (cf. Sect. 4). Therefore, in lines 10 to 23, we explicitly intersected the sort of all generalizations with the sort of all normal forms, dropping the normal-form restriction from the enumeration algorithm. The latter method

Given the series $0, 1, 4, 9, \dots$, and $k = 3$

$[suc^3(0), \quad suc^4(0), \quad suc(0), 0]$	$\frac{suc^9(0)}{suc^4(0)}$	$\ni suc^3(0) * suc^3(0)$
$[suc^2(0), \quad suc(0), \quad 0]$	$\frac{suc^4(0)}{suc(0)}$	$\ni suc^2(0) * suc^2(0)$
$[suc(0), \quad 0 \quad]$	$\frac{suc(0)}{1}$	$\ni suc(0) * suc(0)$
<hr/>		
$[suc(v_1), \quad v_2 \mid \quad v_3]$	$L(s_{1,4,9})$	$\ni suc(v_1) * suc(v_1)$

For example, we have $suc(v_1) * suc(v_1) \in L(s_{1,4,9})$;

the corresponding rewrite rule is:

$$[suc(v_1), \quad v_2 \mid \quad v_3] \rightsquigarrow suc(v_1) * suc(v_1)$$

This rule rewrites:

$$\begin{aligned} [suc(0), \quad 0 \quad] &\rightsquigarrow suc(0) \\ [suc^2(0), \quad suc(0), \quad 0] &\rightsquigarrow suc^4(0) \\ [suc^3(0), \quad suc^4(0), \quad suc(0), 0] &\rightsquigarrow suc^9(0) \\ [suc^4(0), \quad suc^9(0), suc^4(0), suc(0), 0] &\rightsquigarrow suc^{16}(0) \\ \dots & \end{aligned}$$

Fig. 39. Example: Series Guessing

makes sort enumeration dead-end-free, and is thus faster for all but very small examples — cf. lines 9 and 10. An ∞ in column “Nr” indicates that the desired law has not been enumerated among, say, the first 100 terms. In this case, the column “Σ” shows a “>”.

Since t^l is not usually a list of variables, but contains function symbols, m, v_1, v_2, \dots from Fig. 40 do, in fact, stand for proper terms. For example, in line 23, we have $v_1 = suc^2(x)$, and the enumerated law, viz. $suc(x)$, has been abbreviated to $p(v_1)$, the predecessor of v_1 . The case is similar in line 21 — cf. also Fig. 42.

Some remarks on lines 13 to 16:

The desired solution $if(ev(suc(m)), suc^3(m), suc(0))$ has the depth 5. Hence, all solutions of depth 4 are enumerated before the desired one, the first being $ev(x_1) * (x_3 + x) + if(if(x, x_2, x_1), suc(x_3), ev(x))$, which is, in fact, considered a “correct” solution — see Fig. 41, where each column corresponds to a subterm of the solution (the marked column corresponds to the whole term), irrelevant numbers are printed in small type. The solution $ev(x_1) * suc(x_2) + suc(ev(x_1))$ is shorter and of the same depth, but not in normal form. The shortest solution seems to be $suc(ev(x_1) * suc^2(x_2))$, which is deeper and not in normal form. If the depth measure of the enumeration algorithm is changed such that unfolding a sort definition is counted only if the head symbol of the right-hand side is not in $\{0, s\}$, the desired solution is of depth 2 and is thus enumerated first.

Some remarks on line 18:

While the function $ev(\cdot)$ allows one to distinguish series members with even and odd rank, no function is available in background theory (5) to distinguish ranks 0, 1, and 2 mod. 3. For this reason, the series in line 18 was expected by the authors to have no solution at all when given to the algorithm. Note that the trivial solution v_3 was prevented by setting $L = 4$, thus forcing the algorithm to compute the first 1 from the two preceding 0’s. However, the algorithm found the solution $ev(v_1 + v_2)$, which is “correct”. This seems to indicate a possible strength of the algorithm: building series laws from a well-defined limited set of functions rather than predicting the next series member(s).

Figure 42 gives some enumerated series laws. The column “T” indicates the background theory used. In the column “Series”, the semicolon indicates the number of example suffixes: one for each series member to the right of the semicolon. Column “D” shows the depth up to which the shown laws appeared. An “ ∞ ” means that the result sort is finite and is given completely in the column “Laws”;

	T	Series	Law	L	Nr	A	D	N	E	Σ	
1	1	0, 1, 4	$m * m$	3?	1.	0			0	0	gstv
2	1	0, 1, 4	$m * m$	3	1.	0	0		0	0	dgs v
3	1	0, 2, 4, 6	$suc(suc(v_1))$	3?	1.	15			20	35	gstv
4	1	0, 2, 4, 6	$suc(suc(v_1))$	3	1.	35	85		0	120	dgs v
5	1	0, 2, 4, 6	$m + m$	3?	3.	15			405	420	gstv
6	1	0, 2, 4, 6	$m + m$	3	3.	35	85		0	120	dgs v
7	1	1, 1, 2, 3, 5	$v_1 + v_2$	3?	1.	10			225	235	gstv
8	1	1, 1, 2, 3, 5	$v_1 + v_2$	3	1.	15	15		0	30	dgs v
9	5	1, 1, 2, 3, 5	$v_1 + v_2$	3		45	10		∞	∞	dgs v
10	5	1, 1, 2, 3, 5	$v_1 + v_2$	3	1.	45	15	150	0	210	dgs v
11	5	1, 1, 2, 3, 5	$v_1 + v_2$	3	1.	45	5	100	0	150	bdgs v
12	5	1, 1, 2, 3, 5	$v_1 + v_2$	4	∞	200	55	610	-	>865	bdgs v
13	5	0, 1, 2, 1, 4, 1, 6	$if(ev(m), m, 1)$	3	∞	60	20	190		>270	dgs v
14	5	0, 1, 2, 1, 4, 1, 6	$if(ev(m), m, 1)$	4	∞	255	285	160		>745	dgs v
15	5	0, 1, 2, 1, 4, 1, 6	$if(ev(m), m, 1)$	4	∞	200	50	100		>350	bdgs v
16	5	0, 1, 2, 1, 4, 1	$if(ev(m), m, 1)$	4	13.	125	25	170	0	320	bdgs v
17	5	0, 0, 1, 1, 0, 0, 1, 1	$ev(v_2)$	5	1.	500	155	1500	0	2155	dgs v
18	5	0, 0, 1, 0, 0, 1	$ev(v_1 + v_2)$	4	1.	75	15	210	0	300	dgs v
19	5	0, 1, 3, 7	$suc(v_1 + v_1)$	3	1.	30	5	95	0	130	dgs v
20	5	1, 2, 2, 3, 3, 3, 4, 4, 4, 4	-	3	∞	60	20	205	-	>285	dgs v
21	5	2, 3, 5, 9	$p(v_1 + v_1)$	3	1.	145	55	525	0	725	dgs v
22	5	1, 2, 3, 4, 5, 6	$suc(m)$	3	1.	125	40	425	0	590	dgs v
23	5	6, 5, 4, 3, 2, 1	$p(v_1)$	3	1.	15	0	40	0	55	dgs v

Fig. 40. Series-Guessing Runtimes

We have

$$m = suc^3(x) \quad + \quad *$$

$$v_1 = suc(x_1) \quad ev \quad + \quad (if \quad s \quad ev)$$

$$v_2 = suc(x_2) \quad (x_1) \quad (x_3 \quad x) \quad (x, x_2, x_1), \quad (x_3), \quad (x)$$

$$v_3 = x_3$$

cf. line 16

in Fig. 42

0	1	0	0	0	0	1	1	1	0	0	1	1	0	1	0
1	0	2	1	2	1	4	2	1	1	1	0	2	1	0	1
0	3	0	2	4	2	1	1	0	2	0	3	3	2	1	2
1	0	4	1	4	3	6	2	3	3	3	0	2	1	0	3
0	5	0	4	8	4	1	1	0	4	0	5	5	4	1	4
1	0	6	1	6	5	8	2	5	5	5	0	2	1	0	5
0	7	0	6	12	6	1	1	0	6	0	7	7	6	1	6
1	0	8	1	8	7	10	2	7	7	7	0	2	1	0	7

Fig. 41. Simplest Series Law for 0, 1, 2, 1, 4, 1, 6, ...

	T	Series	D	$m:v_1, v_2, \dots$	Laws
1	1	;0,1,4	∞	$x_1:$	$x_1 * x_1$
2	1	1,2,3;4,5,6	∞	$suc^3(x_1):suc^3(x_1), \dots$	$suc^4(x_1)$
3	1	0;2,4,6	∞	$suc(x_1):x_2$	$suc^2(x_2)$
4	1	0,1,2,1;4,1,6	∞	$suc^4(x_1):suc(x_2) \dots$	$suc^2(x_1 + x_1)$
5	1	1,1;2,3,5	2	$suc^2(x_1):suc(x_1), suc(x_2)$	$suc^2(x_2 + x_1)$ $suc^2(x_2 * x_2 + x_1)$
6	1	1,1,2;3,5,8	2	$suc^3(x_1):suc^2(x_2), suc(x_1), suc(x_3)$	$suc^3(x_3 + (x_1 + x_1))$ $suc^3(x_2 + x_1)$
7	1	0;1,3,7	5	$suc(x_1):x_2$	$suc(x_1 + x_1 * x_1)$ $suc(x_2 + x_2)$
8	1	2,3;5,9	2	$suc^2(x_1):suc^3(x_2), suc^2(x_1)$	$suc^5(x_2 * x_2)$ $suc^5(x_2 + x_2)$
9	6	0;1,4	4	$suc(x_1):x_1$	$suc(x_1 + x_1 + x_1)$ $x_1 + x_1 + x_1 + (suc(x_1) - x_1)$
10	7	;0,1,4	5	$x_1:x_2$	$x_1 * x_1$ $3 * x_1^3 - x_1^2 - x_1^4$
11	7	0,0,1;0,0,1	3	$suc^3(x_1):x_2, x_3, x_4$	x_4 $(x_4 + x_3) * (x_4 - x_2)$
12	5	;0,1,4	3	$x_1:$	$x_1 * x_1$ $x_1 * if(x_1, x_1, 0)$
13	5	1,1;2,3,5	4	$suc^2(x_1):suc(x_1), suc(x_2)$	$suc(suc(x_1 + x_2))$ $suc(if(x_1, x_1 + x_1, suc(0)))$
14	5	0,0,1,1;0,0,1,1	3	$suc^4(x_1):x_2, x_3, x_4, x_5$	x_5 $ev(x_3)$
15	5	0,0;1,0,0,1	3	$suc^2(x_1):x_2, x_3$	$ev(x_2 + x_3)$ $ev(if(x_2, x_1, x_3))$
16	5	0,1,2,1;4,1,6	4	$suc^4(x_1):suc(x_2), suc(x_3), suc(x_4), x_1$	$(ev(x_1) + if(x_1, x_4, x_3)) * (ev(x_1) + (x_3 + x_1))$
17	8	abc;bca,cab	∞	$suc(x_1):[x_2, x_3, x_4]$	$[x_4, x_2, x_3]$
18	8	abab,aba;ab,a	∞	$suc^2(x_1):[a, b x_2], [a, b, a x_3]$	
19	8	baba,aba;ba,a	2	$suc^2(x_1):[x_2, x_3 x_4], \dots$	$il([x_3], x_4)$ $[x_3 rev(x_4)]$
20	8	a,ba;aba,baba	2	$suc^2(x_1):[x_2, x_3 x_4], [x_3 x_4]$	$[x_3, x_2, x_3 x_4]$ $il([x_3, x_3], app(x_4[x_2]))$

Fig. 42. Some Enumerated Series Laws

in all other cases, an arbitrary selection of the enumerated laws is shown. The column “ $m : v_1, v_2, \dots$ ” shows the result of syntactical anti-unification of examples.

Note that in lines 1 to 4, and 17 to 18, the result sort is finite, and precise statements can be made about the series and their possible continuations. For example, from line 3 we can conclude that there are only two ways to build the series 0, 2, 4, 6 from functions $+$ and $*$. From line 4 we can conclude that there is no way to build 0, 1, 2, 1, 4, 1, 6 from these functions.

8 Other Potential Applications

In this section, we look at some other potential applications of E-anti-unification which we have considered but not implemented.

8.1 Divergence Handling in Knuth–Bendix Completion

The Knuth–Bendix completion procedure is an important tool in equational reasoning. However, its applicability is limited by the possibility of its generating an infinite sequence of term-rewriting rules. Kirchner [Kir87,Kir89] proposes schematizing such a sequence using meta variables; Avenhaus [Ave89] proposes consistently introducing new function symbols and new equations to achieve termination of the Knuth–Bendix completion.

In any approach, it is necessary to recognize schemata in rule or term sequences. Kirchner proposes using generalization modulo an equivalence relation to find useful schemata candidates, but has not developed any approaches to anti-unification modulo equational theories. Nevertheless, useful equational candidates could be suggested by using the mechanisms for anti-unification presented in this paper.

Below, we sketch a possible approach for finding useful schemas using Exm. 2.1 from [Kir89]. Consider the rewrite rules shown in Fig. 43 axiomatizing the “signed binary tree theory”. From these rules, the completion procedure generates two infinite families of rules shown in Fig. 44. The rightmost column shows the origin of each rule. For example, Rule 5 arose from superimposing Rule 4 at position 1 on Rule 2 at the root, denoted by ε . Kirchner states in Example 2.3 that the first and second family of rules can be schematized by the meta rule $\forall X, Y \quad f(-X, f(X, Y)) \rightsquigarrow Y$ and $\forall X, Y \quad f(f(Y, X), -X) \rightsquigarrow Y$, respectively, modulo the equational theory consisting of Rules 1 and 2 from Fig. 43.

$$\begin{array}{ll}
 1 & - - x \rightsquigarrow x \\
 2 & - f(x, y) \rightsquigarrow f(-y, -x) \\
 3 & f(f(y, x), -x) \rightsquigarrow y \\
 4 & f(-x, f(x, y)) \rightsquigarrow y
 \end{array}$$

Fig. 43. Input Rewrite Rules from Kirchner’s Example 2.1

We now focus on the first family and discuss the problems that arise when we try to reduce the detection of the first meta rule to a series-guessing task like the ones described in Sect. 7.1.

Finding appropriate subsequences. A fair completion strategy will not produce the rules in the order shown in Fig. 44; rather, it will produce a sequence of rules interleaving both rule families. Thus we are left with the task of detecting subsequences that may obey common schemes. To achieve this, we propose applying the series-guessing algorithm for the empty equational theory to the origin information.

Each rule generated by the completion process can be assigned an “origin term” constructed from the numbers of the input rules as constants and a 4-ary operator o (for “overlay”). For example, Rules 4, 5, 9, and 10 from Figs. 43 and 44 are represented by the origin terms 4, $o(4, \varepsilon.1, 2, \varepsilon)$, $o(o(4, \varepsilon.1, 2, \varepsilon), \varepsilon.1.1, 2, \varepsilon)$, and $o(o(o(4, \varepsilon.1, 2, \varepsilon), \varepsilon.1.1, 2, \varepsilon), \varepsilon.1.1.1, 2, \varepsilon)$, respectively. Note that the path concatenation operator “.” associates to the left. Applying the series-guessing algorithm to this term sequence, and setting, say, the example count $k = 2$, we obtain one single law, viz. $[suc(suc(v_1)), o(v_2, v_3.1, 2, \varepsilon), v_2[v_4] \rightsquigarrow o(o(v_2, v_3.1, 2, \varepsilon), v_3.1.1, 2, \varepsilon)]$. If, by way of a negative example,

5	$f(f(-x_2, -x_1), f(f(x_1, x_2), y)) \rightsquigarrow y$	$o(4, \varepsilon.1, 2, \varepsilon)$
6	$f(f(-x_2, x_1), f(f(-x_1, x_2), y)) \rightsquigarrow y$	$o(5, \varepsilon.1.2, 1, \varepsilon)$
7	$f(f(x_2, -x_1), f(f(x_1, -x_2), y)) \rightsquigarrow y$	$o(5, \varepsilon.1.1, 1, \varepsilon)$
8	$f(f(x_2, x_1), f(f(-x_1, -x_2), y)) \rightsquigarrow y$	$o(6, \varepsilon.1.1, 1, \varepsilon)$ or $o(7, \varepsilon.1.2, 1, \varepsilon)$
9	$f(f(f(-x_3, -x_2), -x_1), f(f(x_1, f(x_2, x_3)), y)) \rightsquigarrow y$	$o(5, \varepsilon.1.1, 2, \varepsilon)$
	...	
10	$f(f(f(f(-x_4, -x_3), -x_2), -x_1), f(f(x_1, f(x_2, f(x_3, x_4))), y)) \rightsquigarrow y$	$o(9, \varepsilon.1.1.1, 2, \varepsilon)$
	...	
11	$f(f(y, f(x_1, x_2)), f(-x_2, -x_1)) \rightsquigarrow y$	$o(3, \varepsilon.1.2, 2, \varepsilon)$
12	$f(f(y, f(-x_1, x_2)), f(-x_2, x_1)) \rightsquigarrow y$	$o(11, \varepsilon.1.2.1, 1, \varepsilon)$
13	$f(f(y, f(x_1, -x_2)), f(x_2, -x_1)) \rightsquigarrow y$	$o(11, \varepsilon.1.2.2, 1, \varepsilon)$
14	$f(f(y, f(-x_1, -x_2)), f(x_2, x_1)) \rightsquigarrow y$	$o(12, \varepsilon.1.2.2, 1, \varepsilon)$ or $o(13, \dots, 1, \varepsilon)$
15	$f(f(y, f(x_1, f(x_2, x_3))), f(f(-x_3, -x_2), -x_1)) \rightsquigarrow y$	$o(11, \varepsilon.1.2.2, 2, \varepsilon)$
	...	

Fig. 44. Completion of the Theory from Fig. 43

we apply the series-guessing algorithm to Rules 4, 5, 11, 15, we get no law at all, thus indicating that this sequence does not belong to a rule family.

Since series guessing wrt. the empty theory is quite fast, one might check all possible ascending rule sequences whether their origin terms reveal a common law or not. Some optimization may be achieved using the fact that a sequence has no law if any of its prefixes has no law. Moreover, efficient data structures like substitution tree indexes [Gra94] might help to find all rule subsequences that have a law.

Applying series guessing to a selected rule subsequence. Once we have selected one or more rule sequences as described above, we may apply the series-guessing algorithm to the rules. Each series law thus obtained will correspond to a meta rule that generalizes all rules of the subsequence modulo the chosen equational background theory. For example, in a first attempt, we applied series guessing to the rules shown in Fig. 45, however without success. Figure 46 documents a successful attempt at series guessing for both $k = 2$ and $k = 3$, taking 17 and 28 seconds user time, respectively.

Dealing with variable names. Since the E-anti-unification approach presented here is based on regular tree grammars, we treated variables as constants. This is no longer sufficient for the present completion example: if the variables are named as in Figs. 43 and 44, the syntactical anti-unification of length-annotated suffixes will yield the variables shown in Fig. 45. Note that x_4 does not occur in any argument of φ ; in particular, a variable $v = \varphi(-x_3, f(-x_4, -x_3))$ is missing. This is the reason for the non-existence of a series law in Fig. 45. If we number the variables x_1 to x_4 in reverse order, we have a similar problem, since then $\varphi(x_3, x_4)$ is missing.

In a second attempt, we tried to circumvent this problem by modeling variables as functions of natural numbers, i.e. $x(i)$ for x_i , where i denotes $\text{succ}^i(0)$ for the sake of brevity. The rule sequence and the corresponding φ is shown in Fig. 46. Note that we also had to reverse the numbering of variables, since otherwise $\varphi(x(3), f(x(3), x(4)))$ would again be missing.

What is needed, though, is an approach to E-anti-unification that is able to deal with variables rather than just constants.

Setting up the equational background theory's tree grammar. The main problem here consists in determining an appropriate equational background theory in the first place. In the above example,

Input Series:

$$\begin{aligned} & [f(f(f(f(-x_4, -x_3), -x_2), -x_1), f(f(x_1, f(x_2, f(x_3, x_4))) , y)) \rightsquigarrow y , \\ & f(f(f(-x_3, -x_2), -x_1), f(f(x_1, f(x_2, x_3)) , y)) \rightsquigarrow y , \\ & f(f(-x_2, -x_1), f(f(x_1, x_2) , y)) \rightsquigarrow y , \\ & f(-x_1, f(x_1, y)) \rightsquigarrow y] \end{aligned}$$

Variables ($k = 2$):

$$\begin{aligned} v_1 &= \varphi(0, suc(0)) \\ v_2 &= \varphi(-x_2, f(-x_3, -x_2)) \\ v_3 &= \varphi(x_2, f(x_2, x_3)) \\ v_4 &= \varphi(-x_1, f(-x_2, -x_1)) \\ v_5 &= \varphi(x_1, (f x_1, x_2)) \\ v_6 &= \varphi([], [f(-x_1, f(x_1, y)) \rightsquigarrow y]) \end{aligned}$$

No laws found ($k = 2, k = 3$)

Fig. 45. Series-Guessing Approach (A) Applied to the Rule Sequence 4, 5, 9, 10

Input Series:

$$\begin{aligned} & [f(f(f(f(-x(1), -x(2)), -x(3)), -x(4)), f(f(x(4), f(x(3), f(x(2), x(1)))) , y)) \rightsquigarrow y , \\ & f(f(f(-x(2), -x(3)), -x(4)), f(f(x(4), f(x(3), x(2))) , y)) \rightsquigarrow y , \\ & f(f(-x(3), -x(4)), f(f(x(4), x(3)) , y)) \rightsquigarrow y , \\ & f(-x(4), f(x(4), y)) \rightsquigarrow y] \end{aligned}$$

Variables ($k = 2$):

$$\begin{aligned} v_1 &= \varphi(0, 1) \\ v_2 &= \varphi(-x(1), f(-x(1), -x(2))) \\ v_3 &= \varphi(x(1), f(x(2), x(1))) \\ v_4 &= \varphi([], [\dots]) \end{aligned}$$

Variables ($k = 3$):

$$\begin{aligned} v_1 &= \varphi(0, 1, 2) \\ v_2 &= \varphi(-x(1), f(-x(1), -x(2)), f(f(-x(1), -x(2)), -x(3))) \\ v_3 &= \varphi(x(1), f(x(2), x(1)), f(x(3), f(x(2), x(1)))) \\ v_4 &= \varphi([], [\dots], [\dots]) \end{aligned}$$

Laws found ($k = 2$):

$$\begin{aligned} & f(f(f(v_2, -x(suc^2(v_1))), -x(suc^3(v_1))), f(f(x(suc^3(v_1)), f(x(suc^2(v_1)), v_3)), y)) \rightsquigarrow y \\ & f(f(f(v_2, -x(suc^2(v_1))), -x(suc^3(v_1))), f(f(x(suc^3(v_1)), f(x(suc^2(v_1)), -v_2)), y)) \rightsquigarrow y \\ & f(f(f(-v_3, -x(suc^2(v_1))), -x(suc^3(v_1))), f(f(x(suc^3(v_1)), f(x(suc^2(v_1)), v_3)), y)) \rightsquigarrow y \\ & f(f(f(-v_3, -x(suc^2(v_1))), -x(suc^3(v_1))), f(f(x(suc^3(v_1)), f(x(suc^2(v_1)), -v_2)), y)) \rightsquigarrow y \end{aligned}$$

Laws found ($k = 3$):

$$\begin{aligned} & f(f(v_2, -x(suc^2(v_1))), f(f(x(suc^2(v_1)), v_3), y)) \rightsquigarrow y \\ & f(f(v_2, -x(suc^2(v_1))), f(f(x(suc^2(v_1)), -v_2), y)) \rightsquigarrow y \\ & f(f(-v_3, -x(suc^2(v_1))), f(f(x(suc^2(v_1)), v_3), y)) \rightsquigarrow y \\ & f(f(-v_3, -x(suc^2(v_1))), f(f(x(suc^2(v_1)), -v_2), y)) \rightsquigarrow y \end{aligned}$$

Fig. 46. Series-Guessing Approach (B) Applied to the Rule Sequence 4, 5, 9, 10

$$\begin{array}{lcl}
\frac{X \rightsquigarrow Y}{f(X, Y)} & \doteq & \frac{X \rightsquigarrow Y}{f(\underline{X}, \underline{Y})} \mid - - f(X, Y) \mid - f(\underline{-Y}, \underline{-X}) \\
\frac{-f(X, Y)}{- - X} & \doteq & \frac{f(-Y, -X)}{\underline{X}} \\
\frac{-X}{\underline{X}} & \doteq & -\underline{X} \mid - - \underline{-X} \\
\frac{\underline{X}}{\underline{X}} & \doteq & X \mid - - \underline{X}
\end{array}$$

Fig. 47. Tree-Grammar Scheme for Equivalence Classes in Kirchner's Example 2.1

we simply took the theory given in [Kir89], consisting of Rules 1 and 2 from Fig. 43. In general, though, it is not at all clear which background theory is appropriate. In Kirchner's examples, it is always taken from the input rules. In many other examples, it simply defines the repeated application of certain functions; e.g., for the completion example³ in Fig. 48, it is sufficient to define $b^i(x)$ by $b^0(x) = x$ and $b^{s(i)}(x) = b(b^i(x))$, and to define $d^i(x)$ similarly. This is a special case of ω terms which have been investigated to deal with infinite sets of terms occurring in Knuth-Bendix completion or elsewhere [Com95]. It seems to be a good default measure to use a theory of appropriate ω terms as the equational background theory.

Once the background theory itself is fixed, the next problem is how to find an appropriate regular tree grammar for the equivalence classes modulo this theory. In the above example, this was the most time-consuming and error-prone task. Figure 47 shows the tree-grammar scheme (cf. Sect. 9) we eventually came up with; the rules are prioritized PROLOG facts. Note that the approach of [Emm94] mentioned in Sect. 3.1 is not applicable, since it requires a confluent and noetherian term-rewriting system, which is precisely what the completion process itself is about to build. Except for the cases where the background theory is based on ω terms, a satisfactory approach to automatically obtaining a tree grammar for the equivalence classes has yet to be found.

Considering equations in anti-unifying length-annotated suffixes. A closer look at the above completion example reveals a flaw in our approach to series guessing as presented in Sect. 7.1. Consider the variables v_2 and v_3 in Fig. 46. Their pre-images under φ are inverse to each other modulo the background equational theory. Hence, it was desirable to establish the relation $v_2 = -v_3$, and to delete v_3 from the set V of variables that may appear in a series law, thus avoiding certain redundant series laws. In fact, the set of enumerated laws for $k = 2$ collapses into the singleton set

$$\{f(f(f(v_2, -x(suc^2(v_1))), -x(suc^3(v_1))), f(f(x(suc^3(v_1)), f(x(suc^2(v_1)), -v_2)), y)) \rightsquigarrow y\}$$

modulo the above relation; the case is similar for $k = 3$. The reason for the appearance of both v_2 and v_3 is that we did not consider the equational background theory when anti-unifying the length-annotated suffixes. However, by simply E-anti-unifying them using the algorithm from Sect. 3.2 / 3.3, we would still get both v_2 and v_3 in the variables set V .

8.2 Hoare Invariants

When verifying imperative programs, a still unresolved problem is how to generate loop invariants automatically. We show below that E-anti-unification could be used for this purpose.

Consider the following imperative program:

³ This example was provided by Jörg Denzinger, University of Kaiserslautern.

Ordering LPO: $c \succ b \succ a \succ d \succ f \succ e$

Input Equations:

- 1 $a(c(x)) \rightsquigarrow x$
- 2 $c(f(x)) \rightsquigarrow e(x)$
- 3 $a(e(x)) \rightsquigarrow f(x)$

Completion:

- 4 $c(d(x)) \rightsquigarrow b(c(x))$
- 5 $a(b(c(x))) \rightsquigarrow d(x)$
- 6 $a(b(e(x))) \rightsquigarrow d(f(x))$
- 7 $a(b(b(c(x)))) \rightsquigarrow d(d(x))$
- 8 $a(b(b(e(x)))) \rightsquigarrow d(d(f(x)))$
- 9 $a(b(b(b(c(x))))) \rightsquigarrow d(d(d(x)))$
- 10 $a(b(b(b(e(x))))) \rightsquigarrow d(d(d(f(x))))$
- 11 $a(b(b(b(b(c(x))))) \rightsquigarrow d(d(d(d(x))))$
- 12 $a(b(b(b(b(e(x))))) \rightsquigarrow d(d(d(d(f(x)))))$
- 13 $a(b(b(b(b(b(c(x))))) \rightsquigarrow d(d(d(d(d(x)))))$
- 14 $a(b(b(b(b(b(e(x))))) \rightsquigarrow d(d(d(d(d(f(x)))))$
- 15 $a(b(b(b(b(b(b(c(x))))) \rightsquigarrow d(d(d(d(d(d(x)))))$
- 16 $a(b(b(b(b(b(b(e(x))))) \rightsquigarrow d(d(d(d(d(d(f(x)))))$
- ...

Fig. 48. Divergent Completion Schematizable using ω Terms

```

x := 0;
for i:=0 by 1 to n
do  x:=x+2*i
done

```

By anti-unifying sample values of $\langle i, x \rangle$ for a couple of iterations, e.g. $\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 4 \rangle, \langle 3, 9 \rangle$ modulo an appropriate theory (which should define at least $+$ and $*$), we obtain the candidate $x = i * i$ for the loop invariant. Note that $0 \leq i \leq n$ cannot be obtained since it is not an equation.

8.3 Reengineering of Functional Programs

When programming software, it is often difficult to attain an adequate level of abstraction at the first try. Consequently, one often has different procedures that do almost — but not quite — the same thing. E-anti-unification could potentially help in reengineering such programs by computing a most specific generalization.

By way of an example, consider a functional program where two almost identical procedures $f_1(\cdot)$ and $f_2(\cdot)$ are defined:

$$\begin{aligned}
f_1(x) &:= F[x + 1] \\
f_2(x) &:= F[2 * x] \\
g(x) &:= G[f_1(x_1) , f_2(x_2)]
\end{aligned}$$

where $F[\cdot]$ denotes the possibly complex body expression common to both procedures; the third line indicates calls to f_1 and f_2 in the body $G[\cdot]$ of another procedure g . Purely syntactic anti-unification of the procedure bodies of f_1 and f_2 yields $F[y]$, leading to the following reengineered program:

$$\begin{aligned} f(x, y) &:= F[y] \\ g(x) &:= G[f(x_1, x_1 + 1) , f(x_2, 2 * x_2)] \end{aligned}$$

Note that the calls to $+$ and $*$ have been moved outward since they are considered to have nothing in common. Using E-anti-unification modulo the theory of $+$ and $*$, we get a more specific generalization of the bodies, viz. $F[a * x + b]$, leading to a better factorization of the commonalities of f_1 and f_2 :

$$\begin{aligned} f(x, a, b) &:= F[a * x + b] \\ g(x) &:= G[f(x_1, 1, 1) , f(x_2, 2, 0)] \end{aligned}$$

In practice, the literally identical part $F[\cdot]$ of two procedure bodies will usually be small, or even non-existent. It is thus of great importance to be able to deal with commonalities that are revealed only by applying a background theory about programming language semantics. To the extent that this theory is an equational one and admits regular equivalence classes, our E-anti-unification approach can be employed to detect the commonalities.

8.4 Strengthening of Induction Hypotheses

It is well known that sometimes a formula F cannot be proven by induction, while a stronger formula G such that $G \Rightarrow F$ is provable. The reason is that a stronger formula also provides a stronger induction hypothesis. In inductive theorem proving, it is therefore a common technique to strengthen F if its original proof attempt failed. This is usually done by replacing some constants or terms by variables, or by decoupling different occurrences of the same variable; in other words, by building anti-instances of F . The main problem is not to overgeneralize F such that it is no longer valid.

There are several heuristics known from the literature, e.g. [BM79, Hum90, BvHSI90]. Manna and Waldinger [MW80] propose using anti-unification, based on the data acquired during the failed original proof attempt, to find the least strengthening of F that has a chance of being proven inductively. However, they failed to notice that their example does in fact need anti-unification modulo an equational theory.

Consider the equational theory and the induction proof attempt shown in Fig. 49. Manna and Waldinger suggest anti-unifying the actual goal with the induction hypothesis if simple rewriting with it does not lead to success. This way, overgeneralization could be avoided, since anti-unification computes the most specific anti-instance of the induction hypothesis that is just general enough to make it applicable. Backtracking to 9., we obtain by E-anti-unification with the induction hypothesis:

$$\begin{array}{lll} \text{rev_a}(t, [h]) & = \text{app}(\text{rev}(t), [h]) & 9. \\ \frac{\text{rev_a}(l, []) = \text{rev}(l) = \text{app}(\text{rev}(l), []) \quad \text{I.H. (7.), and } l = \text{app}(l, [])}{\text{rev_a}(t', l') = \text{app}(\text{rev}(t'), l')} & \text{strengthened I.H.} \end{array}$$

It is well known that the proof succeeds using the strengthened induction hypothesis.

In order to do the E-anti-unification shown above, we need to define equivalence classes of non-ground terms as regular tree languages. In our example, we get the grammar scheme shown in Fig. 50.

In the example, it is sufficient to syntactically anti-unify the left-hand sides, and to provide the variable set $\{t', l'\}$ where $t' = \varphi(t, l)$ and $l' = \varphi([h], [])$ to rsg_V of the right-hand sides, viz. $\text{rsg}_V(\{t', l'\}, \underline{\text{app}(\text{rev}(t), [h])}, \underline{\text{rev}(l)})$.

1. $app([], l) = l$
 2. $app([h | t], l) = [h | app(t, l)]$
 3. $rev([]) = []$
 4. $rev([h | t]) = app(rev(t), [h])$
-
5. $rev_a([], l) = l$
 6. $rev_a([h | t], l) = rev_a(t, [h] | l)$

Equations 1. to 4. are given as background theory, defining *append* and *reverse*; equations 5. and 6. are the usual optimized version of *reverse* using an accumulator. The goal is to show the correctness of *rev_a*, i.e. to show

7. $rev_a(l, []) = rev(l)$

We try a proof by induction on l ; $l = []$ is trivial; in case $l = [h | t]$, we get the following goal sequence:

8. $rev_a([h | t], []) = rev([h | t])$
9. $rev_a(t, [h]) = app(rev(t), [h])$ by 6. and 4.
10. $rev_a(t, [h]) = app(rev_a(t, []), [h])$ by I.H. (7.)

Fig. 49. Failed Induction–Proof Attempt

$$\begin{aligned}
\frac{rev(A)}{app(A, B)} &\doteq \frac{rev(\underline{A}) | app([], rev(A)) | app(rev(A), [])}{app(\underline{A}, \underline{B}) | app(app(A, B), []) | app([], app(A, B)) | rev(rev(app(A, B)))} \\
&\doteq \underline{A} | app([], \underline{A}) | app(\underline{A}, []) | rev(rev(A)) \quad \text{if } app(\dots) \neq A \neq rev(\dots)
\end{aligned}$$

Fig. 50. Equivalence Classes of Non–Ground Terms mod. *append*, *reverse*

The desired solution is the one enumerated first, and it is found in 1 second user time. We list all enumerated terms at depth level 3, which is the minimum level of the result sort:

$$app(rev(t'), l'), \quad app(rev(t'), rev(l')), \quad app(rev(t'), app(l', [])), \quad app(rev(t'), app([], l')).$$

Note that the second term is not a correct solution, the reason being that instances of l' , viz. $[h]$ and $[]$, are both “trivial” values in the sense of requirement 3. from Sect. 6.3, i.e. they satisfy $rev(x) = x$. The third and fourth solution are correct, but not in normal form; they would have been filtered out by a normal form filter.

9 Dynamic Sort Generation

For many background equational theories, some of the equivalence classes follow a common schema. For example, consider Theory (3) from Fig. 3, defining *append* and *reverse* of lists. The equivalence class of any list $[A, B]$ can be defined by

$$[A, B] \doteq [\underline{A}, \underline{B}] | app([\underline{A}, \underline{B}], []) | app([\underline{A}], [\underline{B}]) | app([], [\underline{A}, \underline{B}]) | rev([\underline{B}, \underline{A}]),$$

irrespective of the value of A and B . We thus get a schema of sort definitions abbreviating any of its instances. This can be implemented in PROLOG by a corresponding fact containing A and B as PROLOG variables.

$$\begin{aligned}
(\underline{0} &\doteq 0 \mid \underline{0} + \underline{0}). \\
(\underline{suc}(N) &\doteq \underline{suc}(\underline{N}) \mid Sums) && \leftarrow all_sums(0, N, Sums). \\
all_sums(I, 0, \underline{I} + \underline{0}). \\
all_sums(I, \underline{suc}(J), \underline{I} + \underline{suc}(J) \mid Sums) &\leftarrow all_sums(\underline{suc}(I), J, Sums).
\end{aligned}$$

Fig. 51. Sort Definition Scheme for Background Theory (0) from Fig. 3

Moreover, we may even use PROLOG rules to dynamically generate sort definitions as needed. For example, the schema shown in Fig. 51 will generate the equivalence class of each natural number modulo the theory of (+). The predicate $all_sums(I, J, Sums)$ computes a sort disjunction

$$Sums = \underline{I} + \underline{J} \mid \underline{I+1} + \underline{J-1} \mid \dots \mid \underline{I+J} + \underline{0}.$$

Similarly, the theory of (+) and (*) has been schematized, requiring a predicate that computes all factors of a given number.

10 Depth-Bounded E-Anti-Unification

In most applications of E-anti-unification, one is interested only in the first few enumerated terms, which are the simplest wrt. the employed depth measure. In such cases, it may be sufficient to cut off the rsg_V algorithm each time it reaches a given depth, thus avoiding the computation of larger solutions that will subsequently be ignored anyway. This way, even large terms can be E-anti-unified.

For example, consider the series 2, 4, 16, 256. The series-prediction algorithm from Sect. 7.1 with example count $k = 2$ will call $rsg_V(\{v_1, v_2, v_3, v_4\}, s_{16}, s_{256})$, where $v_1 = \varphi(16-4, 0)$, $v_2 = \varphi(4-2, 0)$, $v_3 = \varphi([2], [])$, and $v_4 = \varphi(4-3, 0)$.

The first enumerated solution should be $suc^4(v_1) * suc^4(v_1)$, which has depth 5, or depth 1 if suc is not counted. In any case, it is not necessary to descend to depth 256 to obtain this solution.

In this example, however, owing to the growing length of sort definitions caused by (+), the search-tree breadth is too large to get reasonable runtimes, even for depth 2.

11 Equational Theories of Finite Algebras

In this section, we show that for each finite algebra we can always generate a closed representation of all its quantifier-free and variable-bounded theorems.

We first give some sufficient criteria for an equational theory so that the equivalence classes are regular tree languages (Sect. 11.1). We then show that the set $\mathcal{TH}_n(\mathcal{A})$ of formal equations in up to n variables, which are universally valid over a given finite algebra \mathcal{A} , is always a regular tree language. This implies a constructive way of describing $\mathcal{TH}_n(\mathcal{A})$ by a finite term-rewriting system (Sect. 11.3). By extending the approach to typed algebras and including a type *Bool*, we are able to construct an axiom set describing the set of all quantifier-free and variable-bounded theorems of a given finite algebra with arbitrary functions, predicates, and junctors. See Fig. 56 for an axiom system that describes $\mathcal{TH}_2(\langle (N \bmod 3), (+), (<), (=), (\wedge), (\vee) \rangle)$.

11.1 Criteria for Regular Equivalence Classes

Lemma 15. (Sufficient Criterion for Regular Equivalence Classes)

If for each $t \in \mathcal{N}$ and each $f \in \mathcal{F}$ only finitely many $\langle t_1, \dots, t_n \rangle \in \mathcal{N}^n$ exist such that $f(t_1, \dots, t_n) =_E t$, then the equivalence class \underline{t} of each term $t \in \mathcal{T}$ can be represented as a regular sort.

Proof.

For each $t \in \mathcal{N}$, we may define the sorts

$$\underline{t} \doteq \bigsqcup_{f \in \mathcal{F}, t_1, \dots, t_n \in \mathcal{N}, f(t_1, \dots, t_n) =_E t} s_{ft_1 \dots t_n} \quad \text{and} \\ s_{ft_1 \dots t_n} \doteq f(\underline{t_1}, \dots, \underline{t_n}) \quad \text{for each } t_1, \dots, t_n \in \mathcal{N}.$$

We use the induction principle (Thm. 6 from Sect. 2) with

$$P_{\underline{t}}(t') : \Leftrightarrow t' =_E t \quad \text{and} \\ P_{s_{ft_1 \dots t_n}}(t') : \Leftrightarrow t' =_E f(t_1, \dots, t_n).$$

- Let $t' = f(t'_1, \dots, t'_n)$ with $t'_i \in \mathcal{T}$, then

$$\begin{aligned} & P_{\underline{t}}(t') \\ & \Rightarrow t' =_E t && \text{Def. } P \\ & \Rightarrow f(t'_1, \dots, t'_n) =_E t && \text{Def. } t' \\ & \Rightarrow t' =_E f(t_1, \dots, t_n) && \text{where } t_i := nf(t'_i) \\ & \Rightarrow P_{s_{ft_1 \dots t_n}}(t') \text{ for some } f \in \mathcal{F}, t_1, \dots, t_n \in \mathcal{N} \text{ with } f(t_1, \dots, t_n) =_E t \\ & \text{conversely:} \end{aligned}$$

$$\begin{aligned} & P_{s_{ft_1 \dots t_n}}(t') \text{ for some } f(t_1, \dots, t_n) =_E t \\ & \Rightarrow t' =_E f(t_1, \dots, t_n) =_E t && \text{Def. } P \\ & \Rightarrow P_{\underline{t}}(t') && \text{Def. } P \end{aligned}$$

- $P_{s_{ft_1 \dots t_n}}(t')$
 - $\Leftrightarrow t' =_E f(t_1, \dots, t_n)$ Def. P
 - $\Leftrightarrow \exists t'_1, \dots, t'_n \in \mathcal{T} \ t' = f(t'_1, \dots, t'_n) \wedge t'_1 =_E t_1 \wedge \dots \wedge t'_n =_E t_n$ (*)
 - $\Leftrightarrow \exists t'_1, \dots, t'_n \in \mathcal{T} \ t' = f(t'_1, \dots, t'_n) \wedge P_{\underline{t_1}}(t'_1) \wedge \dots \wedge P_{\underline{t_n}}(t'_n)$
 - (*) : “ \Rightarrow ”: choose $t'_i := t_i$
 - “ \Leftarrow ”: $t' =_E f(t'_1, \dots, t'_n) =_E f(t_1, \dots, t_n)$

Hence, the sort \underline{t} contains all term that are equivalent mod. E to t .

Corollary 16. (Sufficient Criterion for Regular Equivalence Classes)

Let \prec be a well-founded ordering on \mathcal{N} with a finite branching degree, and let \preceq be its reflexive closure.

If $t_i \preceq nf(f(t_1, \dots, t_n))$ for all $f \in \mathcal{F}$, $t_1, \dots, t_n \in \mathcal{N}$ and all $i \in \{1, \dots, n\}$, then \underline{t} is representable as a regular sort for all $t \in \mathcal{T}$.

Proof.

Under the above assumptions, we have for fixed $t \in \mathcal{N}$ and $f \in \mathcal{F}$:

$t_1, \dots, t_n \in \mathcal{N} \wedge f(t_1, \dots, t_n) =_E t \Rightarrow t_i \preceq nf(f(t_1, \dots, t_n)) = t$, hence
 $\{\langle t_1, \dots, t_n \rangle \in \mathcal{N}^n \mid f(t_1, \dots, t_n) =_E t\} \subset \{\langle t_1, \dots, t_n \rangle \in \mathcal{N}^n \mid t_1 \preceq t, \dots, t_n \preceq t\}$,
 where the latter set is finite. The conclusion follows by Lemma 15.

Corollary 17.

If $\mathcal{T}/_E$ is finite, each equivalence class can be represented as a regular tree language.

Proof. Trivially, \mathcal{N} has as many elements as $\mathcal{T}/_E$. The conclusion follows by Lemma 15.

11.2 Equational Theories in One Variable

Definition 18.

Let $(:) \notin \mathcal{F}$ be a symbol of arity 2; we call a term of the form $t_1 : t_2$ a (formal) equation if $t_1, t_2 \in \mathcal{T}$. An equation $t_1 : t_2$ is called universally valid if $\sigma t_1 =_E \sigma t_2$ for every substitution σ . Define $\mathcal{TH}_n(\mathcal{T}/_E)$ as the set of all formal equations in up to n variables that are universally valid over $\mathcal{T}/_E$.

Definition 19.

We will assume below that $\mathcal{T}/_E$ is finite, and hence we have only a finite number N of normal forms. Define $B := \{\langle b_1, \dots, b_N \rangle \mid \forall t \in \mathcal{T} \exists i \in \{1, \dots, N\} \ t =_E b_i\}$ the set of all N -tuples that represent all N normal forms. Let $\underline{b} := \langle \underline{b}_1, \dots, \underline{b}_N \rangle$ denote the equivalence class of \mathbf{b} . In this section, let $x := \varphi(\mathbf{b})$ for some $\mathbf{b} \in B$, and let $A := \{\underline{a} \in (\mathcal{T}/_E)^N \mid \mathbb{L}(rsg_V(\{x\}, \underline{a})) \neq \{\}\}$.

Lemma 20. Let $t_1, t_2 \in \mathcal{T}$ with $vars(t_1) \cup vars(t_2) \subset \{x\}$. The equation $t_1 : t_2$ is universally valid iff there exists an $\underline{a} \in A$ such that $t_1, t_2 \in \mathbb{L}(rsg_V(\{x\}, \underline{a}))$.

Proof.

“ \Rightarrow ”:

Consider the σ_i from Lemma 14 in Sect. 3.4. Define $a_i := \sigma_i t_1$; then $t_1 \in \mathbb{L}(rsg_V(\{x\}, \underline{a}))$ by Lemma 14; similarly, $\sigma_i t_2 =_E \sigma_i t_1 = a_i$ for all i implies $t_2 \in \mathbb{L}(rsg_V(\{x\}, \underline{a}))$.

“ \Leftarrow ”:

Let $t_1, t_2 \in \mathbb{L}(rsg_V(\{x\}, \underline{a}))$; consider an arbitrary substitution $\{x \leftarrow t\}$. We have

$$\begin{aligned} & \{x \leftarrow t\} (t_1) \\ =_E & \{x \leftarrow b_i\} (t_1) && \text{since } t =_E b_i \text{ for some } i \in \{1, \dots, N\} \text{ by Def. 19} \\ = & \sigma_i t_1 && \text{by Def. of } \sigma_i \text{ and since } vars(t_1) \subset \{x\} \\ =_E & a_i && \text{by Lemma 14 since } t_1 \in \mathbb{L}(rsg_V(\{x\}, \underline{a})) \\ =_E & \{x \leftarrow t\} (t_2) && \text{similarly} \end{aligned}$$

Hence, the equation $t_1 : t_2$ is universally valid.

Theorem 21. If $\mathcal{T}/_E$ is finite, the set of all universally valid equations in one variable x is a regular tree language.

Proof. Let φ be such that $\varphi(\mathbf{b}) = x$ for some $\mathbf{b} \in B$. By Cor. 17, \underline{a}_i is a regular tree language for each $a \in \mathcal{T}$. Define $s_{\text{valid}} \doteq \bigcap_{\underline{a} \in A} \text{rsg}_V(\{x\}, \underline{a}) : \text{rsg}_V(\{x\}, \underline{a})$. Let $t_1, t_2 \in \mathcal{T}$ with $\text{vars}(t_1) \cup \text{vars}(t_2) \subset \{x\}$. Then,

$$\begin{aligned} & t_1 : t_2 \text{ is universally valid} \\ \Leftrightarrow & \text{exists } \mathbf{a} \in \mathcal{T}^N \text{ such that } t_1, t_2 \in \text{L}(\text{rsg}_V(\{x\}, \underline{\mathbf{a}})) && \text{by Lemma 20} \\ \Leftrightarrow & \text{exists } \underline{\mathbf{a}} \in A \text{ such that } (t_1 : t_2) \in \text{L}(\text{rsg}_V(\{x\}, \underline{\mathbf{a}}) : \text{rsg}_V(\{x\}, \underline{\mathbf{a}})) \\ \Leftrightarrow & (t_1 : t_2) \in \text{L}(s_{\text{valid}}) \end{aligned}$$

Example 22.

Consider $(\mathbb{N} \bmod 2)$ with $+$. We have, letting $v_{01} = \varphi(0, 1)$, and abbreviating $s_0 := \underline{0}$, $s_1 := \underline{1}$, and $s_{ij} := \text{rsg}_V(\{v_{01}\}, s_i, s_j)$:

$$\begin{aligned} s_0 & \doteq 0 & | & s_0 + s_0 & | & s_1 + s_1 \\ s_1 & \doteq 1 & | & s_0 + s_1 & | & s_1 + s_0 \\ s_{00} & \doteq 0 & | & s_{00} + s_{00} & | & s_{01} + s_{01} & | & s_{10} + s_{10} & | & s_{11} + s_{11} \\ s_{01} & \doteq v_{01} & | & s_{00} + s_{01} & | & s_{01} + s_{00} & | & s_{10} + s_{11} & | & s_{11} + s_{10} \\ s_{10} & \doteq & | & s_{00} + s_{10} & | & s_{01} + s_{11} & | & s_{10} + s_{00} & | & s_{11} + s_{01} \\ s_{11} & \doteq 1 & | & s_{00} + s_{11} & | & s_{01} + s_{10} & | & s_{10} + s_{01} & | & s_{11} + s_{00} \\ s_{\text{valid}} & \doteq & | & s_{00} : s_{00} & | & s_{01} : s_{01} & | & s_{10} : s_{10} & | & s_{11} : s_{11} \end{aligned}$$

11.3 Equational Theories in n Variables

Theorem 21 can be generalized to n variables, which we will do in this section. The definitions given below generalize Defs. 18 and 19 to n variables.

Definition 23. Modifying Def. 19, define

$$B := \left\{ \begin{pmatrix} b_{1,1} & \dots & b_{1,N^n} \\ \vdots & & \vdots \\ b_{n,1} & \dots & b_{n,N^n} \end{pmatrix} \middle| \forall t_1, \dots, t_n \in \mathcal{T} \exists j \in \{1, \dots, N^n\} \forall i \in \{1, \dots, n\} b_{i,j} =_E t_i \right\}$$

B is non-empty since $(\mathcal{T}/_E)^n$ is finite, viz. of cardinality N^n . Intuitively, for each matrix $\mathbf{b} \in B$ there is a one-to-one correspondence between its column vectors and the elements of $(\mathcal{T}/_E)^n$.

In this section, let $x_1 := \varphi(b_{1,1}, \dots, b_{1,N^n})$, \dots , $x_n := \varphi(b_{n,1}, \dots, b_{n,N^n})$ for some $\mathbf{b} \in B$, and let $V := \{x_1, \dots, x_n\}$. Let $A := \{\underline{\mathbf{a}} \in (\mathcal{T}/_E)^{(N^n)} \mid \text{L}(\text{rsg}_V(\{x_1, \dots, x_n\}, \underline{\mathbf{a}})) \neq \{\}\}$.

Lemma 24. Let $t_1, t_2 \in \mathcal{T}$ with $\text{vars}(t_1) \cup \text{vars}(t_2) \subset V$. The equation $t_1 : t_2$ is universally valid iff there exists an $\mathbf{a} \in A$ such that $t_1, t_2 \in \text{L}(\text{rsg}_V(\{x_1, \dots, x_n\}, \underline{\mathbf{a}}))$.

Proof.

“ \Rightarrow ”:

Consider the σ_i from Lemma 14 in Sect. 3.4. Define $a_i := \sigma_i t_1$ for $i = 1, \dots, N^n$;

then $t_1 \in \text{L}(\text{rsg}_V(\{x_1, \dots, x_n\}, \underline{\mathbf{a}}))$ by Lemma 14;

similarly, $\sigma_i t_2 =_E \sigma_i t_1 = a_i$ for all i implies $t_2 \in \text{L}(\text{rsg}_V(\{x_1, \dots, x_n\}, \underline{\mathbf{a}}))$.

“ \Leftarrow ”:

Let $t_1, t_2 \in \mathbf{L}(\text{rsg}_V(\{x_1, \dots, x_n\}, \underline{a}))$; consider an arbitrary substitution $\{x_1 \leftarrow t'_1, \dots, x_n \leftarrow t'_n\}$. We have

$$\begin{aligned} & \sigma_j x_i \\ &= \sigma_j \varphi(b_{i,1}, \dots, b_{i,N^n}) && \text{Def. } x_i \\ &= b_{i,j} && \text{Def. } \sigma_j \\ &= \{x_1 \leftarrow b_{1,j}, \dots, x_n \leftarrow b_{n,j}\} (x_i) \end{aligned}$$

By induction on t , it follows that $\sigma_j t = \{x_1 \leftarrow b_{1,j}, \dots, x_n \leftarrow b_{n,j}\} (t)$ for all $t \in \mathcal{T}$ with $\text{vars}(t) \subset \{x_1, \dots, x_n\}$. Thus:

$$\begin{aligned} & \{x_1 \leftarrow t'_1, \dots, x_n \leftarrow t'_n\} (t_1) \\ &=_E \{x_1 \leftarrow b_{1,j}, \dots, x_n \leftarrow b_{n,j}\} (t_1) && \text{since } \exists j \in \{1, \dots, N^n\} \forall i \in \{1, \dots, n\} \ b_{i,j} =_E t'_i \text{ by Def. 23} \\ &= \sigma_j t_1 && \text{as shown above} \\ &=_E a_j && \text{by Lemma 14, since } t_1 \in \mathbf{L}(\text{rsg}_V(\{x_1, \dots, x_n\}, \underline{a})) \\ &=_E \{x_1 \leftarrow t'_1, \dots, x_n \leftarrow t'_n\} (t_2) && \text{similarly} \end{aligned}$$

Hence, the equation $t_1 : t_2$ is universally valid.

Theorem 25. If $\mathcal{T}/_E$ is finite, the set $\mathcal{TH}_n(\mathcal{T}/_E)$ of all universally valid equations in n variables is a regular tree language for each n .

Proof. Let φ be such that $\varphi(b_{1,1}, \dots, b_{1,N^n}) = x_1, \dots, \varphi(b_{n,1}, \dots, b_{n,N^n}) = x_n$ for some $\mathbf{b} \in B$; let $V := \{x_1, \dots, x_n\}$. By Cor. 17, $\underline{a_i}$ is a regular tree language for each $a \in \mathcal{T}$. Define $s_{\text{valid}} \doteq \bigcup_{\underline{a} \in A} \text{rsg}_V(\{x_1, \dots, x_n\}, \underline{a}) : \text{rsg}_V(\{x_1, \dots, x_n\}, \underline{a})$. Let $t_1, t_2 \in \mathcal{T}$ with $\text{vars}(t_1) \cup \text{vars}(t_2) \subset V$. Then

$$\begin{aligned} & t_1 : t_2 \text{ universally valid} \\ & \Leftrightarrow \text{exists } \mathbf{a} \in A \text{ such that } t_1, t_2 \in \mathbf{L}(\text{rsg}_V(\{x_1, \dots, x_n\}, \underline{a})) && \text{by Lemma 24} \\ & \Leftrightarrow \text{exists } \underline{a} \in A \text{ such that } (t_1 : t_2) \in \mathbf{L}(\text{rsg}_V(\{x_1, \dots, x_n\}, \underline{a}) : \text{rsg}_V(\{x_1, \dots, x_n\}, \underline{a})) \\ & \Leftrightarrow (t_1 : t_2) \in \mathbf{L}(s_{\text{valid}}) \end{aligned}$$

Example 26. Consider again $(\mathbb{N} \bmod 2)$ with $+$. Let $v_{0011} = \varphi(0, 0, 1, 1)$, $v_{0101} = \varphi(0, 1, 0, 1)$, $V = \{v_{0011}, v_{0101}\}$, and abbreviate $s_0 := \underline{0}$, $s_1 := \underline{1}$, and $s_{ijkl} := \text{rsg}_V(\{v_{0011}, v_{0101}\}, s_i, s_j, s_k, s_l)$. The sort definitions are shown in Fig. 52; the sorts s_{0001} , s_{0010} , s_{0100} , s_{0111} , s_{1000} , s_{1011} , s_{1101} , and s_{1110} are all empty and have been omitted in the definition of s_{valid} . For example, the commutativity law $v_{0011} + v_{0101} : v_{0101} + v_{0011}$ is contained in $\mathbf{L}(s_{0110} : s_{0110})$.

Corollary 27. If $\mathcal{T}/_E$ is finite, for each finite $V_1 \subset V_2 \subset \mathcal{V}$ and $V_3 \subset V_4 \subset \mathcal{V}$, the set of all universally valid equations $t_1 : t_2$ such that $V_1 \subset \text{vars}(t_1) \subset V_2$ and $V_3 \subset \text{vars}(t_2) \subset V_4$ is a regular tree language.

Proof. Apply Thm. 25 to $V := V_2 \cup V_4$; using the variable filter sorts $s_{V_1}^{V_2}$, $s_{V_3}^{V_4}$ from Sect. 6.2, define $s_{\text{result}} \doteq \bigcup_{\underline{a} \in A} (\text{rsg}_V(V, \underline{a}) \cap s_{V_1}^{V_2}) : (\text{rsg}_V(V, \underline{a}) \cap s_{V_3}^{V_4})$.

Corollary 28. If $\mathcal{T}/_E$ is finite and $n \in \mathbb{N}$ arbitrary, the set $\mathcal{TH}_n(\mathcal{T}/_E)$ of universally valid equations in n variables can be represented as the deductive closure of finitely many equations.

Proof. Using Thm. 25, $\mathcal{TH}_n(\mathcal{T}/_E) = \mathbf{L}(s_{\text{valid}})$ where $s_{\text{valid}} \doteq \bigcup_{\underline{a} \in A} \text{rsg}_V(V, \underline{a}) : \text{rsg}_V(V, \underline{a})$, and $V = \{x_1, \dots, x_n\}$. For each $\underline{a} \in A$, choose some arbitrary normal form $nf_{\underline{a}} \in \mathbf{L}(\text{rsg}_V(V, \underline{a}))$.

$s_{0000} \doteq 0$		$s_{0000} + s_{0000}$		$s_{0011} + s_{0011}$		$s_{0101} + s_{0101}$		$s_{0110} + s_{0110}$
		$s_{1001} + s_{1001}$		$s_{1010} + s_{1010}$		$s_{1100} + s_{1100}$		$s_{1111} + s_{1111}$
$s_{0011} \doteq v_{0011}$		$s_{0000} + s_{0011}$		$s_{0011} + s_{0000}$		$s_{0101} + s_{0110}$		$s_{0110} + s_{0101}$
		$s_{1001} + s_{1010}$		$s_{1010} + s_{1001}$		$s_{1100} + s_{1111}$		$s_{1111} + s_{1100}$
$s_{0101} \doteq v_{0101}$		$s_{0000} + s_{0101}$		$s_{0011} + s_{0110}$		$s_{0101} + s_{0000}$		$s_{0110} + s_{0011}$
		$s_{1001} + s_{1100}$		$s_{1010} + s_{1111}$		$s_{1100} + s_{1001}$		$s_{1111} + s_{1010}$
$s_{0110} \doteq$		$s_{0000} + s_{0110}$		$s_{0011} + s_{0101}$		$s_{0101} + s_{0011}$		$s_{0110} + s_{0000}$
		$s_{1001} + s_{1111}$		$s_{1010} + s_{1100}$		$s_{1100} + s_{1010}$		$s_{1111} + s_{1001}$
$s_{1001} \doteq$		$s_{0000} + s_{1001}$		$s_{0011} + s_{1010}$		$s_{0101} + s_{1100}$		$s_{0110} + s_{1111}$
		$s_{1001} + s_{0000}$		$s_{1010} + s_{0011}$		$s_{1100} + s_{0101}$		$s_{1111} + s_{0110}$
$s_{1010} \doteq$		$s_{0000} + s_{1010}$		$s_{0011} + s_{1001}$		$s_{0101} + s_{1111}$		$s_{0110} + s_{1100}$
		$s_{1001} + s_{0011}$		$s_{1010} + s_{0000}$		$s_{1100} + s_{0110}$		$s_{1111} + s_{0101}$
$s_{1100} \doteq$		$s_{0000} + s_{1100}$		$s_{0011} + s_{1111}$		$s_{0101} + s_{1001}$		$s_{0110} + s_{1010}$
		$s_{1001} + s_{0101}$		$s_{1010} + s_{0110}$		$s_{1100} + s_{0000}$		$s_{1111} + s_{0011}$
$s_{1111} \doteq 1$		$s_{0000} + s_{1111}$		$s_{0011} + s_{1100}$		$s_{0101} + s_{1010}$		$s_{0110} + s_{1001}$
		$s_{1001} + s_{0110}$		$s_{1010} + s_{0101}$		$s_{1100} + s_{0011}$		$s_{1111} + s_{0000}$
$s_{valid} \doteq$		$s_{0000} : s_{0000}$		$s_{0011} : s_{0011}$		$s_{0101} : s_{0101}$		$s_{0110} : s_{0110}$
		$s_{1001} : s_{1001}$		$s_{1010} : s_{1010}$		$s_{1100} : s_{1100}$		$s_{1111} : s_{1111}$

Fig. 52. Sort Definitions in Exm. 26

$s_{ffff} \doteq f$		$s_{ffff} \vee s_{ffff}$		$s_{fttt} \wedge s_{ffff}$		$s_{tttt} \wedge s_{ffff}$		$s_{ffff} \wedge s_{fttt}$		$s_{ffff} \wedge s_{tttt}$		$s_{ffff} \wedge s_{ffff}$
		$s_{ffff} \wedge s_{ffft}$		$s_{ffff} \wedge s_{fftt}$		$s_{ffff} \wedge s_{ftft}$		$s_{fftt} \wedge s_{ffff}$		$s_{fftt} \wedge s_{ffff}$		$s_{ftft} \wedge s_{ffff}$
$s_{ffft} \doteq$		$s_{ffff} \vee s_{ffft}$		$s_{ffft} \vee s_{ffff}$		$s_{ffft} \vee s_{ffft}$		$s_{fttt} \wedge s_{ffft}$		$s_{tttt} \wedge s_{ffft}$		$s_{ffft} \wedge s_{fttt}$
		$s_{ffft} \wedge s_{ffft}$		$s_{ffft} \wedge s_{fftt}$		$s_{ffft} \wedge s_{ftft}$		$s_{fftt} \wedge s_{ffft}$		$s_{fftt} \wedge s_{ftft}$		$s_{ftft} \wedge s_{ffft}$
$s_{fftt} \doteq v_{fftt}$		$s_{ffff} \vee s_{fftt}$		$s_{ffft} \vee s_{fftt}$		$s_{fftt} \vee s_{ffff}$		$s_{fftt} \vee s_{ffft}$		$s_{fftt} \vee s_{fftt}$		
		$s_{fttt} \wedge s_{fftt}$		$s_{tttt} \wedge s_{fftt}$		$s_{fftt} \wedge s_{fttt}$		$s_{fftt} \wedge s_{tttt}$		$s_{fftt} \wedge s_{fftt}$		
$s_{ftft} \doteq v_{ftft}$		$s_{ffff} \vee s_{ftft}$		$s_{ffft} \vee s_{ftft}$		$s_{ftft} \vee s_{ffff}$		$s_{ftft} \vee s_{ffft}$		$s_{ftft} \vee s_{ftft}$		
		$s_{fttt} \wedge s_{ftft}$		$s_{tttt} \wedge s_{ftft}$		$s_{ftft} \wedge s_{fttt}$		$s_{ftft} \wedge s_{tttt}$		$s_{ftft} \wedge s_{ftft}$		
$s_{fttt} \doteq$		$s_{fttt} \vee s_{fttt}$		$s_{fttt} \vee s_{ffff}$		$s_{fttt} \vee s_{ffft}$		$s_{fttt} \vee s_{fftt}$		$s_{fttt} \vee s_{ftft}$		$s_{ffff} \vee s_{fttt}$
		$s_{fftt} \wedge s_{fttt}$		$s_{fftt} \wedge s_{ftft}$		$s_{ftft} \wedge s_{fttt}$		$s_{ftft} \wedge s_{fftt}$		$s_{fttt} \wedge s_{fttt}$		$s_{fttt} \wedge s_{fttt}$
$s_{tttt} \doteq t$		$s_{fttt} \vee s_{tttt}$		$s_{tttt} \vee s_{fttt}$		$s_{tttt} \vee s_{tttt}$		$s_{tttt} \vee s_{ffff}$		$s_{tttt} \vee s_{ffft}$		$s_{tttt} \vee s_{fftt}$
		$s_{tttt} \vee s_{ftft}$		$s_{ffff} \vee s_{tttt}$		$s_{ffft} \vee s_{tttt}$		$s_{fftt} \vee s_{tttt}$		$s_{ftft} \vee s_{tttt}$		$s_{tttt} \wedge s_{tttt}$

Fig. 53. Sort Definitions in Exm. 29

$$\begin{aligned}
& f \wedge x : f \\
& y \wedge x : x \wedge y \\
& x \wedge y \wedge y : x \wedge y \\
& x \wedge y \wedge (x \vee y) : x \wedge y \\
& f \vee x : x \\
& x \vee x : x \\
& x \vee (x \wedge y) : x \\
& t \wedge x : x \\
& x \wedge x : x \\
& x \wedge (x \vee y) : x \\
& y \vee x : x \vee y \\
& x \vee y \vee y : x \vee y \\
& x \vee y \vee (x \wedge y) : x \vee y \\
& t \vee x : t
\end{aligned}$$

Fig. 54. Equational Theory of *Bool* with \wedge and \vee

By Cor. 17 and Lemma 15, the sort definition of each \underline{a}_i is in head normal form and refers only to sort names corresponding to some \underline{a}_j . Hence, by construction of rsg_V , the sort definition of each $rsg_V(V, \underline{a})$ is in head normal form and refers only to sort names corresponding to some $rsg_V(V, \underline{a}')$.

For $\underline{a} \in A$, let $rsg_V(V, \underline{a}) \doteq \prod_{i=1}^m f_i(rsg_V(V, \underline{a}_{i1}), \dots, rsg_V(V, \underline{a}_{in_i}))$ for some f_i, \underline{a}_{ij} . Define $TH_{\underline{a}} := \{f_i(nf_{\underline{a}_{i1}}, \dots, nf_{\underline{a}_{in_i}}) : nf_{\underline{a}} \mid i = 1, \dots, m\}$. Define $TH := \bigcup_{\underline{a} \in A} TH_{\underline{a}}$.

Let $t \in L(rsg_V(V, \underline{a}))$ for some $\underline{a} \in A$; we show by induction on t that the equation $t : nf_{\underline{a}}$ is a deductive consequence of TH . Consider the alternative in the sort definition of $rsg_V(V, \underline{a})$ that leads to t :

- If $t = f_i(t_1, \dots, t_{n_i})$, where $t_j \in L(rsg_V(V, \underline{a}_{ij}))$, then by induction hypothesis the equations $t_j : nf_{\underline{a}_{ij}}$ are deductive consequences of TH . Since $(f_i(nf_{\underline{a}_{i1}}, \dots, nf_{\underline{a}_{in_i}}) : nf_{\underline{a}}) \in TH$, we have finished.
- If $t = f_i$ is a constant or a variable, we immediately have $(f_i : nf_{\underline{a}}) \in TH$.

If we now take an arbitrary equation $(t_1 : t_2) \in L(s_{valid})$, then $t_1, t_2 \in L(rsg_V(V, \underline{a}))$ for some $\underline{a} \in A$; the equation is a consequence of $t_1 : nf_{\underline{a}}$ and $t_2 : nf_{\underline{a}}$.

Remark 1. Note that no proper instances of the equations in TH are needed to derive any equation $t_1 : t_2$ in $L(s_{valid})$. Hence, if we consider all variables in V as constants and choose $nf_{\underline{a}}$ to be of minimal size within $L(rsg_V(V, \underline{a}))$, we obtain a noetherian ground-rewriting system for the set of all universally valid equations in variables from V . Moreover, this rewriting system assigns a unique normal form to each term t that may occur in a universally valid equation, viz. $nf_{\underline{a}}$ if $t \in L(rsg_V(V, \underline{a}))$.

Of course, when permitting proper instantiations, we lose these properties, since the commutativity law, for example, could be among the universally valid equations. On the other hand, we may delete equations that are instances of others, thus reducing the number of equations significantly. To find such subsumed equations, an appropriate indexing technique may be used, see e.g. [Gra92, GM93, Gra94].

Example 29. Consider $Bool = \{f, t\}$ with the operations \wedge and \vee :

$x+x : 0$	$y+z+(x+y) : x+z$	$y+z+(x+1)+x : y+z+1$
$x : x$	$y+z+1+(x+(1+y)) : x+z$	$1+(x+z) : x+(1+z)$
$0+x : x$	$y+z+x+y : x+z$	$y+(y+z+(x+1)) : x+(1+z)$
$x+0 : x$	$y+z+(x+1)+(1+y) : x+z$	$z+(x+1) : x+(1+z)$
$y+(x+y) : x$	$x+y+(x+z) : y+z$	$1+y+(y+z+x) : x+(1+z)$
$x+y+y : x$	$x+z+(x+y) : y+z$	$x+1+z : x+(1+z)$
$y+z+1+(y+z+(x+1)) : x$	$x+(1+y)+(x+(1+z)) : y+z$	$x+y+(y+z+1) : x+(1+z)$
$y+z+(x+1)+(y+z+1) : x$	$x+(1+z)+(x+(1+y)) : y+z$	$x+z+1 : x+(1+z)$
$x+(x+y) : y$	$1+(y+z+(x+1)) : y+z+x$	$x+(1+y)+(y+z) : x+(1+z)$
$x+1+(x+(1+y)) : y$	$y+(x+z) : y+z+x$	$y+z+(x+(1+y)) : x+(1+z)$
$x+y+x : y$	$z+(x+y) : y+z+x$	$y+z+1+(x+y) : x+(1+z)$
$x+z+(y+z+x) : y$	$1+y+(x+(1+z)) : y+z+x$	$y+z+x+(1+y) : x+(1+z)$
$x+(1+y)+(x+1) : y$	$1+z+(x+(1+y)) : y+z+x$	$y+z+(x+1)+y : x+(1+z)$
$x+(1+z)+(y+z+(x+1)) : y$	$x+y+z : y+z+x$	$x+y+(y+z+(x+1)) : 1+z$
$y+z+x+(x+z) : y$	$x+z+y : y+z+x$	$x+(1+y)+(y+z+x) : 1+z$
$y+z+(x+1)+(x+(1+z)) : y$	$x+(1+y)+(1+z) : y+z+x$	$y+z+x+(x+(1+y)) : 1+z$
$1+(x+(1+y)) : x+y$	$x+(1+z)+(1+y) : y+z+x$	$y+z+(x+1)+(x+y) : 1+z$
$y+x : x+y$	$y+z+(x+1)+1 : y+z+x$	$z+(y+z+(x+1)) : x+(1+y)$
$z+(y+z+x) : x+y$	$1+(y+z+x) : y+z+(x+1)$	$1+z+(y+z+x) : x+(1+y)$
$1+z+(y+z+(x+1)) : x+y$	$x+(y+z+1) : y+z+(x+1)$	$x+z+(y+z+1) : x+(1+y)$
$x+z+(y+z) : x+y$	$y+(x+(1+z)) : y+z+(x+1)$	$x+(1+z)+(y+z) : x+(1+y)$
$x+(1+y)+1 : x+y$	$z+(x+(1+y)) : y+z+(x+1)$	$y+z+(x+(1+z)) : x+(1+y)$
$x+(1+z)+(y+z+1) : x+y$	$1+y+(x+z) : y+z+(x+1)$	$y+z+1+(x+z) : x+(1+y)$
$y+z+(x+z) : x+y$	$1+z+(x+y) : y+z+(x+1)$	$y+z+x+(1+z) : x+(1+y)$
$y+z+1+(x+(1+z)) : x+y$	$x+y+(1+z) : y+z+(x+1)$	$y+z+(x+1)+z : x+(1+y)$
$y+z+x+z : x+y$	$x+z+(1+y) : y+z+(x+1)$	$x+z+(y+z+(x+1)) : 1+y$
$y+z+(x+1)+(1+z) : x+y$	$x+(1+y)+z : y+z+(x+1)$	$x+(1+z)+(y+z+x) : 1+y$
$x+y+(y+z+x) : z$	$x+(1+z)+y : y+z+(x+1)$	$y+z+x+(x+(1+z)) : 1+y$
$x+(1+y)+(y+z+(x+1)) : z$	$y+z+1+x : y+z+(x+1)$	$y+z+(x+1)+(x+z) : 1+y$
$y+z+x+(x+y) : z$	$y+z+x+1 : y+z+(x+1)$	$y+(x+(1+y)) : x+1$
$y+z+(x+1)+(x+(1+y)) : z$	$x+(y+z+(x+1)) : y+z+1$	$x+(1+y)+y : x+1$
$y+(y+z+x) : x+z$	$x+y+(x+(1+z)) : y+z+1$	$x+y+(x+(1+y)) : 1$
$1+y+(y+z+(x+1)) : x+z$	$x+z+(x+(1+y)) : y+z+1$	$x+(1+y)+(x+y) : 1$
$x+y+(y+z) : x+z$	$x+(1+y)+(x+z) : y+z+1$	$y+z+x+(y+z+(x+1)) : 1$
$x+(1+y)+(y+z+1) : x+z$	$x+(1+z)+(x+y) : y+z+1$	$y+z+(x+1)+(y+z+x) : 1$

Fig. 55. Equational Theory of $(\mathbb{N} \bmod 2)$ with +

\wedge	f	t
f	f	f
t	f	t

\vee	f	t
f	f	t
t	t	t

Computing s_{valid} for $n = 2$ variables, we obtain the six non-empty sorts shown in Fig. 53. Proceeding as described in the proof of Cor. 28, we obtain 76 equations in TH ; 43 of them are instances of others and can therefore be deleted. Of the remaining 33, a further 17 have been manually deleted since they were subsumed modulo commutativity. The remaining equations are given in Fig. 54.

Example 30. Figure 55 shows an axiom system for the equational theory of $(\mathbb{N} \bmod 2)$ with $+$ in 3 variables. The 105 equations have been extracted from a system of 256 sort definitions 240 of which where empty. Only equations that were syntactically subsumed have been removed (automatically).

Remark 2. In order to estimate the complexity of computing an axiomatization of a finite algebra \mathcal{T}/E , observe the following facts. Using an appropriate caching mechanism when computing $rsg_V(V, \underline{a})$ for all $\underline{a} \in A_T$, we can ensure that rsg_V is called exactly once for each such \underline{a} . If all sort definitions are in head normal form, the cost of one rsg_V call without its recursive sub-calls is precisely the cost of its grouping algorithm. In Sect. 3.5, we estimated the latter to be $\mathcal{O}(N' \cdot m' \cdot (m' + n') \cdot g'^{N'})$, where N' denotes the number of sorts to be anti-unified simultaneously, m' is the maximal number of alternatives of a sort definition's right-hand side, n' is the number of variables in V , and g' is the maximum number of disjuncts of a sort-definition's right-hand side that start with the same function symbol. We have $N' = N^n$ and $n' = n$, leading to an overall complexity of $\mathcal{O}((N \cdot g')^{(N^n)} \cdot N^n \cdot m' \cdot (m' + n))$ for computing all $N^{(N^n)}$ rsg_V calls. Transformation of the sort definitions into an axiom system can be done in linear time.

11.4 Typed Equational Theories in n Variables

Theorem 25 and Corollaries 27 and 28 can be generalized to typed algebras where different variables may have different – disjoint – domains. We will do this in this section. The definitions given below generalize Def. 23 to n typed variables.

Definition 31.

Let \mathcal{TV} be a finite set of types. Assume each $x \in \mathcal{V}$ has a fixed type $type(x) \in \mathcal{TV}$ such that for each $T \in \mathcal{TV}$ there are infinitely many $x \in \mathcal{V}$ with $type(x) = T$. Assume each $f \in \mathcal{F}$ has a fixed signature $f : T_1 \times \dots \times T_n \rightarrow T$ where $T_1, \dots, T_n, T \in \mathcal{TV}$.

For $T \in \mathcal{TV}$, let \mathcal{T}_T be the set of well-typed terms of type T , which is defined as usual. Let $\mathcal{T} := \bigcup_{T \in \mathcal{TV}} \mathcal{T}_T$; for $t \in \mathcal{T}$ define $type(t) := T$ if $t \in \mathcal{T}_T$; $type(t)$ is uniquely determined since any f has only one signature. A substitution $\{x_1 \leftarrow t_1, \dots, x_n \leftarrow t_n\}$ is called well-typed if $type(x_i) = type(t_i)$ for $i = 1, \dots, n$. Consequently, we always have $type(\sigma t) = type(t)$ for well-typed σ, t .

We call a term of the form $t_1 : t_2$ a typed (formal) equation if $t_1, t_2 \in \mathcal{T}_T$ for some $T \in \mathcal{TV}$; in this case we define $type(t_1 : t_2) := T$. An equation $t_1 : t_2$ is called universally valid if $\sigma t_1 =_E \sigma t_2$ for every well-typed substitution σ . For $x_1, \dots, x_n \in \mathcal{V}$ and $T \in \mathcal{TV}$, define $\mathcal{TH}_{x_1, \dots, x_n}^T(\mathcal{T}/E)$ as the set of all typed formal equations of type T with variables x_1, \dots, x_n that are universally valid over \mathcal{T}/E . Define $\mathcal{TH}_{x_1, \dots, x_n}(\mathcal{T}/E) := \bigcup_{T \in \mathcal{TV}} \mathcal{TH}_{x_1, \dots, x_n}^T(\mathcal{T}/E)$ as the set of all possible typed formal equations with variables x_1, \dots, x_n that are universally valid over \mathcal{T}/E . Note that not all x_i must occur in such an equation, but no other variables may occur.

We still assume that \mathcal{T}/E is finite; hence \mathcal{T}_T/E is finite for each $T \in \mathcal{TV}$. We assume in this section that we are given n fixed variables x_1, \dots, x_n of types T_1, \dots, T_n , respectively, such that \mathcal{T}_{T_i}/E is of cardinality N_i . Let $N := N_1 \cdot \dots \cdot N_n$.

Definition 32. Modifying Defs. 19 and 23, define

$$B := \left\{ \left(\begin{array}{ccc} b_{1,1} & \dots & b_{1,N} \\ \vdots & & \vdots \\ b_{n,1} & \dots & b_{n,N} \end{array} \right) \mid \forall t_1 \in \mathcal{T}_{T_1}, \dots, t_n \in \mathcal{T}_{T_n} \ \exists j \in \{1, \dots, N\} \ \forall i \in \{1, \dots, n\} \ b_{i,j} =_E t_i \right\}$$

B is non-empty since $\mathcal{T}_{T_1}/_E \times \dots \times \mathcal{T}_{T_n}/_E$ is finite, viz. of cardinality N . Intuitively, for each matrix $\mathbf{b} \in B$ there is a one-to-one correspondence between its column vectors and the elements of $\mathcal{T}_{T_1}/_E \times \dots \times \mathcal{T}_{T_n}/_E$.

In this section, let $x_1 = \varphi(b_{1,1}, \dots, b_{1,N})$, \dots , $x_n = \varphi(b_{n,1}, \dots, b_{n,N})$ for some $\mathbf{b} \in B$, and let $V := \{x_1, \dots, x_n\}$. Let $A_T := \{\underline{a} \in (\mathcal{T}/_E)^N \mid \mathbb{L}(\text{rsg}_V(V, \underline{a})) \neq \{\}\}$ for $T \in \mathcal{T}\mathcal{Y}$.

Lemma 33. Let $t_1, t_2 \in \mathcal{T}_T$ for some $T \in \mathcal{T}\mathcal{Y}$ with $\text{vars}(t_1) \cup \text{vars}(t_2) \subset V$.

The equation $t_1 : t_2$ is universally valid iff there exists an $\mathbf{a} \in A_T$ such that $t_1, t_2 \in \mathbb{L}(\text{rsg}_V(V, \underline{a}))$.

Proof. The proof is similar to that of Lemma 24, with the following observations:

“ \Rightarrow ”: The σ_i from Lemma 14 in Sect. 3.4 are well-typed, since $\text{type}(x_j) = T_j = \text{type}(b_{j,i})$.

Moreover, $\underline{a} \in A_T$, since $\text{type}(a_i) = \text{type}(\sigma_i t_1) = \text{type}(t_1) = T$ for all i .

“ \Leftarrow ”: It is sufficient to consider an arbitrary well-typed substitution $\{x_1 \leftarrow t'_1, \dots, x_n \leftarrow t'_n\}$ with $\text{type}(t'_i) = \text{type}(x_i) = T_i$.

Theorem 34. If $\mathcal{T}/_E$ is finite, the set $\mathcal{TH}_{x_1, \dots, x_n}^T(\mathcal{T}/_E)$ of all universally valid equations between terms of type T with variables x_1, \dots, x_n of types $T_1, \dots, T_n \in \mathcal{T}\mathcal{Y}$ is a regular tree language.

Proof. Let φ be such that $\varphi(b_{1,1}, \dots, b_{1,N}) = x_1$, \dots , $\varphi(b_{n,1}, \dots, b_{n,N}) = x_n$ for some $\mathbf{b} \in B$; let $V := \{x_1, \dots, x_n\}$. By Cor. 17, \underline{a}_i is a regular tree language for each $a \in \mathcal{T}_T$, $T \in \mathcal{T}\mathcal{Y}$. Define $s_{\text{valid}}^T \doteq \bigcup_{\underline{a} \in A_T} \text{rsg}_V(V, \underline{a}) : \text{rsg}_V(V, \underline{a})$. Let $t_1, t_2 \in \mathcal{T}_T$ with $\text{vars}(t_1) \cup \text{vars}(t_2) \subset V$. Then,

$$\begin{aligned} & t_1 : t_2 \text{ is universally valid} \\ \Leftrightarrow & \text{exists } \mathbf{a} \in A_T \text{ such that } t_1, t_2 \in \mathbb{L}(\text{rsg}_V(V, \underline{a})) && \text{by Lemma 33} \\ \Leftrightarrow & \text{exists } \underline{a} \in A_T \text{ such that } (t_1 : t_2) \in \mathbb{L}(\text{rsg}_V(V, \underline{a}) : \text{rsg}_V(V, \underline{a})) \\ \Leftrightarrow & (t_1 : t_2) \in \mathbb{L}(s_{\text{valid}}^T) \end{aligned}$$

Corollary 35. If $\mathcal{T}/_E$ is finite and $n \in \mathbb{N}$ arbitrary, the set $\mathcal{TH}_{x_1, \dots, x_n}(\mathcal{T}/_E)$ of universally valid equations in variables x_1, \dots, x_n of types $T_1, \dots, T_n \in \mathcal{T}\mathcal{Y}$ can be represented as the deductive closure of finitely many equations.

Proof. The proof is similar to that of Cor. 28:

Using Thm. 34, $\mathcal{TH}_{x_1, \dots, x_n}^T(\mathcal{T}/_E) = \mathbb{L}(s_{\text{valid}}^T)$ where $s_{\text{valid}}^T \doteq \bigcup_{\underline{a} \in A_T} \text{rsg}_V(V, \underline{a}) : \text{rsg}_V(V, \underline{a})$. Let $s_{\text{valid}} \doteq \bigcup_{T \in \mathcal{T}\mathcal{Y}} s_{\text{valid}}^T$. For each $\underline{a} \in A_T$ choose some arbitrary normal form $nf_{\mathbf{a}} \in \mathbb{L}(\text{rsg}_V(V, \underline{a}))$. For $\underline{a} \in A_T$, let $\text{rsg}_V(V, \underline{a}) \doteq \bigcup_{i=1}^m f_i(\text{rsg}_V(V, \underline{a}_{i1}), \dots, \text{rsg}_V(V, \underline{a}_{in_i}))$ be in head normal form. Define $TH_{\underline{a}} := \{f_i(nf_{\mathbf{a}_{i1}}, \dots, nf_{\mathbf{a}_{in_i}}) : nf_{\mathbf{a}} \mid i = 1, \dots, m\}$. Define $TH_T := \bigcup_{\underline{a} \in A_T} TH_{\underline{a}}$ and $TH := \bigcup_{T \in \mathcal{T}\mathcal{Y}} TH_T$.

By analogy with the proof of Thm. 28, we can show that, for arbitrary $t \in \mathbb{L}(\text{rsg}_V(V, \underline{a}))$, $\underline{a} \in A_T$, and $T \in \mathcal{T}\mathcal{Y}$, the equation $t : nf_{\mathbf{a}}$ is a deductive consequence of TH . An arbitrary equation $(t_1 : t_2) \in \mathbb{L}(s_{\text{valid}})$ follows from $t_1 : nf_{\mathbf{a}}$ and $t_2 : nf_{\mathbf{a}}$. Note that an arbitrary equation $(t_1 : t_2)$ in

$L(s_{valid}^T)$ is generally not a deductive consequence of TH_T only, since rewritings on subterms, e.g. of t_1 , that are not of type T may be necessary.

Corollary 36. If $\mathcal{T}/_E$ is finite and $n \in \mathbb{N}$ arbitrary, the set of universally valid quantifier-free formulas in n variables is a regular tree language and can be represented as the deductive closure of finitely many axioms that may use the equality predicate. The same holds if the set of admitted junctors is arbitrarily restricted, as long as it contains logical equivalence (\leftrightarrow).

Proof. This follows immediately from Thm. 34 and Cor. 35 by adding a type *Bool* to \mathcal{TY} , coding each predicate as a function into *Bool*, and coding logical junctors as functions from *Bool* to *Bool*. Equations $(t_1 : t_2)$ of type *Bool* are read as logical equivalences $(t_1 \leftrightarrow t_2)$; all other equations as equality axioms $(t_1 = t_2)$.

Example 37. Consider $(\mathbb{N} \bmod 3)$ with function $(+)$ and predicates $(<)$ and $(=)$:

$+$	0	1	2	$<$	0	1	2	$=$	0	1	2	\wedge	f	t	\vee	f	t
0	0	1	2	0	f	t	t	0	t	f	f	f	f	f	f	f	t
1	1	2	0	1	f	f	t	1	f	t	f	t	f	t	t	t	t
2	2	0	1	2	f	f	f	2	f	f	t						

The deductive closure of the formulas given in Fig. 56 yields the set of all valid formulas in one variable x of type $\mathbb{N} \bmod 3$ and with \wedge and \vee (and \leftrightarrow) as the only logical junctors. Pure ground formulas and formulas that are instances of others have been deleted, as well as variants modulo commutativity or idempotency of \wedge and \vee . Equations of type $(\mathbb{N} \bmod 3)$ are listed first, followed by equations of type *Bool*. Note that the former are not redundant; for example, reducing $x + 1 = 1 + x$ to t requires the equations:

$$\begin{aligned}
 & x + 1 : 1 + x \quad \text{and} \\
 & 1 + x = 1 + x : t, \quad \text{which has been subsumed by} \\
 & x = x : t.
 \end{aligned}$$

As another example, the formula $x = 0 \vee x = 1 \vee x = 2$ reduces to t via the equations:

$$\begin{aligned}
 & (x = 0) : (x < 1), \\
 & (x = 2) : (1 < x), \\
 & (1 < x) \vee (x < 1) : x + x < 2, \quad \text{and} \\
 & (1 = x) \vee (x + x < 2) : t.
 \end{aligned}$$

Remark 3. Since $(=)$ and (\leftrightarrow) are required among the predicates and junctors, respectively, Cor. 36 does not apply to the set of universally valid Horn formulas.

Remark 4. By analogy with the final remark in Sect. 11.3, we can estimate the complexity of computing all $N_1^N + \dots + N_n^N$ *rsgv* calls as $\mathcal{O}((N_1^N + \dots + N_n^N) \cdot g'^N \cdot N \cdot m' \cdot (m' + n))$, where m' denotes the maximal number of alternatives of a sort definition's right-hand side, and g' is the maximum number of disjuncts of a sort definition's right-hand side that start with the same function symbol. Figure 57 shows the runtimes and result statistics for some examples (optimizations “bdgsv” used). Column “Dom” shows the involved domains, “F” shows the functions, “P” the predicates, “J” the junctors, and “V” the number of variables. Columns “S”, “A”, “M”, and “I” show the time for setting-up, anti-unifying, enumerating the minimal terms, and removing the redundant instances,

$1+x+(x+x+2) : 0$	$(x+x < x) : 1 < x$	$(1 < x) \vee (1 = x) : 0 < x$
$x+x+x : 0$	$(x+x+2 < 1) : 1 < x$	$(x < 1) \wedge (x < 2) : x < 1$
$0+x : x$	$(x+x+2 < x) : 1 < x$	$(x < 1) \wedge (x+x < 2) : x < 1$
$1+x+2 : x$	$(x+x+2 < 2+x) : 1 < x$	$(x < 2) \wedge (x+x < 2) : x < 1$
$x+x+(x+x) : x$	$(x+x+2 < x+x+1) : 1 < x$	$(1 < 2+x) : x < 1$
$x+x+1+(x+x+2) : x$	$(x = 2) : 1 < x$	$(1 < x+x+2) : x < 1$
$x+x+1+2 : x+x$	$(1+x = 0) : 1 < x$	$(x < 2+x) : x < 1$
$1+x+x : x+x+1$	$(1+x = x+x+2) : 1 < x$	$(x < x+x+1) : x < 1$
$x+x+2+2 : x+x+1$	$(2+x = 1) : 1 < x$	$(x < x+x+2) : x < 1$
$2+x+(x+x+2) : 1$	$(2+x = x+x) : 1 < x$	$(1+x < x+x+2) : x < 1$
$x+x+1+x : 1$	$(x+x = 1) : 1 < x$	$(x+x < 1) : x < 1$
$x+1 : 1+x$	$(x+x = 2+x) : 1 < x$	$(x+x < 1+x) : x < 1$
$2+x+2 : 1+x$	$(x+x+1 = 2) : 1 < x$	$(x+x < 2+x) : x < 1$
$x+x+1+(x+x) : 1+x$	$(x+x+1 = x) : 1 < x$	$(x+x < x+x+2) : x < 1$
$x+x+2+(x+x+2) : 1+x$	$(x+x+2 = 0) : 1 < x$	$(x+x+1 < 2+x) : x < 1$
$x+2 : 2+x$	$(0 < x) \wedge (x < 2) : 1 = x$	$(x = 0) : x < 1$
$1+x+1 : 2+x$	$(0 < x) \wedge (1 = x) : 1 = x$	$(1+x = 1) : x < 1$
$x+x+(x+x+2) : 2+x$	$(x < 2) \wedge (1 = x) : 1 = x$	$(1+x = x+x+1) : x < 1$
$x+x+1+(x+x+1) : 2+x$	$(1 < 1+x) : 1 = x$	$(2+x = 2) : x < 1$
$2+x+x : x+x+2$	$(1 < x+x) : 1 = x$	$(2+x = x+x+2) : x < 1$
$x+x+1+1 : x+x+2$	$(x < x+x) : 1 = x$	$(x+x = 0) : x < 1$
$x+x+1+(1+x) : 2$	$(2+x < 1) : 1 = x$	$(x+x = x) : x < 1$
$x+x+2+x : 2$	$(2+x < 1+x) : 1 = x$	$(x+x+1 = 1) : x < 1$
$(0 < x) \wedge (x < 1) : f$	$(2+x < x+x) : 1 = x$	$(x+x+2 = 2) : x < 1$
$(1 < x) \wedge (x < 1) : f$	$(2+x < x+x+2) : 1 = x$	$(0 < 2+x) : x+x < 2$
$(1 < x) \wedge (x < 2) : f$	$(x+x+1 < 1) : 1 = x$	$(0 < x+x+1) : x+x < 2$
$(1 < x) \wedge (1 = x) : f$	$(x+x+1 < x) : 1 = x$	$(1+x < 2) : x+x < 2$
$(x < 1) \wedge (1 = x) : f$	$(x+x+1 < 1+x) : 1 = x$	$(1+x < 2+x) : x+x < 2$
$(x+x < 2) \wedge (1 = x) : f$	$(x+x+1 < x+x) : 1 = x$	$(x+x < x+x+1) : x+x < 2$
$(2 < x) : f$	$(x+x+2 < 1+x) : 1 = x$	$(1 < x) \vee (x < 1) : x+x < 2$
$(x < 0) : f$	$(x = 1) : 1 = x$	$(1 < x) \vee (x+x < 2) : x+x < 2$
$(x < x) : f$	$(1+x = 2) : 1 = x$	$(x < 1) \vee (x+x < 2) : x+x < 2$
$(x = 1+x) : f$	$(1+x = x+x) : 1 = x$	$(0 < 1+x) : x < 2$
$(x = 2+x) : f$	$(2+x = 0) : 1 = x$	$(0 < x+x+2) : x < 2$
$(1+x = 2+x) : f$	$(2+x = x+x+1) : 1 = x$	$(x < 1+x) : x < 2$
$(x+x = x+x+1) : f$	$(x+x = 2) : 1 = x$	$(x+x+1 < 2) : x < 2$
$(x+x = x+x+2) : f$	$(x+x+1 = 0) : 1 = x$	$(x+x+1 < x+x+2) : x < 2$
$(x+x+1 = x+x+2) : f$	$(x+x+1 = 2+x) : 1 = x$	$(x < 1) \vee (x < 2) : x < 2$
$(0 < x) \wedge (1 < x) : 1 < x$	$(x+x+2 = 1) : 1 = x$	$(x < 1) \vee (1 = x) : x < 2$
$(0 < x) \wedge (x+x < 2) : 1 < x$	$(x+x+2 = x) : 1 = x$	$(x < 2) \vee (1 = x) : x < 2$
$(1 < x) \wedge (x+x < 2) : 1 < x$	$(0 < x+x) : 0 < x$	$(x = x) : t$
$(1 < x+x+1) : 1 < x$	$(2+x < 2) : 0 < x$	$(0 < x) \vee (x < 1) : t$
$(1+x < 1) : 1 < x$	$(2+x < x) : 0 < x$	$(0 < x) \vee (x < 2) : t$
$(1+x < x) : 1 < x$	$(x+x+2 < 2) : 0 < x$	$(0 < x) \vee (x+x < 2) : t$
$(1+x < x+x) : 1 < x$	$(x+x+2 < x+x) : 0 < x$	$(1 < x) \vee (x < 2) : t$
$(1+x < x+x+1) : 1 < x$	$(0 < x) \vee (1 < x) : 0 < x$	$(x < 2) \vee (x+x < 2) : t$
$(2+x < x+x+1) : 1 < x$	$(0 < x) \vee (1 = x) : 0 < x$	$(1 = x) \vee (x+x < 2) : t$

Fig. 56. Theory of $\mathbb{N} \bmod 3$ with $+$, $<$, $=$, and *Bool* with \wedge and \vee

respectively. “ $> n$ ” means running out of memory after n seconds. Column “ Σ ” shows the total time; column “Sz” shows the number of non-redundant equations (without manual deletions).

Remark 5. The above results refer to finite algebras only, i.e. models with finite \mathcal{T}/E . It remains to be uninvestigated whether, given an arbitrary equational theory, we can find a finite “test set” $\mathcal{T}_{test}/E \subset \mathcal{T}/E$ of ground instances such that an equation is universally valid iff it holds for all instances from the test set.

Dom	F	P	J	V	S	A	M	I	Σ	Sz
$\mathbb{N} \bmod 2$	+			2	4	0	1	5	6	27
$\mathbb{N} \bmod 2$	+			3	0	3881	0	11	3892	105
$\mathbb{N} \bmod 2, Bool$	+	$<, =$	\wedge, \vee	2, 0	0	18	0	41	59	452
$\mathbb{N} \bmod 3, Bool$	+	$<, =$	\wedge, \vee	1, 0	0	30	1	17	48	307
$\mathbb{N} \bmod 3, Bool$	+	$<, =$	\wedge, \vee	2, 0	20	>143				

Fig. 57. Axiomatization Runtimes and Results

References

- AM91. A. Aiken and B. Murphy. Implementing regular tree expressions. In *ACM Conference on Functional Programming Languages and Computer Architecture*, pages 427–447, August 1991.
- Ave89. Jrgen Avenhaus. Transforming infinite rewrite systems into finite rewrite systems by embedding techniques. SEKI-Report SR–89–21, Univ. Kaiserslautern, 1989.
- BM79. R.S. Boyer and J.S. Moore. *A Computational Logic*. Academic, New York, 1979.
- Bur93. Jochen Burghardt. *Eine feinkrnlige Sortendisziplin und ihre Anwendung in der Programmkonstruktion*. PhD thesis, Univ. Karlsruhe, 1993.
- BvHSI90. Alan Bundy, Frank van Harmelen, Alan Smaill, and Andrew Ireland. Extensions to the rippling-out tactic for guiding inductive proofs. In *Proc. 10th CADE*, volume 449 of *LNAI*, pages 132–146. Springer, 1990.
- Com90. Hubert Comon. Equational formulas in order-sorted algebras. In *Proc. ICALP*, 1990.
- Com95. Hubert Comon. On unification of terms with integer exponents. *Math. Systems Theory*, 28:67–88, 1995.
- Emm94. Helmut Emmelmann. *Codeselektion mit regulr gesteuerter Termersetzung*. PhD thesis, University Karlsruhe, 1994.
- GM93. Peter Graf and Christoph Meyer. Extended path-indexing. Technical Report MPI-I-93-253, Max-Planck-Institut fr Informatik, Saarbrcken, Dec 1993.
- Gra92. Peter Graf. Path indexing for term retrieval. Technical Report MPI-I-92-237, Max-Planck-Institut fr Informatik, Saarbrcken, April 1992.
- Gra94. Peter Graf. Substitution tree indexing. Technical Report MPI-I-94-251, Max-Planck-Institut fr Informatik, Saarbrcken, Oct 1994.
- Hei94. Birgit Heinz. Lemma discovery by anti-unification of regular sorts. Technical Report 94–21, TU Berlin, 1994.
- Hei95. Birgit Heinz. *Anti-Unifikation modulo Gleichungstheorie und deren Anwendung zur Lemmagenerierung*. PhD thesis, TU Berlin, Dec 1995.
- HH94. Dieter Hofbauer and Maria Huber. Linearizing term rewriting systems using test sets. *J. Symbolic Computation*, 17:91–129, 1994.
- Hum90. B. Hummel. *Generierung von Induktionsformeln und Generalisierung beim automatischen Beweisen mit vollstndiger Induktion*. PhD thesis, University Karlsruhe, May 1990.
- Kir87. H. Kirchner. Schematization of infinite sets of rewrite rules. Application to the divergence of completion processes. In *Proc. Conf. on Rewriting Techniques and Applications*, volume 256 of *LNCS*, pages 180–191. Springer, May 1987.
- Kir89. Hlène Kirchner. Schematization of infinite sets of rewrite rules generated by divergent completion processes. *Theoretical Computer Science*, 67:303–332, 1989.
- MW80. Zohar Manna and Richard Waldinger. A deductive approach to program synthesis. *ACM Transactions on Programming Languages and Systems*, 2:90–121, Jan 1980.
- Plo70. Gordon D. Plotkin. A note on inductive generalization. *Machine Intelligence*, 5:153–163, 1970.
- Plo71. Gordon D. Plotkin. A further note on inductive generalization. *Machine Intelligence*, 6:101–124, 1971.
- Rey70. John C. Reynolds. Transformational systems and the algebraic structure of atomic formulas. *Machine Intelligence*, 5:135–151, 1970.

Appendix

A PROLOG Source Code

```
:-
    op(910,xfx,:=),          % substitution
    op(831,xfx,sortdef),     % user sort definition
    op(831,xfx,sortdf),      % system sort definition
    op(831,xfx,inf),          % sort infimum
    op(831,yfx,\),           % sort difference
    op(831,fx,def),           % defining equation
    op(812,xfx,->),           % series rewrite rule
    op(810,yfx,!),            % sort disjunction
    op(700,xfy,:).            % user equal

:-
    assert((Sort sortdef SortDef :- Sort sortdf SortDef)).

% %%%%%%%%% functional dependency detection %%%%%%%%% %
% %%%%%%%%% lemma generation %%%%%%%%% %
% %%%%%%%%% series guessing %%%%%%%%% %

find_dependency(Eqns,AuL:Dep) :-
    eval_term_list(Eqns,EqnsE),
    strip_common_constructor(EqnsE,(:),[Lhss,Rhss]),
    equiv_class_list(Rhss,RhssC),
    abolish(origin_v,2),
    sg(Lhss,AuL),
    setof((Ss,var,V),origin_v(Ss,V),OccL),
    rgbv([/*inf*/999999],RhssC,OccL,AuR),
    write(generalized(AuR)), nl,
    %init_red,
    %normalforms_sort(NfS),
    %inf(AuR,NfS,AuR1),
    %write(filtered(AuR1)), nl,
    abolish(redices,1),
    assert(redices([])),
    !,
    find_rhs(AuR,[],Dep).

% %%%%%%%%% lemma generation %%%%%%%%% %
% %%%%%%%%% series guessing %%%%%%%%% %

% given term Lhs, generate a rhs candidate RhS
%<*backtrackable*>
generate_lemma(Lhs,Rhs) :-
    read_instances(Lhs,Eqns),
    write('normalized:'), nl,
    beautify_output(Eqns,EqnsB),
    write_list(EqnsB), nl,
    strip_common_constructor(Eqns,(:),[Lhss,Rhss]),
    abolish(origin_v,2),
    sg(Lhss,AuL),
    setof((Ss,var,V),origin_v(Ss,V),OccL),
    rsgv([/*inf*/999999,Rhss,OccL,AuR),
    write('generalized: '),
    beautify_output(AuL:AuR,AuLAuRB),
    write(AuLAuRB), nl,
    init_red,
    normalforms_sort(NfS),
    inf(AuR,NfS,AuR1),
    abolish(redices,1),
    assert(redices([])),
    !,
    find_rhs(AuR1,[],Rhs).
```

```

guess_series_b(N,M,Series,LawB,PrognoseB) :-
    guess_series(N,Series,Law),
    prognose_series(Series,Law,M,Prognose),
    beautify_output(Law,LawB),
    beautify_output(Prognose,PrognoseB).

% given the list Series and the example count N,
% generate a construction law candidate term LawL->LawR
%<*backtrackable*>
guess_series(N,Series,LawL->LawR) :-
    eval_term_list(Series,SeriesN),
    %: [s(s(s(s(0))))],s(0),0]
    bagof(Suffix,suffix(Suffix,SeriesN),Suffixes),
    %: [[s(s(s(s(0))))],s(0),0],[s(0),0],[0],[[]]
    reverse(Suffixes,[[[]|SuffixesR]]),
    %: [[s(0),0],[s(s(s(s(0))))],s(0),0]]
    merge_lists(Heads,(.),Tails,SuffixesR),
    %: Heads = [s(0),s(s(s(s(0))))]
    %: Tails = [[0],[s(0),0]]
    equiv_class_list(Heads,HeadsE),
    %: HeadsE = [s1,s4]
    length(HeadsE,Lgth),
    make_0s_list_of_length(Lgth,Index),
    merge_lists(Index,(.),Tails,TailsI),
    %: TailsI = [[s(0),0],[s(s(0)),s(0),0]]
    make_varlist_of_length(N,TailsS),
    make_varlist_of_length(N,HeadsS),
    suffix(TailsS,TailsI),
    suffix(HeadsS,HeadsE),
    beautify_output(TailsS:HeadsS,TailsSHeadsSB),
    write('anti-unifying: '), write(TailsSHeadsSB), nl,
    abolish(origin_v,2),
    sg(TailsS,LawL),
    setof((Ss,var,V),origin_v(Ss,V),OccT),
    rsgv(/*inf*/999999,HeadsS,OccT,AuH),
    beautify_output(LawL:AuH,LawLAuHB),
    write('generalized: '), write(LawLAuHB), nl,
    init_red,
    normalforms_sort(NfS),
    inf(AuH,NfS,AuH1),
    abolish(redices,1),
    assert(redices([])),
    !,
    find_rhs(AuH1,[],LawR).

prognose_series(Series,Law,Lgth,Prognose) :-
    length(Series,Index),
    eval_term_t([Index|Series],SeriesE),
    prognose_series1(SeriesE,Law,Lgth,[_IndexP|Prognose]), !.

prognose_series1(Series,_Law,0,Series) :- !.

prognose_series1([Index|Series],LawL->LawR,Lgth,Prog) :-
    try_rewrite([Index|Series],LawL,LawR,P),
    eval_term_t(P,PE),
    Lgth1 is Lgth - 1,
    prognose_series1([s(Index),PE|Series],LawL->LawR,Lgth1,Prog), !.

prognose_series1([Index|Series],LawL->LawR,Lgth,[in|prognose_failed]) :-
    beautify_output(LawL->LawR,LawB),
    beautify_output([Index|Series],SeriesB),
    write(prognose_failed(SeriesB,LawB,Lgth)), nl, !.

prognose_series1(A,B,C,D) :-
    dont_backtrack(prognose_series1(A,B,C,D)).

% %%%%%%%%%%%
% %%%%%%%%%%% theory computation for finite algebras %%%%%%%%%%%
% %%%%%%%%%%%

% given the list of lists Eqcss of equivalence classes
% and the list Ns of variable numbers,
% print an axiom system for the corresponding finite algebra,
% Consts gives the constants forbidden in the result
%: compute_theory([s0,s1,s2],[st,sf],[2,0],[[0,1,2],[t,f]])
%: where s0,s1,s2 denote the equivalence class of 0,1,2, respectively

```

```

%: in N mod 3,
%: st,sf denote the class of true,false in Bool, respectively
compute_theory(Eqcss,Ns,Constss) :-
    trace_write(prof,set_up),
    calc_sort_depths,
    first_groundterms_t_list_list(Eqcss,Nfss),
    % generate variables
    extend_to_length_list(vv,Ns,VVs1),
    new_name_list_list(VVs1,VVs),
    extend_to_same_length_list(0,VVs,Nulls),
    merge_lists_list(VVs,arity,Nulls,Arities),
    assert_list_list(Arities),
    make_varlist_of_same_length_list(VVs,VSorts),
    merge_lists_list(VVs,is_variable,VSorts,IsVariables),
    assert_list_list(IsVariables),
    % generate variable assignments
    setof(Map,(
        mapping_list12(VVs,Nfss,Map)
    ),Maps),
    transpose(Maps,Maps1),
    merge_lists_list(_VNsg,(:=),VSss,Maps1),
    extend_to_same_length_list(var,VVs,Vars),
    merge_lists_list(Vars,(','),VVs,VVs2),
    append_list(VVs2,VVs3),
    merge_lists(VSss,(','),VVs3,Occl),
    % generate rsgv input sorts
    Occl = [(Dom1,var,_V)|_],
    length(Dom1,DomLg),
    make_natlist_of_length(DomLg,Dom),
    bagof(Matrix,Eqcs^Col^Places^Map^(
        member(Eqcs,Eqcss),
        setof(Col,Places^Map^(
            mapping(Dom,Eqcs,Map),
            merge_lists(Places,(:=),Col,Map)
        ),Matrix)
    ),Matrixes),
    % anti-unify
    trace_write(prof,anti_unify),
    rsgv_list_list(/*inf*/999999,Matrixes,Occl,Auss1),
    trace_write(prof,transform_to_axioms),
    remove_constants_list_list12(Auss1,Constss),
    calc_sort_depths,
    append_list(Auss1,Aus1),
    remove_empty_sorts(Aus1,Aus),
    first_groundterms_t_list(Aus,MinTerms),
    merge_lists(Aus,(:),MinTerms,SMinL),
    trace_write(prof,min_terms),
    write_list(SMinL),
    conv_list_to_bbt(SMinL,SMinBBT),
    min_eqns_t_list(Aus,SMinBBT,Eqns),
    trace_write(prof,remove_instances),
    remove_instances([],Eqns,Eqns1),
    beautify_output(Eqns1,EqnsB),
    nl, write_list(EqnsB), nl, !.

% remove all constants in definition of Sort that are in Consts
remove_constants(Sort,Consts) :-
    retract(Sort sortdf Def),
    flatten_op(Def,(!),DefF),
    list_trisection(DefF,Consts,DefRF,_,_),
    { unflatten_op(DefRF,(!),DefR)
    ; DefR = bottom
    },
    assert(Sort sortdf DefR), !.

remove_constants(A,B) :-
    dont_backtrack(remove_constants(A,B)).

remove_constants_list([Sort|Sorts],Consts) :-
    remove_constants(Sort,Consts),
    remove_constants_list(Sorts,Consts), !.

remove_constants_list([],_Consts) :- !.

remove_constants_list_list12([Sorts|Sortss],[Consts|Constss]) :-
    remove_constants_list(Sorts,Consts),
    remove_constants_list_list12(Sortss,Constss), !.

remove_constants_list_list12([],[]) :- !.

% remove all equations from EqnsNew that are subsumed by others in
% EqnsNew or EqnsOld, yielding EqnsR

```

```

remove_instances(EqnsOld, [LN:RN|EqnsNew], EqnsR) :-
    try_delete_from_list(LO:RO, EqnsOld, EqnsOld1),
    ( is_instance_118(RN:LN, RO:LO, _Subst),
      remove_instances(EqnsOld, EqnsNew, EqnsR)
    ; is_instance_118(RO:LO, RN:LN, _Subst),
      remove_instances(EqnsOld1, [LN:RN|EqnsNew], EqnsR)
    ), !.

remove_instances(EqnsOld, [LN:RN|EqnsNew], EqnsR) :-
    remove_instances([LN:RN|EqnsOld], EqnsNew, EqnsR), !.

remove_instances(EqnsOld, [], EqnsR) :-
    reverse(EqnsOld, EqnsR), !.

```

```

% %%%%%%%%%% %
% %%%%%%%%%% syntactic anti-unification %%%%%%%%%% %
% %%%%%%%%%% %

```

```

% calc the sort Au of anti-unifiers of all ground terms in Ss
sg_t(Ss, Au) :-
    trace_enter(sg, sg(Ss)),
    sg(Ss, Au),
    trace_leave(sg, sg(Ss) = Au).

```

```

%- decompose common constructor
sg(Ss, Au) :-
    strip_common_constructor(Ss, Cr, SsArgs),
    make_varlist_of_same_length(SsArgs, AuArgs),
    Au =.. [Cr|AuArgs],
    sg_list(SsArgs, AuArgs), !.

```

```

%- different constructors, return old variable
sg(Ss, Au) :-
    origin_v(Ss, Au), !.

```

```

%- different constructors, return new variable
sg(Ss, Au) :-
    new_name(v, Au),
    assert(origin_v(Ss, Au)),
    assert(is_variable(Au, top)),
    assert(arity(Au, 0)), !.

```

```

sg_list([Ss|Sss], [Au|Aus]) :-
    sg_t(Ss, Au),
    sg_list(Sss, Aus), !.

```

```

sg_list([], []) :- !.

```

```

% %%%%%%%%%% %
% %%%%%%%%%% simple sort anti-unification %%%%%%%%%% %
% %%%%%%%%%% %

```

```

% calc the sort Au of anti-unifiers of all sorts in Ss,
% defining new sorts recursively as needed
hsg(Ss, Au) :-
    hsg_t(Ss, Au0, nil, OccBT),
    conv_bbtlist_to_list([OccBT], Occ),
    postprocess_hsg(Au0, Au1, Occ, SortDefs1, VarSubst),
    extract_defs_from_occ(SortDefs1, SortDefs2, xxx, _Memo),
    apply_subst_sc([Au1, SortDefs2], VarSubst, [Au2, SortDefs3]),
    simplify_variables([Au2, SortDefs3], [Au, SortDefs4]),
    assert_list(SortDefs4), !.

```

```

%- calc the NameAu of anti-unifiers of all sorts in Ss,
%- OccIn, OccOut is the binary tree of triples (Ss, AuSs, NameAu)
%- for which hsg already has been called,
%- AuSs may be uninstantiated until the completion of the call,
%- the meaning of a triple is: NameAu sortdef AuSs, hsg(Ss) = NameAu
hsg_t(Ss, Au, OccIn, OccOut) :-
    trace_enter(hsg, hsg(Ss)),
    hsg(Ss, Au, OccIn, OccOut),
    trace_leave(hsg, hsg(Ss) = Au).

```

```

%- loop check
hsg(Ss, NameAu, OccIn, OccIn) :-
    member_bbt((Ss, _DefAu, NameAu), OccIn),

```

```

{ nonvar(NameAu)
; new_name(sort,NameAu),
  assert(origin_s(NameAu,Ss))
}, !.

%- replace sort names by definitions,
%- then distribute "!"
hsg(Ss,Au,OccIn,OccOut) :-
  replace_defs(Ss,SsDef,OccIn,Occ1,DefAu,NameAu,true),
  { member(_!,SsDef),
    %: [0!s0+s0!s0*top!top*s0,s(s0)!s1+s0!s0+s1!s1*s1]
    flatten_op_list1(SsDef,(!),Ss1),
    %: [[0,s0+s0,s0*top,top*s0],[s(s0),s1+s0,s0+s1,s1*s1]]
    sort_list(Ss1,Ss2),
    %: [[0,s0*top,top*s0,s0+s0],[s(s0),s1*s1,s0+s1,s1+s0]]
    group_cr_sorts(Ss2,[],Groups),
    %: [[s0*top,s1*s1],[top*s0,s1*s1],
      %: [s0+s0,s0+s1],[s0+s0,s1+s0]]
    hsg_list(Groups,Aus,Occ1,Occ2),
    { origin_v(Ss,V),
      OccOut = Occ2,
      unflatten_op([V|Aus],(!),DefAu)
    ; new_name(v,V),
      assert(origin_v(Ss,V)),
      assert(is_variable(V,top)),
      assert(arity(V,0)),
      enter_into_bbt((Ss,var,V),Occ2,OccOut),
      % same sort pairs get same variable
      % necessary for calculating var intersections
      unflatten_op([V|Aus],(!),DefAu)
    }
  ; hsg_t(SsDef,DefAu,Occ1,OccOut)
},
{ var(NameAu),
  Au = DefAu
; Au = NameAu
}, !.

%- decompose common constructor
hsg(Ss,Au,OccIn,OccOut) :-
  strip_common_constructor(Ss,Cr,SsArgs),
  make_varlist_of_same_length(SsArgs,AuArgs),
  Au =.. [Cr|AuArgs],
  hsg_list(SsArgs,AuArgs,OccIn,OccOut), !.

%- different constructors, return new variable
hsg(Ss,Au,OccIn,OccOut) :-
  new_name(v,Au),
  assert(origin_v(Ss,Au)),
  assert(is_variable(Au,top)),
  assert(arity(Au,0)),
  enter_into_bbt((Ss,var,Au),OccIn,OccOut), !.

hsg_list([Ss|Sss],[Au|Aus],OccIn,OccOut) :-
  hsg_t(Ss,Au,OccIn,Occ1),
  hsg_list(Sss,Aus,Occ1,OccOut), !.

hsg_list([],[],OccIn,OccIn) :- !.

% set up variable substitution to consider sort intersections
postprocess_hsg(Au,AuS,Occ,SortDefs,VarSubst) :-
  split_sort_var_defs(Occ,SortDefs1,VarDefs1),
  simplify_sorts_and_defs([Au],[AuS],SortDefs1,SortDefs),
  calc_sort_infs(VarDefs1,Infs),
  conv_list_to_bbt(Infs,InfsBT),
  calc_inf2s(VarDefs1,[],L2,InfsBT),
  calc_varcnt(L2,VarCnt1),
  sort(VarCnt1,VarCnt2),
  sort(VarDefs1,VarDefs1S),
  collect_varcnt(['XXX':0|VarCnt2],VarDefs1S,VarCnt),
  conv_list_to_bbt(VarCnt,VarCntBT),
  conv_list_to_bbt(L2,BT2),
  calc_infs_list(L2,VarCnt,VarCntBT,BT2,[],VarDefs2),
  calc_var_subst(VarDefs2,VarSubst), !.

% calculate all nonempty intersections of two occurring sort names
calc_sort_infs(VarDefs,Infs) :-
  calc_sort_infs1(VarDefs,Sorts),
  sort(Sorts,Sorts1),
  calc_sort_infs2(Sorts1,Infs), !.

% collect all sorts

```

```

%: calc_sort_infs1([(v8,v8,[s0,s1]),(v4,v4,[s0,s2])],[s0,s1,s0,s2])
calc_sort_infs1([(_R,V,Ss)|List],Sorts) :-
    calc_sort_infs1(List,Sorts2),
    append(Ss,Sorts2,Sorts), !.

calc_sort_infs1([],[]) :- !.

% calculate all nonempty intersections of any two sorts in Sorts
%: calc_sort_infs2([s0,s1,s2,top],[(s0,top,s0),(s1,top,s1),(s2,top,s2)])
calc_sort_infs2([Sort|Sorts],Infs) :-
    calc_sort_infs3(Sort,Sorts,Infs1),
    calc_sort_infs2(Sorts,Infs2),
    append(Infs1,Infs2,Infs), !.

calc_sort_infs2([],[]) :- !.

% calc all nonempty intersections of Sort with each element in Sorts
%- intersection of Sort and Sort0 is nonempty
calc_sort_infs3(Sort,[Sort0|Sorts0],[(Sort,Sort0,Sort1)|Infs]) :-
    inf(Sort,Sort0,Sort1),
    inh(Sort1),
    calc_sort_infs3(Sort,Sorts0,Infs), !.

%- intersection of Sort and Sort0 is empty
calc_sort_infs3(Sort,[_Sort0|Sorts0],Infs) :-
    calc_sort_infs3(Sort,Sorts0,Infs), !.

calc_sort_infs3(_Sort,[],[]) :- !.

% calculate all nonempty intersections of two variables,
% InfsBT is a binary tree with an entry (S,S0,S1) for each pair of
% sort names S,S0 with non-empty intersection S1
calc_inf2s([(V,V,Ss)|List],In,Out,InfsBT) :-
    calc_inf2s1([V],V,Ss,List,In,Mid,InfsBT),
    calc_inf2s(List,Mid,Out,InfsBT), !.

calc_inf2s([],In,In,_InfsBT) :- !.

% calculate all nonempty intersections of V with each element in List
%- intersection of V and V0 is nonempty
calc_inf2s1([V],V,Ss,[([V0],V0,S0s)|List],In,Out,InfsBT) :-
    calc_inf2s2(Ss,S0s,S2s,InfsBT),
    { origin_v(S2s,V2),
      ; new_name(v,V2),
        assert(origin_v(S2s,V2)),
        assert(is_variable(V2,top)),
        assert(arity(V2,0))
    },
    sort([V,V0],R2),
    calc_inf2s1([V],V,Ss,List,[(R2,V2,S2s)|In],Out,InfsBT), !.

%- intersection of V0 and V1 is empty
calc_inf2s1([V],V,Ss,[([V0],V0,_S0s)|List],In,Out,InfsBT) :-
    calc_inf2s1([V],V,Ss,List,In,Out,InfsBT), !.

calc_inf2s1([V],V,_Ss,[],In,In,_InfsBT) :- !.

% test all intersections S1s of Ss with S0s
calc_inf2s2([S|Ss],[S|S0s],[S|S1s],InfsBT) :-
    !,
    calc_inf2s2(Ss,S0s,S1s,InfsBT), !.

calc_inf2s2([S|Ss],[S0|S0s],[S1|S1s],InfsBT) :-
    member_bbt((S,S0,S1),InfsBT),
    !,
    calc_inf2s2(Ss,S0s,S1s,InfsBT), !.

calc_inf2s2([S|Ss],[S0|S0s],[S1|S1s],InfsBT) :-
    member_bbt((S0,S,S1),InfsBT),
    !,
    calc_inf2s2(Ss,S0s,S1s,InfsBT), !.

calc_inf2s2([],[],[],_InfsBT) :- !.

% calculate all possible intersection lists containing R0
calc_infs_list([(R0,V0,S0s)|List2],List,ListBT,BT2,In,Out) :-
    length(R0,LgR0),
    calc_infs_list_H1(R0,LgR0,ListBT),
    calc_infs(R0,V0,S0s,List,[],BT2,In,Mid),
    calc_infs_list(List2,List,ListBT,BT2,Mid,Out), !.

calc_infs_list([(R0,V0,S0s)|List2],List,ListBT,BT2,In,Out) :-

```

```

        calc_infs_list(List2,List,ListBT,BT2,[(R0,V0,S0s)|In],Out), !.
calc_infs_list([],_List,_ListBT,_BT2,In,In) :- !.

calc_infs_list_H1([V|Vs],Lg,ListBT) :-
    member_bbt([(V,V,_Ss):Cnt,ListBT),
    Cnt >= Lg,
    !,
    calc_infs_list_H1(Vs,Lg,ListBT), !.
calc_infs_list_H1([],_Lg,_ListBT) :- !.

%- V is already in R0, ignore
calc_infs(R0,V0,S0s,[(V,V,_Ss):_Cnt|List],Lsusp,BT2,In,Out) :-
    member(V,R0),
    calc_infs(R0,V0,S0s,List,Lsusp,BT2,In,Out), !.

%- V does not occur often enough, ignore
calc_infs(R0,V0,S0s,[(V,V,_Ss):Cnt|List],Lsusp,BT2,In,Out) :-
    length(R0,LgR0),
    Cnt < LgR0,
    calc_infs(R0,V0,S0s,List,Lsusp,BT2,In,Out), !.

%- [V|R0] has 2 element sublist with empty intersection, ignore
calc_infs(R0,V0,S0s,[(V,V,_Ss):_Cnt|List],Lsusp,BT2,In,Out) :-
    member(VR0,R0),
    sort([VR0,V],VR0V),
    not member_bbt((VR0V,_,_),BT2),
    calc_infs(R0,V0,S0s,List,Lsusp,BT2,In,Out), !.

%- [V|R0] is sublist of a nonempty intersection list, suspend V
calc_infs(R0,V0,S0s,[(V,V,_Ss):Cnt|List],Lsusp,BT2,In,Out) :-
    sort([V|R0],R2),
    member((RR,_VV,_SSs),In),
    is_sublist(R2,RR),
    calc_infs(R0,V0,S0s,List,[(V,V,_Ss):Cnt|Lsusp],BT2,In,Out), !.

%- [V|R0] has nonempty intersection
calc_infs(R0,V0,S0s,[(V,V,_Ss):_Cnt|List],Lsusp,BT2,In,Out) :-
    sort([V|R0],R2),
    calc_infs2(S0s,Ss,S1s,0ccsS),
    calc_infs3(S1s,0ccsS,S2s),
    { origin_v(S2s,V2)
    ; new_name(v,V2),
      assert(origin_v(S2s,V2)),
      assert(is_variable(V2,top)),
      assert(arity(V2,0))
    },
    append(List,Lsusp,List1),
    calc_infs(R2,V2,S2s,List1,[],BT2,In,Mid),
    calc_infs(R0,V0,S0s,List1,[],BT2,Mid,Out), !.

%- [V|R0] has empty intersection, ignore
calc_infs(R0,V0,S0s,[(V,V,_Ss):_Cnt|List],Lsusp,BT2,In,Out) :-
    calc_infs(R0,V0,S0s,List,Lsusp,BT2,In,Out), !.

%- maximal intersection list subsumed, ignore
calc_infs(R0,_V0,_S0s,[],[],_BT2,In,In) :-
    member((RR,_VV,_SSs),In),
    is_sublist(R0,RR), !.

%- maximal intersection list, report
calc_infs(R0,V0,S0s,[],[],_BT2,In,[(R0,V0,S0s)|In]) :- !.

%- maximal intersection list subsumed, ignore
calc_infs(_R0,_V0,_S0s,[],_Lsusp,_BT2,In,In) :- !.

% test all intersections of Ss with S0s
%<may fail*>
calc_infs2([S|Ss],[S0|S0s],[S1|S1s],[0ccS|0ccsS]) :-
    inf_t(S,S0,S1,nil,0ccSBT),
    conv_bbtlist_to_list([0ccSBT],0ccS),
    inh_t(S1,[],0ccS),
    calc_infs2(Ss,S0s,S1s,0ccsS), !.

calc_infs2([],[],[],[]) :- !.

% simplify all intersections S1s

```



```

calc_infs3([S1|S1s],[OccS|OccsS],[S2|S2s]) :-
    simplify_sort1(OccS,OccsS),
    simplify_sort4(S1,S2),
    extract_defs_from_occ(OccsS,DefsS,memo_inf,Memo),
    assert_list(DefsS),
    assert_list(Memo),
    calc_infs3(S1s,OccsS,S2s), !.

calc_infs3([],[],[]) :- !.

%: calc_v([(a,b),_,_],([a,c],_,_),[a:[a,b],b:[a,b],a:[a,c],c:[a,c]])
calc_varcnt([(R0,_V0,_S0s)|List2],VarCnts) :-
    calc_varcnt1(R0,R0,VarCnts1),
    calc_varcnt(List2,VarCnts2),
    append(VarCnts1,VarCnts2,VarCnts), !.

calc_varcnt([],[]) :- !.

%: calc_varcnt1([a,b],[a,b],[a:[a,b],b:[a,b]])
calc_varcnt1([V|Vs],R0,[V:R0|VarCnts]) :-
    calc_varcnt1(Vs,R0,VarCnts), !.

calc_varcnt1([],_R0,[]) :- !.

%: collect_varcnt(['XXX':0,a:[a,b],a:[a,c],b:[a,b],c:[a,c]],
%                [([a],a,s1),([b],b,s2),([c],c,s3)],
%                [([a],a,s1):2,([b],b,s2):1,([c],c,s3):1])
%- collect multiple occurrences of V
collect_varcnt([V:Cnt,V:_R|Vars],Group,VarCnts) :-
    Cnt1 is Cnt + 1,
    collect_varcnt([V:Cnt1|Vars],Group,VarCnts), !.

%- V1 in Vars and in Group, join
collect_varcnt([V1:Cnt1,V2:_R2|Vars],[([V1],V1,S1)|Group],
               [([V1],V1,S1):Cnt1|VarCnts]) :-
    collect_varcnt([V2:1|Vars],Group,VarCnts), !.

%- V1 not in Group, ignore
collect_varcnt([V1:_Cnt1,V2:_R2|Vars],[([V3],V3,S3)|Group],VarCnts) :-
    V1 @< V3,
    collect_varcnt([V2:1|Vars],[([V3],V3,S3)|Group],VarCnts), !.

%- V3 not in Vars, ignore
collect_varcnt([V1:Cnt1,V2:_R2|Vars],[([V3],V3,_S3)|Group],VarCnts) :-
    V3 @< V1,
    collect_varcnt([V1:Cnt1,V2:1|Vars],Group,VarCnts), !.

%- V in Vars and in Group, join
collect_varcnt([V:Cnt],[([V],V,S)|_Group],[([V],V,S):Cnt]) :- !.

%- V1 not in Group, ignore
collect_varcnt([V1:_Cnt1],[([V3],V3,_S3)|_Group],[]) :-
    V1 @< V3, !.

%- V3 not in Vars, ignore
collect_varcnt([V1:Cnt1],[([V3],V3,_S3)|Group],VarCnt) :-
    V3 @< V1,
    collect_varcnt([V1:Cnt1],Group,VarCnt), !.

collect_varcnt([],_Group,[]) :- !.

collect_varcnt(_Vars,[],[]) :- !.

% build variable substitution from intersection list
calc_var_subst(VarDefs,VarSubst) :-
    calc_var_subst1(VarDefs,[],VarSubst1),
    sort(VarSubst1,VarSubst2),
    calc_var_subst3(VarSubst2,xxx:=xxx,[xxx:=xxx|VarSubst3]),
    conv_list_to_bbt(VarSubst3,VarSubst), !.

calc_var_subst1([(Vs,Vnew,_Ss)|VarDefs],VarSubstIn,VarSubstOut) :-
    calc_var_subst2(Vnew,Vs,VarSubstIn,VarSubst1),
    calc_var_subst1(VarDefs,VarSubst1,VarSubstOut), !.

calc_var_subst1([],VarSubstIn,VarSubstIn) :- !.

calc_var_subst2(Vnew,[V|Vs],VarSubstIn,VarSubstOut) :-
    calc_var_subst2(Vnew,Vs,[V:=Vnew|VarSubstIn],VarSubstOut), !.

calc_var_subst2(_Vnew,[],VarSubstIn,VarSubstIn) :- !.

```

```

calc_var_subst3([V:=Vnew1|VarSubst],V:=Vnew,VarSubst0) :-
    calc_var_subst3(VarSubst,V:=Vnew!Vnew1,VarSubst0), !.

calc_var_subst3([V1:=Vnew1|VarSubst],V:=Vnew,[V:=Vnew|VarSubst0]) :-
    calc_var_subst3(VarSubst,V1:=V1!Vnew1,VarSubst0), !.

calc_var_subst3([],V:=Vnew,[V:=Vnew]) :- !.

% %%%%%%%%%%%%%% variable-restricted sort anti-unification %%%%%%%%%%%%%% %
% %%%%%%%%%%%%%% variable-restricted sort anti-unification %%%%%%%%%%%%%% %
% %%%%%%%%%%%%%% variable-restricted sort anti-unification %%%%%%%%%%%%%% %

:-
    assert(origin_s_bbt([],nil)).

% calc the sort Au of anti-unifiers of all sorts in Ss,
% proceed breadth-first,
% defining new sorts recursively as needed,
% OccVL is a list containing all admitted variables
% <*backtrackable*>
rgbv(Par,Ss,OccVL,Au) :-
    sort(OccVL,OccVL1),
    { origin_s_bbt(OccVL1,BTin)
    ; BTin = nil
    },
    ( member_bbt((Ss,_Def,_Depth,Au),BTin)
    ; build_occ_var_trie(OccVL1,[],OccVT),
      new_name(sort,Au),
      { retract(origin_s_bbt(OccVL1,BTin))
      ; BTin = nil
      },
      enter_into_bbt((Ss,_DefAu,Dp,Au),BTin,BT1),
      rgbv_t([Dp,[Ss:Dp:DEx:Au]]|Par,[Ss:Dp:DEx:Au,layer:0],OccVL1,OccVT,BT1,BTo),
      ( assert(origin_s_bbt(OccVL1,BTo))
      ; retract(origin_s_bbt(OccVL1,_BT)),
        fail
      )
    )
).

rgbv_list(Par,[Ss|Sss],Occ,[Au|Aus]) :-
    rgbv(Par,Ss,Occ,Au),
    rgbv_list(Par,Sss,Occ,Aus), !.

rgbv_list(_Par,[],_Occ,[]) :- !.

rgbv_list_list(Par,[Sss|Ssss],Occ,[Aus|Auss]) :-
    rgbv_list(Par,Sss,Occ,Aus),
    rgbv_list_list(Par,Ssss,Occ,Auss), !.

rgbv_list_list(_Par,[],_Occ,[]) :- !.

simp_sort_depths_list([Layer|Layers],[LayerS|LayersS]) :-
    simp_sort_depths_t(Layer,LayerS,Chg),
    { Chg = no,
      LayersS = Layers
    ; simp_sort_depths_list(Layers,LayersS)
    }, !.

simp_sort_depths_list([],[]) :- !.

simp_sort_depths_t(Layer,LayerS,Chg) :-
    trace_enter(sd,simp(Layer)),
    simp_sort_depths(Layer,LayerS,Chg),
    trace_leave(sd,simp(Chg,LayerS)), !.

simp_sort_depths([Ss:Dp:DpDjs:Au|SssA],[Ss:Dp:DpDjs:Au|SssA1],yes) :-
    nonvar(Dp),
    { sort_depth(Au,_DpAu)
    ; assert(sort_depth(Au,Dp)),
      trace_write(sd1,assd(Au,Dp))
    },
    simp_sort_depths(SssA,SssA1,_Chg), !.

simp_sort_depths([Ss:Dp:DpDjs:Au|SssA],[Ss:Dp:DpDjs:Au|SssA1],Chg) :-
    var(DpDjs),
    simp_sort_depths(SssA,SssA1,Chg), !.

```

```

simp_sort_depths([Ss:Dp:DpDjs: Au|SssA],[Ss:Dp:DpDjs1: Au|SssA1],Chg) :-
  simp_sort_depths1(DpDjs,DpDjs1,DpDisjs2),
  trace_write(sd1,smp1(Au,DpDjs,DpDisjs2)),
  { DpDisjs2 = [],
    Chg = Chg1
  ; sort_depth(Au,_DpAu),
    Chg = Chg1
  ; unflatten_op(DpDisjs2,(min),DpI),
    Dp is DpI + 1,
    assert(sort_depth(Au,Dp)),
    trace_write(sd1,assd(Au,Dp,DpDjs1)),
    Chg = yes
  },
  simp_sort_depths(SssA,SssA1,Chg1), !.

simp_sort_depths([],[],no) :- !.

simp_sort_depths1([DisjI|DisjsI],[[Max]|Disjs0],[Max|Maxs]) :-
  ground(DisjI),
  unflatten_op(DisjI,(max),DisjMax1),
  Max is DisjMax1,
  simp_sort_depths1(DisjsI,Disjs0,Maxs), !.

simp_sort_depths1([DisjI|DisjsI],[DisjI|Disjs0],Maxs) :-
  simp_sort_depths1(DisjsI,Disjs0,Maxs), !.

simp_sort_depths1([],[],[]) :- !.

% Par = [DpExOrig,OldLayers,MaxDepth,MinSolutions]
%- calc the NameAu of anti-unifiers of all sorts in Ss,
%- OccIn, OccOut is the binary tree of triples (Ss,AuSs,NameAu)
%- for which hsg already has been called,
%- AuSs may be uninstantiated until the completion of the call,
%- the meaning of a triple is: NameAu sortdef AuSs, rgbv(Ss) = NameAu,
rgbv_t(Par,SssAus,OccVL,OccVT,OccIn,OccOut) :-
  trace_write(hsg,rgbv(SssAus)),
  rgbv(Par,SssAus,OccVL,OccVT,OccIn,OccOut),
  trace_write(hsg,rgbv_exit(SssAus)).

%- no new tasks, terminate
rgbv(_Par,[layer:_L],_OccVL,_OccVT,OccIn,OccIn) :- !.

%- layer border, re-sort
%<backtrackable*>
rgbv([DpOrig,OldLys|Par],[layer:L|SssAus],OccVL,OccVT,OccI,OccO) :-
  sort(SssAus,SssAus1),
  join_unifiables(SssAus1,SssAus2),
  OldLys1 = [SssAus2|OldLys],
  simp_sort_depths_list(OldLys1,OldLys2),
  L1 is L + 1,
  append(SssAus2,[layer:L1],SssAus3),
  trace_write(layer,layer(DpOrig,L1)),
  ( ground(DpOrig),
    OccO = OccI
  ; rgbv_t([DpOrig,OldLys2|Par],SssAus3,OccVL,OccVT,OccI,OccO)
  ).

%- loop check
rgbv(Par,[_Ss:_Dp:_DpEx:NameAu|SssAus],OccVL,OccVT,OccIn,OccOut) :-
  NameAu sortdef _DefAu,
  !,
  rgbv_t(Par,SssAus,OccVL,OccVT,OccIn,OccOut).

%- replace sort names by definitions,
%- then distribute "!"
rgbv(Par,[Ss:Dp:DpEx: Au|SssAus],OccVL,OccVT,OccIn,OccOut) :-
  replace_defs_star(Ss,SsDef,nil,_Occ,DefAu,_NameAu,true),
  % ..._star necessary if sorts are not in head nf
  ( member(_!,SsDef),
    %: [0!s0+s0!s0*top!top*s0,s(s0)!s1+s0!s0+s1!s1*s1]
    flatten_op_list1(SsDef,(!),Ss1),
    %: [[0,s0+s0,s0*top,top*s0],[s(s0),s1+s0,s0+s1,s1*s1]]
    sort_list(Ss1,Ss2),
    %: [[0,s0*top,top*s0,s0+s0],[s(s0),s1*s1,s0+s1,s1+s0]]
    group_cr_sorts(Ss2,OccVT,Groups),
    %: OccVL = [[0,s(0)],var,v1],[s(0),0],var,v2]]
    %: Groups = [[0,s(s0)],[s0*top,s1*s1],[top*s0,s1*s1],
    %: [s0+s0,s0+s1],[s0+s0,s1+s0]]

  rgbv_H1_list(Groups,DefL,SssAus1,Dp,DpDjs,OccVL,OccIn,Occ1),

  { unflatten_op(DefL,(!),DefAu)
  ; DefAu = bottom

```

```

    },
    { member([], DpDjs),
      DpEx = [],
      Dp = 1
    ; sort_list(DpDjs, DpDjs1),
      sort(DpDjs1, DpEx)
    },
    assert(Au sortdf DefAu),
    append(SssAus, SssAus1, SssAus2),
    rgbv_t(Par, SssAus2, OccVL, OccVT, Occ1, OccOut)
  ; not member(_!, SsDef), %% ???
    append(SssAus, [SsDef:DefAu], SssAus1), %% ???
    rgbv_t(Par, SssAus1, OccVL, OccVT, Occ1, OccOut) %% ???
  ).

%- no tasks, terminate
rgbv_t(_Par, [], _OccVL, _OccVT, OccIn, OccIn) :- !.

rgbv_H1_list([Ss|Sss], Defs, Sss0, Dp, DpDjs, OccVL, OccIn, OccOut) :-
  rgbv_H1(Ss, Defs1, Sss01, DpDj, OccVL, OccIn, Occ1),
  rgbv_H1_list(Sss, Defs2, Sss02, Dp, DpDjs1, OccVL, Occ1, OccOut),
  append(Defs1, Defs2, Defs),
  append(Sss01, Sss02, Sss0),
  { member(Dp1, DpDj),
    Dp1 == Dp, %% skip direct recursive sort
    DpDjs = DpDjs1
  ; member(Dp1, DpDj),
    Dp1 >= /*inf*/999000, %% skip empty sort
    DpDjs = DpDjs1
  ; DpDjs = [DpDj|DpDjs1]
  }, !.

rgbv_H1_list([], [], [], _Dp, [], _OccVL, OccIn, OccIn) :- !.

%- decompose common constructor
rgbv_H1(Ss, [Def], Sss0, DpCjs, _OccVL, OccIn, OccOut) :-
  %: Ss = [s0*top, s1*s1]
  strip_common_constructor(Ss, Cr, SsArgs),
  %: Cr = (*)
  %: SsArgs = [[s0, s1], [top, s1]]
  assign_sort_name_list(SsArgs, Names, Sss0, DpCjs, OccIn, OccOut),
  %: Sss0 = [[s0, s1]:A:_:sort6, [top, s1]:B:_:sort7]
  %: DpCjs = [A, B]
  %: Names = [sort6, sort7]
  Def =.. [Cr|Names], !.

%- different constructors,
%- return a disjunction of all admitted variables whose terms are in Ss
rgbv_H1(Ss, Vs, [], [], OccVL, OccIn, OccIn) :-
  bagof(V, VTs^(
    member((VTs, var, V), OccVL),
    inhmlist(Ss, VTs)
  ), Vs), !.

%- different constructors,
%- no admitted variables whose terms are in Ss, return bottom
rgbv_H1(_Ss, [], [], /*inf*/999999, _OccVL, OccIn, OccIn) :- !.

% get the sort name Name of Ss from BTin,
% or create and enter a new sort name for Ss
assign_sort_name(Ss, Name, Ss0, Dp, BTin, BTout) :-
  enter_into_bbt((Ss, _Def, Dp, Name), BTin, BTout),
  { nonvar(Name)
  ; new_name(sort, Name)
  },
  Ss0 = (Ss:Dp:_DpEx:Name), !.

assign_sort_name_list([Ss|Sss], [Nm|Nms], [Ss0|Sss0], [DC|DCs], BTi, BTo) :-
  assign_sort_name(Ss, Nm, Ss0, DC, BTi, BT1),
  assign_sort_name_list(Sss, Nms, Sss0, DCs, BT1, BTo), !.

assign_sort_name_list([], [], [], [], BTin, BTin) :- !.

% %- catch disjunctions (necessary if sorts are not in head normal form)
% rgbv(Par, [Ss: Au|SssAus], OccVL, OccVT, OccIn, OccOut) :- %% ???

```

```

%      distribute1(Ss,SsDist,true),
%      make_varlist_of_same_length(SsDist,AusDist),
%      merge_lists(SsDist,(:),AusDist,SssAus2),
%      %sort(SssAus2,SssAus3),
%      append(SssAus,SssAus2,SssAus1),
%      rgbv_t(Par,SssAus1,OccVL,OccVT,OccIn,OccOut),
%      unflatten_op(AusDist,(!),Au), !.
%
% %- decompose common constructor
% rgbv(Par,[Ss:Au|SssAus],OccVL,OccVT,OccIn,OccOut) :- %% ???
%      strip_common_constructor(Ss,Cr,SsArgs),
%      make_varlist_of_same_length(SsArgs,AuArgs),
%      Au =.. [Cr|AuArgs],
%      merge_lists(SsArgs,(:),AuArgs,SssAus2),
%      %sort(SssAus2,SssAus3),
%      append(SssAus,SssAus2,SssAus1),
%      rgbv_t(Par,SssAus1,OccVL,OccVT,OccIn,OccOut), !.
%
% %- different constructors,
% %- return a disjunction of all admitted vars whose terms are in Ss
% rgbv(Par,[Ss:Au|SssAus],OccVL,OccVT,OccIn,OccOut) :- %% ???
%      bagof(V,VTs^(
%          member((VTs,var,V),OccVL),
%          inhm_list(Ss,VTs)
%          ),Vs),
%      unflatten_op(Vs,(!),Au),
%      rgbv_t(Par,SssAus,OccVL,OccVT,OccIn,OccOut), !.
%
% %- different constructors,
% %- no admitted variables whose terms are in Ss, return bottom
% rgbv(Par,[_Ss:bottom|SssAus],OccVL,OccVT,OccIn,OccOut) :- %% ???
%      rgbv_t(Par,SssAus,OccVL,OccVT,OccIn,OccOut), !.

```

```

% calc the sort Au of anti-unifiers of all sorts in Ss,
% defining new sorts recursively as needed,
% OccVL is a list containing all admitted variables
rsgv(MaxDepth,Ss,OccVL,Au) :-
    sort(OccVL,OccVL1),
    build_occ_var_trie(OccVL1,[],OccVT),
    rsg_t(MaxDepth,Ss,Au1,OccVL1,OccVT,nil,OccBT),
    conv_bbtlist_to_list([OccBT],Occ),
    split_sort_var_defs(Occ,SortDefs1,_VarDefs1),
    extract_defs_from_occ(SortDefs1,SortDefs2,xxx,_Memo),
    Au = Au1,
    sort(SortDefs2,SortDefs4),
    trace_write(prof,au(Ss)=Au),
    assert_list(SortDefs4), !.
%calc_sort_depths, !.

```

```

rsgv_list(Dp,[Ss|Sss],Occ,[Au|Aus]) :-
    rsgv(Dp,Ss,Occ,Au),
    rsgv_list(Dp,Sss,Occ,Aus), !.

```

```

rsgv_list(_Dp,[],_Occ,[]) :- !.

```

```

rsgv_list_list(Dp,[Sss|Ssss],Occ,[Aus|Auss]) :-
    rsgv_list(Dp,Sss,Occ,Aus),
    rsgv_list_list(Dp,Ssss,Occ,Auss), !.

```

```

rsgv_list_list(_Dp,[],_Occ,[]) :- !.

```

```

%- calc the NameAu of anti-unifiers of all sorts in Ss,
%- OccIn, OccOut is the binary tree of triples (Ss,AuSs,NameAu)
%- for which hsg already has been called,
%- AuSs may be uninstantiated until the completion of the call,
%- the meaning of a triple is: NameAu sortdef AuSs, rsgv(Ss) = NameAu,
rsg_t(MaxDepth,Ss,Au,OccVL,OccVT,OccIn,OccOut) :-
    trace_enter(hsg,rsgv(Ss)),
    rsgv(MaxDepth,Ss,Au,OccVL,OccVT,OccIn,OccOut),
    trace_leave(hsg,rsgv(Ss) = Au).

```

```

%- check cache
rsgv(_MaxDepth,Ss,NameAu,OccVL,_OccVT,OccIn,OccOut) :-

```

```

        origin_s(NameAu,Ss,OccVL), !.

%- depth check
rsgv(0,_Ss,bottom,_OccVL,_OccVT,OccIn,OccIn) :- !.

%- loop check
rsgv(_MaxDepth,Ss,NameAu,OccVL,_OccVT,OccIn,OccIn) :-
    member_bbt((Ss,_DefAu,NameAu),OccIn),
    { nonvar(NameAu)
      ; new_name(sort,NameAu),
        assert(origin_s(NameAu,Ss,OccVL))
    }, !.

%- replace sort names by definitions,
%- then distribute "!"
rsgv(MaxDepth,Ss,Au,OccVL,OccVT,OccIn,OccOut) :-
    replace_defs_star(Ss,SsDef,OccIn,Occ1,DefAu,NameAu,true),
    % ..._star necessary if sorts are not in head nf
    MaxDepth1 is MaxDepth - 1,
    { member(_!,SsDef),
      %: [0!s0+s0!s0*top!top*s0,s(s0)!s1+s0!s0+s1!s1*s1]
      flatten_op_list1(SsDef,!),Ss1),
      %: [[0,s0+s0,s0*top,top*s0],[s(s0),s1+s0,s0+s1,s1*s1]]
      sort_list(Ss1,Ss2),
      %: [[0,s0*top,top*s0,s0+s0],[s(s0),s1*s1,s0+s1,s1+s0]]
      group_cr_sorts(Ss2,OccVT,Groups),
      %: OccVL = [[0,s(0)],var,v1],[s(0),0],var,v2]]
      %: Groups = [[0,s(s0)],[s0*top,s1*s1],[top*s0,s1*s1],
        %: [s0+s0,s0+s1],[s0+s0,s1+s0]]
      rsgv_list(MaxDepth1,Groups,Aus,OccVL,OccVT,Occ1,OccOut),
      { unflatten_op(Aus,!),DefAu)
        ; DefAu = bottom
      }
    ; rsgv(MaxDepth1,SsDef,DefAu,OccVL,OccVT,Occ1,OccOut)
    },
    { var(NameAu),
      new_name(sort,NameAu),
      assert(origin_s(NameAu,Ss,OccVL)),
      Au = NameAu
    ; Au = NameAu
    }, !.

%- catch disjunctions (necessary if sorts are not in head normal form)
rsgv(MaxDepth,Ss,Au,OccVL,OccVT,OccIn,OccOut) :-
    distribute1(Ss,SsDist,true),
    rsgv_list(MaxDepth,SsDist,Aus,OccVL,OccVT,OccIn,OccOut),
    unflatten_op(Aus,!),Au), !.

%- decompose common constructor
rsgv(MaxDepth,Ss,Au,OccVL,OccVT,OccIn,OccOut) :-
    strip_common_constructor(Ss,Cr,SsArgs),
    make_varlist_of_same_length(SsArgs,AuArgs),
    Au =.. [Cr|AuArgs],
    rsgv_list(MaxDepth,SsArgs,AuArgs,OccVL,OccVT,OccIn,OccOut), !.

%- different constructors,
%- return a disjunction of all admitted variables whose terms are in Ss
rsgv(_MaxDepth,Ss,Au,OccVL,_OccVT,OccIn,OccIn) :-
    bagof(V,VTs^(
        member((VTs,var,V),OccVL),
        inhm_list(Ss,VTs)
    ),Vs),
    unflatten_op(Vs,!),Au), !.

%- different constructors,
%- no admitted variables whose terms are in Ss, return bottom
rsgv(_MaxDepth,_Ss,bottom,_OccVL,_OccVT,OccIn,OccIn) :- !.

rsgv_list(MaxDepth,[Ss|Sss],[Au|Aus],OccVL,OccVT,OccIn,OccOut) :-
    rsg_t(MaxDepth,Ss,Au,OccVL,OccVT,OccIn,Occ1),
    rsgv_list(MaxDepth,Sss,Aus,OccVL,OccVT,Occ1,OccOut), !.

rsgv_list(_MaxDepth,[],[],_OccVL,_OccVT,OccIn,OccIn) :- !.

% calc the infimum Inf of sorts S and T
inf(_S,bottom,bottom) :- !.

inf(bottom,_T,bottom) :- !.

inf(S,top,S) :- !.

inf(top,T,T) :- !.

%- different equivalence classes are always disjoint

```

```

inf(S,T,bottom) :-
    equiv_class(SC,S),
    equiv_class(TC,T),
    SC \== TC, !.

%- anti-unify S and T, allowing no variables in the result sort
inf(S,T,Inf) :-
    rsgv(/*inf*/999999,[S,T],[],Inf), !.

% %%%%%%%%%% optimized argument selection %%%%%%%%%% %
% %%%%%%%%%% %%%%%%%%%% %%%%%%%%%% %%%%%%%%%% %
% %%%%%%%%%% %%%%%%%%%% %%%%%%%%%% %%%%%%%%%% %

% given a list of variables with origin sorts, compute a search tree
%: build([([0,s(a)],var,v1),([0,b+c],var,v2)],[],[0:(+):[],s:[]])
build_occ_var_trie([([Ss,var,_V)|OccVL],OccVTIn,OccVTOut) :-
    build_occ_var_trie_H1(Ss,Ss1),
    enter_into_trie(Ss1,OccVTIn,OccVT1),
    build_occ_var_trie(OccVL,OccVT1,OccVTOut), !.

build_occ_var_trie([],OccVTIn,OccVTIn) :- !.

%: build_occ_var_trie_H1([0,b+c],[0,+])
build_occ_var_trie_H1([S|Ss],[Cr|CrS]) :-
    S =.. [Cr|_],
    build_occ_var_trie_H1(Ss,Crs), !.

build_occ_var_trie_H1([],[]) :- !.

% main grouping algorithm
group_cr_sorts(Lists,VT,Groups) :-
    group_cr_sorts_H1_list(Lists,Listss),
    group_cr_sorts1_t(eq,[],Listss,VT,Groups1),
    sort(Groups1,Groups), !.

% partition each list in Lists into sublists of terms starting with
% equal constructors
group_cr_sorts_H1_list([List|Lists],[ListP|ListsP]) :-
    group_cr_sorts_H1([],List,ListP),
    group_cr_sorts_H1_list(Lists,Listss), !.

group_cr_sorts_H1_list([],[]) :- !.

%: gr([], [0+0,1+1,2+2,3*3,s(4),s(5)], [[0+0,1+1,2+2],[3*3],[s(4),s(5)]])
group_cr_sorts_H1([], [CrSort|CrSorts],ListP) :-
    group_cr_sorts_H1([CrSort],CrSorts,ListP), !.

group_cr_sorts_H1([CrS|Grp],[CrSort|CrSorts],ListP) :-
    CrS =.. [Cr|_],
    CrSort =.. [Cr|_],
    group_cr_sorts_H1([CrSort,CrS|Grp],CrSorts,ListP), !.

group_cr_sorts_H1(Grp,[CrSort|CrSorts],[GrpR|ListP]) :-
    reverse(Grp,GrpR),
    group_cr_sorts_H1([CrSort],CrSorts,ListP), !.

group_cr_sorts_H1(Grp,[],[GrpR]) :-
    reverse(Grp,GrpR), !.

group_cr_sorts1_t(Eq,Gr,Listss,VT,Gr0) :-
    trace_enter(grp,grp(Gr,Listss,VT)),
    group_cr_sorts1(Eq,Gr,Listss,VT,Gr0),
    trace_leave(grp,grp(Gr,Listss,VT) = Gr0).

%- join CrSort to current group
group_cr_sorts1(eq,[CrS1s|Gr],[[CrS2s|Ls]|Lss],VT,Gr0) :-
    CrS1s = [CrS1|_],
    CrS2s = [CrS2|_],
    CrS1 =.. [Cr|_],
    CrS2 =.. [Cr|_],
    get_subtrie(VT,Cr,VT1),
    group_cr_sorts1_t(eq,[CrS2s,CrS1s|Gr],Lss,VT1,Gr1),
    group_cr_sorts1_t(eq,[CrS1s|Gr],[Ls|Lss],VT,Gr2),
    append(Gr1,Gr2,Gr0), !.

%- output current group

```

```

group_cr_sorts1(_Eq,Gr,[],_VT,Gr0) :-
    reverse(Gr,GrR),
    setof(CrSs,member_list12(CrSs,GrR),Gr0), !.

%- abort current group due to empty VT
group_cr_sorts1(ne,_Gr,_Listss,[],[]) :- !.

%- abort current group due to empty VT
group_cr_sorts1(eq,[[CrS1|_|_Gr],[[CrS2|_|_Ls]|_Lss],[],[]) :-
    CrS1 @< CrS2, !.

%- follow VT
group_cr_sorts1(Eq,[CrS1s|Gr],[[CrS2|CrS2s]|Ls]|Lss],VT,Gr0) :-
    CrS2 =.. [Cr|_],
    member(Cr:VT1,VT),
    group_cr_sorts1_t(ne,[[CrS2|CrS2s],CrS1s|Gr],Lss,VT1,Gr1),
    group_cr_sorts1_t(Eq,[CrS1s|Gr],[Ls|Lss],VT,Gr2),
    append(Gr1,Gr2,Gr0), !.

%- skip CrSorts
group_cr_sorts1(Eq,[CrS1s|Gr],[[_CrS2s|Ls]|Lss],VT,Gr0) :-
    group_cr_sorts1_t(Eq,[CrS1s|Gr],[Ls|Lss],VT,Gr0), !.

%- initialize new group
group_cr_sorts1(_Eq,[],[[CrS2|CrS2s]|Ls]|Lss],VT,Gr0) :-
    CrS2 =.. [Cr|_],
    get_subtrie(VT,Cr,VT1),
    group_cr_sorts1_t(eq,[[CrS2|CrS2s]],Lss,VT1,Gr1),
    group_cr_sorts1_t(eq,[],[Ls|Lss],VT,Gr2),
    append(Gr1,Gr2,Gr0), !.

%- abort current group due to empty CrS2s
group_cr_sorts1(_Eq,_Pre,[[]|_Listss],_VT,[]) :- !.

% split Occ into SortDefs and VarDefs
split_sort_var_defs([(Ss,var,V)|Occ],SortDefs,[(V,V,Ss)|VarDefs]) :-
    split_sort_var_defs(Occ,SortDefs,VarDefs), !.

split_sort_var_defs([(Ss,D,N)|Occ],[[Ss,D,N)|SortDefs],VarDefs) :-
    split_sort_var_defs(Occ,SortDefs,VarDefs), !.

split_sort_var_defs([],[],[]) :- !.

% %%%%%%%%%%%%%%% sort elements enumeration %%%%%%%%%%%%%%% %
% %%%%%%%%%% sort elements enumeration %%%%%%%%%% %
% %%%%%%%%%% sort elements enumeration %%%%%%%%%% %

% enumerate all ground terms Rhs of sort Sort in depth first order,
% iteratively deepening the bound limit
%<backtrackable>
%- no bound limit if sort is finite
find_rhs(Sort,Red,Rhs) :-
    finite(Sort),
    !,
    write('--- enumerating finite '), write(Sort), nl,
    all_groundterms_t(Sort,/*inf*/999999,_,Red,Rhs).

find_rhs(Sort,Red,Rhs) :-
    nl, write('--- enumerating '), write(Sort), nl,
    { sort_depth(Sort,Dp)
    ; Dp = 1
    },
    find_rhs1(Sort,Dp,Red,Rhs).

find_rhs1(Sort,Dp,Red,Rhs) :-
    write('--- new depth bound '), write(Dp), nl,
    all_groundterms_t(Sort,Dp,_,Red,Rhs).

find_rhs1(Sort,Dp,Red,Rhs) :-
    Dp1 is Dp + 1,
    find_rhs1(Sort,Dp1,Red,Rhs).

% enumerate all ground terms Term of sort Sort
% with max number MaxRepIn of replacements
% of sort names by their definitions,
% enumerate only terms that do not contain instances of any
% redex term in Red,
%<backtrackable>

```



```

all_groundterms_t(Sort,MaxRepIn,MaxRepOut,Red,Term) :-
    trace_enter(all,Sort:MaxRepIn),
    all_groundterms(Sort,MaxRepIn,MaxRepOut,Red,Term),
    trace_leave(all,Sort : Term).

%- empty sort, cut
all_groundterms(bottom,_MaxRepIn,_MaxRepOut,_Red,_Term) :-
    !, fail.

%- sort name, replace by definition
all_groundterms(Sort,MaxRepIn,MaxRepIn,Red,Term) :-
    %% dont accumulate sort name
    %% replacements on different branches
    Sort sortdef SortDef,
    !,
    sort_depth(Sort,Depth),
    MaxRepIn >= Depth,
    MaxRepIn1 is MaxRepIn - 1,
    all_groundterms_t(SortDef,MaxRepIn1,_MaxRepOut1,Red,Term1),
    Term = Term1.

%- yet undefined sort name, ignore
all_groundterms(Sort,MaxRepIn,MaxRepIn,_Red,_Term) :-
    atom(Sort),
    name(Sort,[115,111,114,116|_Rest]),
    !, fail.

%- sort disjunction, try all alternatives
all_groundterms(Sort1!Sort2,MaxRepIn,MaxRepOut,Red,Term) :-
    !,
    flatten_op(Sort1!Sort2,(!),Sorts),
    all_groundterms_list_o(Sorts,MaxRepIn,MaxRepOut,Red,Term).

%- variable, output
all_groundterms(Var,MaxRepIn,MaxRepIn,_Red,Var) :-
    atomic(Var),
    name(Var,[118|_Nr]), !.

%- constant, output
all_groundterms(Const,MaxRepIn,MaxRepIn,Red,Const) :-
    atomic(Const), !,
    not member(Const,Red), !.

%- constructor sort, enumerate arguments in lexicographical order
all_groundterms(Sort,MaxRepIn,MaxRepOut,Red,Term) :-
    Sort =.. [Cr|Args],
    !,
    length(Args,L),
    check_redices(Cr,Red,Red1),
    update_redices(Cr,1,L,Red1,Reds),
    check_depth_list(Args,MaxRepIn),
    all_groundterms_list(Args,MaxRepIn,MaxRepOut,Reds,Terms),
    Term =.. [Cr|Terms].

all_groundterms_list([Sort|Sorts],MRI,MRO,[Red|Reds],[T|Ts]) :-
    all_groundterms_t(Sort,MRI,MRI,Red,T),
    all_groundterms_list(Sorts,MRI,MRO,Reds,Ts).

all_groundterms_list([],MRI,MRI,[],[]) :- !.

all_groundterms_list_o([Sort|Sorts],MRI,MRO,Red,Term) :-
    all_groundterms_t(Sort,MRI,MRO,Red,Term).

all_groundterms_list_o([_Sort|Sorts],MRI,MRO,Red,Term) :-
    all_groundterms_list_o(Sorts,MRI,MRO,Red,Term).

% find first ground term of each sort in Sorts
first_groundterms_t_list([Sort|Sorts],[Term|Terms]) :-
    sort_depth(Sort,Depth),
    all_groundterms_t(Sort,Depth,_MaxRepOut,[],Term), !,
    first_groundterms_t_list(Sorts,Terms), !.

first_groundterms_t_list([],[]) :- !.

first_groundterms_t_list_list([Sorts|Sortss],[Terms|Termss]) :-
    first_groundterms_t_list(Sorts,Terms),
    first_groundterms_t_list_list(Sortss,Termss), !.

first_groundterms_t_list_list([],[]) :- !.

```

```

% for each sort in Sorts return the set of minimal equations
min_eqns_t_list([Sort|Sorts],SMinBBT,Eqns) :-
    member_bbt(Sort:MinTerm,SMinBBT),
    Sort sortdf SortDef,
    { setof(Term:MinTerm,(
        min_lhs_t(SortDef,SMinBBT,Term)
    ),Eqns1)
    ; Eqns1 = []
    },
    min_eqns_t_list(Sorts,SMinBBT,Eqns2),
    append(Eqns1,Eqns2,Eqns), !.

min_eqns_t_list([],_SMinBBT,[]) :- !.

% enumerate all minimal ground terms Term of sort Sort,
% i.e., replace sort names by their minimal terms
% which are obtained from SMinBBT
%<*backtrackable*>
min_lhs_t(Sort,SMinBBT,Term) :-
    trace_enter(all,Sort),
    min_lhs(Sort,SMinBBT,Term),
    trace_leave(all,Sort : Term).

%- sort name, return its minimal term
min_lhs(Sort,SMinBBT,Term) :-
    Sort sortdef _SortDef,
    !,
    member_bbt(Sort:Term,SMinBBT), !.

%- sort disjunction, try all alternatives
min_lhs(Sort1!Sort2,SMinBBT,Term) :-
    !,
    flatten_op(Sort1!Sort2,(!),Sorts),
    min_lhs_list_o(Sorts,SMinBBT,Term).

%- variable, output
min_lhs(Var,_SMinBBT,Var) :-
    atomic(Var),
    name(Var,[118|_Nr]), !.

%- constant, output
min_lhs(Const,_SMinBBT,Const) :-
    atomic(Const), !.

%- constructor sort, enumerate arguments in lexicographical order
min_lhs(Sort,SMinBBT,Term) :-
    Sort =.. [Cr|Args],
    !,
    min_lhs_list_a(Args,SMinBBT,Terms),
    Term =.. [Cr|Terms].

% enumerate all minimal ground terms for each sort in Sorts
%<*backtrackable*>
min_lhs_list_a([Sort|Sorts],SMinBBT,[T|Ts]) :-
    min_lhs_t(Sort,SMinBBT,T),
    min_lhs_list_a(Sorts,SMinBBT,Ts).

min_lhs_list_a([],_SMinBBT,[]) :- !.

% enumerate a minimal ground term for some sort in Sorts
%<*backtrackable*>
min_lhs_list_o([Sort|_Sorts],SMinBBT,Term) :-
    min_lhs_t(Sort,SMinBBT,Term).

min_lhs_list_o([_Sort|Sorts],SMinBBT,Term) :-
    min_lhs_list_o(Sorts,SMinBBT,Term).

% succeed if each sort name in Sorts has a depth less than Depth,
% ignore proper sort expressions in Sorts
check_depth_list([Sort|Sorts],Depth) :-
    check_depth(Sort,Depth),
    check_depth_list(Sorts,Depth), !.

check_depth_list([],_Depth) :- !.

%- sort name, check depth

```

```

check_depth(Sort,Depth) :-
    sort_depth(Sort,SortDepth),
    !,
    SortDepth =< Depth, !.

%- proper sort expression, ignore
check_depth(Sort,_Depth) :-
    not (Sort sortdef _Def),
    !.

% succeed if Cr(X1,...,Xn) is not contained in RedIn,
% return all redices of RedIn compatible with Cr as RedOut
check_redices(Cr,[R|RedIn],[R|RedOut]) :-
    R =.. [Cr|Args],
    !,
    not var_list(Args),
    check_redices(Cr,RedIn,RedOut), !.

check_redices(Cr,[_R|RedIn],RedOut) :-
    check_redices(Cr,RedIn,RedOut), !.

check_redices(_Cr,[],[]) :- !.

% update RedIn corresponding to descending along Cr, yielding RedOut
update_redices(_Cr,I,L,_RedIn,[]) :-
    I > L, !.

update_redices(Cr,I,L,RedIn,[RedOut|RedsOut]) :-
    update_redices1(Cr,I,RedIn,RedOut1),
    redices(RedOut2),
    append(RedOut1,RedOut2,RedOut3),
    sort(RedOut3,RedOut),
    I1 is I + 1,
    update_redices(Cr,I1,L,RedIn,RedsOut), !.

% get I.th argument term of each redex in RedIn, yielding RedOut,
update_redices1(Cr,I,[RI|RedIn],[RO|RedOut]) :-
    RI =.. [Cr|Args],
    { get_ith_elem(Args,I,RO) },
    nonvar(RO),
    update_redices1(Cr,I,RedIn,RedOut), !.

update_redices1(Cr,I,[_RI|RedIn],RedOut) :-
    update_redices1(Cr,I,RedIn,RedOut), !.

update_redices1(_Cr,_I,[],[]) :- !.

% %%%%%%%%%%%%%%%
% %%%%%%%%%%% sort depth %%%%%%%%%%%
% %%%%%%%%%%%
% %%%%%%%%%%%

% calc bottom up N times, where N is the number of sort definitions
calc_sort_depths :-
    setof(NewSort:_NewDepth,NewDef^(
        NewSort sortdf NewDef,
        not sort_depth(NewSort,_NewDepth)
    ),NewSortsDepths),
    { setof(OldSort:OldDepth,OldDef^(
        OldSort sortdf OldDef,
        sort_depth(OldSort,OldDepth)
    ),OldSortsDepths)
    ; OldSortsDepths = []
    },
    append(NewSortsDepths,OldSortsDepths,SortsDepths),
    conv_list_to_bbt(SortsDepths,SsDsBBT),
    calc_sort_depths1_list(SsDsBBT,NewSortsDepths,ExprsN,Exprs0),
    length(NewSortsDepths,MaxCycle),
    calc_sort_depths5(ExprsN,2,MaxCycle,_ExprsN1,Exprs01),
    append(Exprs0,Exprs01,Exprs02),
    merge_lists(Sorts2,(:),Depths2,Exprs02),
    merge_lists(Sorts2,(sort_depth),Depths2,Exprs03),
    sort(Exprs03,Exprs04),
    assert_list(Exprs04),
    write_list(Exprs04),
    calc_sort_depths3(NewSortsDepths),
    calc_sort_depths4(NewSortsDepths), !.

%- no new sort defs found

```

```

calc_sort_depths :- !.

% set up depth dependency expressions for all sorts in SsDs,
% Exprs0 contains all new sorts that have been assigned a depth,
% they have the form Sort:Depth where Depth is ground,
% ExprsN contains the other new sorts,
% they have the form Sort:Depth: [[A,B],[C,D,E],[F]],
% meaning Depth = (A max B) min (C max D max E) min F,
% SsDsBBT contains expressions of the form Sort:Depth
% where Depth is instantiated if known
calc_sort_depths1_list(SsDsBBT, [Sort: /*inf*/999999|SsDs], ExprsN,
                      [Sort: /*inf*/999999|Exprs0]) :-
    Sort sortdf bottom,
    trace_write(sd, calc1(Sort, bottom)),
    calc_sort_depths1_list(SsDsBBT, SsDs, ExprsN, Exprs0), !.

calc_sort_depths1_list(SsDsBBT, [Sort:Depth|SsDs], ExprsN, Exprs0) :-
    Sort sortdf Def,
    distribute1([Def], DefL1, _Success),
    append_list(DefL1, DefL),
    calc_sort_depth1(Sort, SsDsBBT, DefL, Expr),
    sort(Expr, Expr1),
    { member([], Expr1),
      trace_write(sd, calc1(Sort, const)),
      Depth = 1,
      ExprsN = ExprsN1,
      Exprs0 = [Sort:Depth|Exprs01]
    ; ExprsN = [Sort:Depth:Expr1|ExprsN1],
      Exprs0 = Exprs01
    },
    calc_sort_depths1_list(SsDsBBT, SsDs, ExprsN1, Exprs01), !.

calc_sort_depths1_list(_SsDsBBT, [], [], []) :- !.

calc_sort_depth1(Own, SsDsBBT, [SortExpr|SortExprs], IsExprs) :-
    get_leaves(SortExpr, [], _V, _F, _C, _0, Sorts),
    calc_sort_depth2(Own, SsDsBBT, Sorts, IsExpr, Rec),
    { Rec = yes,
      IsExprs = IsExprs1
    ; IsExprs = [IsExpr|IsExprs1]
    },
    calc_sort_depth1(Own, SsDsBBT, SortExprs, IsExprs1), !.

calc_sort_depth1(_Own, _SsDsBBT, [], []) :- !.

% Rec indicates whether the disjunct corresponding to Sorts is a
% recursion in sort definition, i.e. contains the sort Own,
% such disjuncts can be ignored
calc_sort_depth2(Own, _SsDsBBT, [Own|_Sorts], [], yes) :- !.

calc_sort_depth2(Own, SsDsBBT, [Sort|Sorts], [Depth|IsExpr], Rec) :-
    member_bbt(Sort:Depth, SsDsBBT),
    calc_sort_depth2(Own, SsDsBBT, Sorts, IsExpr, Rec), !.

%- Sort without depth is empty sort
calc_sort_depth2(_Own, _SsDsBBT, [_Sort|_Sorts], [], yes) :- !.

calc_sort_depth2(_Own, _SsDsBBT, [], [], no) :- !.

% perform Max bottom-up cycles of depth computation,
%- stop if I geq Max
calc_sort_depths5(ExprsI, Max, Max, ExprsI, []) :- !.

calc_sort_depths5(ExprsI, I, Max, ExprsN, Exprs0) :-
    calc_sort_depths6_list(ExprsI, I, Max, ExprsN1, Exprs01, Change),
    trace_write(sd, calc6list(I, Change)),
    { % stop if no change
      Change = no,
      ExprsN = ExprsN1,
      Exprs0 = Exprs01
    ; I1 is I + 1,
      calc_sort_depths5(ExprsN1, I1, Max, ExprsN, Exprs02),
      append(Exprs01, Exprs02, Exprs0)
    }, !.

% perform one bottom-up cycle of depth computation
% splitting ExprsI into Exprs0, which could be assigned a depth in this
% cycle, and ExprsN, which could only be simplified somewhat,
% Chg is "yes" if at least one sort could be assigned a depth
calc_sort_depths6_list([Sort:Dp:Expr|ExprsI], I, Max, ExprsN, Exprs0, Chg) :-
    calc_sort_depths6(Dp, Expr, I, Expr1),
    trace_write(sd, calc6(Sort, Dp)),

```

```

{ % Dp still uninstantiated, add to incomplete
  var(Dp),
  ExprsN = [Sort:Dp:Expr1|ExprsN1],
  Exprs0 = Exprs01,
  Chg = Chg1
; % Dp instantiated, add to complete
  ExprsN = ExprsN1,
  Exprs0 = [Sort:Dp|Exprs01],
  Chg = yes
},
calc_sort_depths6_list(ExprsI,I,Max,ExprsN1,Exprs01,Chg1), !.

calc_sort_depths6_list([],_I,_Max,[],[],no) :- !.

% try to compute a depth from DisjsI,
% at least, simplify DisjsI to Disjs0
%- all necessary depths available, replace by their maximum
calc_sort_depths6(Dp,[DisjI|DisjsI],I,[DisjMax|Disjs0]) :-
  ground(DisjI),
  unflatten_op(DisjI,(max),DisjMax1),
  DisjMax is DisjMax1,
  { % minimum depth in current disjunct, stop search
    I is DisjMax + 1,
    Dp = I,
    Disjs0 = []
  ; % continue search
    calc_sort_depths6(Dp,DisjsI,I,Disjs0)
  }, !.

%- not all necessary depths available, skip
calc_sort_depths6(Dp,[DisjI|DisjsI],I,[DisjI|Disjs0]) :-
  calc_sort_depths6(Dp,DisjsI,I,Disjs0), !.

calc_sort_depths6(_Dp,[],_I,[]) :- !.

% all sorts that still dont have assigned a depth are empty
%- dont change the definition of bottom itself
calc_sort_depths3([bottom:_Depth|SortsDepths]) :-
  calc_sort_depths3(SortsDepths), !.

calc_sort_depths3([Sort:_Depth|SortsDepths]) :-
  var(Depth),
  retract(Sort sortdf _SortDef),
  assert(Sort sortdf bottom),
  assert(sort_depth(Sort,/*inf*/999999)),
  calc_sort_depths3(SortsDepths), !.

calc_sort_depths3([_Sort:_Depth|SortsDepths]) :-
  calc_sort_depths3(SortsDepths), !.

calc_sort_depths3([]) :- !.

% delete occurrences of empty sorts from other sort defs
%- dont change the definition of bottom
calc_sort_depths4([bottom:_Depth|SortsDepths]) :-
  calc_sort_depths4(SortsDepths), !.

calc_sort_depths4([Sort:_Depth|SortsDepths]) :-
  Sort sortdf SortDef,
  calc_sort_depths4_H1(SortDef,SortDef0),
  SortDef0 \== SortDef,
  retract(Sort sortdf SortDef),
  assert(Sort sortdf SortDef0),
  calc_sort_depths4(SortsDepths), !.

calc_sort_depths4([_Sort:_Depth|SortsDepths]) :-
  calc_sort_depths4(SortsDepths), !.

calc_sort_depths4([]) :- !.

calc_sort_depths4_H1(Var,Var) :-
  var(Var), !.

calc_sort_depths4_H1(Sort1!Sort2,Sort0) :-
  flatten_op(Sort1!Sort2,(!),SortL),
  calc_sort_depths4_H1_list(SortL,SortL0),
  sort(SortL0,SortL01),
  list_trisection(SortL01,[bottom],SortL02,_,_),
  { unflatten_op(SortL02,(!),Sort0)
  ; Sort0 = bottom
  }

```

```

    }, !.

calc_sort_depths4_H1(Sort,bottom) :-
    Sort sortdf bottom, !.

calc_sort_depths4_H1(Sort,Sort0) :-
    Sort =.. [Cr|Args],
    calc_sort_depths4_H1_list(Args,Args0),
    { member(Bottom,Args0),
      Bottom == bottom,
      Sort0 = bottom
    }; Sort0 =.. [Cr|Args0]
}, !.

calc_sort_depths4_H1_list([Sort|Sorts],[Sort0|Sorts0]) :-
    calc_sort_depths4_H1(Sort,Sort0),
    calc_sort_depths4_H1_list(Sorts,Sorts0), !.

calc_sort_depths4_H1_list([],[]) :- !.

% %%%%%%%%%% sort difference %%%%%%%%%% %
% %%%%%%%%%% sort difference %%%%%%%%%% %
% %%%%%%%%%% sort difference %%%%%%%%%% %

% calc the difference Diff (= S\T) of sorts S and T,
% defining new sorts recursively as needed
diff(S,T,Diff) :-
    flatten_op(T,(!),Ts),
    sort(Ts,Ts1),
    diff_t(S,Ts1,STs,nil,OccBT),
    conv_bbtlist_to_list([OccBT],Occ),
    Diff = STs,
    OccS = Occ,
    extract_defs_from_occ(OccS,DefsS,memo_diff,Memo),
    assert_list(DefsS),
    calc_sort_depths,
    assert_list(Memo), !.

%- calc the difference NameSTs of sorts S and Ts,
%- OccIn, OccOut is the bin tree of triples ([S|Ts],DiffSTs,NameSTs)
%- for which diff already has been called,
%- DiffST may be uninstantiated until the completion of the call,
%- the meaning of a triple is: NameSTs sortdef DiffSTs,
%- S\T1!...!Tn = NameSTs, where Ts = [T1,...,Tn] is a sorted list
diff_t(S,Ts,STs,OccIn,OccOut) :-
    trace_enter(diff,S\Ts),
    diff(S,Ts,STs,OccIn,OccOut),
    trace_leave(diff,S\Ts=STs).

%- no difference
diff(S,[],S,OccIn,OccIn) :- !.

diff(S,Ts,STs,OccIn,OccIn) :-
    memo_diff(S,Ts,STs), !.

%- loop check
diff(S,Ts,NameSTs,OccIn,OccIn) :-
    member_bbt([S|Ts],_DefSTs,NameSTs),OccIn,
    { nonvar(NameSTs)
    ; new_name(sort,NameSTs)
    }, !.

%- sort disjunction
diff(S!S2,Ts,STs,OccIn,OccOut) :-
    flatten_op(S!S2,(!),SL),
    diff_list2(SL,Ts,STsL,OccIn,OccOut),
    unflatten_op(STsL,(!),STs), !.

%- T subsumes S
diff(S,Ts,bottom,OccIn,OccIn) :-
    member(S,Ts), !.

%- T is top
diff(_S,Ts,bottom,OccIn,OccIn) :-
    member(top,Ts), !.

%- replace sort definitions
diff(S,Ts,STs,OccIn,OccOut) :-
    replace_defs([S|Ts],[SDf|TsDf],OccIn,Occ1,DefSTs,NameSTs,true),
    !,
    flatten_op_list(TsDf,(!),TsDf1),
    sort(TsDf1,TsDf2),

```

```

diff_t(SDf,TsDf2,DefSTs,Occ1,OccOut),
{ var(NameSTs),
  STs = DefSTs
; STs = NameSTs
}, !.

%- constructor sorts
diff(S,Ts,STs,OccIn,OccOut) :-
  diff_H1([S],Ts,DiffList),
  optimize_diff_list(DiffList,DiffList1),
  diff_list(DiffList1,STsL,OccIn,OccOut),
  { unflatten_op(STsL,(!),STs)
; STs = bottom
}, !.

% set up diff calls for constructor case
%: d_H1([a+b],[c+d,e+f],[(a\c\e)+b,(a\c)+(b\f),(a\e)+(b\d),a+(b\d\f)])
diff_H1(Ss,[T|Ts],DiffList) :-
  diff_H2(Ss,T,SsT),
  diff_H1(SsT,Ts,DiffList), !.

diff_H1(Ss,[],Ss) :- !.

%: diff_H2([a+b],c+d,[a\c)+b,a+(b\d)])
diff_H2([S|Ss],T,DiffList) :-
  diff_H3(S,T,ST),
  diff_H2(Ss,T,SsT),
  append(ST,SsT,DiffList), !.

diff_H2([],_T,[]) :- !.

%: diff_H3(a+b,c+d,[a\c)+b,a+(b\d)])
diff_H3(S,T,ST) :-
  S =.. [Cr|SArgs],
  T =.. [Cr|TArgs],
  diff_H4(1,SArgs,TArgs,Cr,ST), !.

%: diff_H3(a+b,c*d,[a+b])
diff_H3(S,_T,[S]) :- !.

%: diff_H4([a,b],[c,d],(+),[(a\c)+b,a+(b\d)])
diff_H4(I,SArgs,[TArgI|TArgs],Cr,[CrI|ST]) :-
  get_ith_elem(SArgs,I,SArgI),
  replace_ith_elem(SArgs,I,SArgI\TArgI,SArgsI),
  CrI =.. [Cr|SArgsI],
  I1 is I + 1,
  diff_H4(I1,SArgs,TArgs,Cr,ST), !.

diff_H4(_I,_SArgs,[],_Cr,[]) :- !.

% perform optimizations on the diff list
optimize_diff_list(DiffList,DiffList0) :-
  optimize_diff_list1(DiffList,DiffList1),
  sort(DiffList1,DiffList2),
  optimize_diff_list2(DiffList2,[],DiffList0), !.

% apply a\b\b=a\b
%: optimize_diff_list1([bottom,(a\top)+c,(a\b\c\b)+c],[a\ [b,c])+(c\ [])])
optimize_diff_list1([bottom|DiffList],DiffList1) :-
  optimize_diff_list1(DiffList,DiffList1), !.

optimize_diff_list1([SCr|DiffList],[SCr0|DiffList1]) :-
  SCr =.. [Cr|Args],
  optimize_diff_list1_H1(Args,Args0),
  SCr0 =.. [Cr|Args0],
  optimize_diff_list1(DiffList,DiffList1), !.

%- optimize_diff_list1_H1 failed
optimize_diff_list1([_SCr|DiffList],DiffList1) :-
  optimize_diff_list1(DiffList,DiffList1), !.

optimize_diff_list1([],[]) :- !.

% apply a\b\b=a\b, flatten 2nd argument of \
% fail on a\b\ a, a\b\top
%: optimize_diff_list1_H1([a\b\c\b,c],[a\ [b,c],c\ [])])
optimize_diff_list1_H1([Arg|Args],[S\Ts1|Args0]) :-
  flatten_op(Arg,(\),[S|Ts]),
  sort(Ts,Ts1),
  not_member(S,Ts1),
  not_member(top,Ts1),
  optimize_diff_list1_H1(Args,Args0), !.

```

```

optimize_diff_list1_H1([],[]) :- !.

% remove elements that are subsumed by others
%: optimize_diff_list2([(a\[b,c])+(c\[ ]),(a\[b])+(c\[ ])],[],[(a\[b])+c])
optimize_diff_list2([SCr|DfListIn],DfListMax,DfListOut) :-
    SCr =.. [Cr|Args],
    diff_subsumed(Cr,Args,DfListMax),
    optimize_diff_list2(DfListIn,DfListMax,DfListOut), !.

optimize_diff_list2([SCr|DfListIn],DfListMax,DfListOut) :-
    SCr =.. [Cr|Args],
    diff_subsumed(Cr,Args,DfListIn),
    optimize_diff_list2(DfListIn,DfListMax,DfListOut), !.

optimize_diff_list2([SCr|DfListIn],DfListMax,DfListOut) :-
    optimize_diff_list2(DfListIn,[SCr|DfListMax],DfListOut), !.

optimize_diff_list2([],DfListMax,DfListMax) :- !.

% succeed if Cr(Args) is subsumed by an element of DfList
diff_subsumed(Cr,Args,DfList) :-
    member(DfM,DfList),
    DfM =.. [Cr|ArgsM],
    diff_subsumed_H1(Args,ArgsM), !.

diff_subsumed_H1([S\[Ts|Args],[S\[TsM|ArgsM]) :-
    is_sublist(TsM,Ts),
    diff_subsumed_H1(Args,ArgsM), !.

diff_subsumed_H1([],[]) :- !.

% evaluate diff calls for constructor case
diff_list([SCr|Ss],[SCrE|SsE],OccIn,OccOut) :-
    SCr =.. [Cr|SArgs],
    diff_list1(SArgs,SArgsE,OccIn,Occ1),
    SCrE =.. [Cr|SArgsE],
    diff_list(Ss,SsE,Occ1,OccOut), !.

diff_list([],[],OccIn,OccIn) :- !.

diff_list1([S\[Ts|SArgs],[STs|SArgsE],OccIn,OccOut) :-
    diff_t(S,Ts,STs,OccIn,Occ1),
    diff_list1(SArgs,SArgsE,Occ1,OccOut), !.

diff_list1([],[],OccIn,OccIn) :- !.

% list extension of diff for sort disjunction case
diff_list2([S|Ss],T,[ST|STs],OccIn,OccOut) :-
    diff_t(S,T,ST,OccIn,Occ1),
    diff_list2(Ss,T,STs,Occ1,OccOut), !.

diff_list2([],_T,[],OccIn,OccIn) :- !.

% %%%%%%%%%%%%%%% sort inhabitation %%%%%%%%%%%%%%%
% %%%%%%%%%%%%%%% sort inhabitation %%%%%%%%%%%%%%%
% %%%%%%%%%%%%%%% sort inhabitation %%%%%%%%%%%%%%%

% succeed if sort T is non-empty
inh(T) :-
    inh_t(T,[],[]), !.

inh_t(T,_Occ,_Defs) :-
    memo_inh(T,TrueFalse),
    !,
    TrueFalse = true.

inh_t(T,Occ,Defs) :-
    trace_enter(inh,inh(T,Occ)),
    inh(T,Occ,Defs),
    trace_leave(inh,inh(T,Occ)), !.

inh(bottom,_Occ,_Defs) :-
    !, fail.

inh(top,_Occ,_Defs) :- !.

inh(T,Occ,_Defs) :-

```



```

    member(T,Occ),
    !, fail.

inh(T,Occ,Defs) :-
    T sortdef TD,
    !,
    use(T,Defs,Used),
    list_trisection([T|Occ],Used,_,OccUsed,_),
    { inh_t(TD,OccUsed,Defs),
      assert(memo_inh(T,true)), !
    ; OccUsed = [T],
      assert(memo_inh(T,false)),
      fail
    }.

inh(T,Occ,Defs) :-
    member( (_,TD,T1),Defs),
    T == T1,
    !,
    inh_t(TD,[T|Occ],Defs), !.

inh(S ! T,Occ,Defs) :-
    !,
    flatten_op(S!T,(!),ST1),
    sort(ST1,ST),
    inh_list_o(ST,Occ,Defs), !.

inh(T,Occ,Defs) :-
    !,
    T =.. [_Cr|Sorts1],
    sort(Sorts1,Sorts),
    inh_list(Sorts,Occ,Defs), !.

inh_list([T|Ts],Occ,Defs) :-
    inh_t(T,Occ,Defs),
    inh_list(Ts,Occ,Defs), !.

inh_list([],_Occ,_Defs) :- !.

inh_list_o([T|_Ts],Occ,Defs) :-
    inh_t(T,Occ,Defs), !.

inh_list_o([_T|Ts],Occ,Defs) :-
    inh_list_o(Ts,Occ,Defs), !.

% succeed if term I is member of sort T
inhm_t(T,I) :-
    trace_enter(inhm,inhm(T,I)),
    inhm(T,I),
    trace_leave(inhm,inhm(T,I)).

inhm(bottom,_I) :-
    !, fail.

inhm(top,_I) :- !.

inhm(T,I) :-
    T sortdef TD,
    !,
    { memo_inhm(T,I,YN),
      !,
      YN = yes
    ; inhm_t(TD,I),
      assert(memo_inhm(T,I,yes))
    ; assert(memo_inhm(T,I,no)),
      !, fail
    }.

inhm(S!T,I) :-
    !,
    flatten_op(S!T,(!),ST1),
    sort(ST1,ST),
    inhm_list_o(ST,I), !.

inhm(T,I) :-
    !,
    T =.. [Cr|Sorts],
    I =.. [Cr|Is],
    inhm_list(Sorts,Is), !.

inhm_list([T|Ts],[I|Is]) :-
    inhm_t(T,I), !,
    inhm_list(Ts,Is), !.

```

```

inhm_list([],[]) :- !.

inhm_list_o([T|Ts],I) :-
    inhm_t(T,I), !.

inhm_list_o([_T|Ts],I) :-
    inhm_list_o(Ts,I), !.

% succeed if sort S contains no definition cycles,
% i.e., is finite,
% assuming that empty sorts have been replaced by "bottom"
finite(S) :-
    finite(S,nil), !.

finite(bottom,_Occ) :- !.

finite(top,_Occ) :-
    !, fail.

finite(S ! T,Occ) :-
    !,
    flatten_op(S!T,(!),ST1),
    sort(ST1,ST),
    finite_list(ST,Occ), !.

finite(S,Occ) :-
    member_bbt(S,Occ),
    !, fail.

finite(S,Occ) :-
    S sortdef SD,
    !,
    enter_into_bbt(S,Occ,Occ1),
    finite(SD,Occ1), !.

finite(S,Occ) :-
    !,
    S' =.. [_Cr|Ss1],
    sort(Ss1,Ss),
    finite_list(Ss,Occ), !.

finite_list([S|Ss],Occ) :-
    finite(S,Occ), !,
    finite_list(Ss,Occ), !.

finite_list([],_Occ) :- !.

% compute the set UsedSet of all sorts that occur directly or
% indirectly in the definition of Sort
use(Sort,_Defs,UsedSet) :-
    memo_use(Sort,UsedSet), !.

use(Sort,Defs,UsedSet) :-
    use_t(Sort,[],Defs,UsedSet),
    assert(memo_use(Sort,UsedSet)), !.

use_list([Sort|Sorts],Defs,[UsedSet|UsedSets]) :-
    use(Sort,Defs,UsedSet),
    use_list(Sorts,Defs,UsedSets), !.

use_list([],_Defs,[]) :- !.

use_t(Sort,Occ,Defs,UsedSet) :-
    trace_enter(use,use(Sort,Occ)),
    use(Sort,Occ,Defs,UsedSet),
    trace_leave(use,use(Sort,Occ)=UsedSet), !.

use(Sort,Occ,_Defs,UsedSet) :-
    member(Sort,Occ),
    sort(Occ,UsedSet), !.

use(Sort,Occ,Defs,UsedSet) :-
    atom(Sort),
    Sort sortdef SortDef,
    use_t(SortDef,[Sort|Occ],Defs,UsedSet), !.

use(Sort,Occ,Defs,UsedSet) :-
    atom(Sort),
    member( (_,SortDef,Sort),Defs),
    use_t(SortDef,[Sort|Occ],Defs,UsedSet), !.

```

```

use(Sort,Occ,_Defs,Occ) :-
    atomic(Sort), !.

use(Sort,Occ,Defs,UsedSet) :-
    get_leaves(Sort,Defs,_Vars,_Wars,_Consts,Sorts),
    list_trisection(Sorts,Occ,NewSorts,-,_),
    use_list(NewSorts,Occ,Defs,UsedSet), !.

use_list([Sort|Sorts],Occ,Defs,UsedSet) :-
    use_t(Sort,Occ,Defs,UsedSet1),
    use_list(Sorts,UsedSet1,Defs,UsedSet), !.

use_list([],Occ,_Defs,Occ) :- !.

% %%%%%%%%%%% constructor dependent sort generation %%%%%%%%%%% %
% %%%%%%%%%%% constructor dependent sort generation %%%%%%%%%%% %
% %%%%%%%%%%% constructor dependent sort generation %%%%%%%%%%% %

% build the top sort, the sort of reducible terms,
% and the sort of irreducible terms wrt. redices_f,
% all redices have to be linear
init_red :-
    calc_top_sort(TopS),
    redices_f(Red),
    calc_reducible_terms_sort(Red,TopS,RedS),
    diff(TopS,RedS,NfS),
    calc_sort_depths,
    abolish(top_sort,1),
    abolish(redices_sort,1),
    abolish(normalforms_sort,1),
    assert(top_sort(TopS)),
    assert(redices_sort(RedS)),
    assert(normalforms_sort(NfS)), !.

% build the top sort Sort, considering all variables etc.
calc_top_sort(Sort) :-
    new_name(sort,Sort),
    setof(Cr/Ar,(
        arity(Cr,Ar)
    ),CrArs),
    calc_top_sort1(Sort,CrArs,SortDefL),
    sort(SortDefL,SortDefL1),
    unflatten_op(SortDefL1,(!),SortDef),
    assert(Sort sortdf SortDef), !.

%: calc_top_sort1(top,[+/2,s/1,0/0],[top+top,s(top),0])
calc_top_sort1(Sort,[Cr/Ar|CrArs],[SortExpr|SortExprs]) :-
    extend_to_length(Sort,Ar,Sorts),
    SortExpr =.. [Cr|Sorts],
    calc_top_sort1(Sort,CrArs,SortExprs), !.

calc_top_sort1(_Sort,[],[]) :- !.

% build the sort Sort of all terms reducible wrt. Redices,
% Redices may contain only linear terms
calc_reducible_terms_sort(Redices,Top,Sort) :-
    new_name(sort,Sort),
    calc_reducible_terms_sort1_list(Redices,Top,SortDefL1),
    setof(Cr/Ar,(
        arity(Cr,Ar)
    ),CrArs),
    calc_reducible_terms_sort2_list(Sort,Top,CrArs,SortDefL2),
    append(SortDefL1,SortDefL2,SortDefL),
    unflatten_op(SortDefL,(!),SortDef),
    assert(Sort sortdf SortDef), !.

% replace each prolog variable by the sort top,
% redices are assumed to be linear terms
calc_reducible_terms_sort1_list([Red|Reds],Top,[SortExpr|SortExprs]) :-
    calc_reducible_terms_sort1(Red,Top,SortExpr),
    calc_reducible_terms_sort1_list(Reds,Top,SortExprs), !.

calc_reducible_terms_sort1_list([],_Top,[]) :- !.

```

```

%: calc_reducible_terms_sort1(X+s(Y),top,top+s(top))
calc_reducible_terms_sort1(Var,Top,Top) :-
    var(Var), !.

calc_reducible_terms_sort1(Redex,Top,SortExpr) :-
    Redex =.. [Optr|Opnds],
    calc_reducible_terms_sort1_list(Opnds,Top,SortExprs),
    SortExpr =.. [Optr|SortExprs], !.

%: calc(red,top,[+/2,s/1,0/0],[red+top,top+red,s(red)])
calc_reducible_terms_sort2_list(Sort,Top,[_Cr/0|CrArs],SortExprs) :-
    calc_reducible_terms_sort2_list(Sort,Top,CrArs,SortExprs), !.

calc_reducible_terms_sort2_list(Sort,Top,[Cr/Ar|CrArs],SortExprs) :-
    extend_to_length(Top,Ar,Tops),
    calc_reducible_terms_sort2(Sort,Tops,Cr,1,Ar,SortExprs1),
    calc_reducible_terms_sort2_list(Sort,Top,CrArs,SortExprs2),
    append(SortExprs1,SortExprs2,SortExprs), !.

calc_reducible_terms_sort2_list(_Sort,_Top,[],[]) :- !.

%: calc_reducible_terms_sort2(red,[top,top],(+),1,2,[red+top,top+red])
calc_reducible_terms_sort2(Sort,Tops,Cr,Ar,Ar,[SortExpr]) :-
    replace_ith_elem(Tops,Ar,Sort,TopsS),
    SortExpr =.. [Cr|TopsS], !.

calc_reducible_terms_sort2(Sort,Tops,Cr,I,Ar,[SortExpr|SortExprs]) :-
    replace_ith_elem(Tops,I,Sort,TopsS),
    SortExpr =.. [Cr|TopsS],
    I1 is I + 1,
    calc_reducible_terms_sort2(Sort,Tops,Cr,I1,Ar,SortExprs), !.

% %%%%%%%%%%%%%%
% %%%%%%%%%% variable sorts %%%%%%%%%%
% %%%%%%%%%%

% generate the sort of all terms t with Min subset vars(t) subset Max
generate_var_sort(Min,Max,Sort) :-
    var_sort_name(Min,Max,Sort), !.

generate_var_sort([],Max,Sort) :-
    new_name(sort,Sort),
    assert(var_sort_name([],Max,Sort)),
    setof(is_cr(Cr,Ar),arity(Cr,Ar),AllCrS),
    generate_var_sort_H1(AllCrS,Sort,CrSorts),
    append(Max,CrSorts,SortDef1),
    sort(SortDef1,SortDef2),
    unflatten_op(SortDef2,(!),SortDef),
    assert(Sort sortdef SortDef), !.

generate_var_sort([V],Max,Sort) :-
    new_name(sort,Sort),
    assert(var_sort_name([V],Max,Sort)),
    setof(Ar,Cr^(
        arity(Cr,Ar),
        Ar > 0
    ),Arities),
    setof(CrSort,Ar^Part^Parts^Cr^CrArgs^(
        member(Ar,Arities),
        all_set_partitions([V],Ar,Parts),
        member(Part,Parts),
        arity(Cr,Ar),
        generate_var_sort_list1(Part,Max,CrArgs),
        CrSort =.. [Cr|CrArgs]
    ),CrSorts),
    sort([V|CrSorts],SortDef1), % <---- include V
    unflatten_op(SortDef1,(!),SortDef),
    assert(Sort sortdef SortDef), !.

generate_var_sort(Min,Max,Sort) :-
    new_name(sort,Sort),
    assert(var_sort_name(Min,Max,Sort)),
    setof(Ar,Cr^(
        arity(Cr,Ar),
        Ar > 0
    ),Arities),
    setof(CrSort,Ar^Part^Parts^Cr^CrArgs^(
        member(Ar,Arities),
        all_set_partitions(Min,Ar,Parts),

```

```

        member(Part,Parts),
        arity(Cr,Ar),
        generate_var_sort_list1(Part,Max,CrArgs),
        CrSort =.. [Cr|CrArgs]
            ),CrSorts),
        sort(CrSorts,SortDef1), % <----
        unflatten_op(SortDef1,(!),SortDef),
        assert(Sort sortdef SortDef), !.

generate_var_sort_list1([Min|Mins],Max,[Sort|Sorts]) :-
    generate_var_sort(Min,Max,Sort),
    generate_var_sort_list1(Mins,Max,Sorts), !.

generate_var_sort_list1([],_Max,[]) :- !.

%- dont include variables
generate_var_sort_H1([is_cr(Cr,0)|AllCrS],Sort,CrSorts) :-
    { name(Cr,[118|_]) % "v"
      ; name(Cr,[119|_]) % "w"
    },
    generate_var_sort_H1(AllCrS,Sort,CrSorts), !.

generate_var_sort_H1([is_cr(Cr,Ar)|AllCrS],Sort,[CrSort|CrSorts]) :-
    extend_to_length(Sort,Ar,Args),
    CrSort =.. [Cr|Args],
    generate_var_sort_H1(AllCrS,Sort,CrSorts), !.

generate_var_sort_H1([],_Sort,[]) :- !.

% %%%%%%%%%% %
% %%%%%%%%%% term evaluation %%%%%%%%%% %
% %%%%%%%%%% %
% %%%%%%%%%% %

% evaluate Term according to the equations for non-constructor-functions
%- pure constructor term
eval_term_t(Term,Term) :-
    get_leaves(Term,[],_V,[],_C,_O,_S), !.

eval_term_t(Term,TermE) :-
    trace_enter(eval,Term),
    eval_term(Term,TermE),
    trace_leave(eval,TermE), !.

%- non-constructor function at top
eval_term(Term,TermE) :-
    Term =.. [F|Args],
    is_function(F),
    eval_term_list(Args,Args1),
    Term1 =.. [F|Args1],
    { def Lhs=Rhs,
      try_rewrite(Term1,Lhs,Rhs,Term2),
      eval_term_t(Term2,TermE)
    ; TermE = Term1
    }, !.

%- constructor at top
eval_term(Term,TermE) :-
    Term =.. [Cr|Args],
    eval_term_list(Args,ArgsE),
    TermE =.. [Cr|ArgsE], !.

eval_term_list([Term|Terms],[TermE|TermsE]) :-
    eval_term_t(Term,TermE),
    eval_term_list(Terms,TermsE), !.

eval_term_list([],[]) :- !.

% %%%%%%%%%% %
% %%%%%%%%%% sort simplification %%%%%%%%%% %
% %%%%%%%%%% %
% %%%%%%%%%% %

% simplify a list Sorts of sorts and a list TOcc of sort definitions
% TOcc is a list of triples (_,TDef,T)
simplify_sorts_and_defs(Sorts,SortsS,TOcc,TOccS) :-

```

```

contract_sortdef_cycles(Sorts,Sorts0,TOcc,TOcc0),
simplify_sort1(TOcc0,TOcc1),
simplify_sort2([],TOcc1,TOcc2),
simplify_sort3_list(Sorts0,Sorts3,TOcc2),
simplify_sort4_list(Sorts3,Sorts4),
Use = not_needed,
simplify_sort5(Use,[],TOcc2,TOcc5),
simplify_sort6_list(TOcc5,Sorts4,Sorts6),
SortsS = Sorts6,
TOccS = TOcc5, !.

% contract dead cycles in sort definitions
contract_sortdef_cycles(Sorts,SortsC,TOcc,TOccC) :-
split_contains_vars(Sorts,Sorts1,Sorts2),
split_contains_vars(TOcc,TOcc1,TOcc2),
contract_sortdef_cycles1(TOcc2,TOcc3,Subst),
apply_subst_c(Sorts2,Subst,Sorts3),
append(Sorts1,Sorts3,SortsC),
append(TOcc1,TOcc3,TOccC), !.

contract_sortdef_cycles1(TOcc,TOccC,Subst) :-
find_sortdef_cycle(TOcc,Cycle),
contract_sortdef_cycle2(TOcc,Cycle,Cycle,NewDef1),
contract_sortdef_cycle3(TOcc,Cycle,Subst1),
{ NewDef1 = [],
  NewDef2 = bottom
; sort(NewDef1,NewDef2),
  unflatten_op(NewDef2,(!),NewDef3)
},
apply_subst_c(NewDef3,Subst1,NewDef),
contract_sortdef_cycle4(TOcc,Cycle,Subst1,NewDef,TOcc1),
contract_sortdef_cycles1(TOcc1,TOccC,Subst2),
append(Subst2,Subst1,Subst), !.

contract_sortdef_cycles1(TOcc,TOcc,[]) :- !.

% find a dead sortdef cycle in TOcc,
% return a list Cycle of its member sort names
find_sortdef_cycle(TOcc,Cycle) :-
member( (_,_TDef,T),TOcc),
find_sortdef_cycle1(T,TOcc,[],Cycle), !.

find_sortdef_cycle1(T,_TOcc,CycleIn,CycleOut) :-
try_get_prefix_to(CycleIn,T,CycleOut), !.

find_sortdef_cycle1(T,TOcc,CycleIn,CycleOut) :-
member( (_,TDef,T),TOcc),
flatten_op(TDef,(!),TDefF),
member(T1,TDefF),
find_sortdef_cycle1(T1,TOcc,[T|CycleIn],CycleOut), !.

% set up the contracted new sort definition for the cycle
contract_sortdef_cycle2(TOcc,[TName|TNames],Cycle,NewDef) :-
member( (_,TDef,TName),TOcc),
flatten_op(TDef,(!),TDefF),
contract_sortdef_cycle2_H1(TDefF,Cycle,NewDef1),
contract_sortdef_cycle2(TOcc,TNames,Cycle,NewDef2),
append(NewDef1,NewDef2,NewDef), !.

contract_sortdef_cycle2(_TOcc,[],_Cycle,[]) :- !.

contract_sortdef_cycle2_H1([TExpr|TDefF],Cycle,NewDef) :-
member(TExpr,Cycle),
contract_sortdef_cycle2_H1(TDefF,Cycle,NewDef), !.

contract_sortdef_cycle2_H1([TExpr|TDefF],Cycle,[TExpr|NewDef]) :-
contract_sortdef_cycle2_H1(TDefF,Cycle,NewDef), !.

contract_sortdef_cycle2_H1([],_Cycle,[]) :- !.

% set up the contracting substitution for the cycle
contract_sortdef_cycle3(_TOcc,[NewSort|Cycle],Subst) :-
extend_to_same_length(NewSort,Cycle,Range),
merge_lists(Cycle,(=),Range,Subst), !.

% do the contracting replacement

```

```

% TD=TDef, TN=TName, ND=NewDef, NN=NewName, TO=TOcc, Cy=Cycle, Sb=Subst
contract_sortdef_cycle4([(X,_TD,NN)|TO],[NN|Cy],Sb,ND,[(X,ND,NN)|TOC]) :-
    contract_sortdef_cycle4(TO,[NN|Cy],Sb,ND,TOC), !.

contract_sortdef_cycle4([(X,_TD,TN)|TO],[NN|Cy],Sb,ND,[(X,NN,TN)|TOC]) :-
    member(TN,Cy),
    contract_sortdef_cycle4(TO,[NN|Cy],Sb,ND,TOC), !.

contract_sortdef_cycle4([(X,TD,TN)|TO],[NN|Cy],Sb,ND,[(X,TDC,TN)|TOC]) :-
    apply_subst_c(TD,Sb,TDC),
    contract_sortdef_cycle4(TO,[NN|Cy],Sb,ND,TOC), !.

contract_sortdef_cycle4([],[_NN|_Cy],_Sb,_ND,[]) :- !.

% apply simplifies-laws in defs
simplify_sort1([(X,D,T)|T0s],[X,DS,T)|T0sS]) :-
    simplify_sort4(D,DS),
    simplify_sort1(T0s,T0sS), !.

simplify_sort1([],[]) :- !.

% unfold trivial defs (bottom,SortName,Constant) in defs
simplify_sort2(TOccL,[(X,TD,T)|TOccR],TOccS) :-
    atomic(TD),
    nonvar(T),
    apply_subst_c(TOccL,[T := TD],TOccL1),
    apply_subst_c(TOccR,[T := TD],TOccR1),
    simplify_sort2([(X,TD,T)|TOccL1],TOccR1,TOccS), !.

simplify_sort2(TOccL,[(X,TD,T)|TOccR],TOccS) :-
    simplify_sort2([(X,TD,T)|TOccL],TOccR,TOccS), !.

simplify_sort2(TOccL,[],TOccL) :- !.

% unfold trivial defs in sorts
simplify_sort3_list(Sorts,SortsS,[(_,TD,T)|TOcc]) :-
    atomic(TD),
    nonvar(T),
    apply_subst_c(Sorts,[T := TD],Sorts1),
    simplify_sort3_list(Sorts1,SortsS,TOcc), !.

simplify_sort3_list(Sorts,SortsS,[_|TOcc]) :-
    simplify_sort3_list(Sorts,SortsS,TOcc), !.

simplify_sort3_list(Sorts,Sorts,[]) :- !.

simplify_sort4(Sort,SortS) :-
    simplify_sort4_H1(Sort,SortS), !.

simplify_sort4_list([Sort|Sorts],[SortS|SortsS]) :-
    simplify_sort4(Sort,SortS),
    simplify_sort4_list(Sorts,SortsS), !.

simplify_sort4_list([],[]) :- !.

simplify_sort4_H1(Sort1!Sort2,SortS) :-
    flatten_op(Sort1!Sort2,(!),SortF),
    simplify_sort4_list(SortF,SortF1),
    sort(SortF1,SortF2),
    simplify_sort4_H2(SortF2,SortF3),
    { unflatten_op(SortF3,(!),SortS)
    ; SortS = bottom
    }, !.

simplify_sort4_H1(Sort,SortS) :-
    Sort =.. [OpPtr|Opnds],
    simplify_sort4_H1_list(Opnds,OpndsS),
    Sort1 =.. [OpPtr|OpndsS],
    { simplifies_sort(Sort1,SortS)
    ; SortS = Sort1
    }, !.

simplify_sort4_H1(Sort,Sort) :- !.

simplify_sort4_H1_list([Sort|Sorts],[SortS|SortsS]) :-
    simplify_sort4_H1(Sort,SortS),
    simplify_sort4_H1_list(Sorts,SortsS), !.

simplify_sort4_H1_list([],[]) :- !.

```

```

simplify_sort4_H2([bottom|Sorts],SortsS) :-
    simplify_sort4_H2(Sorts,SortsS), !.

simplify_sort4_H2([Sort|Sorts],[Sort|SortsS]) :-
    simplify_sort4_H2(Sorts,SortsS), !.

simplify_sort4_H2([],[]) :- !.

simplifies_sort(Fbottom,bottom) :-
    Fbottom =.. [_F|Args],
    member(bottom,Args), !.

% fold defs in defs
simplify_sort5(Use,TOccL,[(X,TD,T)|TOccR],TOccS) :-
    ( member( (_,SortDefFold,SortFold),TOccL)
      ; member( (_,SortDefFold,SortFold),TOccR)
    ),
    nonvar(SortDefFold),
    nonvar(SortFold),
    not atomic(SortDefFold),
    apply_subst_c(TD,[SortDefFold:=SortFold],TD1),
    TD \== TD1,
    reverse(TOccL,TOccL1),
    append(TOccL1,TOccR,TOcc),
    simplify_sort5(Use,[],[(X,TD1,T)|TOcc],TOccS), !.

simplify_sort5(Use,TOccL,[(X,TD,T)|TOccR],TOccS) :-
    simplify_sort5(Use,[(X,TD,T)|TOccL],TOccR,TOccS), !.

simplify_sort5(_Use,TOccL,[],TOccL) :- !.

% fold defs in sorts
simplify_sort6_list(SortOcc,[Sort|Sorts],[SortS|SortsS]) :-
    member( (_,SortDefFold,SortFold),SortOcc),
    nonvar(SortDefFold),
    nonvar(SortFold),
    not atomic(SortDefFold),
    apply_subst_c(Sort,[SortDefFold:=SortFold],Sort1),
    Sort \== Sort1,
    simplify_sort6_list(SortOcc,[Sort1|Sorts],[SortS|SortsS]), !.

simplify_sort6_list(SortOcc,[Sort|Sorts],[Sort|SortsS]) :-
    simplify_sort6_list(SortOcc,Sorts,SortsS), !.

simplify_sort6_list(_SortOcc,[],[]) :- !.

% eliminate each variable that occurs at exactly the same places
% or a subset of another one
%: simplify_variables(v1!v2!(v3!v4)*s9*(v3!v4),v1!v3*s9*v3)
simplify_variables(Sort,SortS) :-
    simplify_variables_H1(1,_0,Sort,Classes),
    sort(Classes,Classes1),
    simplify_variables_H3(Classes1,xxx,[],Classes2),
    sort(Classes2,[[]:xxx|Classes3]),
    simplify_variables_H4(Classes3,_Classes4,ElimVars1),
    ElimVars = ElimVars1,
    simplify_variables_H5(Sort,ElimVars,SortS), !.

% get classes of variables that occur simultaneously,
% i.e. connected by "!", in Sort
%: si(1,4,(v1!v2!(v3!v4)*s9*(v3!v4))+[],[v1:1,v2:1,v3:2,v4:2,v3:3,v4:3])
simplify_variables_H1(I,0,Sort1,!Sort2,Classes) :-
    flatten_op(Sort1!Sort2,(!),SortF),
    sort(SortF,SortF1),
    simplify_variables_H2(I,SortF1,VarsF1,ExprsF1),
    I1 is I + 1,
    simplify_variables_H1_list(I1,0,ExprsF1,Classes1),
    append(VarsF1,Classes1,Classes), !.

simplify_variables_H1(I,0,Sort,Classes) :-
    Sort =.. [_Optr|Opnds],
    simplify_variables_H1_list(I,0,Opnds,Classes), !.

simplify_variables_H1_list(I,0,[Sort|Sorts],Classes) :-
    simplify_variables_H1(I,M,Sort,Classes1),

```



```

        simplify_variables_H1_list(M,O,Sorts,Classes2),
        append(Classes1,Classes2,Classes), !.

simplify_variables_H1_list(I,I,[],[]) :- !.

% separate Sorts into Vars and Exprs, all Vars beginning with "v"
simplify_variables_H2(I,[Var|Sorts],[Var:I|Vars],Exprs) :-
    atom(Var),
    name(Var,[118|_Nr]),
    simplify_variables_H2(I,Sorts,Vars,Exprs), !.

simplify_variables_H2(I,[Expr|Sorts],Vars,[Expr|Exprs]) :-
    simplify_variables_H2(I,Sorts,Vars,Exprs), !.

simplify_variables_H2(_I,[],[],[]) :- !.

% collect classes of occurrences of each variable
%: simplify_variables_H3([v1:1,v2:1,v3:2,v3:3,v4:2,v4:3],xxx,[],
%: [[:xxx,[1]:v1,[1]:v2,[2,3]:v3,[2,3]:v4])
simplify_variables_H3([V:C|Rel],V,Class,Classes) :-
    simplify_variables_H3(Rel,V,[C|Class],Classes), !.

simplify_variables_H3([V1:C|Rel],V2,Class,[ClassS:V2|Classes]) :-
    sort(Class,ClassS),
    simplify_variables_H3(Rel,V1,[C],Classes), !.

simplify_variables_H3([],V,Class,[ClassS:V]) :-
    sort(Class,ClassS), !.

% calculate all variables whose occurrences are equal to that
% of another, calculate the reduces occurrence list with lengths
%: si([[:1]:v1,[1]:v2,[2,3]:v3,[2,3]:v4],[1:[1]:v1,2:[2,3]:v3],[v2,v4])
simplify_variables_H4([C:V1,C:V2|Classes],ClassesR,[V2|ElimVars]) :-
    simplify_variables_H4([C:V1|Classes],ClassesR,ElimVars), !.

simplify_variables_H4([C1:V1,C2:V2|C1],[L1:C1:V1|C1R],ElimVars) :-
    length(C1,L1),
    simplify_variables_H4([C2:V2|C1],C1R,ElimVars), !.

simplify_variables_H4([C:V],[L:C:V],[]) :-
    length(C,L), !.

simplify_variables_H4([],[],[]) :- !.

% group variable occurrences in classes by their length
%: si([[:1]:v1,2:[1,2]:v3],1,[[:0]:v0],[[:1]:v1,[0]:v0],[[1,2]:v3]])
simplify_variables_H6([L:C:V|C1In],L,C1,C1Out) :-
    simplify_variables_H6(C1In,L,[C:V|C1],C1Out), !.

simplify_variables_H6([L1:C:V|C1In],_L2,C1,[C1|C1Out]) :-
    simplify_variables_H6(C1In,L1,[C:V],C1Out), !.

simplify_variables_H6([],_L,C1,[C1]) :- !.

% calculate all variables whose occurrences form a subset of another,
% occurrences are ordered in classes by their length
%: simplify_variables_H7([[:1]:v1,[0]:v0],[[1,2]:v3],[v1])
simplify_variables_H7([C1:V1|C11|Classes],[V1|ElimVars]) :-
    member(C12,Classes),
    member(C2:_V2,C12),
    is_sublist(C1,C2),
    simplify_variables_H7([C11|Classes],ElimVars), !.

simplify_variables_H7([[_C1:_V1|C11|Classes],ElimVars) :-
    simplify_variables_H7([C11|Classes],ElimVars), !.

simplify_variables_H7([[]|Classes],ElimVars) :-
    simplify_variables_H7(Classes,ElimVars), !.

simplify_variables_H7([],[]) :- !.

% do the elimination
%: si((v1!v2!(v3!v4)*s9*(v3!v4))+[],[v2,v4],[v1!v3*s9*v3]+[])
simplify_variables_H5(Sort1!Sort2,ElimVars,SortS) :-
    flatten_op(Sort1!Sort2,(!),SortF),
    list_trisection(SortF,ElimVars,SortF1,_,_),

```

```

simplify_variables_H5_list(SortF1,ElimVars,SortF2),
unflatten_op(SortF2,(!),SortS), !.

simplify_variables_H5(Sort,ElimVars,SortS) :-
Sort =.. [Optr|Opnds],
simplify_variables_H5_list(Opnds,ElimVars,OpndsS),
SortS =.. [Optr|OpndsS], !.

simplify_variables_H5_list([Sort|Sorts],ElimVars,[SortS|SortsS]) :-
simplify_variables_H5(Sort,ElimVars,SortS),
simplify_variables_H5_list(Sorts,ElimVars,SortsS), !.

simplify_variables_H5_list([],_ElimVars,[]) :- !.

% %%%%%%%%%%%
% %%%%%%%%%%% administration %%%%%%%%%%%
% %%%%%%%%%%%

% remove all empty sorts from Sorts, yielding SortsR
remove_empty_sorts([bottom|Sorts],SortsR) :-
remove_empty_sorts(Sorts,SortsR), !.

remove_empty_sorts([Empty|Sorts],SortsR) :-
Empty sortdf bottom,
remove_empty_sorts(Sorts,SortsR), !.

remove_empty_sorts([Sort|Sorts],[Sort|SortsR]) :-
remove_empty_sorts(Sorts,SortsR), !.

remove_empty_sorts([],[]) :- !.

% replace all sort names in Ss by their definitions as long as not each
% disjunct starts with a constructor
replace_defs_star(Ss,SsDef,OccI,OccO,Def,Name,true) :-
%% link all Def's and Name's together
replace_defs(Ss,SsD1,OccI,Occ1,Def,Name,true),
replace_defs_star(SsD1,SsDef,Occ1,OccO,Def,Name,_Succ1), !.

replace_defs_star(Ss,Ss,OccI,OccI,_Def,_Name,false) :- !.

% replace all sort names in Ss by their definitions, yielding SsDef,
% update OccO from OccI,
% Success is true when any definition was actually replaced,
replace_defs(Ss,SsDef,OccI,OccO,Def,Name,true) :-
replace_defs2(Ss,SsDef,SsN,true),
enter_into_bbt((SsN,Def,Name),OccI,OccO), !.

replace_defs(Ss,Ss,OccIn,OccIn,_Def,_Name,false) :- !.

replace_defs2([Sa!Sb|Ss],[SDef|SsDef],[SN|SsN],Success) :-
flatten_op(Sa!Sb,(!),S1),
sort(S1,S2),
unflatten_op(S2,(!),SN),
replace_defs1(S2,S3,Success1),
unflatten_op(S3,(!),SDef),
replace_defs2(Ss,SsDef,SsN,Success2),
calc_or(Success1,Success2,Success), !.

replace_defs2([S|Ss],[SDef|SsDef],[S|SsN],Success) :-
replace_defs1([S],[SDef],Success1),
replace_defs2(Ss,SsDef,SsN,Success2),
calc_or(Success1,Success2,Success), !.

replace_defs2([],[],[],false) :- !.

replace_defs1([T|Ts],[TDef|TDefs],true) :-
T sortdf TDef,
replace_defs1(Ts,TDefs,_Success), !.

replace_defs1([T|Ts],[T|TDefs],Success) :-
replace_defs1(Ts,TDefs,Success), !.

replace_defs1([],[],false) :- !.

% extract sort definitions from occurrence list in range,inf,diff
extract_defs_from_occ([(A,D,N)|Occ],[N sortdf D|Dfs],MName,[Mm|Mms]) :-

```

```

    nonvar(N),
    append(A,[N],AN),
    Mm =.. [MName|AN],
    extract_defs_from_occ(Occ,Dfs,MName,Mms), !.

extract_defs_from_occ([(_,_D,_N)|Occ],Dfs,MName,Memos) :-
    extract_defs_from_occ(Occ,Dfs,MName,Memos), !.

extract_defs_from_occ([],[],_MName,[]) :-
    !.

% merge List1 and List2, using Optr, into List
% modes: (>List1,>Optr,>List2,List>), (List1>,Optr>,List2>,>List)
merge_lists([Elem1|List1],Optr,[Elem2|List2],[Elem|List]) :-
    Elem =.. [Optr,Elem1,Elem2],
    merge_lists(List1,Optr,List2,List), !.

merge_lists([],_Optr,[],[]) :-
    !.

merge_lists_list([List1|Lists1],Optr,[List2|Lists2],[List|Lists]) :-
    merge_lists(List1,Optr,List2,List),
    merge_lists_list(Lists1,Optr,Lists2,Lists), !.

merge_lists_list([],_Optr,[],[]) :-
    !.

%: distribute1(f(a!b,g(c!d)),[f(a,g(c)),f(a,g(d)),f(b,g(c)),f(b,g(d))])
distribute1(S,[S],false) :-
    var(S), !.

distribute1(S,SD,true) :-
    S = (_S1 ! _S2),
    flatten_op(S,!),SF),
    distribute1_list_a(SF,SD,_Success), !.

distribute1(S,SD,Success) :-
    S =.. [F|Args],
    distribute1_list_c(Args,ArgsD,Success),
    distribute1_H1(F,ArgsD,SD), !.

%: distribute1_list_a([a,b,h(c!d)],[a,b,h(c),h(d)])
distribute1_list_a([S|Ss],SSsD,Success) :-
    distribute1(S,SD,Success1),
    distribute1_list_a(Ss,SsD,Success2),
    append(SD,SsD,SSsD),
    calc_or(Success1,Success2,Success), !.

distribute1_list_a([],[],false) :-
    !.

%: distribute1_list_c([a!b,g(c!d)],[[a,g(c)],[a,g(d)],[b,g(c)],[b,g(d)]]
distribute1_list_c([S|Ss],SSsD,Success) :-
    distribute1(S,SD,Success1),
    distribute1_list_c(Ss,SsD,Success2),
    combine(SD,SsD,SSsD),
    calc_or(Success1,Success2,Success), !.

distribute1_list_c([],[],false) :-
    !.

%: distribute1_H1(f,[a,g(c)],[a,g(d)],[f(a,g(c)),f(a,g(d))])
distribute1_H1(F,[Args|Argss],[Term|Terms]) :-
    Term =.. [F|Args],
    distribute1_H1(F,Argss,Terms), !.

distribute1_H1(_F,[],[]) :-
    !.

%: co([a,b,c],[[1,2],[3]],[[a,1,2],[a,3],[b,1,2],[b,3],[c,1,2],[c,3]])
combine([H1|T1],LL2,L1LL2) :-
    combine1(H1,LL2,H1LL2),
    combine(T1,LL2,T1LL2),
    append(H1LL2,T1LL2,L1LL2), !.

combine([],_LL2,[]) :-
    !.

%: combine1(a,[1,2],[3]],[[a,1,2],[a,3]])
combine1(H1,[HL2|TL2],[[H1|HL2]|H1TL2]) :-
    combine1(H1,TL2,H1TL2), !.

combine1(_H1,[],[]) :-
    !.

```

% read some example instances of Term and return the equations that

```

% normalize them
%: read_instances(v1+v2,[0+s(0):s(0),s(s(0))+s(0):s(s(s(0)))])
%: input:
%:      [0,s(0)].
%:      [s(s(0)),s(0)].
%:      end.
read_instances(Term,Eqns) :-
    get_leaves(Term,[],Vars,_F,_C,_O,_S),
    read_subst_list(Vars,Substs),
    apply_subst_c_list2(Term,Substs,GndTerms),
    eval_term_list(GndTerms,NormalForms),
    equiv_class_list(NormalForms,EqClasses),
    merge_lists(GndTerms,(:),EqClasses,Eqns), !.

% for each term in Terms, return its equivalence class sort
equiv_class_list([Term|Terms],[EqClass|EqClasses]) :-
    equiv_class(Term,EqClass),
    equiv_class_list(Terms,EqClasses), !.

equiv_class_list([Term|_Terms],_EqClasses) :-
    nl, write('+++++ cant find equivalence class for '),
    write(Term), nl,
    !, abort.

equiv_class_list([],[]) :- !.

% succeed if a variant of Subj is an instance of Pattern
is_instance_118(Subj,Pattern,Subst) :-
    is_instance1_118(Subj,Pattern,Subst1),
    sort(Subst1,Subst),
    is_valid_substitution(Subst), !.

is_instance1_118(Subj,Var,[Var:=Subj]) :-
    atomic(Var),
    name(Var,[118|_]), !.

is_instance1_118(Subj,Pattern,Subst) :-
    Subj =.. [Optr|OpndsS],
    Pattern =.. [Optr|OpndsP],
    is_instance1_118_list(OpndsS,OpndsP,Subst), !.

is_instance1_118_list([Subj|Subjs],[Pattern|Patterns],Subst) :-
    is_instance1_118(Subj,Pattern,Subst1),
    is_instance1_118_list(Subjs,Patterns,Subst2),
    append(Subst1,Subst2,Subst), !.

is_instance1_118_list([],[],[]) :- !.

% succeed if Subst does not assign different image terms
% to the same variable
is_valid_substitution(Subst) :-
    sort(Subst,Subst1),
    is_valid_substitution1(Subst1), !.

is_valid_substitution1([X:=_T1,X:=_T2|_Subst]) :-
    !, fail.

is_valid_substitution1([_X:=_T1,Y:=T2|Subst]) :-
    is_valid_substitution1([Y:=T2|Subst]), !.

is_valid_substitution1([_X:=_T1]) :- !.

is_valid_substitution1([]) :- !.

% succeed if Subj is an instance of Pattern via Subst,
% variables are atoms declared by "is_variable/2"
root_match(Subj,Var,Subst) :-
    root_match1(Subj,Var,Subst,SubjVars),
    merge_lists(SubstVars,:=,_SubstTerms,Subst),
    list_trisection(SubstVars,SubjVars,_,[],_), !.

root_match1(Subj,_Pattern,_,_) :-
    var(Subj),
    !, fail.

root_match1(Var,Var,[],[Var]) :-
    is_variable(Var,_), !.

```

```

root_match1(Const,Const,[],[]) :-
    atomic(Const), !.

root_match1(Subj,Var,[Var:=Subj],SubjVars) :-
    is_variable(Var,_),
    get_variables(Subj,SubjVars), !.

root_match1(Subj,Pattern,Subst,SubjVars) :-
    Pattern =.. [Optr|PatList],
    Subj =.. [Optr|SubjList],
    root_match1_list(SubjList,PatList,[],Subst,SubjVars), !.

root_match1_list([Subj|Subjs],[Pat|Pats],SubstI,SubstO,SubjVars) :-
    apply_subst(Pat,SubstI,PatSI),
    root_match1(Subj,PatSI,SubstA,SubjVarsA),
    append(SubstI,SubstA,SubstIA),
    root_match1_list(Subjs,Pats,SubstIA,SubstO,SubjVarsB),
    append(SubjVarsA,SubjVarsB,SubjVars), !.

root_match1_list([],[],SubstI,SubstI,[]) :- !.

% apply the list of elementary substitutions SubstL to Term,
% yielding TermS,
% SubstL = [S1,...,Sn] denotes  $S_n * \dots * S_1$ ,
% does not handle bound variables correctly
% leaves prolog variables unchanged,
% variables are atoms declared by "is_variable/2"
apply_subst(Term,[Subst|SubstL],TermS) :-
    apply_subst1(Term,Subst,Term1),
    apply_subst(Term1,SubstL,TermS), !.

apply_subst(Term,[],Term) :- !.

%- apply the elementary substitution Subst to Term, yielding TermS
apply_subst1(PrologVar,_Subst,PrologVar) :-
    var(PrologVar), !.

apply_subst1(UserVar,(UserVar:=Term),Term) :-
    is_variable(UserVar,_), !.

apply_subst1(Atom,_Subst,Atom) :-
    atomic(Atom), !.

apply_subst1(Term,Subst,TermS) :-
    Term =.. [Optr|TermL],
    apply_subst1_list(TermL,Subst,TermSL),
    TermS =.. [Optr|TermSL], !.

apply_subst1_list([Term|TermL],Subst,[TermS|TermSL]) :-
    apply_subst1(Term,Subst,TermS),
    apply_subst1_list(TermL,Subst,TermSL), !.

apply_subst1_list([],_Subst,[]) :- !.

% succeed if the rule Lhs***>Rhs is applicable to Old, yielding New
try_rewrite(Old,Lhs,Rhs,New) :-
    root_match(Old,Lhs,Subst),
    apply_subst(Rhs,Subst,New), !.

% like apply_subst, but substitute also constants
apply_subst_c(Term,_SubstL,Term) :-
    contains_vars(Term), !.

apply_subst_c(Term,SubstL,Term) :-
    contains_vars(SubstL), !.

apply_subst_c(Term,SubstL,TermS) :-
    apply_subst_c1(Term,SubstL,TermS), !.

apply_subst_c1(Term,[Subst|SubstL],TermS) :-
    apply_subst_c2(Term,Subst,Term1),
    apply_subst_c1(Term1,SubstL,TermS), !.

apply_subst_c1(Term,[],Term) :- !.

%- apply the elementary substitution Subst to Term, yielding TermS
apply_subst_c2(Lhs,(Lhs:=Rhs),Rhs) :- !.

apply_subst_c2(Term,Subst,TermS) :-
    Term =.. [Optr|TermL],

```

```

        apply_subst_c2_list(TermL,Subst,TermSL),
        TermS =.. [Optr|TermSL], !.

apply_subst_c2_list([Term|TermL],Subst,[TermS|TermSL]) :-
    apply_subst_c2(Term,Subst,TermS),
    apply_subst_c2_list(TermL,Subst,TermSL), !.

apply_subst_c2_list([],_Subst,[]) :- !.

apply_subst_c_list2(Term,[Subst|Substs],[TermS|TermsS]) :-
    apply_subst_c(Term,Subst,TermS),
    apply_subst_c_list2(Term,Substs,TermsS), !.

apply_subst_c_list2(_Term,[],[]) :- !.

% apply the substitution SubstS in set form to Term, yielding TermS,
% substitute also constants, but not composite terms
apply_subst_sc(Lhs,Subst,Rhs) :-
    atomic(Lhs),
    member_bbt(Lhs:=Rhs,Subst), !.

apply_subst_sc(Term,Subst,TermS) :-
    Term =.. [Optr|TermL],
    apply_subst_sc_list(TermL,Subst,TermSL),
    TermS =.. [Optr|TermSL], !.

apply_subst_sc_list([Term|TermL],Subst,[TermS|TermSL]) :-
    apply_subst_sc(Term,Subst,TermS),
    apply_subst_sc_list(TermL,Subst,TermSL), !.

apply_subst_sc_list([],_Subst,[]) :- !.

% %%%%%%%%%%% basic functions %%%%%%%%%%% %
% %%%%%%%%%%% basic functions %%%%%%%%%%% %
% %%%%%%%%%%% basic functions %%%%%%%%%%% %

% find a list Parts of all partitions of of Set into L disjoint subsets
all_set_partitions(Set,L,Parts) :-
    make_nat_list_of_length(L,Ran),
    extend_to_length([],L,Init),
    setof(Part,Map^Part1^(
        mapping(Set,Ran,Map),
        partition_from_mapping(Map,Init,Part1),
        sort_list(Part1,Part)
    ),Parts), !.

% append all lists in Lists, yielding ListsA
append_list(Lists,ListsA) :-
    append_list1(Lists,[],ListsA), !.

append_list1([List|Lists],CollectIn,CollectOut) :-
    append(CollectIn,List,Collect1),
    append_list1(Lists,Collect1,CollectOut), !.

append_list1([],Collect,Collect) :- !.

assert_list([H|T]) :-
    assert(H),
    assert_list(T), !.

assert_list([]) :- !.

assert_list_list([List|Lists]) :-
    assert_list(List),
    assert_list_list(Lists), !.

assert_list_list([]) :- !.

%: beautify_output(s(s(s(0)))+s(s(v1)),3+s2(v1))
beautify_output(s(Term),TermB) :-
    beautify_output(Term,Term1),
    { integer(Term1), !,

```

```

        TermB is Term1 + 1
    ; Term1 = s(Opnd), !,
    TermB = s2(Opnd)
    ; Term1 =.. [S|Opnd],
    name(S,[115|ExpN]),
    name(Exp,ExpN),
    integer(Exp),
    Exp1 is Exp + 1,
    name(Exp1,Exp1N),
    name(S1,[115|Exp1N]),
    TermB =.. [S1|Opnd]
    ; TermB = s(Term1)
}, !.

beautify_output(Term,TermB) :-
    Term =.. [Optr|Opnds],
    beautify_output_list(Opnds,OpndsB),
    TermB =.. [Optr|OpndsB], !.

beautify_output_list([Term|Terms],[TermB|TermsB]) :-
    beautify_output(Term,TermB),
    beautify_output_list(Terms,TermsB), !.

beautify_output_list([],[]) :- !.

% calculate the or junctor
calc_or(false,false,false).
calc_or(true,_,true).
calc_or(_,true,true).

% calculate the and junctor
calc_and(true,true,true).
calc_and(false,_,false).
calc_and(_,false,false).

% cons each element of L with the corresponding list of LList,
% yielding LLList
cons_list12([H|L],[HList|LList],[[H|HList]|LList]) :-
    cons_list12(L,LList,LList), !.

cons_list12([],[],[]) :- !.

% succeed if Term contains prolog variables
% dont instantiate anyone of them
contains_vars(Term) :-
    var(Term), !.

contains_vars(Term) :-
    Term =.. [_Optr|Opnds],
    contains_vars_list(Opnds), !.

contains_vars_list([Term|_Terms]) :-
    contains_vars(Term), !.

contains_vars_list([_Term|Terms]) :-
    contains_vars_list(Terms), !.

% convert each balanced binary tree in BBTs to list form,
% append the lists
conv_bbtlist_to_list([f(L,_Lh,E,_Rh,R)|BBTs],List) :-
    conv_bbtlist_to_list([L,[E],R|BBTs],List), !.
conv_bbtlist_to_list([nil|BBTs],List) :-
    conv_bbtlist_to_list(BBTs,List), !.

conv_bbtlist_to_list([[_E]|BBTs],[E|List]) :-
    conv_bbtlist_to_list(BBTs,List), !.

conv_bbtlist_to_list([],[]) :- !.

% convert List to balanced binary tree BBTree
% nodes: f(Left,LeftHg,Val,RightHg,Right), nil
conv_list_to_bbt(List,BBTree) :-
    sort(List,ListS),
    length(ListS,Lg),

```

```

conv_list_to_bbt1(ListS,Lg,BBTree,_Hg), !.

conv_list_to_bbt1([E1],1,f(nil,0,E1,0,nil),1) :- !.

conv_list_to_bbt1([],0,nil,0) :- !.

conv_list_to_bbt1(List,Lg,f(BBTreeL,HgL,V,HgR,BBTreeR),Hg) :-
    LgL is Lg // 2,
    LgR is Lg - LgL - 1,
    conv_list_to_bt2(List,LgL,ListL,V,ListR),
    conv_list_to_bbt1(ListL,LgL,BBTreeL,HgL),
    conv_list_to_bbt1(ListR,LgR,BBTreeR,HgR),
    Hg is max(HgL,HgR) + 1, !.

conv_list_to_bt2([H|T],0,[],H,T) :- !.

conv_list_to_bt2([H|T],LgL,[H|ListL],V,ListR) :-
    Lg1 is LgL - 1,
    conv_list_to_bt2(T,Lg1,ListL,V,ListR), !.

% enter Elem into balanced binary tree BBT
%- Elem already in BBT
enter_into_bbt(Elem,f(L,Lh,Elem,Rh,R),f(L,Lh,Elem,Rh,R)) :- !.

%- enter into left subtree
enter_into_bbt(Elem,f(L,_Lh,E,Rh,R),BBT1) :-
    Elem @< E,
    enter_into_bbt(Elem,L,f(LL,LLh,LE,LRh,LR)),
    { LLh > Rh,
      Rh1 is max(LRh,Rh) + 1,
      BBT1 = f(LL,LLh,LE,Rh1,f(LR,LRh,E,Rh,R))
    }; LRh > Rh,
    LR = f(LRL,LRLh,LRE,LRRh,LRR),
    Lh1 is max(LLh,LRLh) + 1,
    Rh1 is max(LRRh,Rh) + 1,
    BBT1 = f(f(LL,LLh,LE,LRLh,LRL),Lh1,LRE,Rh1,f(LRR,LRRh,E,Rh,R))
    ; Lh1 is max(LLh,LRh) + 1,
    BBT1 = f(f(LL,LLh,LE,LRh,LR),Lh1,E,Rh,R)
    }, !.

%- enter into right subtree
enter_into_bbt(Elem,f(L,Lh,E,_Rh,R),BBT1) :-
    E @< Elem,
    enter_into_bbt(Elem,R,f(RL,RLh,RE,RRh,RR)),
    { RRh > Lh,
      Lh1 is max(Lh,RLh) + 1,
      BBT1 = f(f(L,Lh,E,RLh,RL),Lh1,RE,RRh,RR)
    }; RLh > Lh,
    RL = f(RLL,RLLh,RLE,RLRh,RLR),
    Lh1 is max(Lh,RLLh) + 1,
    Rh1 is max(RLRh,RRh) + 1,
    BBT1 = f(f(L,Lh,E,RLLh,RLL),Rh1,RLE,Lh1,f(RLR,RLRh,RE,RRh,RR))
    ; Rh1 is max(RLh,RRh) + 1,
    BBT1 = f(L,Lh,E,Rh1,f(RL,RLh,RE,RRh,RR))
    }, !.

%- enter into empty tree
enter_into_bbt(Elem,nil,f(nil,0,Elem,0,nil)) :- !.

enter_into_bbt(A,B,C) :-
    dont_backtrack(enter_into_bbt(A,B,C)).

%: enter([a,b,d], [a:[b:[c:[ ]]],b:[ ]], [a:[b:[c:[ ],d:[ ]]],b:[ ]])
enter_into_trie(List,TrieIn,TrieOut) :-
    enter_into_trie1(List,[],TrieIn,TrieOut), !.

enter_into_trie1([V|Vs],TrieInL,[V:TrieV|TrieInR],TrieOut) :-
    enter_into_trie1(Vs,[],TrieV,TrieV1),
    reverse(TrieInL,TrieInL1),
    append(TrieInL1,[V:TrieV1|TrieInR],TrieOut), !.

enter_into_trie1([V1|Vs],TrieInL,[V2:TrieV2|TrieInR],TrieO) :-
    V2 @< V1,
    enter_into_trie1([V1|Vs],[V2:TrieV2|TrieInL],TrieInR,TrieO), !.

enter_into_trie1([V1|Vs],TrieInL,[V2:TrieV2|TrieInR],TrieOut) :-
    V1 @< V2,
    enter_into_trie1(Vs,[],[],TrieV1),
    reverse(TrieInL,TrieInL1),
    append(TrieInL1,[V1:TrieV1,V2:TrieV2|TrieInR],TrieOut), !.

```



```

enter_into_trie1([V1|Vs],TrieInL,[],TrieOut) :-
    enter_into_trie1(Vs,[],[],TrieV1),
    reverse(TrieInL,TrieInL1),
    append(TrieInL1,[V1:TrieV1],TrieOut), !.

enter_into_trie1([],[],TrieInR,TrieInR) :- !.

% build a list List2 of length I, each element being Elem
extend_to_length(_Elem,0,[]) :- !.

extend_to_length(Elem,I,[Elem|List]) :-
    I1 is I - 1,
    extend_to_length(Elem,I1,List), !.

extend_to_length_list(Elem,[Length|Lengths],[List|Lists]) :-
    extend_to_length(Elem,Length,List),
    extend_to_length_list(Elem,Lengths,Lists), !.

extend_to_length_list(_Elem,[],[]) :- !.

% build a list List2 of same length as List1, each element being Elem
extend_to_same_length(Elem,[_Head1|Tail1],[Elem|Tail2]) :-
    extend_to_same_length(Elem,Tail1,Tail2), !.

extend_to_same_length(_Elem,[],[]) :- !.

extend_to_same_length_list(Elem,[List1|Lists1],[List2|Lists2]) :-
    extend_to_same_length(Elem,List1,List2),
    extend_to_same_length_list(Elem,Lists1,Lists2), !.

extend_to_same_length_list(_Elem,[],[]) :- !.

% operator Op is associative and commutative,
% enter all its top level operands in Tree into List
flatten_op(Tree,Op,List) :-
    Tree =.. [Op|Opnds],
    flatten_op_list(Opnds,Op,List), !.

flatten_op(Tree,_Op,[Tree]) :- !.

flatten_op_list([Head|Tail],Op,List) :-
    flatten_op(Head,Op,HeadList),
    flatten_op_list(Tail,Op,TailList),
    append(HeadList,TailList,List), !.

flatten_op_list([],_Op,[]) :- !.

% flatten each tree in Trees, yielding Lists
flatten_op_list1([Tree|Trees],Op,[List|Lists]) :-
    flatten_op(Tree,Op,List),
    flatten_op_list1(Trees,Op,Lists), !.

flatten_op_list1([],_Op,[]) :- !.

% return the I'th element of List
get_ith_elem([Head|_Tail],1,Head) :- !.

get_ith_elem([_Head|Tail],I,Elem) :-
    I1 is I - 1,
    get_ith_elem(Tail,I1,Elem), !.

% calculate a list Vars of all variables in Term starting with "v",
% Fcts of non-constructor functions, Crs of constructors,
% and a list Optrs of all other symbols except sort names
get_leaves(Term,Defs,Vars,Fcts,Crs,Optrs,Sorts) :-
    get_leaves1(Term,Defs,Vars1,Fcts1,Crs1,Optrs1,Sorts1),
    sort(Vars1,Vars),
    sort(Fcts1,Fcts),
    sort(Crs1,Crs),
    sort(Optrs1,Optrs),
    sort(Sorts1,Sorts), !.

get_leaves1(Var,_Defs,[],[],[],[],[]) :-
    var(Var), !.

```

```

get_leaves1(Var,_Defs,[Var],[],[],[],[]) :-
    atom(Var),
    name(Var,[118|_Nr]), !.

get_leaves1(Sort,_Defs,[],[],[],[],[Sort]) :-
    atomic(Sort),
    Sort sortdef _SortDef, !.

get_leaves1(Sort,Defs,[],[],[],[],[Sort]) :-
    atomic(Sort),
    member(("_,_SortDef",Sort),Defs), !.

get_leaves1(Term,Defs,Vars,[Fct|Fcts],Crs,Optrs,Sorts) :-
    Term =.. [Fct|Opnds],
    is_function(Fct),
    get_leaves1_list(Opnds,Defs,Vars,Fcts,Crs,Optrs,Sorts), !.

get_leaves1(Term,Defs,Vars,Fcts,[Cr|Crs],Optrs,Sorts) :-
    Term =.. [Cr|Opnds],
    arity(Cr,_Ar),
    get_leaves1_list(Opnds,Defs,Vars,Fcts,Crs,Optrs,Sorts), !.

get_leaves1(Term,Defs,Vars,Fcts,Crs,[Optr|Optrs],Sorts) :-
    Term =.. [Optr|Opnds],
    get_leaves1_list(Opnds,Defs,Vars,Fcts,Crs,Optrs,Sorts), !.

get_leaves1_list([Term|Terms],Defs,Vars,Fcts,Crs,Optrs,Sorts) :-
    get_leaves1(Term,Defs,Vars1,Fcts1,Crs1,Optrs1,Sorts1),
    get_leaves1_list(Terms,Defs,Vars2,Fcts2,Crs2,Optrs2,Sorts2),
    append(Vars1,Vars2,Vars),
    append(Fcts1,Fcts2,Fcts),
    append(Crs1,Crs2,Crs),
    append(Optrs1,Optrs2,Optrs),
    append(Sorts1,Sorts2,Sorts), !.

get_leaves1_list([],_Defs,[],[],[],[],[]) :- !.

get_max(Int1,Int2,Int1) :-
    Int1 >= Int2, !.

get_max(_Int1,Int2,Int2) :- !.

get_min(Int1,Int2,Int1) :-
    Int1 <= Int2, !.

get_min(_Int1,Int2,Int2) :- !.

get_subtrie(Trie,E,TrieE) :-
    member(E:TrieE,Trie), !.

get_subtrie(_Trie,_E,[]) :- !.

% calculate a list Vars of all variables in Term
% declared by "is_variable/2",
get_variables(Term,Vars) :-
    get_variables1(Term,Vars1),
    sort(Vars1,Vars), !.

get_variables1(Var,[Var]) :-
    atom(Var),
    is_variable(Var,_), !.

get_variables1(Term,Vars) :-
    Term =.. [_|Opnds],
    get_variables1_list(Opnds,Vars), !.

get_variables1_list([Term|Terms],Vars) :-
    get_variables1(Term,Vars1),
    get_variables1_list(Terms,Vars2),
    append(Vars1,Vars2,Vars), !.

get_variables1_list([],[]) :- !.

% succeed if each element of List1 is contained in List2
is_sublist([Elem|List1],[Elem|List2]) :-
    is_sublist(List1,List2), !.

```

```

is_sublist([Elem1|List1],[Elem2|List2]) :-
    Elem2 @< Elem1,
    is_sublist([Elem1|List1],List2), !.

is_sublist([],_List2) :- !.

% list L is sorted, join all succeeding unifiable terms together,
% yielding LJ
join_unifiables([H,H|T],LJ) :-
    join_unifiables([H|T],LJ), !.

join_unifiables([H1,H2|T],[H1|LJ]) :-
    join_unifiables([H2|T],LJ), !.

join_unifiables([H],[H]) :- !.

join_unifiables([],[]) :- !.

% calculate intersection and both differences of List1 and List2
list_trisection(List1,List2,Only1,Both,Only2) :-
    sort(List1,List1s),
    sort(List2,List2s),
    list_trisection1(List1s,List2s,Only1,Both,Only2), !.

list_trisection1([E12|List1],[E12|List2],Only1,[E12|Both],Only2) :-
    list_trisection1(List1,List2,Only1,Both,Only2), !.

list_trisection1([E1|List1],[E2|List2],[E1|Only1],Both,Only2) :-
    E1 @< E2,
    list_trisection1(List1,[E2|List2],Only1,Both,Only2), !.

list_trisection1([E1|List1],[E2|List2],Only1,Both,[E2|Only2]) :-
    E2 @< E1,
    list_trisection1([E1|List1],List2,Only1,Both,Only2), !.

list_trisection1(List1,[],List1,[],[]) :- !.

list_trisection1([],List2,[],[],List2) :- !.

%: make_0s_list_of_length(3,[0,s(0),s(s(0))])
make_0s_list_of_length(0,[]) :- !.

make_0s_list_of_length(I,[0|List]) :-
    I1 is I - 1,
    make_0s_list_of_length(I1,List1),
    make_0s_list_of_length_H1(List1,List), !.

make_0s_list_of_length_H1([N|Ns],[s(N)|SNs]) :-
    make_0s_list_of_length_H1(Ns,SNs), !.

make_0s_list_of_length_H1([],[]) :- !.

%: make_natlist_of_length(5,[1,2,3,4,5])
make_natlist_of_length(0,[]) :- !.

make_natlist_of_length(N,List) :-
    make_natlist_of_length1(1,N,List), !.

make_natlist_of_length1(N,N,[N]) :- !.

make_natlist_of_length1(I,N,[I|List]) :-
    I1 is I + 1,
    make_natlist_of_length1(I1,N,List), !.

make_natlist_of_length_list([Length|Lengths],[List|Lists]) :-
    make_natlist_of_length(Length,List),
    make_natlist_of_length_list(Lengths,Lists), !.

make_natlist_of_length_list([],[]) :- !.

%: make_varlist_of_length(5,[A,B,C,D,E])
make_varlist_of_length(0,[]) :- !.

make_varlist_of_length(I,[_|List]) :-
    I1 is I - 1,

```

```

make_varlist_of_length(I1,List), !.

% generate a list V with the same length as L
% but consisting entirely of different prolog variables
make_varlist_of_same_length([_E|L],[_V|V]) :-
    make_varlist_of_same_length(L,V), !.

make_varlist_of_same_length([],[]) :- !.

make_varlist_of_same_length_list([List1|Lists1],[List2|Lists2]) :-
    make_varlist_of_same_length(List1,List2),
    make_varlist_of_same_length_list(Lists1,Lists2), !.

make_varlist_of_same_length_list([],[]) :- !.

% succeed if Map is a mapping from the bag Dom to the bag Ran
%<backtrackable>
mapping([X|Dom],Ran,[X:=Y|Map]) :-
    member(Y,Ran),
    mapping(Dom,Ran,Map).

mapping([],_Ran,[]) :- !.

% succeed if Map maps each Dom in Doms to the respective Ran in Rans
%<backtrackable>
mapping_list12([Dom|Doms],[Ran|Rans],Map) :-
    mapping(Dom,Ran,Map1),
    mapping_list12(Doms,Rans,Map2),
    { append(Map1,Map2,Map)
    }.

mapping_list12([],[],[]) :- !.

% succeed if E is a member of the balanced binary tree BBT
member_bbt(E,f(_L,_Lh,E,_Rh,_R)) :- !.

member_bbt(E,f(L,_Lh,V,_Rh,_R)) :-
    E @< V,
    !,
    member_bbt(E,L), !.

member_bbt(E,f(_L,_Lh,V,_Rh,R)) :-
    V @< E,
    !,
    member_bbt(E,R), !.

% succeed if each term in Elems is a member of the corresponding
% list in Lists
member_list12([Elem|Elems],[List|Lists]) :-
    member(Elem,List),
    member_list12(Elems,Lists).

member_list12([],[]) :- !.

merge_lists_list([List1|Lists1],OpPtr,[List2|Lists2],[List|Lists]) :-
    merge_lists(List1,OpPtr,List2,List),
    merge_lists_list(Lists1,OpPtr,Lists2,Lists), !.

merge_lists_list([],_OpPtr,[],[]) :- !.

:-
    abolish(cur_name_no,1),
    assert(cur_name_no(1)).

% create a new name with the prefix Prefix
new_name(Prefix,NewName) :-
    name(Prefix,PrefixL),
    retract(cur_name_no(CurNo)),
    NewNo is CurNo + 1,
    assert(cur_name_no(NewNo)),
    name(CurNo,CurNoL),
    append(PrefixL,CurNoL,NewNameL),
    name(NewName,NewNameL), !.

```

```

new_name_list([Prefix|Prefixes],[NewName|NewNames]) :-
    new_name(Prefix,NewName),
    new_name_list(Prefixes,NewNames), !.

new_name_list([],[]) :- !.

new_name_list_list([Prefixes|Prefixess],[NewNames|NewNames]) :-
    new_name_list(Prefixes,NewNames),
    new_name_list_list(Prefixess,NewNames), !.

new_name_list_list([],[]) :- !.

% return the partition induced on the domain set by the mapping Map
partition_from_mapping([X:=I|Map],In,Out) :-
    partition_from_mapping_H1(X,I,In,In1),
    partition_from_mapping(Map,In1,Out), !.

partition_from_mapping([],In,In) :- !.

partition_from_mapping_H1(X,I,[Part|Parts],[[X|Part]|Parts]) :- !.

partition_from_mapping_H1(X,I,[Part|Parts],[Part|PartsX]) :-
    I1 is I - 1,
    partition_from_mapping_H1(X,I1,Parts,PartsX), !.

% succeed once for each atom that is read in, stop if 'end' is entered
read_list(Elem) :-
    repeat,
    read(Elem),
    { Elem == end,
      !,
      fail
    }; true
}.

read_subst_list(Vars,Substs) :-
    bagof(Subst,SubstRanges^(
        read_list(SubstRanges),
        merge_lists(Vars,(=),SubstRanges,Subst)
    ),Substs), !.

% replace the I'th element of List by NewElem, yielding ListR
replace_ith_elem([_Head|Tail],I,NewElem,[NewElem|Tail]) :- !.

replace_ith_elem([Head|Tail],I,NewElem,[Head|ListR]) :-
    I1 is I - 1,
    replace_ith_elem(Tail,I1,NewElem,ListR), !.

% reverse List, yielding ListR
reverse(List,ListR) :-
    reverse1(List,[],ListR), !.

reverse1([Elem1|List1],List2,ListR) :-
    reverse1(List1,[Elem1|List2],ListR), !.

reverse1([],List2,List2) :- !.

sort_list([H|T],[Hs|Ts]) :-
    sort(H,Hs),
    sort_list(T,Ts), !.

sort_list([],[]).

% split List into ListV and ListNV, such that each element in ListV
% contains prolog variables, and none in ListNV does
split_contains_vars([Elem|List],[Elem|ListV],ListNV) :-
    contains_vars(Elem),
    split_contains_vars(List,ListV,ListNV), !.

split_contains_vars([Elem|List],ListV,[Elem|ListNV]) :-
    split_contains_vars(List,ListV,ListNV), !.

```

```

split_contains_vars([],[],[]) :- !.

%: strip_common_constructor([c(a,b),c(d,e),c(f,g)],c,[[a,d,f],[b,e,g]])
strip_common_constructor([S],Cr,SsArgs) :-
    S =.. [Cr|SArgs],
    extend_to_same_length([],SArgs,Nils),
    cons_list12(SArgs,Nils,SsArgs), !.

strip_common_constructor([S|Ss],Cr,SsArgs) :-
    S =.. [Cr|SArgs],
    strip_common_constructor(Ss,Cr,SsArgs1),
    cons_list12(SArgs,SsArgs1,SsArgs), !.

strip_common_constructor_list([Ss|Sss],[Cr|Crs],[SsArgs|SssArgs]) :-
    strip_common_constructor(Ss,Cr,SsArgs),
    strip_common_constructor_list(Sss,Crs,SssArgs), !.

strip_common_constructor_list([],[],[]) :- !.

% succeed if Suffix is a suffix of List
%<*backtrackable*>
suffix(List,List).

suffix(Suffix,[_Head|Tail]) :-
    suffix(Suffix,Tail).

%: transpose([[a,b,c],[d,e,f]],[[a,d],[b,e],[c,f]])
transpose(Rows,[Col|Cols]) :-
    cons_list12(Col,Rows1,Rows),
    transpose(Rows1,Cols), !.

transpose(_Rows,[]) :- !.

% succeed if Elem is in List and remove it from List, yielding ListD
%<*backtrackable*>
try_delete_from_list(Elem,[Elem|List],List).

try_delete_from_list(Elem,[Other|List],[Other|ListD]) :-
    try_delete_from_list(Elem,List,ListD).

% succeed if Elem is member of List,
% return the prefix of List, upto, and including Elem
try_get_prefix_to([Elem|_List],Elem,[Elem]) :- !.

try_get_prefix_to([Head|List],Elem,[Head|Prefix]) :-
    try_get_prefix_to(List,Elem,Prefix), !.

% connect all elements of List by Op yielding Term
% eg. unflatten_op([a,b,c],(&),(a&b)&c).
unflatten_op([Term],_Op,Term) :- !.

unflatten_op([Term1,Term2|List],Op,Term) :-
    NewTerm =.. [Op,Term1,Term2],
    unflatten_op([NewTerm|List],Op,Term), !.

% succeed if List is a list of prolog variables
var_list([H|T]) :-
    var(H),
    !,
    var_list(T), !.

var_list([]) :- !.

write_list([H|T]) :-
    write(H), nl,
    write_list(T), !.

write_list([]) :-
    nl, !.

```

```
% %%%%%%%%%% debugging functions %%%%%%%%%% %
% %%%%%%%%%% debugging functions %%%%%%%%%% %
% %%%%%%%%%% debugging functions %%%%%%%%%% %
```

```
% report clause call / fail
trace_enter(Event,Value) :-
    % call
    { trace_wanted(Event),
      { retract(trace_tab(TabL))
      ; TabL = []
      },
      assert(trace_tab([32,32|TabL])),
      name(Tab,TabL),
      write(Tab), write(Value),
      trace_break(Event,call,Ctrl),
      !, Ctrl \== fail
    }, !
;
    % fail
    { trace_wanted(Event),
      retract(trace_tab([32,32|TabL])),
      assert(trace_tab(TabL)),
      name(Tab,TabL),
      write(Tab), write(Value), write(' FAILED'), nl
    }, !, fail.
```

```
% report clause exit / back to
trace_leave(Event,Value) :-
    % exit
    { trace_wanted(Event),
      retract(trace_tab([32,32|TabL])),
      assert(trace_tab(TabL)),
      name(Tab,TabL),
      write(Tab), write(Value), nl
    }, !
;
    % back to
    { trace_wanted(Event),
      retract(trace_tab(TabL)),
      assert(trace_tab([32,32|TabL])),
      name(Tab,TabL),
      write(Tab), write(Value), write(' RETRY'),
      trace_break(Event,retry,Ctrl)
    }, !, fail.
```

```
trace_break(Event,Port,Ctrl) :-
    trace_break_wanted(Event,Port),
    write(' '),
    get0(P),
    { P = 10
    ; get0(_),
      { [P] = "a", abolish(trace_tab,1),
        assert(trace_tab([])), abort
      ; [P] = "b", retract(trace_break_wanted(Event,Port))
      ; [P] = "B", ( retract(trace_break_wanted(Event,_Any)),
                    fail
                  ; true
                )
      ; [P] = "f", Ctrl = fail
      ; [P] = "n", retract(trace_wanted(Event)),
        { retract(trace_tab([32,32|TabL])),
          assert(trace_tab(TabL))
        }, !
      ; [P] = "t", trace
    }, !
; nl, !.
```

```
% provide a means for additional information
trace_write(Event,Value) :-
    trace_wanted(Event),
    trace_tab(TabL),
    name(Tab,TabL),
    write(Tab), write(Value),
    trace_break(Event,write,Ctrl),
    !, Ctrl \== fail
; !.
```

```
% report unwanted backtracking
```

```

dont_backtrack(Predicate) :-
    nl, write('+++++ unwanted backtrack occurred (')',
    write(Predicate), write(') +++++'), nl,
    trace, nl, nl, stop, !.

```

```

:-
    abolish(trace_tab,1),
    assert(trace_tab([])),
    abolish(trace_wanted,1),
    abolish(trace_break_wanted,2),

    assert(trace_wanted(hsg)),
    assert(trace_wanted(layer)),
    assert(trace_wanted(prof)),
    assert(trace_wanted(sd)),
    assert(trace_wanted(sd1)),

    assert(trace_break_wanted(hsg,call)),
    assert(trace_break_wanted(hsg,write)),
    assert(trace_break_wanted(hsg,retry)),

    assert(trace_break_wanted(layer,call)),
    assert(trace_break_wanted(layer,write)),
    assert(trace_break_wanted(layer,retry)),

    assert(trace_break_wanted(prof,call)),
    assert(trace_break_wanted(prof,write)),
    assert(trace_break_wanted(prof,retry)),

    assert(trace_break_wanted(sd,call)),
    assert(trace_break_wanted(sd,write)),
    assert(trace_break_wanted(sd,retry)),

    assert(trace_break_wanted(sd1,call)),
    assert(trace_break_wanted(sd1,write)),
    assert(trace_break_wanted(sd1,retry)).

```