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## 5 Sheaves of Modules

### 5.1 $\mathcal{O}(X)$ – modules

Fix a scheme  $X$

sheaf of  $\mathcal{O}_X$ -modules: A (pre)sheaf of  $\mathcal{O}(X)$ -modules is a (pre)sheaf of abelian groups  $\mathcal{F}$  on  $X$  together with a morphism of presheaves of sets  $\mathcal{O}_X \times \mathcal{F} \rightarrow \mathcal{F}$ , which turns  $\mathcal{F}(U)$  into an  $\mathcal{O}_X(U)$ -module for every open  $U \subset X$

A homomorphism of such is a morphism of presheaves  $\mathcal{F} \rightarrow \mathcal{G}$  such that  $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is  $\mathcal{O}_X(U)$ -linear for each  $U$

analogously define composition, identity morphism etc.  $\leadsto$  this forms a category.

Take  $\mathcal{O}_X$ -module  $\mathcal{F}_i$  for  $i \in I$

direct product: The direct product  $\prod \mathcal{F}_i$  is defined by  $U \mapsto \prod(\mathcal{F}_i(U))$  with component-wise structure

direct sum: the direct sum  $\bigoplus \mathcal{F}_i$  is the sheafification the presheaf  $U \mapsto \bigoplus \mathcal{F}_i(U)$ . This is a subsheaf of  $\prod \mathcal{F}_i$  with equality if  $|I|$  is finite.

tensor product: The tensor product  $\mathcal{F} \otimes \mathcal{G}$  of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  &  $\mathcal{G}$  is the sheafification of the presheaf  $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$

inner Hom: The inner Hom is the sheaf of homomorphisms of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  and  $\mathcal{G}$   
 $U \mapsto \text{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{G}|_U) =: \mathcal{H}om_{\mathcal{O}(X)}(\mathcal{F}, \mathcal{G})(U)$

### 5.2 Locally free $\mathcal{O}_X$ -modules

Abbreviate:  $\mathcal{O}_X^{(I)} := \bigoplus_{i \in I} \mathcal{O}_X$   
 $\mathcal{O}_X^I := \prod_{i \in I} \mathcal{O}_X$   
 $\mathcal{O}_X^r := \bigoplus_{i=1}^r \mathcal{O}_X$

free $\mathcal{O}_X$ -module :	An $\mathcal{O}_X$ -module isomorphic to $\mathcal{O}_X^{(I)}$ for some $I$ is called free
locally free $\mathcal{O}_X$ -module :	<p>(a) An <math>\mathcal{O}_X</math>-module is locally free if <math>\forall x \exists x \in U \subset X \text{ open } \exists I : \mathcal{F} _U \cong \mathcal{O}_X^{(I)} _U</math></p> <p>(b) It is called locally free of rank <math>r</math> if <math>\exists I :  I  = r</math> and <math>\forall x \exists x \in U \subset X : \mathcal{F} _U \cong \mathcal{O}_X^{(I)} _U</math></p>
dual sheaf:	For $\mathcal{F}$ locally free of finite rank, $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is called the dual sheaf, also locally free of rank $r$ .
invertible sheaf:	A locally free sheaf of rank 1 is called an invertible sheaf.
Picard group:	The set of isomorphism classes of invertible sheaves on $X$ form an abelian group, $Pic(X)$ , called the Picard group of $X$ .

### 5.3 $\mathcal{O}_X$ -modules on affine schemes

### 5.4 Quasicoherent $\mathcal{O}_X$ -modules

Let  $X$  be an arbitrary scheme

Notation for global sections:  $\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$

Any system of global sections  $s_i \in \mathcal{F}(X)$  determines a homomorphism  $\mathcal{O}_X^{(I)} \rightarrow \mathcal{F}$ ,  $(f_i)_{i \in I} \mapsto \sum f_i \cdot \text{res}_U^X$

$\mathcal{F}$  is called generated by global sections if  $\exists I : \exists$  a surjective homomorphism:  $\mathcal{O}_X^{(I)} \twoheadrightarrow \mathcal{F}$

quasicoherent $\mathcal{O}_X$ -module :	An $\mathcal{O}_X$ -module $\mathcal{F}$ is called quasicoherent if $\forall x \in X \exists$ open neighbourhood $U \subset X \exists I \exists J \exists$ exact sequence
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$$\mathcal{O}_X^{(I)}|_U \rightarrow \mathcal{O}_X^{(J)}|_U \rightarrow \mathcal{F}|_U \rightarrow 0$$

locally free  $\implies$  quasicoherent

### 5.5 Coherent sheaves

finitely generated $\mathcal{O}_X$ -module :	An $\mathcal{O}_X$ -module $\mathcal{F}$ is called:
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(a) finitely generated (or of finite type) if  $\forall x \exists x \in U \subset X$  open:  $\exists n < \infty \exists \mathcal{O}_X^n|_U \twoheadrightarrow \mathcal{F}|_U$   $\mathcal{O}_X$ -module homomorphism.

coherent $\mathcal{O}_X$ -module	(b) coherent if it is finitely generated and $\forall U \subset X$ open $\forall n < \infty \forall$ homomorphisms $\varphi : \mathcal{O}_X^n _U \rightarrow \mathcal{F} _U$ $\ker(\varphi)$ is finitely generated.
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### 5.6 Functoriality

Consider a morphism  $f : X \rightarrow Y$  with sheaves  $\mathcal{F}$  on  $X$  and  $\mathcal{G}$  on  $Y$ .

Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module and  $\mathcal{G}$  be an  $\mathcal{O}_Y$ -module.

$f$  comes with a homomorphism of sheaves of rings:

$$f^\flat : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X \iff f^\sharp : f^* \mathcal{O}_Y \longrightarrow \mathcal{O}_X$$

push-forward of a sheaf: Make  $f_* \mathcal{F}$  into an  $\mathcal{O}_Y$ -module by

$$\begin{array}{ccccc} \mathcal{O}_Y(V) \times (f_* \mathcal{F})(V) & \longrightarrow & (f_* \mathcal{F})(V) \\ \downarrow & \downarrow = & \downarrow \\ \mathcal{O}_X(f^{-1}(V)) \times \mathcal{F}(f^{-1}(V)) & \xrightarrow{\text{mult.}} & \mathcal{F}(f^{-1}(V)) \end{array}$$

inverse image:

the inverse image of an  $\mathcal{O}_X$ -module is  $f^* \mathcal{G} := f^{-1} \mathcal{G} \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$ .  
i.e. the sheafification of the presheaf

$$U \longmapsto \varinjlim_{f(U) \subset V \subset Y} (\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(V)} \mathcal{G}(V))$$

scheme-theoretic support of  $\mathcal{F}$ :

Let  $\mathcal{F}$  be a quasicoherent, finitely generated  $\mathcal{O}_X$ -module. The scheme-theoretic support is the smallest closed subscheme  $i : Y \hookrightarrow X$  such that  $\mathcal{F} \cong i_* i^* \mathcal{F}$ . Moreover  $Y = \{x \in X \mid \mathcal{F}_x \neq 0\}$

## 5.7 $\mathcal{O}_X$ -modules on a projective scheme

For any graded  $R$ -module  $M$  and any  $n \in \mathbb{Z}$  we set  $M(n) := M$  as  $R$ -module with grading  $M(n)_d = M_{n+d}$

$$\widetilde{R(n)} =: \mathcal{O}_X(n)$$

clear:  $\mathcal{O}_X(0) \cong \mathcal{O}_X$

twisting sheaf:

$\mathcal{O}_X(1)$  is called the twisting sheaf on  $X = \text{Proj } R$   
for any  $\mathcal{O}_X$ -module we set  $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$

## 5.8 Morphisms to projective spaces

very ample sheaf:

An invertible sheaf  $\mathcal{L}$  is called very ample (over  $\text{Spec } R$ ) if  $\mathcal{L} \cong f^* \mathcal{O}_X(1)$  for some locally closed embedding over  $R$   $f : X \hookrightarrow \mathbb{P}_R^n$  for some  $n$ .

ample sheaf:

An invertible sheaf  $\mathcal{L}$  on a quasicompact scheme  $X$  is called ample if for any finitely generated quasicoherent sheaf  $\mathcal{F}$  on  $X$  there exists  $n_0$  such that  $n \geq n_0$  the sheaf  $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$  is generated by global sections.

relatively ample / very ample sheaf

$\mathcal{L}$  an invertible sheaf with respect to  $f : X \longrightarrow Y$  iff there exists an affine open covering  $Y = \bigcup_{i \in I} V_i$  such that  $\forall i \mathcal{L}|_{f^{-1}(V_i)}$  is ample / very ample over  $V_i$

## 5.9 Divisors

Assume  $X$  integral with function field  $K$ .

field of rational functions on $X$	the constant sheaf $(K)_X := \underline{K}$ on $X$ is called the sheaf of rational functions on $X$
group of Cartier divisors	$Div(X) := \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$ $Div(X) = \{([f_x]) \in \prod_{x \in X} K^\times / \mathcal{O}_{X,x}^\times \mid \forall x \exists U \ni x \exists f \in K^\times \forall y \in U : f_y \cdot \mathcal{O}_{X,y}^\times = f \mathcal{O}_{X,y}^\times\}$ $= \{ \text{collections of } f_i \in K^\times \text{ for all } i \in I \text{ and an open covering } X = \bigcup U_i \text{ such that } \forall i, j : f_i / f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times) \} / \text{ modulo some equivalence relation}$ convention: The group law on $Div(X)$ is written additively.
$\mathcal{O}_X(D)(U)$	For any Cartier divisor $D = ([f_x])_{x \in X}$ and any open $U \subset X$ set: $\mathcal{O}_X(D)(U) = \begin{cases} \{0\} & \text{if } U = \emptyset \\ \bigcap_{x \in U} f_x^{-1} \mathcal{O}_{X,x} & \text{else} \end{cases}$
locally factorial scheme	A noetherian integral scheme such that $\forall x \in X \mathcal{O}_{X,x}$ is factorial is called locally factorial.
effective cartier divisor	A cartier divisor $D = ([f_x])_x$ is called effective if $\forall x \in X : f_x \in \mathcal{O}_{X,x}$ equiv: $\forall i : f_i \in \mathcal{O}_X(u_i)$ $\iff \mathcal{O}_X(-D) \subset \mathcal{O}_X \iff \mathcal{O}_X \subset \mathcal{O}_X(D)$ $\iff \mathcal{O}_X(-D)$ is a quasicoherent sheaf of ideals of $\mathcal{O}_X$ $\iff D$ corresponds to a closed subscheme locally given by one equation $f_i$ i.e. locally principal. this subscheme determines $D$ .
principal cartier divisor	A cartier divisor of the form $div(f) := (f) := ([f])_{x \in X}$ for some $f \in K^\times$ is called principal.
cartier divisor class group of $X$	The factor group $DivCl(X) := Div(X) / \text{principal divisors}$
prime cycle	an integral closed subscheme is called a prime cycle. (equiv: irred closed subset)
codimension of a prime cycle	A prime cycle's codimension is $\dim \mathcal{O}_{X,y}$ for the generic point $y \in Y$ .

cycle	A finite formal $\mathbb{Z}$ -linear combination of prime cycles $\sum_y n_y y$ is called a cycle.
codimension of a cycle	If all these $Y$ have codim $d$ the cycle has codim $d$
Weil divisor	A cycle of dimension 1 is called a Weil divisor.
principal weil divisor	A Weil divisor of the form $\text{cyc}(\text{div}(f)) = \sum \text{ord}_y(f) \cdot \overline{\{y\}}$ is called principal.
Weil divisor class group	The factor group $Z(X)/\{\text{principal}\} = Cl(X)$ is called the Weil divisor group.
effective weil divisor	A weil divisor $\sum n_y Y$ is effective if all $n_y \geq 0$ equiv: associated cartier divisor is effective equiv: $\mathcal{O}_X(-D)$ is an ideal sheaf of $\mathcal{O}_X$
ample/very ample divisor	A divisor is called ample/very ample iff $\mathcal{O}_X(D)$ is dito.

## 5.10 Differentials

Let  $X = \text{Spec } B$ ,  $Y = \text{Spec } A$  and  $f : Y \rightarrow X$

A-derivation of B to M    An A-derivation of B to a B-module M is a map  $d : B \rightarrow M$  with  $\forall b, b' \in B \forall a \in A$   
 (a)  $d(b + b') = d(b) + d(b')$   
 (b)  $d(b \cdot b') = b \cdot d(b') + d(b) \cdot b'$   
 (c)  $d(a \cdot 1_B) = 0$

$\Omega_{B/A}$  module of (relative) differential (form)s    A module of relative differential forms of B over A is a B-module  $\Omega_{B/A}$  with a derivation  $d : B \rightarrow \Omega_{B/A}$  over A which satisfies the universal property: for all B-modules M and all derivations  $\delta : B \rightarrow M$  over A there exists exactly one B-module homomorphism  $f : \Omega_{B/A} \rightarrow M$  with  $f \circ d = \delta$

Consider a morphism  $f : Y \rightarrow X$ . then for all open affine subsets we have:

$$\begin{array}{ccc}
 Y & \longrightarrow & X \\
 \cup & & \cup \\
 \text{Spec } B & \longrightarrow & \text{Spec } A \\
 \cup & & \cup \\
 \text{Spec } B_{ab} & \longrightarrow & \text{Spec } A_a
 \end{array}$$

Then:

So there is a unique sheaf of  $\mathcal{O}_X$ -modules  $\Omega_{Y/X}$  with  $\Omega_{Y/X}(\text{Spec } B) = \Omega_{B/A}$

$\Omega_{Y/X}$	$\Omega_{Y/X}$ is the sheaf of (relative) differentials of $Y$ over $X$ . It comes with a "universal derivation" $d : \mathcal{O}_Y \longrightarrow \Omega_{Y/X}$
sheaf of relative differential forms of degree $d$ over $Y$	for any $d \geq 0$ set $\Omega_{X/Y}^d := \Lambda_{\mathcal{O}_X}^d \Omega_{X/Y}$ the sheaf of relative differential forms of degree $d$ over $Y$ .
$\omega_{X/Y}$ Canonical sheaf of $X$ over $Y$	if $\Omega_{X/Y}$ is locally free of rank $n$ then $\Omega_{X/Y}^d$ is too, of rank $\binom{n}{d}$ . In particular $\Omega_{X/Y}^n$ is an invertible sheaf, called the canonical sheaf of $X$ over $Y$ denoted $\omega_{X/Y}$ .

## 6 cohomology

### 6.1 Some (quick) homological algebra

Let  $\mathcal{C}$  be the category of  $R$ -modules/ sheaves of abelian groups/  $\mathcal{O}_X$ -modules on a scheme  $X$  or any abelian category.

(cochain) complex	a cochain complex consists of morphisms $\dots \xrightarrow{d_{n-1}} X^n \xrightarrow{d_n} X^{n+1} \xrightarrow{d_{n+1}} \dots$ with $d_{n+1} \circ d_n = 0$ Elements of $X^n$ are called $n$ -cochains. Elements of $Z^n := \ker(d_n)$ $n$ -cocycles. Elements of $B^n := \operatorname{im}(d_{n-1})$ $n$ -coboundaries
$H^n(X)$ $n$ -th cohomology	$H^n(X) := Z^n/B^n$ is the $n$ -th cohomology of $(X^\bullet, d_\bullet)$
acyclic	$X$ is called acyclic if $\forall n : H^n(X) = 0$
homomorphism of complexes	of A homomorphism of complexes $f : X \rightarrow Y$ is a collection of homomorphisms $f^n : X^n \rightarrow Y^n$ such that for all $n$ the following diagram commutes: $\begin{array}{ccc} X^n & \xrightarrow{d_n} & X^{n+1} \\ \downarrow f_n & & \downarrow f_{n+1} \\ Y^n & \xrightarrow{d_n} & Y^{n+1} \end{array}$
augmentation complex	of a An augmentation of a complex $(X^\bullet, d_\bullet)$ is a homo $\Xi \xrightarrow{a} X^0$ such that $d_0 \circ a = 0$ . i.e. $0 \rightarrow \Xi \xrightarrow{a} X^0 \xrightarrow{d_0} X^1 \xrightarrow{d_1} \dots$ is a complex. get natural homo $\Xi \xrightarrow{a} Z^0(X) = H^0(X)$

(cochain) homotopy	consider two homos of complexes $f, g : X \longrightarrow Y$ . A cochain homotopy from $f$ to $g$ is a collection of homos $h : X^{n+1} \rightarrow Y^n$ such that $f - g = d \circ h + h \circ d$ If such $h$ exists, $f$ and $g$ are called homotopic and we write $f \simeq g$
$X$ contractible	$X$ is contractible if $id_X$ is homotopic to 0
quasi-isomorphism of complexes	A homo of complexes $f : X^\bullet \longrightarrow Y^\bullet$ is called a quasi-isomorphism if $\forall n \in \mathbb{Z} : H^n(f) : H^n(X^\bullet) \rightarrow H^n(Y^\bullet)$ is an isomorphism.

## 6.2 Cech Cohomology

Let  $\mathcal{F}$  be a sheaf of abelian groups of a topological space  $X$ . Take an open covering  $\mathcal{U} := (U_i)_{i \in I}$  of  $X$ . The  $U_i$  need not be distinct or nonempty.

$C^\bullet(\mathcal{U}, \mathcal{F})$ (total) cech complex For any p set:

$$C^p(\mathcal{U}, \mathcal{F}) := \begin{cases} \prod_{i_0, \dots, i_p \in I} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p}) & \text{for } p \geq 0 \\ 0 & \text{for } p < 0 \end{cases}$$

$$d : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$$

$$(f_{i_0 \dots i_p})_{i_0 \dots i_p} \longmapsto (\sum_{k=0}^{p+1} (-1)^k f_{i_0, \dots, \hat{i}_k, \dots, i_{p+1}} |_{U_{i_0 \dots i_{p+1}}})_{i_0 \dots i_{p+1}}$$

this is the (total) cech complex.

$$H^n(\mathcal{U}, \mathcal{F}) \quad H^n(\mathcal{U}, \mathcal{F}) := H^n(C^\bullet(\mathcal{U}, \mathcal{F}))$$

$C_{alt}^\bullet(\mathcal{U}, \mathcal{F})$  alternating cech complex

$$C_{alt}^n(\mathcal{U}, \mathcal{F}) := \{(f_{i_0 \dots i_p}) \in C^p(\mathcal{U}, \mathcal{F}) \mid \forall i_0 \dots i_p : f_{i_0 \dots i_p} \text{ if not all } i_0 \dots i_p \text{ are}$$

$C_{ord}^\bullet(\mathcal{U}, \mathcal{F})$  ordered cech complex As alternating cech complex but now assume  $I$  comes with a total order  $<$   
Then  $C_{ord}^\bullet(\mathcal{U}, \mathcal{F})$  with  $i_0 < \dots < i_p$

refinement of open covering For  $\mathcal{U} := (U_i)_{i \in I}$  and  $\mathcal{V} := (V_j)_{j \in J}$  we say  $\mathcal{V}$  is a refinement if there exists a map  $\sigma : I \longrightarrow J$  such that  $\forall j \in J : V_j \subset U_{\sigma j}$

This defines a homo of complexes  $\sigma^* : C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{V})$ . Hence a natural homo  $H^n(\sigma) : H^n(\mathcal{U}, \mathcal{F}) \longrightarrow H^n(\mathcal{V}, \mathcal{F})$ .

Let  $\mathcal{C}$  be the category whose objects are all open coverings and  $Mor_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) := \{\sigma \text{ as above}\}$ .

Restrict to  $\mathcal{U} = (U_i)_{i \in I}$  for all  $U_i$  distinct  $\implies$  small cofinal subcategory.

$H^n(X, \mathcal{F})$  the n-th cohomology of  $\mathcal{F}$   $H^n(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^n(\mathcal{U}, \mathcal{F})$  is called the n-th Čech cohomology.

### 6.3 Cohomology of projective space

### 6.4 Higher direct images

Take  $f : X \rightarrow Y$  sheaves of modules  $\mathcal{F}$  on  $X$  and  $\mathcal{G}$  on  $Y$  We have a natural homo:  
 $H^p(Y, f_* \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$  and  $H^p(Y, \mathcal{G}) \rightarrow H^p(X, f^* \mathcal{G})$

Special case:  $U \subset X$  open natural "restriction map"  $H^p(X, \mathcal{F}) \longrightarrow H^p(U, \mathcal{F}|_U)$ .

Special case: For all open  $V' \subset V \subset Y$  the restriction map for  $f^{-1}(V') \subset f^{-1}(V)$  makes this into a presheaf of  $\mathcal{O}_Y$ -modules on  $Y$ .

$R^p f_* \mathcal{F}$  the p-th higher direct image of  $\mathcal{F}$  The associated sheaf to the sheaf above is called the p-th higher direct image (sheaf) of  $\mathcal{F}$  with respect to  $f$