

for errors or questions email: nicolatr@student.ethz.ch

5 Sheaves of Modules

5.1 $\mathcal{O}(X)$ – modules

Fix a scheme X

sheaf of \mathcal{O}_X -modules: A (pre)sheaf of $\mathcal{O}(X)$ -modules is a (pre)sheaf of abelian groups \mathcal{F} on X together with a morphism of presheaves of sets $\mathcal{O}_X \times \mathcal{F} \rightarrow \mathcal{F}$, which turns $\mathcal{F}(U)$ into an $\mathcal{O}_X(U)$ -module for every open $U \subset X$

A homomorphism of such is a morphism of presheaves $\mathcal{F} \rightarrow \mathcal{G}$ such that $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is $\mathcal{O}_X(U)$ -linear for each U

analogously define composition, identity morphism etc. \leadsto this forms a category.

Take \mathcal{O}_X -module \mathcal{F}_i for $i \in I$

direct product: The direct product $\prod \mathcal{F}_i$ is defined by $U \mapsto \prod(\mathcal{F}_i(U))$ with component-wise structure

direct sum: the direct sum $\bigoplus \mathcal{F}_i$ is the sheafification the presheaf $U \mapsto \bigoplus \mathcal{F}_i(U)$. This is a subsheaf of $\prod \mathcal{F}_i$ with equality if $|I|$ is finite.

tensor product: The tensor product $\mathcal{F} \otimes \mathcal{G}$ of \mathcal{O}_X -modules \mathcal{F} & \mathcal{G} is the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$

inner Hom: The inner Hom is the sheaf of homomorphisms of \mathcal{O}_X -modules \mathcal{F} and \mathcal{G}
 $U \mapsto \text{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{G}|_U) =: \mathcal{H}om_{\mathcal{O}(X)}(\mathcal{F}, \mathcal{G})(U)$

5.2 Locally free \mathcal{O}_X -modules

Abbreviate: $\mathcal{O}_X^{(I)} := \bigoplus_{i \in I} \mathcal{O}_X$
 $\mathcal{O}_X^I := \prod_{i \in I} \mathcal{O}_X$
 $\mathcal{O}_X^r := \bigoplus_{i=1}^r \mathcal{O}_X$

free \mathcal{O}_X -module :	An \mathcal{O}_X -module isomorphic to $\mathcal{O}_X^{(I)}$ for some I is called free
locally free \mathcal{O}_X -module :	<p>(a) An \mathcal{O}_X-module is locally free if $\forall x \exists x \in U \subset X \text{ open } \exists I : \mathcal{F} _U \cong \mathcal{O}_X^{(I)} _U$</p> <p>(b) It is called locally free of rank r if $\exists I : I = r$ and $\forall x \exists x \in U \subset X : \mathcal{F} _U \cong \mathcal{O}_X^{(I)} _U$</p>
dual sheaf:	For \mathcal{F} locally free of finite rank, $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is called the dual sheaf, also locally free of rank r .
invertible sheaf:	A locally free sheaf of rank 1 is called an invertible sheaf.
Picard group:	The set of isomorphism classes of invertible sheaves on X form an abelian group, $Pic(X)$, called the Picard group of X .

5.3 \mathcal{O}_X -modules on affine schemes

5.4 Quasicoherent \mathcal{O}_X -modules

Let X be an arbitrary scheme

Notation for global sections: $\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$

Any system of global sections $s_i \in \mathcal{F}(X)$ determines a homomorphism $\mathcal{O}_X^{(I)} \rightarrow \mathcal{F}$, $(f_i)_{i \in I} \mapsto \sum f_i \cdot \text{res}_U^X$

\mathcal{F} is called generated by global sections if $\exists I : \exists$ a surjective homomorphism: $\mathcal{O}_X^{(I)} \twoheadrightarrow \mathcal{F}$

quasicoherent \mathcal{O}_X -module :	An \mathcal{O}_X -module \mathcal{F} is called quasicoherent if $\forall x \in X \exists$ open neighbourhood $U \subset X \exists I \exists J \exists$ exact sequence
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$$\mathcal{O}_X^{(I)}|_U \rightarrow \mathcal{O}_X^{(J)}|_U \rightarrow \mathcal{F}|_U \rightarrow 0$$

locally free \implies quasicoherent

5.5 Coherent sheaves

finitely generated \mathcal{O}_X -module :	An \mathcal{O}_X -module \mathcal{F} is called:
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(a) finitely generated (or of finite type) if $\forall x \exists x \in U \subset X$ open: $\exists n < \infty \exists \mathcal{O}_X^n|_U \twoheadrightarrow \mathcal{F}|_U$ \mathcal{O}_X -module homomorphism.

coherent \mathcal{O}_X -module	(b) coherent if it is finitely generated and $\forall U \subset X$ open $\forall n < \infty \forall$ homomorphisms $\varphi : \mathcal{O}_X^n _U \twoheadrightarrow \mathcal{F} _U$ $\ker(\varphi)$ is finitely generated.
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5.6 Functoriality

Consider a morphism $f : X \rightarrow Y$ with sheaves \mathcal{F} on X and \mathcal{G} on Y .

Let \mathcal{F} be an \mathcal{O}_X -module and \mathcal{G} be an \mathcal{O}_Y -module.

f comes with a homomorphism of sheaves of rings:

$$f^\flat : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X \iff f^\sharp : f^* \mathcal{O}_Y \longrightarrow \mathcal{O}_X$$

push-forward of a sheaf: Make $f_* \mathcal{F}$ into an \mathcal{O}_Y -module by

$$\begin{array}{ccccc} \mathcal{O}_Y(V) \times (f_* \mathcal{F})(V) & \longrightarrow & (f_* \mathcal{F})(V) \\ \downarrow & \downarrow = & \downarrow \\ \mathcal{O}_X(f^{-1}(V)) \times \mathcal{F}(f^{-1}(V)) & \xrightarrow{\text{mult.}} & \mathcal{F}(f^{-1}(V)) \end{array}$$

inverse image:

the inverse image of an \mathcal{O}_X -module is $f^* \mathcal{G} := f^{-1} \mathcal{G} \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$.
i.e. the sheafification of the presheaf

$$U \longmapsto \varinjlim_{f(U) \subset V \subset Y} (\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(V)} \mathcal{G}(V))$$

scheme-theoretic support of \mathcal{F} :

Let \mathcal{F} be a quasicoherent, finitely generated \mathcal{O}_X -module. The scheme-theoretic support is the smallest closed subscheme $i : Y \hookrightarrow X$ such that $\mathcal{F} \cong i_* i^* \mathcal{F}$. Moreover $Y = \{x \in X \mid \mathcal{F}_x \neq 0\}$

5.7 \mathcal{O}_X -modules on a projective scheme

For any graded R -module M and any $n \in \mathbb{Z}$ we set $M(n) := M$ as R -module with grading $M(n)_d = M_{n+d}$

$$\widetilde{R(n)} =: \mathcal{O}_X(n)$$

clear: $\mathcal{O}_X(0) \cong \mathcal{O}_X$

twisting sheaf:

$\mathcal{O}_X(1)$ is called the twisting sheaf on $X = \text{Proj } R$
for any \mathcal{O}_X -module we set $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$

5.8 Morphisms to projective spaces

very ample sheaf:

An invertible sheaf \mathcal{L} is called very ample (over $\text{Spec } R$) if $\mathcal{L} \cong f^* \mathcal{O}_X(1)$ for some locally closed embedding over R $f : X \hookrightarrow \mathbb{P}_R^n$ for some n .

ample sheaf:

An invertible sheaf \mathcal{L} on a quasicompact scheme X is called ample if for any finitely generated quasicoherent sheaf \mathcal{F} on X there exists n_0 such that $n \geq n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.

relatively ample / very ample sheaf

\mathcal{L} an invertible sheaf with respect to $f : X \longrightarrow Y$ iff there exists an affine open covering $Y = \bigcup_{i \in I} V_i$ such that $\forall i \mathcal{L}|_{f^{-1}(V_i)}$ is ample / very ample over V_i

5.9 Divisors

Assume X integral with function field K .

field of rational functions on X	the constant sheaf $(K)_X := \underline{K}$ on X is called the sheaf of rational functions on X
group of Cartier divisors	$Div(X) := \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$ $Div(X) = \{([f_x]) \in \prod_{x \in X} K^\times / \mathcal{O}_{X,x}^\times \mid \forall x \exists U \ni x \exists f \in K^\times \forall y \in U : f_y \cdot \mathcal{O}_{X,y}^\times = f \mathcal{O}_{X,y}^\times\}$ $= \{ \text{collections of } f_i \in K^\times \text{ for all } i \in I \text{ and an open covering } X = \bigcup U_i \text{ such that } \forall i, j : f_i / f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times) \} / \text{ modulo some equivalence relation}$ convention: The group law on $Div(X)$ is written additively.
$\mathcal{O}_X(D)(U)$	For any Cartier divisor $D = ([f_x])_{x \in X}$ and any open $U \subset X$ set: $\mathcal{O}_X(D)(U) = \begin{cases} \{0\} & \text{if } U = \emptyset \\ \bigcap_{x \in U} f_x^{-1} \mathcal{O}_{X,x} & \text{else} \end{cases}$
locally factorial scheme	A noetherian integral scheme such that $\forall x \in X \mathcal{O}_{X,x}$ is factorial is called locally factorial.
effective cartier divisor	A cartier divisor $D = ([f_x])_x$ is called effective if $\forall x \in X : f_x \in \mathcal{O}_{X,x}$ equiv: $\forall i : f_i \in \mathcal{O}_X(u_i)$ $\iff \mathcal{O}_X(-D) \subset \mathcal{O}_X \iff \mathcal{O}_X \subset \mathcal{O}_X(D)$ $\iff \mathcal{O}_X(-D)$ is a quasicoherent sheaf of ideals of \mathcal{O}_X $\iff D$ corresponds to a closed subscheme locally given by one equation f_i i.e. locally principal. this subscheme determines D .
principal cartier divisor	A cartier divisor of the form $div(f) := (f) := ([f])_{x \in X}$ for some $f \in K^\times$ is called principal.
cartier divisor class group of X	The factor group $DivCl(X) := Div(X) / \text{principal divisors}$
prime cycle	an integral closed subscheme is called a prime cycle. (equiv: irred closed subset)
codimension of a prime cycle	A prime cycle's codimension is $\dim \mathcal{O}_{X,y}$ for the generic point $y \in Y$.

cycle	A finite formal \mathbb{Z} -linear combination of prime cycles $\sum_y n_y y$ is called a cycle.
codimension of a cycle	If all these Y have codim d the cycle has codim d
Weil divisor	A cycle of dimension 1 is called a Weil divisor.
principal weil divisor	A Weil divisor of the form $\text{cyc}(\text{div}(f)) = \sum \text{ord}_y(f) \cdot \overline{\{y\}}$ is called principal.
Weil divisor class group	The factor group $Z(X)/\{\text{principal}\} = Cl(X)$ is called the Weil divisor group.
effective weil divisor	A weil divisor $\sum n_y Y$ is effective if all $n_y \geq 0$ equiv: associated cartier divisor is effective equiv: $\mathcal{O}_X(-D)$ is an ideal sheaf of \mathcal{O}_X
ample/very ample divisor	A divisor is called ample/very ample iff $\mathcal{O}_X(D)$ is dito.

5.10 Differentials

Let $X = \text{Spec } B$, $Y = \text{Spec } A$ and $f : Y \rightarrow X$

A-derivation of B to M An A-derivation of B to a B-module M is a map $d : B \rightarrow M$ with $\forall b, b' \in B \forall a \in A$
 (a) $d(b + b') = d(b) + d(b')$
 (b) $d(b \cdot b') = b \cdot d(b') + d(b) \cdot b'$
 (c) $d(a \cdot 1_B) = 0$

$\Omega_{B/A}$ module of (relative) differential (form)s A module of relative differential forms of B over A is a B-module $\Omega_{B/A}$ with a derivation $d : B \rightarrow \Omega_{B/A}$ over A which satisfies the universal property: for all B-modules M and all derivations $\delta : B \rightarrow M$ over A there exists exactly one B-module homomorphism $f : \Omega_{B/A} \rightarrow M$ with $f \circ d = \delta$

Consider a morphism $f : Y \rightarrow X$. then for all open affine subsets we have:

$$\begin{array}{ccc}
 Y & \longrightarrow & X \\
 \cup & & \cup \\
 \text{Spec } B & \longrightarrow & \text{Spec } A \\
 \cup & & \cup \\
 \text{Spec } B_{ab} & \longrightarrow & \text{Spec } A_a
 \end{array}$$

Then:

So there is a unique sheaf of \mathcal{O}_X -modules $\Omega_{Y/X}$ with $\Omega_{Y/X}(\text{Spec } B) = \Omega_{B/A}$

$\Omega_{Y/X}$	$\Omega_{Y/X}$ is the sheaf of (relative) differentials of Y over X . It comes with a "universal derivation" $d : \mathcal{O}_Y \longrightarrow \Omega_{Y/X}$
sheaf of relative differential forms of degree d over Y	for any $d \geq 0$ set $\Omega_{X/Y}^d := \Lambda_{\mathcal{O}_X}^d \Omega_{X/Y}$ the sheaf of relative differential forms of degree d over Y .
$\omega_{X/Y}$ Canonical sheaf of X over Y	if $\Omega_{X/Y}$ is locally free of rank n then $\Omega_{X/Y}^d$ is too, of rank $\binom{n}{d}$. In particular $\Omega_{X/Y}^n$ is an invertible sheaf, called the canonical sheaf of X over Y denoted $\omega_{X/Y}$.

6 cohomology

6.1 Some (quick) homological algebra

Let \mathcal{C} be the category of R -modules/ sheaves of abelian groups/ \mathcal{O}_X -modules on a scheme X or any abelian category.

(cochain) complex	a cochain complex consists of morphisms $\dots \xrightarrow{d_{n-1}} X^n \xrightarrow{d_n} X^{n+1} \xrightarrow{d_{n+1}} \dots$ with $d_{n+1} \circ d_n = 0$ Elements of X^n are called n -cochains. Elements of $Z^n := \ker(d_n)$ n -cocycles. Elements of $B^n := \operatorname{im}(d_{n-1})$ n -coboundaries
$H^n(X)$ n -th cohomology	$H^n(X) := Z^n/B^n$ is the n -th cohomology of (X^\bullet, d_\bullet)
acyclic	X is called acyclic if $\forall n : H^n(X) = 0$
homomorphism of complexes	of A homomorphism of complexes $f : X \rightarrow Y$ is a collection of homomorphisms $f^n : X^n \rightarrow Y^n$ such that for all n the following diagram commutes: $\begin{array}{ccc} X^n & \xrightarrow{d_n} & X^{n+1} \\ \downarrow f_n & & \downarrow f_{n+1} \\ Y^n & \xrightarrow{d_n} & Y^{n+1} \end{array}$
augmentation complex	of a An augmentation of a complex (X^\bullet, d_\bullet) is a homo $\Xi \xrightarrow{a} X^0$ such that $d_0 \circ a = 0$. i.e. $0 \rightarrow \Xi \xrightarrow{a} X^0 \xrightarrow{d_0} X^1 \xrightarrow{d_1} \dots$ is a complex. get natural homo $\Xi \xrightarrow{a} Z^0(X) = H^0(X)$

(cochain) homotopy	consider two homos of complexes $f, g : X \longrightarrow Y$. A cochain homotopy from f to g is a collection of homos $h : X^{n+1} \rightarrow Y^n$ such that $f - g = d \circ h + h \circ d$ If such h exists, f and g are called homotopic and we write $f \simeq g$
X contractible	X is contractible if id_X is homotopic to 0
quasi-isomorphism of complexes	A homo of complexes $f : X^\bullet \longrightarrow Y^\bullet$ is called a quasi-isomorphism if $\forall n \in \mathbb{Z} : H^n(f) : H^n(X^\bullet) \rightarrow H^n(Y^\bullet)$ is an isomorphism.

6.2 Cech Cohomology

Let \mathcal{F} be a sheaf of abelian groups of a topological space X . Take an open covering $\mathcal{U} := (U_i)_{i \in I}$ of X . The U_i need not be distinct or nonempty.

$C^\bullet(\mathcal{U}, \mathcal{F})$ (total) cech complex For any p set:

$$C^p(\mathcal{U}, \mathcal{F}) := \begin{cases} \prod_{i_0, \dots, i_p \in I} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p}) & \text{for } p \geq 0 \\ 0 & \text{for } p < 0 \end{cases}$$

$$d : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$$

$$(f_{i_0 \dots i_p})_{i_0 \dots i_p} \longmapsto (\sum_{k=0}^{p+1} (-1)^k f_{i_0, \dots, \hat{i}_k, \dots, i_{p+1}} |_{U_{i_0 \dots i_{p+1}}})_{i_0 \dots i_{p+1}}$$

this is the (total) cech complex.

$$H^n(\mathcal{U}, \mathcal{F}) \quad H^n(\mathcal{U}, \mathcal{F}) := H^n(C^\bullet(\mathcal{U}, \mathcal{F}))$$

$C_{alt}^\bullet(\mathcal{U}, \mathcal{F})$ alternating cech complex

$$C_{alt}^n(\mathcal{U}, \mathcal{F}) := \{(f_{i_0 \dots i_p}) \in C^p(\mathcal{U}, \mathcal{F}) \mid \forall i_0 \dots i_p : f_{i_0 \dots i_p} \text{ if not all } i_0 \dots i_p \text{ are}$$

$C_{ord}^\bullet(\mathcal{U}, \mathcal{F})$ ordered cech complex As alternating cech complex but now assume I comes with a total order $<$
Then $C_{ord}^\bullet(\mathcal{U}, \mathcal{F})$ with $i_0 < \dots < i_p$

refinement of open covering For $\mathcal{U} := (U_i)_{i \in I}$ and $\mathcal{V} := (V_j)_{j \in J}$ we say \mathcal{V} is a refinement if there exists a map $\sigma : I \longrightarrow J$ such that $\forall j \in J : V_j \subset U_{\sigma j}$

This defines a homo of complexes $\sigma^* : C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{V})$. Hence a natural homo $H^n(\sigma) : H^n(\mathcal{U}, \mathcal{F}) \longrightarrow H^n(\mathcal{V}, \mathcal{F})$.

Let \mathcal{C} be the category whose objects are all open coverings and $Mor_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) := \{\sigma \text{ as above}\}$.

Restrict to $\mathcal{U} = (U_i)_{i \in I}$ for all U_i distinct \implies small cofinal subcategory.