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5 Sheaves of Modules

5.1 $\mathcal{O}(X) - modules$

Fix a scheme X

sheaf of $\mathcal{O}_X$-modules:	A (pre)sheaf of $\mathcal{O}(X)$-modules is a (pre)sheaf of abelian groups $\mathcal{F}$ on X together with a morphism of presheaves of sets $\mathcal{O}_X \times \mathcal{F} \longrightarrow \mathcal{F}$, which turns $\mathcal{F}(U)$ into an $\mathcal{O}_X(U)$-module for every open U$\subset$ X

A homomorphism of such is a morphism of presheaves $\mathcal{F} \rightarrow \mathcal{G}$ such that $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear for each U

analogously define composition, identity morphism etc. $\rightsquigarrow$ this forms a category.

Take $\mathcal{O}_X$-module $\mathcal{F}_i$ for $i \in I$

direct product:	The direct product $\prod \mathcal{F}_i$ is defined by $U \longmapsto \prod(\mathcal{F}_i(U))$ with component-wise structure
direct sum:	the direct sum $\bigoplus \mathcal{F}_i$ is the sheafification the presheaf $U \longmapsto \bigoplus \mathcal{F}_i(U)$. This is a subsheaf of $\prod \mathcal{F}_i$ with equality if $\|I\|$ is finite.
tensor product:	The tensor product $\mathcal{F} \otimes \mathcal{G}$ of $\mathcal{O}_X$-modules$\mathcal{F}$ & $\mathcal{G}$ is the sheafification of the presheaf $U \longmapsto \mathcal{F}(U) \otimes_{O(U)} \mathcal{G}(U)$
inner Hom:	The inner Hom is the sheaf of homomorphisms of $\mathcal{O}_X$-modules$\mathcal{F}$ and $\mathcal{G}$ $U \longmapsto Hom_{O_X}(\mathcal{F}\|_U, \mathcal{G}\|_U) =: \mathcal{H}om_{\mathcal{O}(X)}(\mathcal{F}, \mathcal{G})(U)$

5.2 Locally free $\mathcal{O}_X$-modules

Abbreviate: $\mathcal{O}_X^{(I)} := \bigoplus_{i \in I} \mathcal{O}_X$

$\mathcal{O}_X^I := \prod_{i \in I} \mathcal{O}_X$

$\mathcal{O}_X^r := \bigoplus_{i=1}^r \mathcal{O}_X$

free $\mathcal{O}_X$-module :	An $\mathcal{O}_X$-module isomorphic to $\mathcal{O}_X^{(I)}$ for some I is called free
locally free $\mathcal{O}_X$-module :	(a) An $\mathcal{O}_X$-module is locally free if $\forall x \exists x \in U \subset X open \, \exists I : \mathcal{F}|_U \cong \mathcal{O}_X^{(I)}|_U$
	(b) It is called locally free of rank r if $\exists I : \lvert I \rvert = r$ and $\forall x \exists x \in U \subset X : \mathcal{F}|_U \cong \mathcal{O}_X^{(I)}|_U$
dual sheaf:	For $\mathcal{F}$ locally free of finite rank, $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is called the dual sheaf, also locally free of rank r.
invertible sheaf:	A locally free sheaf of rank 1 is called an invertible sheaf.
Picard group:	The set of isomorphism classes of invertible sheaves on X form an abelian group, $Pic(X)$, called the Picard group of X.

5.3 $\mathcal{O}_X$-module s on affine schemes

5.4 Quasicoherent $\mathcal{O}_X$-module s

Let X be an arbitrary scheme
Notation for global sections: $\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$

Any system of global sections $s_i \in \mathcal{F}(X)$ determines a homomorohism $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{F}$, $(f_i)_{i \in I} \longmapsto \sum f_i \cdot res_U^X$

$\mathcal{F}$ is called generated by global sections if $\exists I$: $\exists$ a surjective homomorphism: $\mathcal{O}_X^{(I)} \twoheadrightarrow \mathcal{F}$

quasicoherent $\mathcal{O}_X$-module :	An $\mathcal{O}_X$-module $\mathcal{F}$ is called quasicoherent if $\forall x \in X \exists$ open neighbourhood $x \in U \subset X \, \exists I \, \exists J \, \exists$ exact sequence $\mathcal{O}_X^{(I)}|_U \longrightarrow \mathcal{O}_X^{(J)}|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0$

locally free $\Longrightarrow$ quasicoherent

5.5 Coherent sheaves

finitely generated $\mathcal{O}_X$-module :	An $\mathcal{O}_X$-module $\mathcal{F}$ is called:
	(a) finitely generated (or of finite type) if $\forall \exists x \in U \subset X$ open: $\exists n < \infty \, \exists \, \mathcal{O}_X^n|_U \twoheadrightarrow \mathcal{F}|_U$ $\mathcal{O}_X$-module homomorphism.
coherent $\mathcal{O}_X$-module	(b) coherent if it is finitely generated and $\forall U \subset X$ open $\forall n < \infty \, \forall$ homomorphisms $\varphi : \mathcal{O}_X^n|_U \twoheadrightarrow \mathcal{F}|_U \, ker(\varphi)$ is finitely generated.

5.6 Functoriality

Consider a morphism $f : X \longrightarrow Y$ with sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y.
Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{G}$ be an $\mathcal{O}_Y$-module.
f comes with a homomorphism of sheaves of rings:

$f^\flat : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \Longleftrightarrow f^\sharp : f^*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$

push-forward of a sheaf:	Make $f_*\mathcal{F}$ into an $\mathcal{O}_Y$-module by $$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[d] & \times & (f_*\mathcal{F})(V) \arrow[d, "="] \arrow[r] & (f_*\mathcal{F})(V) \arrow[d] \\ \mathcal{O}_X(f^{-1}(V)) & \times & \mathcal{F}(f^{-1}) \arrow[r, "mult."] & \mathcal{F}(f^{-1}(V)) \end{tikzcd}$$
inverse image:	the inverse image of an $\mathcal{O}_X$-module is $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. i.e.the sheafification of the presheaf $$U \longmapsto \varinjlim_{f(U)\subset V\subset Y} (\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(V)} \mathcal{G}(V))$$
scheme-theoretic support of $\mathcal{F}$:	Let $\mathcal{F}$ be a quasicoherent, finitely generated $\mathcal{O}_X$-module . the scheme-theoretic support is the smallest closed subscheme $i : Y \hookrightarrow X$ such that $\mathcal{F} \cong i_*i^*\mathcal{F}$. Moreover $Y = \{x \in X \vert \mathcal{F}_x \neq 0\}$

5.7 $\mathcal{O}_X$-moduleson a projective scheme

For any graded R-module M and any $n \in \mathbb{Z}$ we set $M(n) := M$ as R-module with grading $M(n)_d = M_{n+d}$

$\widetilde{R(n)} =: \mathcal{O}_X(n)$
clear: $\mathcal{O}_X(0) \cong \mathcal{O}_X$

twisting sheaf:	$\mathcal{O}_X(1)$ is called the twisting sheaf on $X = ProjR$ for any $\mathcal{O}_X$-module we set $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$

5.8 Morphisms to projective spaces

very ample sheaf:	An invertible sheaf $\mathcal{L}$ is called very ample (over SpecR) if $\mathcal{L} \cong f^*\mathcal{O}_X(1)$ for some locally closed embedding over R $f : X \hookrightarrow \mathbb{P}^n_R$ for some n.
ample sheaf:	An invertible sheaf $\mathcal{L}$ on a quasicompact scheme X is called ample if for any finitely generated quasicoherent sheaf $\mathcal{F}$ on X there exists n_0 such that $n \geq n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.
relatively ample /very ample sheaf	$\mathcal{L}$ an invertible sheaf with respect to $f : X \longrightarrow Y$ iff there exists an affine open covering $Y = \bigcup_{i\in I} V_i$ such that $\forall i \mathcal{L}\vert_{f^{-1}(V_i)}$ is ample/ very ample over V_i

5.9 Divisors

Assume X integral with function field K.

field of rational functions on X	the constant sheaf $(K)_X := \underline{K}$ on X is called the sheaf of rational functions on X
group of Cartier divisors	$Div(X) := \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ $Div(X) = \{([f_x]) \in \prod_{x\in X} K^\times/\mathcal{O}_{X,x}^\times \mid \forall x \exists U \exists f \in K^\times \forall y \in U : f_y \cdot \mathcal{O}_{X,y}^\times = f\mathcal{O}_{X,y}^\times\}$ $= \{$ collections of $f_i \in K^\times$ for all $i \in I$ and an open covering $X = \bigcup U_i$ such that $\forall i, j : f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)\}/$ modulo some equivalence relation convention: The group law on Div(X) is written additively.
$\mathcal{O}_X(D)(U)$	For any Cartier divisor $D = ([f_x])x \in X$ and any open $U \subset X$ set: $\mathcal{O}_X(D)(U) = \begin{cases} \{0\} & \text{if } U = \emptyset \\ \bigcap_{x\in U} f_x^{-1}\mathcal{O}_{X,x} & \text{else} \end{cases}$
locally factorial scheme	A noetherian integral scheme such that $\forall x \in X\mathcal{O}_{X,x}$ is factorial is called locally factorial.
effective cartier divisor	A cartier divisor $D = ([f_x])_x$ is called effective if $\forall x \in X : f_x \in \mathcal{O}_{X,x}$ equiv: $\forall i : f_i \in \mathcal{O}_X(u_i)$ $\iff \mathcal{O}_X(-D) \subset \mathcal{O}_X \iff \mathcal{O}_X \subset \mathcal{O}_X(D)$ $\iff \mathcal{O}_X(-D)$ is a quasicoherent sheaf of ideals of $\mathcal{O}_X$ $\iff$ D corresponds to a closed subscheme locally given by one equation f_i i.e. locally principal. this subscheme determines D.
principal cartier divisor	A cartier divisor of the form $div(f) := (f) := ([f])_{x\in X}$ for some $f \in K^\times$ is called principal.
cartier divisor class group of X	The factor group $DivCl(X) := Div(X)/principaldivisors$
prime cycle	an integral closed subscheme is called a prime cycle. (equiv: irred closed subset)
codimension of a prime cycle	A prime cycle's codimension is $dim\mathcal{O}_{X,y}$ for the generic point $y \in Y$.

cycle	A finite formal $\mathbb{Z}$-linear combination of prime cycles $\sum_y n_y y$ is called a cycle.
codimension of a cycle	If all these Y have codim d the cycle has codim d
Weil divisor	A cycle of dimension 1 is called a Weil divisor.
principal weil divisor	A Weil divisor of the form $cyc(div(f)) = \sum ord_y(f) \cdot \overline{\{y\}}$ is called principal.
Weil divisor class group	The factor group $Z(X)/\{principal\} = Cl(X)$ is called the Weil divisor group.
effective weil divisor	A weil divisor $\sum n_y Y$ is effective if all $n_y \geq 0$ equiv: associated cartier divisor is effective equiv: $\mathcal{O}_X(-D)$ is an ideal sheaf of $\mathcal{O}_X$
ample/very ample divisor	A divisor is called ample/very ample iff $\mathcal{O}_X(D)$ is dito.

5.10 Differentials

Let X=Spec B , Y=Spec A and $f : Y \longrightarrow X$

A-derivation of B to M	An A-derivation of B to a B-module M is a map $d : B \longrightarrow M$ with $\forall b, b' \in B \ \forall a \in A$ (a)$d(b + b') = d(b) + d(b')$ (b)$d(b \cdot b') = b \cdot b' + d(b) \cdot b'$ (c)$d(a \cdot 1_B) = 0$
$\Omega_{B/A}$ module of (relative) differential (form)s	A module of relative differential forms of B over A is a B-module $\Omega_{B/A}$ with a derivation $d : B \longrightarrow \Omega_{B/A}$ over A which satisfies the universal property: for all B-modules M and all derivations $\delta : B \longrightarrow M$ over A there exists exactly one B-module homomorphism $f : \Omega_{B/A} \longrightarrow M$ with $f \circ d = \delta$

Consider a morphism $f : Y \longrightarrow X$. then for all open affine subsets we have:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, phantom, "\cup"] & X \arrow[d, phantom, "\cup"] \\
SpecB \arrow[r] \arrow[d, phantom, "\cup"] & SpecA \arrow[d, phantom, "\cup"] \\
SpecB_{ab} \arrow[r] & SpecA_a
\end{tikzcd}$$

Then:
So there is a unique sheaf of $\mathcal{O}_X$-modules$\Omega_{Y/X}$ with $\Omega_{Y/X}(SpecB) = \Omega_{B/A}$

$\Omega_{Y/X}$	$\Omega_{Y/X}$ is the sheaf of (relative) differentials of Y over X. It comes with a "universal derivation" $d : \mathcal{O}_Y \longrightarrow \Omega_{Y/X}$
sheaf of relative differential forms of degree d over Y	for any $d \geq 0$ set $\Omega^d_{X/Y} := \Lambda^d_{\mathcal{O}_X}\Omega_{X/Y}$ the sheaf of relative differential forms of degree d over Y.
$\omega_{X/Y}$ Canonical sheaf of X over Y	if $\Omega_{X/Y}$ is locally free of rank n then $\Omega^d_{X/Y}$ is too, of rank $\binom{n}{d}$. In particular $\Omega^n_{X/Y}$ is an invertible sheaf, called the canonical sheaf of x over Y denoted $\omega_{X/Y}$.

6 cohomology

6.1 Some (quick) homological algebra

Let $\mathcal{C}$ be the category of R-modules/ sheaves of abelian groups/ $\mathcal{O}_X$-moduleson a scheme X or any abelian category.

(cochain) complex	a cochain complex consists of morphisms $\ldots \xrightarrow{d_{n-1}} X^n \xrightarrow{d_n} X^{n+1} \xrightarrow{d_{n+1}} \ldots$ with $d_{n+1} \circ d_n = 0$ Elements of X^n are called n-cochains. Elements of $Z^n := ker(d_n)$ n-cocycles. Elements of $B^n := im(d_{n-1})$ n-coboundaries
$H^n(X)$ n-th cohomology	$H^n(X) := Z^n/B^n$ is the n-th cohomology of $(X^\bullet, d_\bullet)$
acyclic	X is called acyclic if $\forall n : H^n(X) = 0$
homomorphism of complexes	A homomorphism of complexes $f : X \to Y$ is a collection of homomorphisms $f^n : X^n \to Y^n$ such that for all n the following diagramm commutes: $$\begin{tikzcd} X^n \arrow[r, "d_n"] \arrow[d, "f_n"] & X^{n+1} \arrow[d, "f_{n+1}"] \\ Y^n \arrow[r, "d_n"] & Y^{n+1} \end{tikzcd}$$
augmentation of a complex	An augmentation of a complex $(X^\bullet, d_\bullet)$ is a homo $\Xi \xrightarrow{a} X^0$ such that $d_0 \circ a = 0$. i.e. $0 \to \Xi \xrightarrow{a} X^0 \xrightarrow{d_0} X^1 \xrightarrow{d_1} \ldots$ is a complex. get natural homo $\Xi \xrightarrow{a} Z^0(X) = H^0(X)$
(cochain) homotopy	consider two homos of complexes $f, g : X \longrightarrow Y$. A cochain homotopy from f to g is a collection of homos $h : X^{n+1} \to Y^n$ such that $f - g = d \circ h + h \circ d$

	If such h exists, f and g are called homotopic and we write $f \simeq g$
X contractible	X is contractible if id_X is homotopic to 0
quasi-isomorphism of complexes	A homo of complexes $f : X^\bullet \longrightarrow Y^\bullet$ is called a quasi-isomorphism if $\forall n \in \mathbb{Z} : H^n(f) : H^n(X^\bullet) \to H^n(Y^\bullet))$ is an isomorphism.

6.2 Cech Cohomology

Let $\mathcal{F}$ be a sheaf of abelian groups of a topological space X. Take an open covering $\mathcal{U} := (U_i)_{i \in I}$ of X. The U_i need not be distict or nonempty.

$C^\bullet(\mathcal{U}, \mathcal{F})$(total) cech complex	For any p set: $C^p(\mathcal{U}, \mathcal{F}) := \begin{cases} \prod_{i_0, \dots, i_p \in I} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p}) & for p \geq 0 \\ 0 & for p < 0 \end{cases}$ $d : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$ $(f_{i_0 \dots i_p})_{i_0 \dots i_p} \longmapsto (\sum_{k=0}^{p+1} (-1)^k \dot{f}_{i_0, \dots, \hat{i_k}, \dots, i_{p+1}} \|_{U_{i_0 \dots i_{p+1}}})_{i_0 \dots i_{p+1}}$ this is the (total) cech complex.
$H^n(\mathcal{U}, \mathcal{F})$	$H^n(\mathcal{U}, \mathcal{F}) := H^n(C^\bullet(\mathcal{U}, \mathcal{F}))$
$C^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ alternating cech complex	$C^n_{alt}(\mathcal{U}, \mathcal{F}) := \{(f_{i_0 \dots i_p}) \in C^p(\mathcal{U}, \mathcal{F}) \quad \| \quad \forall i_0 \dots i_p : f_{i_0 \dots i_p}$ if not all $i_0 \dots i_p$ are
$C^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ ordered cech complex	As alternating cech complex but now assume I comes with a total order $<$ Then $C^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ with $i_0 < \dots < i_p$
refinement of open covering	For $\mathcal{U} := (U_i)_{i \in I}$ and $\mathcal{V} := (V_j)_{j \in J}$ we say $\mathcal{V}$ is a refinement if there exists a map $\sigma : I \longrightarrow J$ such that $\forall j \in J : V_j \subset U_{\sigma j}$

This defines a homo of complexes $\sigma^* : C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{V})$. Hence a natural homo $H^n(\sigma) : H^n(\mathcal{U}, \mathcal{F}) \longrightarrow H^n(\mathcal{V}, \mathcal{F})$.

Let $\mathcal{C}$ be the category whose objects are all open coverings and $Mor_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) := \{\sigma as above\}$. Restrict to $\mathcal{U} = (U_i)_{i \in I}$ for all U_i distinct $\Longrightarrow$ small cofinal subcategory.

$H^n(X, \mathcal{F})$ the n-th cech cohomology of $\mathcal{F}$	$H^n(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^n(\mathcal{U}, \mathcal{F})$ is called the n-th cech cohomology.

6.3 Cohomology of projective space

6.4 Higher direct images

Take $f : X \to Y$ sheaves of modules $\mathcal{F}$ on X and $\mathcal{G}$ on Y We have a natural homo: $H^p(Y, f_*\mathcal{F}) \to H^p(X, \mathcal{F})$ and $H^p(Y, \mathcal{G}) \to H^p(X, f^*\mathcal{G})$

Special case: $U \subset X$ open $\rightsquigarrow$ natural "restriction map" $H^p(X, \mathcal{F}) \longrightarrow H^p(U, \mathcal{F}|_U)$.

Special case: For all open $V' \subset V \subset Y$ the restriction map for $f^{-1}(V') \subset f^{-1}(V)$ makes this into a presheaf of $\mathcal{O}_Y$-moduleson Y.

$R^pf_*\mathcal{F}$ the p-th higher direct image of $\mathcal{F}$	The associated sheaf to the sheaf above is called the p-th higher direct image (sheaf) of $\mathcal{F}$ with respect to f Note: $R^pf_*\mathcal{F} = 0$ for $p < 0$ Fact: $R^0f_*\mathcal{F} = f_*\mathcal{F}$ (no sheafification required)

6.5 Duality

Construction: Take $\mathcal{O}_X$-modules$\mathcal{F}$, $\mathcal{G}$

$$\begin{aligned} H^0(X, \mathcal{F}) \times H^p(X, \mathcal{G}) &\longrightarrow H^p(X, \mathcal{F} \otimes \mathcal{G}) \\ (f, \xi) &\longmapsto H^p(L_f)(\xi) \\ f : L_f : \mathcal{G} \to \mathcal{F} \otimes \mathcal{G}, g \mapsto f \otimes g & \end{aligned}$$

more generally $H^p \times H^q \longrightarrow H^{p+q}$ (cup product)

perfect pairing	An A-bilinear map $M \times N \longrightarrow L$ for A-modules M, N, L of finite rank with L of rank 1 is a perfect pairing if: the representing matrix is invertible w.r.t. any bases b induces an iso $M \xrightarrow{\sim} Hom_A(N, L)$ $m \mapsto b(m, -)$ b induces an iso $N \xrightarrow{\sim} Hom_A(M, L)$

Fix a proj morphism $f : X \longrightarrow Y$ with Y locally noetherian fix $r \geq 0$ such that all fibers of f have dimension $\leq r$.

(r-)duality sheaf for f	A duality sheaf is a quasi-coherent sheaf ω_f on X together with a homo of $\mathcal{O}_X$-modules$tr_f : R^rf_*\omega_f \longrightarrow \mathcal{O}_Y$ (trace map) such that for every quasicoherent sheaf $\mathcal{F}$ on X the natural bilinear map: $$f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_f) \times R^rf_*\mathcal{F} \longrightarrow R^rf_*\omega_f \xrightarrow{tr_f} \mathcal{O}_Y$$ induces an iso $f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_f) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(R^rf_*\mathcal{F}, \mathcal{O}_Y)$

Note: For $Y = SpecA$ this means: $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_f) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(H^r(X, \mathcal{F}), A)$

6.6 Flat base change

Recall: An A-module M is flat iff $((A-mods)) \longrightarrow ((A-mods))N \mapsto M \otimes_A N$ is exact.

flat $\mathcal{O}_X$-module	A sheaf of $\mathcal{O}_X$-modules$\mathcal{F}$ is flat if $\forall x \in X : \mathcal{F}_x$ is a flat $\mathcal{O}_X$-module
flat morphism	A morphism $f : X \longrightarrow Y$ is flat iff $\forall x \in X : \mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$ modules

Consider an arbitrary commutative diagram:

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] & \supset & f^{-1}(V) \arrow[d] \\
Y' \arrow[r, "g"] & Y & \supset & V
\end{tikzcd}$$

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. for any open $V \subset Y$:

$$H^p(f^{-1}(V), \mathcal{F}(f^{-1}(V))) \xrightarrow{6.4} H^p(g'^{-1}(f^{-1}(V)), g'^{*}\mathcal{F}|_{g'^{-1}(f^{-1}(V))})$$

Varying V this defines a homo of presheaves $\Rightarrow$ sheafify.

Let

$$(R^p f_* \mathcal{F})(V) \longrightarrow (R^p f'_*)(g'^* \mathcal{F})(g^{-1}(V))$$

be the resulting homo of sheaves. Take its adjoint.

base change homo	this is the base change homo: $g^* R^p f_* \mathcal{F} \xrightarrow{BC} R^p f'_* g'^* \mathcal{F}$

7 Riemann-Roch and Serre duality

7.1 Divisors on curves

Let X be regular integral projective scheme of dimension 1 over a field k ($\Rightarrow$ curve)

degree of a Weil divisor	For $D = \sum_{P \in \lvert X \rvert} n_P \cdot \bar{P}$ is $deg_k(D) := \sum_P n_P \cdot [k(P)/k] \in \mathbb{Z}$

7.2 Riemann-Roch

Let X be projective over a field k. Let $\mathcal{F}$ be a coherent sheaf on X.
6.3 $\Rightarrow \forall : H^p(X, \mathcal{F})$ is a fin dim k-Vector space, zero for $p \notin [0, dimX]$
Abbreviate: $h^p(X, \mathcal{F}) := dim_k H^p(X, \mathcal{F})$

$\chi(X, \mathcal{F})$ Euler characteristic of $\mathcal{F}$	$\chi(X, \mathcal{F}) := \sum_{p \in \mathbb{Z}} (-1)^p h^p(X, \mathcal{F})$
$deg(\mathcal{L})$	$deg(\mathcal{L}) := deg(D)$ for any D with $\mathcal{L} \cong \mathcal{O}_X(D)$ $dim(X) = 1 \Rightarrow \chi(X, \mathcal{F}) = h^0(X, \mathcal{F}) - h^1(X, \mathcal{F})$
genus of a curve	$g := h^1(X, \mathcal{O}_X) \in \mathbb{Z}$ is called the genus of X, so $\chi(X, \mathcal{O}_X) = 1 - g$

7.3 Residues

k any field, t variable $F := k((t)) = k[[t]][t^{-1}]$ for $k[[t]] = \varprojlim_n k[t]/(t^n)$.

Set $\Omega^{\hat{}}_{F/k} = (\varprojlim_n \Omega_{k[t]/(t^n)/k})[t^{-1}] \cong F \cdot dt$

residue	For any $\omega = \sum_{i \geq -n} a_i t^i \cdot dt$ we set $res_t \omega := a_{-1}$ $\rightsquigarrow$ k-linear $res_t : \Omega_{F/k} \longrightarrow k$

7.4 Serre duality

7.5 Some consequences of Riemann-Roch

7.6 Embeddings in projective space

hyperelliptic	X is called hyperelliptic if $g \geq 1$ and it posesses a separated morph $f : X \longrightarrow \mathbb{P}^1_k$ of degree 2

7.7 Hyperelliptic curves

7.8 Coverings