for errors or questions email: nicolatr@student.ethz.ch

5 Sheaves of Modules

5.1 $\mathcal{O}(X) - modules$

Fix a scheme X

sheaf of $\mathcal{O}_X$-modules: A (pre)sheaf of $\mathcal{O}(X)$-modules is a (pre)sheaf of abelian groups $\mathcal{F}$ on X together with a morphism of presheaves of sets $\mathcal{O}_X \times \mathcal{F} \longrightarrow \mathcal{F}$, which turns $\mathcal{F}(U)$ into an $\mathcal{O}_X(U)$-module for every open U$\subset$ X

A homomorphism of such is a morphism of presheaves $\mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear for each U

analogously define composition, identity morphism etc. $\rightsquigarrow$ this forms a category.

Take $\mathcal{O}_X$-module $\mathcal{F}_i$ for $i \in I$

direct product: The direct product $\prod \mathcal{F}_i$ is defined by $U \longmapsto \prod(\mathcal{F}_i(U))$ with component-wise structure

direct sum: the direct sum $\bigoplus \mathcal{F}_i$ is the sheafification the presheaf $U \longmapsto \bigoplus \mathcal{F}_i(U)$. This is a subsheaf of $\prod \mathcal{F}_i$ with equality if $|I|$ is finite.

tensor product: The tensor product $\mathcal{F} \otimes \mathcal{G}$ of $\mathcal{O}_X$-modules$\mathcal{F}$ & $\mathcal{G}$ is the sheafification of the presheaf $U \longmapsto \mathcal{F}(U) \otimes_{O(U)} \mathcal{G}(U)$

inner Hom: The inner Hom is the sheaf of homomorphisms of $\mathcal{O}_X$-modules$\mathcal{F}$ and $\mathcal{G}$
$U \longmapsto Hom_{O_X}(\mathcal{F}|_U, \mathcal{G}|_U) =: \mathcal{H}om_{\mathcal{O}(X)}(\mathcal{F}, \mathcal{G})(U)$

5.2 Locally free $\mathcal{O}_X$-modules

Abbreviate: $\mathcal{O}_X^{(I)} := \bigoplus_{i \in I} \mathcal{O}_X$
$\mathcal{O}_X^I := \prod_{i \in I} \mathcal{O}_X$
$\mathcal{O}_X^r := \bigoplus_{i=1}^r \mathcal{O}_X$

free $\mathcal{O}_X$-module :	An $\mathcal{O}_X$-module isomorphic to $\mathcal{O}_X^{(I)}$ for some I is called free
locally free $\mathcal{O}_X$-module :	(a) An $\mathcal{O}_X$-module is locally free if $\forall x \exists x \in U \subset X open \; \exists I : \mathcal{F}\|_U \cong \mathcal{O}_X^{(I)}\|_U$ (b) It is called locally free of rank r if $\exists I : \|I\| = r$ and $\forall x \exists x \in U \subset X : \mathcal{F}\|_U \cong \mathcal{O}_X^{(I)}\|_U$
dual sheaf:	For $\mathcal{F}$ locally free of finite rank, $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is called the dual sheaf, also locally free of rank r.
invertible sheaf:	A locally free sheaf of rank 1 is called an invertible sheaf.
Picard group:	The set of isomorphism classes of invertible sheaves on X form an abelian group, $Pic(X)$, called the Picard group of X.

5.3 $\mathcal{O}_X$-module s on affine schemes

5.4 Quasicoherent $\mathcal{O}_X$-module s

Let X be an arbitrary scheme
Notation for global sections: $\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$

Any system of global sections $s_i \in \mathcal{F}(X)$ determines a homomorohism $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{F}$, $(f_i)_{i \in I} \longmapsto \sum f_i \cdot res_U^X$

$\mathcal{F}$ is called generated by global sections if $\exists I$: $\exists$ a surjective homomorphism: $\mathcal{O}_X^{(I)} \twoheadrightarrow \mathcal{F}$

quasicoherent $\mathcal{O}_X$-module :	An $\mathcal{O}_X$-module $\mathcal{F}$ is called quasicoherent if $\forall x \in X \exists$ open neighbourhood $x \in U \subset X \; \exists I \; \exists J \; \exists$ exact sequence $$\mathcal{O}_X^{(I)}\|_U \longrightarrow \mathcal{O}_X^{(J)}\|_U \longrightarrow \mathcal{F}\|_U \longrightarrow 0$$

locally free $\Longrightarrow$ quasicoherent

5.5 Coherent sheaves

finitely generated $\mathcal{O}_X$-module :	An $\mathcal{O}_X$-module $\mathcal{F}$ is called: (a) finitely generated (or of finite type) if $\forall \exists x \in U \subset X$ open: $\exists n < \infty \; \exists \, \mathcal{O}_X^n\|_U \twoheadrightarrow \mathcal{F}\|_U$ $\mathcal{O}_X$-module homomorphism.
coherent $\mathcal{O}_X$-module	(b) coherent if it is finitely generated and $\forall U \subset X$ open $\forall n < \infty \, \forall$ homomorphisms $\varphi : \mathcal{O}_X^n\|_U \twoheadrightarrow \mathcal{F}\|_U \; ker(\varphi)$ is finitely generated.

5.6 Functoriality

Consider a morphism $f : X \longrightarrow Y$ with sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y.
Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{G}$ be an $\mathcal{O}_Y$-module.
f comes with a homomorphism of sheaves of rings:

$f^\flat : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \Longleftrightarrow f^\sharp : f^*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$

push-forward of a sheaf: Make $f_*\mathcal{F}$ into an $\mathcal{O}_Y$-module by

$$\begin{array}{ccc} \mathcal{O}_Y(V) \times (f_*\mathcal{F})(V) & \longrightarrow & (f_*\mathcal{F})(V) \\ \downarrow \quad\quad\quad \downarrow = & & \downarrow \\ \mathcal{O}_X(f^{-1}(V)) \times \mathcal{F}(f^{-1}) & \xrightarrow{mult.} & \mathcal{F}(f^{-1}(V)) \end{array}$$

inverse image: the inverse image of an $\mathcal{O}_X$-module is $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. i.e.the sheafification of the presheaf

$$U \longmapsto \varinjlim_{f(U)\subset V\subset Y} (\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(V)} \mathcal{G}(V))$$

scheme-theoretic support of $\mathcal{F}$: Let $\mathcal{F}$ be a quasicoherent, finitely generated $\mathcal{O}_X$-module . the scheme-theoretic support is the smallest closed subscheme $i : Y \hookrightarrow X$ such that $\mathcal{F} \cong i_*i^*\mathcal{F}$. Moreover $Y = \{x \in X | \mathcal{F}_x \neq 0\}$

5.7 $\mathcal{O}_X$-moduleson a projective scheme

For any graded R-module M and any $n \in \mathbb{Z}$ we set $M(n) := M$ as R-module with grading $M(n)_d = M_{n+d}$

$\widetilde{R(n)} =: \mathcal{O}_X(n)$
clear: $\mathcal{O}_X(0) \cong \mathcal{O}_X$

twisting sheaf: $\mathcal{O}_X(1)$ is called the twisting sheaf on $X = ProjR$
for any $\mathcal{O}_X$-module we set $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$

5.8 Morphisms to projective spaces

very ample sheaf: An invertible sheaf $\mathcal{L}$ is called very ample (over SpecR) if $\mathcal{L} \cong f^*\mathcal{O}_X(1)$ for some locally closed embedding over R $f : X \hookrightarrow \mathbb{P}^n_R$ for some n.

ample sheaf: An invertible sheaf $\mathcal{L}$ on a quasicompact scheme X is called ample if for any finitely generated quasicoherent sheaf $\mathcal{F}$ on X there exists n_0 such that $n \geq n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.

relatively ample /very ample sheaf: $\mathcal{L}$ an invertible sheaf with respect to $f : X \longrightarrow Y$ iff there exists an affine open covering $Y = \bigcup_{i\in I} V_i$ such that $\forall i \mathcal{L}|_{f^{-1}(V_i)}$ is ample/ very ample over V_i

5.9 Divisors

Assume X integral with function field K.

field of rational functions on X	the constant sheaf $(K)_X := \underline{K}$ on X is called the sheaf of rational functions on X
group of Cartier divisors	$Div(X) := \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$ $Div(X) = \{([f_x]) \in \prod_{x \in X} K^\times / \mathcal{O}_{X,x}^\times \mid \forall x \exists U \exists f \in K^\times \forall y \in U : f_y \cdot \mathcal{O}_{X,y}^\times = f \mathcal{O}_{X,y}^\times\}$ $=$ { collections of $f_i \in K^\times$ for all $i \in I$ and an open covering $X = \bigcup U_i$ such that $\forall i,j : f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)\}/$ modulo some equivalence relation convention: The group law on Div(X) is written additively.
$\mathcal{O}_X(D)(U)$	For any Cartier divisor $D = ([f_x])x \in X$ and any open $U \subset X$ set: $$\mathcal{O}_X(D)(U) = \begin{cases} \{0\} if U = \emptyset \\ \bigcap_{x \in U} f_x^{-1} \mathcal{O}_{X,x} else \end{cases}$$
locally factorial scheme	A noetherian integral scheme such that $\forall x \in X \mathcal{O}_{X,x}$ is factorial is called locally factorial.
effective cartier divisor	A cartier divisor $D = ([f_x])_x$ is called effective if $\forall x \in X : f_x \in \mathcal{O}_{X,x}$ equiv: $\forall i : f_i \in \mathcal{O}_X(u_i)$ $\iff \mathcal{O}_X(-D) \subset \mathcal{O}_X \iff \mathcal{O}_X \subset \mathcal{O}_X(D)$ $\iff \mathcal{O}_X(-D)$ is a quasicoherent sheaf of ideals of $\mathcal{O}_X$ $\iff$ D corresponds to a closed subscheme locally given by one equation f_i i.e. locally principal. this subscheme determines D.
principal cartier divisor	A cartier divisor of the form $div(f) := (f) := ([f])_{x \in X}$ for some $f \in K^\times$ is called principal.
cartier divisor class group of X	The factor group $DivCl(X) := Div(X)/principaldivisors$
prime cycle	an integral closed subscheme is called a prime cycle. (equiv: irred closed subset)
codimension of a prime cycle	A prime cycle's codimension is $dim \mathcal{O}_{X,y}$ for the generic point $y \in Y$.

cycle	A finite formal $\mathbb{Z}$-linear combination of prime cycles $\sum_y n_y y$ is called a cycle.
codimension of a cycle	If all these Y have codim d the cycle has codim d
Weil divisor	A cycle of dimension 1 is called a Weil divisor.
principal weil divisor	A Weil divisor of the form $cyc(div(f)) = \sum ord_y(f) \cdot \overline{\{y\}}$ is called principal.
Weil divisor class group	The factor group $Z(X)/\{principal\} = Cl(X)$ is called the Weil divisor group.
effective weil divisor	A weil divisor $\sum n_y Y$ is effective if all $n_y \geq 0$ equiv: associated cartier divisor is effective equiv: $\mathcal{O}_X(-D)$ is an ideal sheaf of $\mathcal{O}_X$
ample/very ample divisor	A divisor is called ample/very ample iff $\mathcal{O}_X(D)$ is dito.

5.10 Differentials

Let X=Spec B , Y=Spec A and $f : Y \longrightarrow X$

A-derivation of B to M	An A-derivation of B to a B-module M is a map $d : B \longrightarrow M$ with $\forall b, b' \in B\ \forall a \in A$ (a)$d(b + b') = d(b) + d(b')$ (b)$d(b \cdot b') = b \cdot b' + d(b) \cdot b'$ (c)$d(a \cdot 1_B) = 0$
$\Omega_{B/A}$ module of (relative) differential (form)s	A module of relative differential forms of B over A is a B-module $\Omega_{B/A}$ with a derivation $d : B \longrightarrow \Omega_{B/A}$ over A which satisfies the universal property: for all B-modules M and all derivations $\delta : B \longrightarrow M$ over A there exists exactly one B-module homomorphism $f : \Omega_{B/A} \longrightarrow M$ with $f \circ d = \delta$

Consider a morphism $f : Y \longrightarrow X$. then for all open affine subsets we have:

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \cup & & \cup \\ SpecB & \longrightarrow & SpecA \\ \cup & & \cup \\ SpecB_{ab} & \longrightarrow & SpecA_a \end{array}$$

Then:
So there is a unique sheaf of $\mathcal{O}_X$-modules$\Omega_{Y/X}$ with $\Omega_{Y/X}(SpecB) = \Omega_{B/A}$

$\Omega_{Y/X}$	$\Omega_{Y/X}$ is the sheaf of (relative) differentials of Y over X. It comes with a "universal derivation" $d : \mathcal{O}_Y \longrightarrow \Omega_{Y/X}$
sheaf of relative differential forms of degree d over Y	for any $d \geq 0$ set $\Omega^d_{X/Y} := \Lambda^d_{\mathcal{O}_X} \Omega_{X/Y}$ the sheaf of relative differential forms of degree d over Y.
$\omega_{X/Y}$ Canonical sheaf of X over Y	if $\Omega_{X/Y}$ is locally free of rank n then $\Omega^d_{X/Y}$ is too, of rank $\binom{n}{d}$. In particular $\Omega^n_{X/Y}$ is an invertible sheaf, called the canonical sheaf of x over Y denoted $\omega_{X/Y}$

6 cohomology