Contents

Propositions and Theorems

5 Sheaves of Modules

5.1 $\mathcal{O}_X$-modules

Fact. *For any $\mathcal{O}_X$-module $\mathcal{F}$ and any $x \in X$ the stalk $\mathcal{F}_x = \varinjlim \mathcal{F}(U)$ is a module over $\mathcal{O}_{X,x}$*

Lemma. *For any presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the sheafification $\tilde{\mathcal{F}}$ is naturally a sheaf of $\mathcal{O}_X$-modules and it satisfies the analogous universal property:*
For all sheaves of $\mathcal{O}_X$-modules $\mathcal{G}$: $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{\mathcal{F},\mathcal{G}})$

Fact. *for $\mathcal{F}_i$ $\mathcal{O}_X$-modules , $\prod \mathcal{F}_i$ is a sheaf (since $\prod$ is a $\varprojlim$)*

Fact. *$\bigoplus \mathcal{F}_i$ is the sheafification of $U \mapsto \bigoplus \mathcal{F}_i(U)$*
It is a subsheaf of $\prod \mathcal{F}_i$ with equality if $|I| < \infty$

Universal Property. $\mathrm{Hom}_{\mathcal{O}_X}(\bigoplus \mathcal{F}_i, \mathcal{G}) \cong \times \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{G})$ *for all* $\mathcal{O}_X$*-modules* $\mathcal{G}$

Fact. *There is a natural homomorphism* $\mathcal{F}(U) \otimes \mathcal{G}(U) \to \mathcal{F} \otimes \mathcal{G}(U)$

Prop. *Basic Properties:*

1. $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{G} \otimes \mathcal{F}$
2. $(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H} \cong \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H})$
3. $\mathcal{O}_X \otimes \mathcal{F} \cong \mathcal{F}$
4. $(\bigoplus \mathcal{F}_i) \otimes \mathcal{G} \cong \bigoplus (\mathcal{F}_i \otimes \mathcal{G})$

Prop. *for any homomorphism of* $\mathcal{O}_X$*-modules* $\varphi : \mathcal{F} \to \mathcal{G}$ *,* $ker(\varphi)$ *,* $im(\varphi)$ *and* $coker(\varphi)$ *are* $\mathcal{O}_X$*-modules*

5.2 locally free $\mathcal{O}_X$-modules

Prop. $\mathcal{F}, \mathcal{G}$ *locally free of rank* $r, s \Rightarrow \mathcal{F} \oplus \mathcal{G}$ *free of rank* $r + s$ *and* $\mathcal{F} \otimes \mathcal{G}$ *free of rank* $r \cdot s$

Prop.

1. $\mathcal{L}$ *invertible* $\Rightarrow \mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{O}_X$
2. *The set of isomorphism classes of invertible sheaves on* X *form an abelian group* $Pic(X)$ *called the picard group of* X

5.3 $\mathcal{O}_X$-modules on affine schemes

Take $X = \operatorname{Spec} R$ and M an R-module and $\mathcal{B} = \{D_f | f \in R\}$ the basis of the Zariski topology.
For all $f \in R$ set $\tilde{M}(D_f) := M_f = M \otimes_R R_f$
and all $f, g \in R$ with $D_g \subset D_f \rightsquigarrow R_f \to R_g$
$\rightsquigarrow \tilde{M}(D_f) = R_f \otimes_R M \to R_g \otimes_R M = \tilde{M}(D_g)$
$\Rightarrow \tilde{M}$ is a presheaf on $\mathcal{B}$

Prop. $\tilde{M}$ *is a sheaf and extends uniquely to a sheaf of abelian groups on* X*.*
$R \times M \to M$ *induces a scalar* $\mathcal{O}_X \times \tilde{M} \to \tilde{M} \Rightarrow \tilde{M}$ *is an* $\mathcal{O}_X$*-module*

Fact. $\tilde{M}|_{D_f} \cong \tilde{M}_f$
$\forall x \in X : (\tilde{M})_x = \varinjlim \tilde{M}(U) = \varinjlim_{f \in R \setminus f} M_f = M_f$

Functoriality:
Any R-homomorphism $M \xrightarrow{\varphi} N$ induces $id \otimes \varphi : R_f \otimes M \to R_f \otimes N$
$\rightsquigarrow \mathcal{O}_X$-module homomorphism $\tilde{\varphi} : \tilde{M} \to \tilde{N}$
$\Rightarrow$ Functor $((R - mod)) \to ((\mathcal{O}_X - mod))$

Lemma. *For any* $\mathcal{O}_X$*-module* $\mathcal{G}$ *there is a natural isomorphism:*
$\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathcal{G}(X))$
$\psi \mapsto (\psi(X) : \tilde{M}(X) = M \to \mathcal{G}(X))$ *is an* $\mathcal{O}_X(X)$*-module homomorphism i.e. an* R*-mod hom*

Prop. *The Functor $M \to \tilde{M}$ is fully faithful*

Prop. 1. *the functor is exact*

2. *commutes with direct sum*

3. *commutes with tensor product, i.e.* $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \cong (M \otimes_R N)^{\sim}$

Important: $\tilde{M}(X) = M$ so the global sections functor is exact on sheaves of the form $\tilde{M}$

5.4 Quasicoherent $\mathcal{O}_X$-modules

Let X be an arbitrary scheme.

Theorem. *TFAE:*

1. *For all open affine* $U = \operatorname{Spec} R \subset X$ *there exists an* R*-module* M *such that* $\mathcal{F}|_U \cong \tilde{M}$

2. *There exists an open affine covering of such* U

3. $\mathcal{F}$ *is quasicoherent*

4. *for all open affine* $U = \operatorname{Spec} R \subset X$ *and all* $f \in R$ *the homomorphism* $R_f \otimes_r \mathcal{F}(U) \xrightarrow{res,mult} \mathcal{F}(D_f^U)$ *is an isomorphism*

Theorem. 1. *For all homomorphisms of quasicoherent* $\mathcal{O}_X$*-modules , ker, im , coker are quasicoherent*

2. *Direct sum of quasicoherent* $\mathcal{O}_X$*-modules are quasicoherent.*

3. *Let* $\mathcal{F}_i$ *be quasicoherent submodules of a quasicoherent* $\mathcal{O}_X$*-module* $\mathcal{F}$ *then* $\sum \mathcal{F}_i, \bigcap \mathcal{F}_i$ *are quasicoherent.*

4. *if* $\mathcal{F}, \mathcal{G}$ *are quasicoherent then* $\mathcal{F} \otimes \mathcal{G}$ *is quasicoherent and for all open affine* $U : \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \xrightarrow{\sim} (\mathcal{F} \otimes \mathcal{G})(U)$ *is an iso*

Corollary. *The closed subschemes of* X *correspond to the quasicoherent ideal sheaves of* $\mathcal{O}_X$

Theorem. *For* $X = \operatorname{Spec} R$ *we have an equivalence of categories*

$$((R - mod)) \rightleftarrows ((quasicoherent\ \mathcal{O}_X - modules))$$
$$M \longmapsto \tilde{M}$$
$$\mathcal{F}(x) \longleftarrow \mathcal{F}$$

and both functors are exact

5.5 Coherent sheaves

Prop. *For any quasicoherent $\mathcal{O}_X$-module we have the following implications:*

1. *$\mathcal{F}$ is coherent*

   $\Downarrow$

2. *$\mathcal{F}$ is finitely generated*

   $\Updownarrow$

3. *$\forall U \subset X$ open affine $\mathcal{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module*

If X is locally neotherian all are equivalent.

Prop. *Assume X is locally noetherian. Then for all coherent sheaves $\mathcal{F}, \mathcal{G}$: $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ as well as ker, im and coker of any homomorphism are coherent.*

5.6 Functoriality

Consider a morphism $f : X \longrightarrow Y$ and sheaves $\mathcal{F}, \mathcal{G}$ on X and Y respectively.

Prop. *If f is affine* [1] *and $\mathcal{F}$ is quasicoherent then $f_* \mathcal{F}$ is quasicoherent.*

Prop. *If f is finite , X, Y are locally noetherian and $\mathcal{F}$ is coherent then $f_* \mathcal{F}$ is coherent.*

Prop. *f_* naturally extends to a functor which commutes with products and is left exact.*

Prop. *There exists a natural isomorphism $(f^* \mathcal{G})_x \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{G}_{f(x)}$*

Prop. *f^* naturally extends to a functor which commutes with $\oplus$ and is right exact and satisfies:*

$$f^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{H}) \cong (f^* \mathcal{G}) \otimes_{\mathcal{O}_X} (f^* \mathcal{H})$$

Prop. *If $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ and $\mathcal{G} = \tilde{N}$ on Y then $f^* \mathcal{G} \cong (R \otimes_S N)^{\sim}$*

Prop. *If $\mathcal{G}$ is*

- *quasicoherent*
- *finitely generated*
- *free*
- *locally free*
- *coherent (if X, Y are locally noetherian)*

then so is $f^* \mathcal{G}$

Prop. *Adjunction:*

$$\operatorname{Hom}_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F})$$

this is functorial in $\mathcal{F}$ and $\mathcal{G}$.

[1] f is affine if the inverse image of every open affine is open affine

5.7 $\mathcal{O}_X$-modules on a projective scheme

Take $X = \operatorname{Proj} R$ for a graded ring $R = \oplus_{d\geqslant 0} R_d$

Prop. *There exists a functor*

$$((graded\, R - mod)) \longrightarrow ((quasicoherent\, \mathcal{O}_X - module))$$

$$M \longrightarrow \tilde{M}$$

this functor is exact and commutes with $\bigoplus$ and $\bigotimes$

Prop. *1. $\mathcal{O}_X(n)$ is an invertible sheaf*

2. $\tilde{M(n)} \cong \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

Corollary. $\forall m, n \in \mathbb{Z} \, : \, \mathcal{O}_X(m+n) \cong \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$
$\Rightarrow \quad \mathcal{O}_X(n) \cong \mathcal{O}_X(1)^{\otimes n}$ *for all* $n \geqslant 1$
$\mathcal{O}_X(-1) \cong \mathcal{O}_X(1)^\vee$

Lemma. *Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module and $f \in \mathcal{F}(X)$*

1. *If X is quasicompact and $f|_{D_f} = 0$ there exists an $n \geqslant 0$ such that $f \otimes s^n = 0$ in $(\mathcal{F} \otimes \mathcal{L}^{\otimes n})(X)$*
2. *If X is quasicompact and separated for any $g \in \mathcal{F}(D_s)$ there exists n_0 such that $\forall n \geqslant n_0 \qquad g \otimes s^n|_{D_s} \in (\mathcal{F} \otimes \mathcal{L}^n)(D_s)$ extends to an element of $(\mathcal{F} \otimes \mathcal{L}^n)(X)$*

(if $X = \operatorname{Spec} R$ $\mathcal{L} = \mathcal{O}_X$, $\mathcal{F} = \tilde{M}$)

1. *$f = m \in M$ if $\frac{m}{1} = 0$ in M_s then $\exists n \quad m \cdot s^n = 0 \in M$*
2. *Take $n \in M_s$ write $n = \frac{m}{s^{n_0}}$*

Theorem. *For any finitely generated quasicoherent sheaf $\mathcal{F}$ on X:*

1. *$\exists n_0 \, \forall n \geqslant n_0 \quad \mathcal{F}(n)$ is generated by global sections.*
2. *$\exists m \, \exists r \geqslant 0 \, \exists$ surjective homomorphism $\mathcal{O}_X(m)^{\oplus r} \twoheadrightarrow \mathcal{F}$*

<u>Construction:</u>

1. for a graded R-module M and $n \in \mathbb{Z}$ there exists a natural map
   $M_n \longrightarrow \tilde{M}(n)(X) = \widetilde{M(n)}(X)$
   this map is R_0-linear
2. this is multiplicative
3. Set $R' = \bigoplus_{d\geqslant 0} \mathcal{O}_X(d)(X)$. it is a graded ring with a graded ring homomorphism $R \to R'$ and a graded R-module homomorphism $M \to M' = \bigoplus_{n\in\mathbb{Z}} \widetilde{M}(n)(X)$

4. for any $\mathcal{O}_X$-module $\mathcal{F}$ and any $n_0 \in \mathbb{Z} \cup \{-\infty\}$
   set: $M' := \bigoplus_{n \geqslant n_0} \mathcal{F}(n)(X)$
   This is naturally a graded R'-module; hence a graded R-module

Theorem. *For any quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ there is a natural isomorphism*
$\widetilde{M'} \longrightarrow \mathcal{F}$ *for M' as in (4.)*
If $\mathcal{F} = \widetilde{M}$ this is the inverse to the homomorphism
$\widetilde{M} \longrightarrow \widetilde{M'}$ *induced by (3.)*

Corollary. *Every quasicoherent $\mathcal{O}_X$-module on* Proj R *is isomorphic to $\widetilde{M}$ for some graded R-module M*

5.8 Morphisms to projective spaces

Prop. *For all schemes and all $n \geqslant 0$ there is a natural bijection:*

$$\mathrm{Mor}(X, \mathbb{P}^n) \cong \{(\mathcal{L}, l_0, \ldots, l_n) | \mathcal{L} \textit{invertible sheaf on } X.\ l_0, \ldots, l_n \in \mathcal{L}(X) \textit{ generate } \mathcal{L}\}$$

$$\cong \textit{ isos of } \mathcal{L} \xrightarrow{\sim} \mathcal{L}' \textit{ mapping } l_i \mapsto l_i'$$

Prop. *Let $(\mathcal{L}, l_0, \ldots, l_n)$ be as above and let $l_{n+1}, \ldots l_m \in \mathcal{L}(X)$ then we have the following commutative diagram:*

$$\begin{tikzcd}
X \arrow[r, "(\mathcal{L}l_0 \ldots l_n)"] \arrow[dr, "(\mathcal{L}l_0 l_m)"'] & \mathbb{P}^n \\
& \bigcup_{i=0}^n \overline{D}^{p^m}_{X_i} \subset \mathbb{P}^m \arrow[u, "proj"']
\end{tikzcd}$$

Now let X be a scheme over Spec R

Prop. *For X noetherian we have for all very ample sheaves $\mathcal{L}$ and all invertible sheaves $\mathcal{L}'$ on X that is generated by global sections: $\mathcal{L} \otimes \mathcal{L}'$ is very ample.*

Note. *very ample $\Rightarrow$ generated by global sections*

Corollary. *For $\mathcal{L}, \mathcal{L}'$ very ample $\Rightarrow$ $\mathcal{L} \otimes \mathcal{L}'$ and $\mathcal{L}^{\otimes n}$ $\forall n \geqslant 1$ are very ample*

Prop. *TFAE:*

1. *$\mathcal{L}$ ample*
2. *$\forall m \geqslant 1 : \mathcal{L}^{\otimes m}$ ample*
3. *$\exists m \geqslant 1 : \mathcal{L}^{\otimes m}$ ample*

Prop. *very ample $\Rightarrow$ ample*

Theorem. *If X is separated of finite type over* $\operatorname{Spec} R$ *then:*
$\mathcal{L}$ *ample* $\Rightarrow \exists m \geqslant 1 \,:\, \mathcal{L}^{\otimes m}$ *very ample*

Corollary. *same assumptions* $\Rightarrow$ *TFAE:*

1. $\mathcal{L}$ *ample*
2. $\exists m \geqslant 1 \,:\, \mathcal{L}^{\otimes m}$ *very ample*
3. $\exists m_0 \, \forall m \geqslant m_0 \,:\, \mathcal{L}^{\otimes m}$ *very ample*

Note. *very ample is a relative notion for X over* $\operatorname{Spec} R$. *Ample is an absolute notion*
want: true relative notions for X over arbitrary scheme S

Prop. *If $X \to Y$ is separated of finite type with Y noetherian, TFAE:*

1. $\mathcal{L}$ *is relatively ample over Y*
2. $\exists m \geqslant 1 \,:\, \mathcal{L}^{\otimes m}$ *is relatively ample over Y*
3. $\exists m_0 \,:\, \forall m \geqslant m_0 \,:\, \mathcal{L}^{\otimes m}$ *is relatively very ample over Y*

Prop. *let the following diagram be cartesian:*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with all schemes noetherian and f separated of finite type $\mathcal{L}$ invertible on X which is relatively $\{$ample/ very ample$\}$ over Y then $g'^*\mathcal{L}$ is as well over Y'

Lemma. *Take morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z = \operatorname{Spec} R$ which are separated of finite type with R noetherian. Let $\mathcal{L}$ be an invertible sheaf on X which is relatively very ample over Y and let $\mathcal{M}$ be an invertible sheaf on Y which is very ample.*
Then: $\exists l_0 \, \forall l \geqslant l_0$:

1. $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes l} \cong \varepsilon^* \mathrm{pr}_1{}^* \mathcal{O}(1)$ *for some locally closed embedding $\varepsilon : X \hookrightarrow \mathbb{P}^N \times Y$*
2. $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes(l+1)}$ *is very ample for X over S.*

Prop. *Let $f : X \to Y$ be a quasicompact morphism and $\mathcal{L}$ an invertible sheaf on X Let f be separated of finite type with Y noetherian then relatively $\{$ample / very ample$\}$ is equivalent to saying for all open coverings fulfill the respective ampleness property.*

Prop. *Take $X \xrightarrow{f} Y \xrightarrow{g} Z$ separated of finite type with Z noetherian, $\mathcal{L}$ an invertible sheaf on X relatively $\{$ample / very ample$\}$ over Y and $\mathcal{M}$ ditto over Z*
Then $\exists l_0 \, \forall l > l_0 \,:\, \mathcal{L} \otimes f^*\mathcal{M}^{\otimes l}$ is ditto over Z

Prop. *$X \xrightarrow{f} Y \xrightarrow{g} Z$ both $\{$projective/ quasiprojective$\}$ with Z noetherian. Then $g \circ f$ ditto.*

5.9 Divisors

Prop. *$\mathcal{O}_X(D)$ is an invertible sheaf and every invertible sheaf, which is a subsheaf of $\mathcal{K}_X$, arises like this from a unique cartier divisor.*

Prop. 1. *$\forall D, D' \,:\, \mathcal{O}_X(D+D') \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$*

2. *$\forall D \,:\, \mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^\vee$*

Theorem. *The map $D \longmapsto \mathcal{O}_X(S)$ induces an isomorphism*

$$Div\, Cl(X) \longrightarrow Pic(X)$$

Prop. *X regular $\implies$ locally factorial*

Now assume X is noetherian integral and locally factorial

Prop. *For any Cartier divisor $D = ([f_x])_{x\in X}$ on X the sum $cyc(D) := \sum\limits_{\substack{y\in X \\ \dim \mathcal{O}_{X,y}=1}} \operatorname{ord}_y(f_y)\cdot \overline{\{y\}}^{red}$ is finite, so is $\mathrm{Z}^1(X)$.*

Theorem. *there is a group isomorphism $Div(X) \to \mathrm{Z}^1(X) \quad D \mapsto cyc(D)$*

Prop. *For any field k, $\mathrm{Cl}(\mathbb{A}^n_k) = 0 \implies \mathrm{Pic}(\mathbb{A}^n_k) = 0 \; \forall n \geqslant 0$*

Theorem. *For any field k and any $n \geqslant 1$*

$$Cl(\mathbb{P}^n_k) \cong Pic(\mathbb{P}^n_k) \cong \mathbb{Z}$$

generated by $[H] \longleftrightarrow [\mathcal{O}_X(1)]$

Prop. *D is equivalent to an effective divisor iff $\Gamma(X, \mathcal{O}_X(D)) \neq 0$*

Prop. *If X possesses a very ample divisor, every divisor is a difference of very ample divisors.*

Prop. *$\varphi^* : Div(Y) \to Div(X) \quad ([f_y])_y \mapsto ([\varphi^\flat f_{\varphi(x)}])_{x\in X}$
is a well defined homomorphism and $\varphi^*\mathcal{O}_Y(D) \cong \mathcal{O}_X(\varphi^* D)$*

Prop. *If $D$is the effective divisor corresponding to the locally principal closed subscheme $T \subset Y$ then $\varphi^* D$ corresponds to $\varphi^{-1}(T) := X \times_Y T$*

5.10 Differentials

Fact. *$\mathrm{Der}_A(B, M)$ is a B-module by the action of B on M.*

Fact. *Any B-module homomorphism $f : M \to N$ induces a B-module homomorphism $\mathrm{Der}_A(B, M) \to \mathrm{Der}_A(B, N)$ $d \mapsto f \circ d$.*

Fact. *There is a natural bijection
$\mathrm{Der}_A(B, M) \cong \{\varphi : B \to B \ltimes M$ A-algebra homos with $\pi \circ \varphi = \mathrm{id}_B\} \quad d \leftrightarrow (\mathrm{id}, d)$*

Prop. *$(\Omega_{B/A}, d)$ is unique up to unique isomorphism.*

Prop. *I/I^2 with d is a module of differentials of B over A.*

Prop. *$\Omega_{A[X_1,\dots,X_n]/A}$ is a free $A[X_1,\dots,X_n]$ module with basis $dX_1,\dots,dX_n$*

Fact. $\Omega_{A/A} = 0$

Prop. *If $B' = B \otimes_A A'$ then $B' \otimes_B \Omega_{B/A} \xrightarrow{\sim} \Omega_{B'/A'}$*

Prop. *For any multiplicative system $S \subset B$: $S^{-1}\Omega_{B/A} \cong \Omega_{S^{-1}B/A}$.*

Prop. *For any ring homomorphisms $A \to B \to C$ there exists a natural exact sequence $C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$*

Prop. *For all ideals $J \subset B$ set $C := B/J$ $\Rightarrow$ there is a natural exact sequence $J/J^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0$*

Corollary. *If B is a finitely generated A-algebra, then $\Omega_{B/A}$ is a finitely generated B-module.*

Prop. *L/K finite field extension $\Omega_{L/K} = 0 \Leftrightarrow$ separable.*

Prop. *For any local k- algebra B with $B/\Updownarrow \cong k$ there exists a natural isomorphism $\Updownarrow/\Updownarrow^2 \to (B/\Updownarrow) \otimes_B \Omega_{B/k}$*

Theorem. *Let $B = k \oplus \mathfrak{m}$ local ring, k perfect, B is a localization of a finitely generated k-algebra.*
Then $\Omega_{B/k}$ is a free B-module of finite rank $dim(B)$ iff B is a regular local ring.

Prop. *$Y \to X$ locally of finite type $\Rightarrow$ $\Omega_{Y/X}$ finitely generated.*

Prop. $\begin{array}{ccc} Y & \longrightarrow & X \\ \cup & & \cup \\ V & \longrightarrow & U \end{array}$ *open* $\Rightarrow$ $\Omega_{V/U} = \Omega_{Y/X}|_V$

Prop. $\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \uparrow{\scriptstyle g'} & \square & \uparrow \\ Y' & \longrightarrow & X' \end{array}$ $\Rightarrow$ $\Omega_{Y'/X'} \cong g'^*\Omega_{Y/X}$

Prop. $Z \xrightarrow{g} Y \xrightarrow{f} X$ $\Rightarrow$ *exact sequence:* $g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0$

Prop. *If $Z \overset{i}{\hookrightarrow} Y$ is a closed embedding with sheaf of ideals J $\Rightarrow$ exact sequence $J/J^2 \to i^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0$*

Prop. $Y = Y_1 \times_X Y_2$ $\Rightarrow \Omega_{Y/X} \cong \mathrm{pr}_1^*\Omega_{Y_1/X} \oplus \mathrm{pr}_2^*\Omega_{Y_2/x}$

Theorem. *There is a natural short exact sequence for $X := \mathbb{P}^n_Y$:*

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0$$

Theorem. *Any localization of a regular local ring at a prime ideal is regular.*

Theorem. *Let X be connected of finite type over an algebraically closed field k. Then X is regular iff $\Omega_{X/k}$ is locally free of rank $dim(X)$*

Corollary. *X reduced of finite type over k algebraically closed $\Rightarrow$ there is an open dense subscheme $U \subset X$ which is regular*

Theorem. *Let X be regular of finite type over k algebraically closed. Let $Y \subset X$ be the closed subscheme associated to $J \subset \mathcal{O}_X$*
Then Y is regular iff $\Omega_{Y/k}$ is locally free and the second sequence

$$0 \to J/J^2 \to i^*\mathcal{O}_{X/k} \to \Omega_{Y/k} \to 0$$

is also leftexact.
Then J is locally generated by $codim(Y) =: r$ elements and J/J^2 is locally free of rank r

Prop. $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *short exact sequence of locally free sheaves of rank n', n, n''.*
There is a natural isomorphism $\bigwedge^n \mathcal{F} \cong \bigwedge^{n'} \mathcal{F}' \cong \bigwedge^{n''} \mathcal{F}$

Lemma. $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *short exact sequence of locally free sheaves $\Rightarrow$*
There is a natural isomorphism $\bigwedge^{top} \mathcal{F} \cong \bigwedge^{top} \mathcal{F}' \otimes \bigwedge^{top} \mathcal{F}''$

Corollary. $i^*\omega_{X/Y} \cong i^* \bigwedge^{top}(J/J^2) \otimes \omega_{Y/k}$

Prop. *For any nonsingular closed hypersurface $Y \subset \mathbb{P}^n_k$ for k algebraically closed of degree d i.e. $Y = V(f)$ for $f \neq 0$ homogeneous of degree d*
Then $\omega_{Y/k} \cong i^*\mathcal{O}(-n-1+d)$

Corollary. $\omega_{Y/k} \cong \mathcal{O}_Y$ *iff* $d = n+1$

Prop. *Let $Y \subset \mathbb{P}^2_k$ be a nonsingular closed curve of degree d with k algebraically closed. Then:*

1. $\deg \omega_{Y/k} = d(d-3)$
2. $\omega_{Y/k}$ *is ample iff* $d \geqslant 4$
3. $Y \cong \mathbb{P}^1_k$ *iff* $d \leqslant 2$

6 Cohomology

6.1 Some (quick) homological algebra

Fact. *Any homomorphism $f : X \to Y$ induces a homomorphism in homology:*
$H^n(f) : H^n(X) \to H^n(Y)$

Prop. $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ *short exact sequence of complexes $\Rightarrow$ exists natural homomorphisms δ_n yielding a long exact sequence in cohomology:*

$$\ldots \xrightarrow{\delta_{n-1}} H^nX \xrightarrow{H^nf} H^nY \xrightarrow{H^ng} H^nZ \xrightarrow{\delta_n} H^{n+1}X \to \ldots$$

Prop. *f, g homotopic* $\implies \forall n : H^nf = H^ng$

6.2 Cech cohomology

Prop. *The projection $C^\bullet_{alt}(\mathcal{U},\mathcal{F}) \to C^\bullet_{ord}(\mathcal{U},\mathcal{F})$ is an isomorphism of complexes.*

Prop. *The inclusion $C^\bullet_{alt}(\mathcal{U},\mathcal{F}) \overset{\varepsilon}{\hookrightarrow} C^\bullet(\mathcal{U},\mathcal{F})$ is a quasiisomorphism.*

Corollary.

$$H^n(\mathcal{U},\mathcal{F}) \underset{H^n\varepsilon}{\overset{\sim}{\longleftarrow}} H^n(C^\bullet_{alt}(\mathcal{U},\mathcal{F})) \underset{H^n\pi}{\overset{\sim}{\longrightarrow}} H^n(C^\bullet_{ord}(\mathcal{U},\mathcal{F}))$$

Prop. *$H^n(\sigma^*)$ is independent of σ^**

Prop. *If $\mathcal{U}$ and $\mathcal{V}$ are refinements of each other then $H^n(\sigma^*)$ is an isomorphism.*

Prop. *Any homomorphism of sheaves of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ induces a natural homomorphism in cohomology $H^n\varphi : H^n(X,\mathcal{F}) \to H^n(x,\mathcal{G})$*

Now let $\mathcal{F}$ be a quasicoherent sheaf on a scheme X

Theorem. *For X affine $\Rightarrow H^n(X,\mathcal{F}) = 0$ for all $n > 0$*

Prop. *For any closed embedding $i : Y \hookrightarrow X$ and any $\mathcal{O}_X$-module $\mathcal{F}$ and any $p : H^p(Y,\mathcal{F}) \cong H^p(X, i_*\mathcal{F})$*

Theorem. *X projective over* $\operatorname{Spec} A$ *with A noetherian, $\mathcal{L}$ very ample invertible sheaf on X , $\mathcal{F}$ coherent sheaf on X*

1. *$\forall$: $H^p(X,\mathcal{F})$ is a finitely generated A-module*
2. *$\exists m_0\, \forall m \geqslant m_0\, \forall p > 0$: $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$*

Theorem. *Let $\mathcal{F}$ be a quasicoherent sheaf on a separated scheme X. Then for any affine open covering $\mathcal{U}$ of X:*

$$H^p(\mathcal{U},\mathcal{F}) \xrightarrow{\sim} H^p(X,\mathcal{F})$$

Theorem. *Let X be separated which posesses an open covering by $d+1$ open affines. Then $\forall p > d\, \forall \mathcal{F}$ quasicoherent on X : $H^p(X,\mathcal{F}) = 0$*

Prop. *X quasiprojective of dimension d over $k \Rightarrow X$ can be covered by $d+1$ open affines.*

Corollary. *X quasiprojective over k, $\mathcal{F}$ quasicoherent $\Rightarrow \forall p > \dim(X) : H^p(X,\mathcal{F}) = 0$*

Prop. *X separated $0 \to \mathcal{F}' \xrightarrow{\varepsilon} \mathcal{F} \xrightarrow{\pi} \mathcal{F}'' \to 0$ short exact sequence of quasicoherent sheaves $\Rightarrow \exists$ natural long exact sequence in cohomology*

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to H^1(X,\mathcal{F}') \to \dots$$
$$\dots \to H^p(X,\mathcal{F}') \xrightarrow{H^p\varepsilon} H^p(X,\mathcal{F}) \xrightarrow{H^p\pi} H^p(X,\mathcal{F}'') \to H^{p+1}(X,\mathcal{F}')$$

Theorem. *For X separated and quasicoherent: TFAE:*

1. *X is affine*
2. *$\forall \mathcal{F}$ quasicoherent on X : $\forall p > 0$: $H^p(X,\mathcal{F}) = 0$*
3. *dito for $p = 1$*

6.3 Cohomology of projective space

Theorem.

$$H^p(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = \begin{cases} \bigoplus\limits_{\substack{\underline{d}\in\mathbb{Z}^{n+1} \\ all\ d_i \geqslant 0}} A \cdot \underline{X}^{\underline{d}} = A[\underline{X}]_m & \text{if } p = 0 \\ \bigoplus\limits_{\substack{\underline{d}\in\mathbb{Z}^{d+1} \\ \sum d_i = m \\ all\ d_i < 0}} A \cdot \underline{X}^{\underline{d}} & \text{if } p = n \\ 0 & \text{else} \end{cases}$$

Theorem. *Let X be proper over $\operatorname{Spec} A$ for A noetherian and $\mathcal{L}$ an invertible sheaf on X. Then*
TFAE:

1. *$\mathcal{L}$ is ample*
2. *$\forall$ coherent $\mathcal{F} \exists m_0\ \forall m \geqslant m_0\ \forall p \geqslant 1\ :\ H^p(X, \mathcal{F}\mathcal{L}^m) = 0$*
3. *$\forall$ coherent sheaves of ideals $J \subset \mathcal{O}_X\ \exists m_0\ \forall m \geqslant m_0\ :\ H^1(X, J(m)) = 0$*

6.4 Higher direct images

Take $f : X \to Y$ and sheaves of modules $\mathcal{F}$ on X and $\mathcal{G}$ on Y

Fact.
1. *There is a natural homo : $H^p(Y, f_*\mathcal{F}) \to H^p(X, \mathcal{F})$*
2. *There is a natural homo*

$$\begin{tikzcd} H^p(Y,\mathcal{G}) \arrow[rr] \arrow[dr, "H^p(adj)"] & & H^p(X, f^*\mathcal{G}) \\ & H^p(Y, f_*f^*\mathcal{G}) \arrow[ur, "(1.)"'] & \end{tikzcd}$$

3. *Both are compatible with composition and functorial in $\mathcal{F}, \mathcal{G}$*

Prop. *f separated, quasicompact, $\mathcal{F}$ quasicoherent $\Rightarrow$ each $\mathrm{R}^p f_*\mathcal{F}$ is quasicoherent and for all open affine $V \subset Y$:*

$$(\mathrm{R}^p f_*\mathcal{F})(V) = H^p(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$$

Theorem (Direct Image Theorem)**.** *Let f separated, Y locally noetherian, $\mathcal{F}$ coherent $\Rightarrow \forall p\ :\ \mathrm{R}^p f_*\mathcal{F}$ is coherent.*

Fact. *f affine, $\mathcal{F}$ quasicoherent $\Rightarrow \forall p > 0\ :\ \mathrm{R}^p f_*\mathcal{F} = 0$*

Prop. *For $X \xrightarrow{f} Y \xrightarrow{g} Z$ there is a relation between $\mathrm{R}^p g_* \mathrm{R}^p f_*\mathcal{F}$ and $\mathrm{R}^r (g \circ f)_*\mathcal{F}$ (spectral sequences)*

6.5 Duality

Prop. *For $X = \mathbb{P}^n_A$ and all $m \in \mathbb{Z}$ the map*

$$H^0(X; \mathcal{O}_X(m)) \otimes H^n(X, \mathcal{O}_X(-n-1-m)) \longrightarrow H^n(X, \mathcal{O}_X(-n-1)) \cong A$$

is a perfect pairing.

Fact. *The isomorphism $H^n(\mathbb{P}^n_A, \mathcal{O}(-n-1)) \cong A$ is not canonical but after $\mathcal{O}(-n-1) \cong \omega_{\mathbb{P}^n_A/A}$ the isomorphism*

$$H^n(\mathbb{P}^n_A, \omega_{\mathbb{P}^n_A/A}) \cong A$$

is

Prop. *If an r-dualizing sheaf $(\omega_f, \mathrm{tr}_f)$ exists it is unique up to unique isomorphism.*

Theorem. *Y locally noetherian $\implies$ $(\omega_{\mathbb{P}^n_Y/Y, \mathrm{tr}_f})$ is an n-dualizing sheaf for $f : \mathbb{P}^n_Y \longrightarrow Y$*

Theorem. *For any projective morphism $f : X \to Y$ with Y locally noetherian and all fibers of $\dim \leqslant r$ the r-dualizing sheaf for f exists.*

Theorem. *$f : X \to Y$ projective smooth of relative dimension r , Y locally noetherian $\implies$ $\omega_f := \omega_{X/Y} := \Lambda^r \Omega_{X/Y}$ together with some tr_f is an r-dualizing sheaf.*

6.6 Flat base change

Prop. *$\widetilde{M}$ on $\operatorname{Spec} A$ is flat iff M is flat over A.*

Prop.

1. *$\mathcal{F}$ locally free $\Rightarrow$ $\mathcal{F}$ flat*
2. *$\mathcal{F}$ finitely generated , X locally noetherian*

Prop. *$f : \operatorname{Spec} B \to \operatorname{Spec} A$ is flat iff B is a flat A-module.*

Prop. $\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$ *For a cartesian diagramm with f separated and quasicompact and $\mathcal{F}$ quasicoherent and g flat then base change is an isomorphism.*

Prop. *$f : X \to Y$ projective all fibers of dimension $\leqslant d$; $\mathcal{F}$ quasicoherent $\Rightarrow$*

1. *$\forall p > d\ :\ \mathrm{R}^p f_* \mathcal{F} = 0$*
2. *$\mathrm{R}^d f_*$ is rightexact and commutes with base change*

7 Riemann-Roch and Serre duality

7.1 Divisors on curves

Let X be regular integral projective scheme of dim 1 over a field k

Prop. *If D is the effective divisor associated to a finite closed subscheme*

1. $T \subset X$ *then* $D = \sum_{P \in T} length(\mathcal{O}_{T,P}) \cdot P$
2. $\deg_k(D) = \sum_{P \in T} length(\mathcal{O}_{T,P}) \cdot [k(P)/k] = \dim_k \Gamma(T, \mathcal{O}_T) =: \deg T$

Prop. *If k is a finite extension of k'* $\Rightarrow \deg_{k'}(D) = \deg_k(D) \cdot [k/k']$

Prop. *For any field extension k'/k. Let $X' := X_{k'} = X \times_{\operatorname{Spec} k} \operatorname{Spec} k' \xrightarrow{\pi} X$. Assume X' integral then*

$$\deg_{k'}(\pi^* D) = \deg_k(D)$$

Lemma. *For all $P, Q \in |X|$ $\exists R \in |X|$: $P + Q + R \sim 3P_0$*

Theorem. *The map $|X| \to Cl^0(X)$ $\quad P \mapsto [P - P_0]$ is bijective.*

7.2 Riemann-Roch

Prop. *For any short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of coherent sheaves*

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'')$$

Lemma. *For any exact sequence of finite dimensional k-vector spaces V^i for $i \in \mathbb{Z}$ and almost all zero:*

$$\sum_i (-1)^i \dim_k V^i = 0$$

Fact. *$\forall L/k$ field extension: $X_L \xrightarrow{\pi} X \rightsquigarrow \chi(X, \mathcal{F}) = \chi(X_L, \pi^* \mathcal{F})$*

Theorem (Riemann-Roch Version 1). *For any invertible sheaf$\mathcal{L}$ on X:*

$$\chi(X, \mathcal{L}) = 1 - g + \deg(\mathcal{L})$$

Theorem (Serre Duality). *The sheaf $\Omega_{X/k}$ is a dualizing sheaf for X over k i.e. there exists a natural* $\operatorname{tr} : H^1(X, \Omega_{X/k}) \to k$ *such that for all quasicoherent $\mathcal{F}$ we have perfect duality*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \Omega_{X/k}) \times H^1(X, \mathcal{F}) \longrightarrow H^1(X, \Omega_{X/k}) \xrightarrow{\operatorname{tr}} k$$

Theorem (Riemann-Roch Version 2). *$\mathcal{L}$ invertible sheaf on X* $\implies$

$$h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}^\vee \otimes \Omega_{X/k}) = 1 - g + \deg(\mathcal{L})$$

7.3 Residues

Prop. $\forall \omega \in \widehat{\Omega}_{\mathcal{F}/k} \ : \ \mathrm{res}_t(\omega) = \mathrm{res}_t(\omega \circ \varphi)$

Prop. $\forall \omega \in \widehat{\Omega}_{\mathcal{E}/k} \ : \ \mathrm{res}_u(\omega) = \mathrm{res}_t(\mathrm{tr}_{\mathcal{E}/\mathcal{F}}\omega)$

Prop. *The composite map* $\Omega_{K/k} \hookrightarrow \widehat{\Omega}_{\mathcal{F}/k} \xrightarrow{\mathrm{res}_t} k$ *is independent of* t

Prop. *Let* $\varphi : X \to Y$ *be a finite separated morphism of such curves corresponding to the finite separated extension of function fields* K/L.

Define $\mathrm{tr}_{K/L}$ *:*

$$\begin{tikzcd}
\Omega_{K/k} \arrow[r] \arrow[d, phantom, "\cong"] & \Omega_{L/K} \\
K \otimes_L \Omega_{K/L} \arrow[ur, "\mathrm{tr}_{K/L} \otimes \mathrm{id}_{\Omega_{L/k}}"'] &
\end{tikzcd}$$

Then $\forall \omega \in \Omega_{K/k}\ \forall Q \in |Y|$

$$\sum_{P \in \varphi^{-1}(Q)} \mathrm{res}_P(\omega) = \mathrm{res}_Q(\mathrm{tr}_{K/L}\omega)$$

Theorem (Residue Theorem)**.**

$$\forall \omega \in \Omega_{K/k} \ : \ \sum_{P \in |X|} \mathrm{res}_P \omega = 0$$

7.4 Serre duality

Theorem (Serre Duality)**.** *The sheaf* $\Omega_{X/k}$ *is a dualizing sheaf for* X *over* k *i.e. there exists a natural* $\mathrm{tr} : H^1(X, \Omega_{X/k}) \to k$ *such that for all quasicoherent* $\mathcal{F}$ *we have perfect duality*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \Omega_{X/k}) \times H^1(X, \mathcal{F}) \longrightarrow H^1(X, \Omega_{X/k}) \xrightarrow{\mathrm{tr}} k$$

7.5 Some concequences of Riemann-Roch

Now always: X smooth connected projective curve over k algebraically closed field.
<u>Abbreviate</u>: $h^i(\mathcal{L}) := h^i(X, \mathcal{L})$ $\Omega_X := \Omega_{X/k}$

Prop. $\deg(\Omega_X) = 2g - 2$

Prop. *Any smooth closed curve of degree* d *in* $\mathbb{P}^2_k$ *has* $g = \frac{(d-1)(d-2)}{2}$

Prop. *TFAE:*

1. $X \cong \mathbb{P}^1_k$
2. $g = 0$
3. $\deg \Omega_X < 0$

4. *$\forall \mathcal{L}$ of degree 1: $h^0(\mathcal{L}) > 1$*

5. *$\exists \mathcal{L}$ of degree 1: $h^0(\mathcal{L}) > 1$*

Prop. 1. *$\forall P \in |X| \, \exists f \in K(X)$ f has a pole at P and no ther pole*

2. *$\forall P \in |X| : X \backslash \{P\}$ is affine*

3. *$\exists$ finite morphism $X \to \mathbb{P}^1_k$ of degree $\leqslant g+1$*

Prop. *Any locally free coherent sheaf on $X = \mathbb{P}^1_k$ is isomorphic to $\bigoplus_{i=1}^r \mathcal{O}_X(n_i)$ for unique $n_1 \geqslant \cdots \geqslant n_r$*

7.6 Embeddings in projective space

Let $\mathcal{L}$ be an invertible sheaf on X.

Prop. 1. *$\mathcal{L}$ is generated by global sections iff $\forall P \in |X| : h^0(\mathcal{L}(-P)) = h^0(\mathcal{L}) - 1$*

2. *$\mathcal{L}$ is very ample iff : $\forall P, Q \in |X| : h^0(\mathcal{L}(-P-Q)) = h^0(\mathcal{L}) - 2$*

Fact. $\forall \mathcal{L}$:
$\deg(\mathcal{L}) < 0 \Rightarrow h^0(\mathcal{L}) = 0$
$\deg(\mathcal{L}) > 2g-2 \Rightarrow h^1(\mathcal{L}) = h^0(\mathcal{L}^\vee \otimes \Omega_X) = 0$

Prop. *$\forall \mathcal{L}$:*

1. $\deg(\mathcal{L}) \geqslant 2g \Rightarrow$ *$\mathcal{L}$ is generated by global sections.*

2. $\deg(\mathcal{L}) \geqslant 2g+1 \Rightarrow$ *$\mathcal{L}$ is very ample.*

Corollary. *$\mathcal{L}$ is ample iff* $\deg(\mathcal{L}) > 0$

Prop. *$\forall g \geqslant 1 : \Omega_X$ is generated by global sections.*

Corollary. *$g \geqslant 1 \Rightarrow$ get the canonical morphism $X \to \mathbb{P}^{g-1}_k$ (unique up to $PGl_g(k)$)*
If $g \geqslant 2$ this is non constant and hence finite

Prop. *X is hyperelliptic iff $\exists \mathcal{L}$ with $h^0(\mathcal{L}) = \deg(\mathcal{L}) = 2$*

Prop. *For $g \geqslant 2$ the canonical morphism $X \to \mathbb{P}^{g-1}_k$ is a closed embedding iff X is not hyperelliptic*

7.7 Hyperelliptic curves

Prop. $\Omega_{X/k} = f^*\mathcal{O}_{\mathbb{P}^1_k}((g-1)\infty) \cdot \omega$ *and*
$H^0(X, \Omega_{X/k}) = H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}((g-1)\infty)) \cdot \omega$ *for* $\omega := \frac{dx}{2y-a(x)} \in \Omega_{K/k}$ *for* $K = k(X)$

Corollary. *The canonical morphism factors as $(g-1)$-uple embedding if $g \geqslant 2$*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & \mathbb{P}^{g-1}_k \\ \mathbb{P}^1_k \arrow[ur, dashed] & \end{tikzcd}$$

Prop. *char$k \neq 2$:*

1. *for any separated $c \in k[X]$ of degree $\{2g+1, 2g+2\}$ the equation $y^2 = c(x)$ defines a hyperelliptic curve of genus g.*
2. *in particular there exists a hyperelliptic curve of any genus $\geqslant 1$*

7.8 Coverings

Let $f : X \to Y$ be a finite separated morphism of smooth irreducible projective curves over k algebraically closed

Prop. $\forall Q \in |Y|$: $\sum_{P \in f^{-1}(Q)} e_p = \deg(f)$

Prop. *exact sequence*

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

Prop. $\forall P \in |X|$:

$$length(\Omega_{X/Y,P}) = \begin{cases} e_p - 1 & if\ char(k) \nmid e_p \\ \geqslant e_p & if\ char(k) | e_p \end{cases}$$

Theorem (Hurwitz)**.**

$$2g(X) - 2 = \deg(f) \cdot (2g(Y) - 2) + \deg(\mathrm{Ram}_f)$$

Prop. *$Aut_k(X)$ is finite.*