On Theorem by Moore about Vanishing Matrix Coefficients

Nicolas Trutmann

Abstract

In this paper we'll showcase a theorem in ergodic theory by Howe and Moore [1]. On the way there, we'll touch many different fields, from measure theory, over functional analysis, representation theory and ergodic theory of course.

Contents

This paper is based on the book “Ergodic Theory and semisimple Lie Groups” by Robert Zimmer [7], in particular the first two chapters, which contain the theorem itself (Theorem 2.2.20) and surrounding material concerning ergodic theory.

The main aim of the book by Zimmer is focused on two theorems by Mostow and Margulis. The “arithmeticity theorem” and the “rigidity theorem”, which show how Lie groups and lattices in them interact.

The techniques of the proof show a nice interplay between fields and their different approaches, while staying relatively simple. We assume the reader to have an undergraduate level understanding of the prerequisites in algebra and representation theory, but will state foundational information regardless, and provide references in all cases. We furthermore take care to clarify notation before use.

The theorem, which we will state shortly, is historically at home in the development of ergodic theory, which in turn is a relatively new field of mathematics. The original definition of ergodicity was given in 1928 in a paper by P. Smith and G. Birkhoff on dynamical systems. The concept gained importance in 1931 when von Neumann and Birkhoff nearly simultaneously proved the mean and pointwise ergodic theorems. These may be regarded as the starting point of the subject.

The paper by Moore [6] was published in 1966. Margulis’ Theorems were published in Initially for dynamical systems, with physics applications, here however actions of more general groups are studied with respect to ergodicity.

Sources for the historical background: [4](chapter 1. Introduction) [7](chapter 1. Introduction)

The theorem itself does not directly involve ergodicity, but is instead used to prove ergodicity.

The theorem itself is rather simple to state:

[[Moore’s Ergodicity Theorem]]

To clarify some points, note that we have specified non-compact groups. This allows us to talk about “infinity” at all. Next, what is an invariant vector? Simply, for all $g \in G$, and a vector v, we have that $\pi(g)v = v$, or, that v is preserved by any linear map given by the representation.

Introduction

- historical context -> up in first section. maybe move down
- where this theorem comes from -> [1]
- what it does
- why we care
- how we’re gonna go about it

question: when is an action ergodic?

Instead of verifying ergodicity for any given action, space and measure individually, can we find criteria for ergodicity that are easier to evaluate? The Moore’s theorem sits in the middle of an argument that answers the following question.

Let G be a semisimple Lie group and S an ergodic G-space. If $H \subset G$ is a closed subgroup, when is H ergodic on S.

action, lattices in ss groups, asymptotic behavior in non-compact groups [1] Now that we have a concrete question, let us try to get our hands dirty on an example. We'll use the action of fractional linear transforms on the upper half plane, which is nice, because we can look at hyperbolic geometry and draw meaningful pictures of the maps and spaces involved. It'll bring intuition about the question and why one would care to answer the question.

I get the first map now. The action, let's name it for now, $\alpha : SL(2,\mathbb{R}) \curvearrowright \mathbb{H} \to \mathbb{H}$, wich acts by fractional linear transform. ## Lemma 1. $K := SO(2,\mathbb{R})$ is the stabilizer of $i \in \mathbb{H}$. 2. therefore, $G/K \cong AN$ with $KAN \cong G$ being the Iwasawa decomp.

proof

1. from [5](Theorem 1.1.3) map to Klein disk; use Schwarz lemma; map back.

How does the second map work? Using the same fractional linear transform but we take a real value instead of a complex one. It is easy to visualize as a regular matrix product with $\begin{pmatrix} x \\ 1 \end{pmatrix}$ and projecting it to the projective line.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ cx+d \end{pmatrix} \quad \to \quad \begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix}$$

next we care about the behavior of a lattice $\Gamma \subset G$. If G acts transitively on a space X, then there is an isomorphism of G-spaces $G/G_x \to X$, where $G_x = Stab_G(x)$ for $x \in X$, given by the map $gG_x \mapsto gx$. In the case of our example $G = SL(2,\mathbb{R})$, and, as we've shown in the preceding lemma, we know the stabilizer of i to be $SO(2,\mathbb{R})$. ## where we want to go We want to show that the action of Γ on $\bar{\mathbb{R}}$ is ergodic

from book

[unoriginal] To see why ergodicity is relevant, and in fact to say a word about what it is, let us consider a classical example. Let $G = SL(2,\mathbb{R})$, and let X be the upper half plane, $X = \{z \in \mathbb{C} | lm(z) > 0\}$. As is well known[todo], G acts on X via fractional linear transformations, i.e.,

$$g \cdot z = \frac{(az+b)}{(cz+d)} \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Suppose now that $\Gamma \subset G$ is a lattice, which we assume to be torsion free for simplicity. Since the action of G on X allows an identification of X with G/K, where $K = SO(2)$ (the stabilizer of $i \in X$), and K is compact, it follows that the action of Γ on X is properly discontinuous, and so $\Gamma\backslash X$ will be a manifold, in fact a finite volume Riemann surface. On the other hand, via the same fractional linear formula, G acts on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and $\bar{\mathbb{R}}$ can be identified with G/P, where P is the group of upper triangular matrices and the stabilizer of $\infty \in \bar{\mathbb{R}}$. Once again, we can consider the action of Γ on $\bar{\mathbb{R}}$, but now the action will be very far from being properly discontinuous. In fact, every Γ-orbit in $\bar{\mathbb{R}}$ will be a (countable) dense set. In particular, if we try taking the quotient $\Gamma\backslash\bar{\mathbb{R}}$, we obtain a space with the trivial topology. On the other hand, $\bar{\mathbb{R}}$ provides a natural compactification of X, and in fact $\bar{\mathbb{R}}$ can be identified with asymptotic equivalence classes of geodesics in X, where X has the essentially unique G-invariant metric. Thus, it is certainly reasonable to expect the action of Γ on $\bar{\mathbb{R}}$ to yield useful information. However, a thorough understanding requires us to come to grips with actions in which the orbits are very complicated (e.g. dense) sets. Ergodic theory is (in large part) the study of complicated orbit structure in the presence of a measure. Not only are there no non-constant Γ-invariant continuous real-valued functions on $\bar{\mathbb{R}}$, but the same is true for measurable functions. This is embodied in the following definition.

Definition

Suppose G acts on a measure space (S, μ) so that the action map $S \times G \to S$ is measurable and μ is quasi-invariant, i.e., $\mu(A) = 0$ if and only if $\mu(Ag) = 0$. The action is called ergodic if $A \subset S$ is measurable and G-invariant implies $\mu(A) = 0$ or $\mu(S \setminus A) = 0$.

Definitions and Notation

Now that we have stated the goal of the paper, let us immediately make a detour. We will state definitions and relevant theorems (without proof) in compact form with ample references so that a reader can catch up if necessary. The advanced reader can skip this section and move straight to the next topic without issue.

[todo] (put references for everything in each section)

Throughout the whole text, unless otherwise stated, G is a countable discrete group. Its identity element will always be denoted by e.

Measure Spaces

A *measurable space* is a pair $(X, \mathscr{B})$ where X is a set and $\mathscr{B}$ is a σ-algebra of subsets of X. Elements of $\mathscr{B}$ are called *measurable sets.* A function of measurable spaces $f : X \to Y$ is called *measurable* if $f^{-1}(A)$ is a measurable set in X for all measurable sets A of Y.

A *measure* on a measurable space $(X, \mathscr{B})$ is a map $\mu : \mathscr{B} \to [0, \infty]$ such that - $\mu(\emptyset) = 0$, and - $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for every countable collection $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathscr{B}$ (countable additivity).

The Borel σ-algebra of a topological space X is the σ-algebra $\mathscr{B}$ generated by the open subsets of X , and the members of $\mathscr{B}$ are called Borel sets (we may also refer to them as measurable sets if we are viewing $(X, \mathscr{B})$ abstractly as a measurable space). A Borel measure on X is a probability measure on the Borel σ-algebra of X.

Representations

The notation(s) in representation theory are sometimes confusing, so here we clarify which words we will use to mean which objects. We will revisit representations in detail in the following chapter, so we will be brief.

Definition 1 *A representation is a group-homomorphism from a group into the general linear group of a vector space.*

$$\pi : G \to GL(V)$$

We consistently use lowercase Greek letters to refer to representations. Most often π and λ.

The vector space V is often not just a vector space but a topological vector space and in particular a Hilbert space.

[todo] (all of this) repr: a map dim of a repr agree with topology. unitary repr. A unitary representation

"direct difference" notation

Zimmer, and we, use the symbol "$\ominus$" to denote "subtraction" of linear subspaces of Hilbert spaces. If $A \subset B$ are linear subspaces of a Hilbert space, $B \ominus A = \{x \in B : (x, y) = 0 \text{ for all } y \in A\}$. ## Group Actions By an action of the group G on a set X we mean a map $\alpha : G \times X \to X$ such that, writing the first argument as a subscript, $\alpha_s(\alpha_t(x)) = \alpha_{st}(x)$ and $\alpha_e(x) = x$ for all $x \in X$ and $s, t \in G$. Most of the time we will not give this map a name and write the image of a pair (s, x) written as sx, or as $s \cdot x$ if there is a chance of notational confusion. For sets $A \subset X$ and $K \subset G$ and an $s \in G$ we write

$$sA = \{sx : x \in A\}, \quad Kx = \{sx : s \in K\}, \quad KA = \{sx : x \in A \text{ and } s \in K\}.$$

The *G-orbit* of a point $x \in X$ is the set Gx.

Definition 2 *quasi-invariant A measure μ is quasi-invariant under a group action of G if it preserves null sets. If $A = gA$ then either $\mu(A) = 0$ or $\mu(S \setminus A) = 0$.*

Ergodicity

We have successfully made our way back to ergodicity. We will try to illuminate the definition a bit by examples and non-examples.

To reiterate

Definition 3 *Ergodicity For a group G, a measurable separable space S, and a G-invariant measure μ. An action is called ergodic if all G-invariant subsets $A \subset S$ are either null or conull. Which means*

$$\forall g \in G : \; gA = A \quad \Rightarrow \quad \mu(A) = 0 \text{ or } \mu(S \setminus A) = 0$$

definition; explanation of definition; Examples; why the prerequisites come in, like quasi-invariance; clarify edge cases. summarize by "complicated orbits" argument (could use 2.1.7 as example of complicatedness).

The Direct Integral and Unitary Representations

what do we need actually? We have to take a detour into unitary representations and define the direct integral to make statements about certain subgroups. These lead to a theorem (Zimmer 2.2.5) about vanishing matrix coefficients, which we will use to prove the central theorem in question. This is a great example of the usefulness of representation theory, where we transform a problem of groups to a problem of linear algebra. So instead of asking about invariant vectors of a group action we look at the behavior of matrices.

The way there will lead us through the direct integral, unitary representations and in particular the representation of $\mathbb{R}^n$,

The Direct Integral

In simple terms, the direct integral is a way to patch together locally defined functions into a function on the whole domain. Let us first consider the simple case where we have global functions on a measure space M, that takes values in some Hilbert space $\mathscr{H}$, $f : M \to \mathscr{H}$. The 'sensible' space to put these functions into is the space of square integrable functions on M, denoted $L^2(M, \mathscr{H})$. The word 'sensible' here is justified by being again a Hilbert space by integration $\langle f, g\rangle = \int_M \langle f(x), g(x)\rangle$.

The next step towards locality is to use two function, by defining $L^2(M_1 \sqcup M_2, \mathscr{H}_1 \oplus \mathscr{H}_2)$, where every function is defined separately on each M_i, and taking values in $\mathscr{H}_i$.

clear. and say that the intuition works the same later on)

Suppose we have a measure space M, and for each $x \in M$ a Hilbert space $\mathscr{H}_x$ such that $x \mapsto \mathscr{H}_x$ is piecewise constant, that is, we have a disjoint decomposition of M into $\cup_{i=1}^{\infty} M_i$ such that for $x, y \in M_i$, $\mathscr{H}_x = \mathscr{H}_y$. Interesting aside: the condition that the assignment $x \mapsto \mathscr{H}_x$ be piecewise constant is not necessary. We can allow the Hilbert spaces to be arbitrary, and in fact uncountably infinite. Short answer: magic; slightly less short answer: von Neumann. A *section* on M is an assignment $x \mapsto f(x)$, where $f(x) \in \mathscr{H}_x$. Since $\mathscr{H}_x$ is piecewise constant, the notion of measurability carries over in an obvious manner, namely that a measurable function on M is measurable on each M_i into the appropriate Hilbert space. Let $L^2(M, \{\mathscr{H}_x\})$ be the set of square integrable sections $\int \|f\|^2 < \infty$ where we identify two sections if they agree almost everywhere. This set is then also a Hilbert space with the inner product $\langle f|g\rangle = \int_M \langle f(x)|g(x)\rangle$.

Suppose now we have for each $x \in M$ a unitary representation π_x of a group G on $\mathscr{H}_x$. We say this is measurable when for $g \in G$, $\pi_x(g)$ is a measurable function on each $M_i \times G$.

This allows us to define the relevant representation we intermediately care about.

Unitary Representations

irreducible unitary representations to understand the action(s) of $SL(n, \mathbb{R})$.

Theorem

Theorem 1 (Zimmer 2.3.3)
- *For any unitary representation π of $\mathbb{R}^n$, there exist $\mu, \mathscr{H}_\lambda$, on $\hat{\mathbb{R}}^n$ such that $\pi \cong \pi_{\mu,\mathscr{H}_\lambda}$.*
- *$\pi_{\mu,\mathscr{H}_\lambda}$ and $\pi_{\mu',\mathscr{H}'_\lambda}$ are unitarily equivalent if and only if*
  - *$\mu \sim \mu'$, i.e., they are in the same measure class*
  - *and $dim\mathscr{H}_\lambda = dim\mathscr{H}'_\lambda$ a.e.*

Theorem

Theorem 2 (Zimmer Proposition 2.3.5, from [3]) *Suppose $\mathbb{R}^n \subset G$ is a normal subgroup and π is a unitary representation of G. Write $\pi|\mathbb{R}^n \cong \pi_{(\mu,\mathscr{H}_\lambda)}$ for some $(\mu, \mathscr{H}_\lambda)$ by 2.3.3. Then*

- *μ is quasi-invariant under the action of G on $\hat{\mathbb{R}}^n$.*
- *If $E \subset \mathbb{R}^n$ is measurable, let $\mathscr{H}_E = L^2(E, \mu, \{\mathscr{H}_\lambda\})$. Then $\pi(g)\mathscr{H}_E = \mathscr{H}_{g \cdot E}$*
- *If π is irreducible, then μ is ergodic and $dim\mathscr{H}_\lambda$ is constant on a μ-conull set.*

proof 1 *### proof*

Representation of R^n

All the irreducible unitary representations of $\mathbb{R}^n$ are one-dimensional.

It turns out that the group unitary representations on $\mathbb{R}^n$ are isomorphic to $\mathbb{R}^n$. So we define a map from $\mathbb{R}^n$ to $\mathcal{U}(\mathbb{C})$ and show that it's in fact bijective. Let θ. t be in $\mathbb{R}^n$ and let $\lambda_\theta(t) = e^{i\langle\theta|t\rangle}$. This is in fact a unitary automorphism on $\mathbb{C}$ by multiplication. To clarify, for every $\theta \in \mathbb{R}^n$ we have a representation given by

$$\begin{aligned} \lambda_\theta : \mathbb{R}^n &\to \mathcal{U}(\mathbb{C}) \\ t &\mapsto e^{i\langle\theta|t\rangle} \end{aligned}$$

We denote the group of representations by $\hat{\mathbb{R}}^n$. It is in fact a group under pointwise multiplication.

This definition is maybe a bit dense, so here is the assignment formatted in pseudo code. Note here that lambda denotes the programming term of a lambda function, an unfortunate notation collision.

$$\begin{array}{l} \text{func } \ \pi_{\mu,\mathscr{H}_\lambda}(t : \mathbb{R}^n) \to \mathcal{U}(L^2(\hat{\mathbb{R}}^n)) \ \{ \\ \qquad \text{return lambda}(f : \ L^2(\hat{\mathbb{R}}^n)) \to L^2(\hat{\mathbb{R}}^n) \ \{ \\ \qquad\qquad \text{return lambda}(\lambda : \ \hat{\mathbb{R}}^n) \to \mathscr{H}_\lambda \ \{ \\ \qquad\qquad\qquad \text{return } \lambda(t) f(\lambda) \\ \qquad\qquad \} \\ \qquad \} \\ \} \end{array}$$

The Connection between Ergodicity and Unitary Representations

approach: - char func - char func in L2(S) and non-trivial - if A invariant then char func invariant as a vector in L2(S) - due diligence: make sure measure works

To see why we care about unitary representations at all if we really want ergodicity, we neeedd to make the folllowing connection. We use the characteristic function of a set to connect the set to a vector in $L^2(S)$. The characteristic function of a subset $A \subset S$, is defined as $\chi_A(x) = 1$ for $x \in A$ and 0 otherwise.

This representation allows us to pass from talking about sets to talking about vectors, while retaining the properties we care about.

Theorem 3 () *An action $G \curvearrowright S$, with **finite** invariant measure is ergodic on S if and only if the restriction of the above representation to in $L^2(S) \ominus \mathbb{C}$ has no invariant vectors.*

Since S has finite measure, assume $\mu(S) = 1$.

proof 2 *"$\Leftarrow$": Proof by contrapositive: If $A \subset S$ is G-invariant with measure $0 < \mu(A) < \mu(S) = 1$ then χ_A is also G-invariant in $L^2(S)$ as well as the projection $\chi_A - \mu(A) \cdot 1$ in $L^2(S) \ominus \mathbb{C}$. Therefore there exists an invariant vector in $L^2(S) \ominus \mathbb{C}$. "$\Rightarrow$": ([2](Prop 2.7)) Suppose the action is ergodic and $f \in L^2(S) \ominus \mathbb{C}$ is G-invariant. We can find a measurable set $D \subset \mathbb{C}$ such that $0 < \mu(f^{-1}(D)) < 1$ and denote $\widetilde{A} = f^{-1}$. Now we verify ergodicity. For every $g \in G$ the symmetric difference $g\widetilde{A}\Delta\widetilde{A}$, for which all points are in the set $\{x \in X | \ |f(x) - sf(x)| > 0\}$, which has measure zero because $\|f - sf\|_2 = 0$. Therefore the action fails to be ergodic.*

The adjective “finite” on the measure is necessary, because for a set A of infinite measure the statement is no longer true as χ_A will no longer be in L^2.

If $A \subset S$ is G-invariant then $\chi_A \in L^2(S)$ will also be G-invariant. For A neither null nor conull then χ_A, $f_A \neq 0$, where f_A is the projection of χ_A onto $L^2(S) \ominus \mathbb{C}$.

Proof for $SL(2, \mathbb{R})$

We start here because it is an easy example of the theorem and a general group G has many subgroups locally isomorphic to $SL(2, \mathbb{R})$. Later we extend the proof, first to $SL(n, \mathbb{R})$ and then to a general G.

To state our intentions: we first show that either the matrix coefficients vanish as we want, or there exist invariant vectors. Then we show that there are no invariant vectors, completing the statement.

We're going to use the following decomposition, which we take for granted The so called Iwasawa decomposition of $SL(2, \mathbb{R})$ into three matrices K, A, and N, defined as

$$\begin{aligned}
K = &\quad \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \subset SL(2, \mathbb{R}) \mid \theta \in \mathbb{R} \right\} && (1) \\
A = &\quad \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \subset SL(2, \mathbb{R}) \mid r > 0 \right\} && (2) \\
N = &\quad \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \subset SL(2, \mathbb{R}) \mid x \in \mathbb{R} \right\} && (3) \\
& && (4)
\end{aligned}$$

We look at the subgroup

$$P \subset SL(2, \mathbb{R}) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

of upper triangular matrices. Together with the lower diagonal matrices $\bar{P}$, they generate $SL(2, \mathbb{R})$. To see this, decompose as follows:

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & \beta x \\ \alpha x & \alpha\beta x + 1/x \end{pmatrix}$$

For any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \mathbb{R})$ with matrix coefficient $a \neq 0$, we can solve for x, α, β. In the case of $a = 0$ we can use the following construction:

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + \beta\gamma & \delta(1 + \beta\gamma) + \beta \\ \alpha(1 + \beta\gamma) + \gamma & \alpha\delta(1 + \beta\gamma) + \alpha\beta + \gamma\delta + 1 \end{pmatrix}$$

If $1 + \beta\gamma = 0$, the above product becomes $\begin{pmatrix} 0 & \beta \\ \gamma & 1 + \alpha\beta + \gamma\delta \end{pmatrix}$ and we can make suitable choices for $\alpha, \beta, \gamma, \delta$ to construct A.

Theorem for P

The upper triangular group can be decomposed into

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = P = AN = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

Theorem (Zimmer 2.3.6) Let π be a unitary representation of $P = AN$. Then either - $\pi|N$ has a nontrivial invariant vector or - The matrix coefficients of $\pi(g)$ as $g \to \infty$.

Note first, that N is normal in P. To see this, first calculate that the inverse of a matrix $\left(\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix}\right)$ in P is $\left(\begin{smallmatrix} a^{-1} & -x \\ 0 & a \end{smallmatrix}\right)$. Next note that the result of conjugation with an element in P is again in N: $\left(\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a^{-1} & -x \\ 0 & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a^2 x \\ 0 & 1 \end{smallmatrix}\right)$. This defines a group action $P \curvearrowright N \to N$ by multiplication with a^2.

proof 3

We apply 2.3.5, identifying N ~ IR. Let n IN = n<ll.K ,> · If Jl($\{0\}$) > 0, then n IN has invariant vectors (namely Jt' 0). We now show that if Jl($\{0\}$) $= 0$, then assertion (ii) in the theorem is satisfied. To see this, consider the action of P on N. An elementary calculation shows thatErgodic theory and semisimple groups 28 acts on fJ ~ IR via multiplication by a 2 • Hence, given any compact subsets E, F c IR- $\{0\}$, for gEA outside a sufficiently large compact set we have Jl.(gE n F) = 0. Given any two unit vectors f, hE L 2 (1R, Jl., { Jf;.}), and e > 0 we can choose compact subsets E, F c IR- $\{0\}$ such that Then I< n(g)fl h) I :'£ 2e + I(n(g)(XEf)I(XF" h))l. But n(g)(xEf)EJf9 E by 2.3.5 (ii) and by our above remark, choosing gEA outside a sufficiently large compact subset of A we can ensure Jf gE .1 Jt' p, and hence that I< n(g)fl h) I ;£ 2e. This completes the proof of the theorem. Theorem 2.3.6 gives a vanishing theorem for the matrix coefficients of repre- sentations of P. In the next section we will see how to use this to prove Moore's theorem.

Theorem for Cartan decomposition

Polar decomposition to Cartan

$T = US$ for some unitary U and a sym pos def S. S can be diagonalized into $U_0 D U_0^{-1}$ so we can write $T = UU_0 D U_0^{-1} = U_1 D U_2$ for $U_i \in SO(2, \mathbb{R})$. Then $SL(2, \mathbb{R}) = KAK$ for $K = SO_2$ and A the diagonal group. This is the Cartan decomposition.

Lemma

If π is a unitary representation of a Group G and we can write $G = KAK$, then it suffices to check that the matrix coefficients vanish on A as $g \to \infty$.

proof

The proof works by observing that K is compact, and so the only part of G that can go to infinity is A. We take vectors v, w and write $g \in G$ as $g = k_1 a k_2$. Then the corresponding matrix coefficient can be written as $\langle \pi(g)v|w\rangle = \langle \pi(a)\pi(k_2)v|\pi(k_1)^{-1}w\rangle$. Since $g \to \infty$ we can find a sequence $g_n = k_{1,n} g_n k_{2,n} \to \infty$ as $n \to \infty$ with $|\langle \pi(g_n)v|w\rangle| \geq \varepsilon$ for some $\varepsilon > 0$. Suppose $k_{1,n} \to k$ and $k_{2,n}^{-1} \to k'$, then for n sufficiently large n $|\langle \pi(a_n)\pi(k)v|\pi(k')w\rangle| \geq \varepsilon/2$. But since K is compact and $g_n \to \infty$, we must have $a_n \to \infty$. This shows that the must be a matrix coefficient in $\pi|A$ that fails to vanish at infinity.

Theorem for SL(2, R) this time

If π is a unitary representation of $G = SL(2,\mathbb{R})$ with no invariant vectors, then all matrix coefficients of π vanish at ∞.

proof

Proof: By the preceding lemma, it suffices to see that the matrix coefficients vanish at infinity along A and by Theorem 2.3.6, it suffices to see that there are no N invariant vectors. Suppose to the contrary that v =f. 0 is N-invariant. Let f(g) = (n(g)vlv). Thenfis continuous and hi-invariant under N, i.e.,f lifts from a continuous N-invariant function on G/N. Now N is exactly the stabilizer of a vector (namely (1, 0)) in IR 2 under the natural SL(2, IR) action. Thus, we can identify G/N with IR 2 - {0}. The action of Non G/N is therefore identified with the action on IR 2 - {0} given by ordinary matrix multiplication. Thus there are two types of orbits, namely all horizontal lines except the x-axis, and each point on the x-axis (except the origin, of course). Clearly any continuous function on IR 2 - {0} ~ G/N which is constant along these orbits must actually be constant on the x-axis. But the x-axis is identified with P/N c G/N under the identification of G/N with IR 2 - {0}. Hencef(g) is constant on P. However, since n is unitary, if f(g) = (n(g)vlv) is constant on P, it follows that v must be P-invariant. Therefore f is actually hi-invariant under P. But P has a dense orbit in GjP. (For example, identify G/P with projective space of IR 2 under ordinary matrix multiplication.) Thusfis actually a constant function, and as above, this implies that v is G-invariant. We are now ready to prove 2.2.20.

Proof for $SL(n,\mathbb{R})$

In this section we'll prove the statement for $G = SL(n,\mathbb{R})$ and later show how the proof is extended to a general group G.

$$\begin{pmatrix} 1 & b_{1,2} & \cdots & b_{1,n} \\ 0 & & & \\ \vdots & & \mathrm{Id}_{n-1} & \\ 0 & & & \end{pmatrix}$$

Note: in the case of $n = 2$, which reduces this to $SL(2,\mathbb{R})$and the above matrix to N from the previous proof.

Following our remark in the preface, we shall prove this in detail for G = SL(n, IR), and then indicate how the proof carries over to general G. Let A c SL(n, IR) be the group of diagonal matrices. We denote an element aEA by (at, ... , an), where these are to be interpreted as the diagonal elements of a matrix. We note Ila; = 1. Let B be the set of matrices (cii) with cu = 1, and cii = 0 fori =f. j and i ~ 2. We denote an element bEB by b = (1, b 2 , •.• , bn) where this is to be interpreted as the first row of the corresponding matrix. Then30 Ergodic theory and semisimple groups aBa- 1 = B for aEA, and hence H = AB is a subgroup of G, and B c H is normal. We observe B ~ IRn- 1 . As with SL(2, IR), by Lemma 2.4.1, it suffices to show that the matrix coefficients of n: IA vanish at oo. For SL(2, IR) we obtained this using knowledge of the representation of P. In our more general situation, we will examine the representation of H. (Note that H = P for n = 2.) Express n: IB ~ n:<~.x .) (by 2.3.3) via the above identification of B with IRn- 1 . Matrix multiplication shows that for aEA, bEB, aba- 1 = (1, a 1 ai 1 b2, ... , a 1 a;; 1 bn)EB. The adjoint action on !Rn- 1 will be given by the same expression, replacing b; by the dual variables h i = 2, ... , n. Therefore, if E, F c !Rn- 1 are compact subsets which are disjoint from the union of the hyperplanes).; = 0, i = 2, ... , n then for aEA outside a sufficiently large compact set, we have a · En F = 0. Therefore, arguing exactly as in the proof of Theorem 2.3.6, we deduce that if f.J. assigns measure 0 to the union of the hyperplanes A.; = 0, then all matrix coefficients vanish along A, and by our comments above, this suffices to prove the theorem. Therefore, it remains to show that f.J.({A.; = 0}) > 0 is impossible. If f.J.({J.; = 0}) > 0, then by definition of f.J.< 11 .x,J, the subgroup B; c B, B; = {bEBibi = 0 fori #j} leaves non-trivial vectors invariant (namely, the subspace .#p.;=o 1.) However B; c H; c G where H; ~ SL(2, IR) and is defined as follows H; = {(cik)ESL(n, IR)Icjj = 1 for j # 1, i, and for j # k and {1, i} # {j, k}, Cjk = 0}. From the vanishing of matrix coefficients for SL(2, IR), (2.4.2), the existence of a B;-invariant vector implies the existence of a H;-invariant vector (since B; is clearly non-compact). In particular, A;= H; n A has non-trivial invariant vectors. Let W= {vEYl'ln:(a)v = vforallaEA;}.Itsufficestoshowthat WisG-invariant. For then the representation n:w ofG on Whas kernel (n:w) ::::J A; which by simplicity of G implies that kernel(n:w) = G, so that G itself leaves all vectors in W fixed, contradicting our assumptions. (For the analogous argument in the semisimple case the fact that dim(kernel n:w) > 0 contradicts the assumption that no simple factor of G leaves vectors invariant.) We now turn to G-invariance of W. For k # j, let Bki c G be the one- dimensional subgroup defined by Bki = {(c,.)lc„ = 1, and for r #sand (r, s) # (k, j), c,. = 0}. We consider two possibilities. (i) k # i or 1 andj # i or 1. Then Bki commutes with A;, and hence Bki leaves W invariant. (ii) If { k, j} n { i, 1} # 0 then A; normalizes Bki · Hence A;Bki is a 2-dimensional subgroup and is isomorphic to P in such a way that A;+-+(diagonal matricesMoore's ergodicity theorem 31 in P), Bki- N. By Corollary 2.3.7, all A;-invariant vectors are also Bki invariant. Hence in this case, too, Bki leaves W invariant. Finally, we remark that since A; c A, A abelian, A also leaves W invariant. However, A and all Bki together generate G. Therefore G leaves W invariant, completing the proof.

Proof for a general *G*

In concluding this section, we indicate the modifications necessary in the above argument for a general semisimple G. Let A c G be a maximal IR-split torus. Then A c G' c G where G' is semisimple and split over IR, and A is the maximal IR-split torus of G'. Choose a maximal linearly independent set S of positive roots of G' relative to A such that for a, {3ES, a+ {3 is not a root. Then the direct sum of the root spaces is the Lie algebra of an abelian subgroup B c G', with dim B =dim A, and B is normalized by A. The representations of AB can be analyzed exactly as in the case of SL(n, IR), and since the relevant copies of s1(2, IR) are present, we deduce that either we are done, or some one-dimensional subgroup A 0 c A leaves a non-trivial vector fixed. (Actually to obtain this we may need to use the universal covering G of SL(2, IR) rather than SL(2, IR) itself. Namely, we need that for N c SL(2, IR) as in the proof of2.4.2, N c G the connected component of the lift of N to G (so that N ~ N), that N invariant vectors are G-invariant. However, this follows by elementary covering space arguments applied to the picture in the proof of 2.4.2. If G is algebraic, which will be our main concern, consideration of SL(2, IR) suffices.) The proof then proceeds as in the case of SL(n, IR); G is generated by elements that either commute with Ao or lie in a suitable copy of the group P.

Outro

The return of the initial example

circle back to fractional linear transforms. hyperbolas! 3 cases comp eucl and non-comp. if we want to go to infinity and don't want boring examples, hyperbolic geometry is necessary. fractional linear transforms. riemann sphere model?

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