On Theorem by Moore about Vanishing Matrix Coefficients

Nicolas Trutmann

Abstract

In this paper we'll showcase a theorem in ergodic theory by R. Howe and C. Moore [1], as it is presented in the book by R. Zimmer in his book "*Ergodic Theory and Semisimple Groups*" [7] On the way there, we'll touch many different fields, from measure theory, over functional analysis, representation theory and of course ergodic theory.

Contents

This paper is based on the book "Ergodic Theory and semisimple Lie Groups" by Robert Zimmer [7], in particular the first two chapters, which contain the theorem itself (Theorem 2.2.20) and surrounding material concerning ergodic theory.

The techniques of the proof show a nice interplay between fields and their different approaches, while staying relatively simple. We assume the reader to have an undergraduate level understanding of the prerequisites in algebra and representation theory, but will state foundational information regardless, and provide references in all cases. We furthermore take care to clarify notation before use.

The theorem, which we will state shortly, is historically at home in the development of ergodic theory, which in turn is a relatively new field of mathematics. The original definition of ergodicity was given in 1928 in a paper by P. Smith and G. Birkhoff on dynamical systems. The concept gained importance in 1931 when von Neumann and Birkhoff nearly simultaneously proved the mean and pointwise ergodic theorems. These may be regarded as the starting point of the subject.

The theory presented here is almost entirely due to a single mathematical lineage. The root of this lineage is G.D. Birkhoff, who, on one side was the (biological) father of G. Birkhoff, which in turn was the advisor of G. Mostow, known for his rigidity theory which was instrumental to G. Margulis' rigidity and arithmeticity theorem. These theorems are a central part of Zimmer's book, although we will not cover them. On the other side, G.D. Birkhoff was advisor to M.H. Stone who was advisor to Mackey, whose work on representations will feature prominently in the chapter on unitary representations. And Mackey was the advisor of R. Zimmer, the author of our main reference, as well as C.C. Moore, who, together with his student R. Howe, worked out the theorem we are talking about in this paper.

The main aim of the book by Zimmer is focused on two theorems by Mostow and Margulis. The "arithmeticity theorem" and the "rigidity theorem", which show how Lie groups and lattices in them interact.

The paper by Moore [6] was published in 1966. Margulis' Theorems were published in

Sources for the historical background: [4](chapter 1. Introduction) [7](chapter 1. Introduction)

The theorem itself does not directly involve ergodicity, but is instead used to prove ergodicity.

The theorem itself is rather simple to state:

[[Moore's Ergodicity Theorem]]

To clarify some points, note that we have specified non-compact groups. This allows us to talk about "infinity" at all. Next, what is an invariant vector? Simply, for all $g \in G$, and a vector v, we have that $\pi(g)v = v$, or, that v is preserved by any linear map given by the representation.

Introduction

- historical context -> up in first section. maybe move down
- where this theorem comes from -> [1]
- what it does
- why we care
- how we're gonna go about it

question: when is an action ergodic?

Instead of verifying ergodicity for any given action, space and measure individually, can we find criteria for ergodicity that are easier to evaluate? The Moore's theorem sits in the middle of an argument that answers the following question.

Let G be a semisimple Lie group and S an ergodic G-space. If $H \subset G$ is a closed subgroup, when is H ergodic on S.

action, lattices in ss groups, asymptotic behavior in non-compact groups [1] Now that we have a concrete question, let us try to get our hands dirty on an example. We'll use the action of fractional linear transforms on the upper half plane, which is nice, because we can look at hyperbolic geometry and draw meaningful pictures of the maps and spaces involved. It'll bring intuition about the question and why one would care to answer the question.

I get the first map now. The action, let's name it for now, $\alpha : SL(2, \mathbb{R}) \curvearrowright \mathbb{H} \to \mathbb{H}$, wich acts by fractional linear transform. ## Lemma 1. $K := SO(2, \mathbb{R})$ is the stabilizer of $i \in \mathbb{H}$. 2. therefore, $G/K \cong AN$ with $KAN \cong G$ being the Iwasawa decomp.

proof

1. from [5](Theorem 1.1.3) map to Klein disk; use Schwarz lemma; map back.

How does the second map work? Using the same fractional linear transform but we take a real value instead of a complex one. It is easy to visualize as a regular matrix product with $\begin{pmatrix} x \\ 1 \end{pmatrix}$ and projecting it to the projective line.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ cx+d \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix}$$

next we care about the behavior of a lattice $\Gamma \subset G$. If G acts transitively on a space X, then there is an isomorphism of G-spaces $G/G_x \to X$, where $G_x = Stab_G(x)$ for $x \in X$, given by the map $gG_x \mapsto gx$. In the case of our example $G = SL(2, \mathbb{R})$, and, as we've shown in the preceding lemma, we know the stabilizer of i to be $SO(2, \mathbb{R})$. ## where we want to go We want to show that the action of Γ on $\bar{\mathbb{R}}$ is ergodic

Definition 1
Ergodicity For a group G, a measurable separable space S, and a G-invariant measure μ. An action is called ergodic if all G-invariant subsets $A \subset S$ are either null or conull. Which means

$$\forall g \in G: \; gA = A \quad \Rightarrow \quad \mu(A) = 0 \text{ or } \mu(S \setminus A) = 0$$

from book

[unoriginal] To see why ergodicity is relevant, and in fact to say a word about what it is, let us consider a classical example. Let $G = SL(2, \mathbb{R})$, and let X be the upper half plane, $X = \{z \in \mathbb{C} | lm(z) > 0\}$. As is well known[todo], G acts on X via fractional linear transformations, i.e.,

$$g \cdot z = \frac{(az+b)}{(cz+d)} \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Suppose now that $\Gamma \subset G$ is a lattice, which we assume to be torsion free for simplicity. Since the action of G on X allows an identification of X with G/K, where $K = SO(2)$ (the stabilizer of $i \in X$), and K is compact, it follows that the action of Γ on X is properly discontinuous, and so $\Gamma \backslash X$ will be a manifold, in fact a finite volume Riemann surface. On the other hand, via the same fractional linear formula, G acts on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and $\bar{\mathbb{R}}$ can be identified with G/P, where P is the group of upper triangular matrices and the stabilizer of $\infty \in \bar{\mathbb{R}}$. Once again, we can consider the action of Γ on $\bar{\mathbb{R}}$, but now the action will be very far from being properly discontinuous. In fact, every Γ-orbit in $\bar{\mathbb{R}}$ will be a (countable) dense set. In particular, if we try taking the quotient $\Gamma \backslash \bar{\mathbb{R}}$, we obtain a space with the trivial topology. On the other hand, $\bar{\mathbb{R}}$ provides a natural compactification of

X, and in fact $\bar{\mathbb{R}}$ can be identified with asymptotic equivalence classes of geodesics in X, where X has the essentially unique G-invariant metric. Thus, it is certainly reasonable to expect the action of Γ on $\bar{\mathbb{R}}$ to yield useful information. However, a thorough understanding requires us to come to grips with actions in which the orbits are very complicated (e.g. dense) sets. Ergodic theory is (in large part) the study of complicated orbit structure in the presence of a measure. Not only are there no non-constant Γ-invariant continuous real-valued functions on $\bar{\mathbb{R}}$, but the same is true for measurable functions. This is embodied in the following definition.

Definition

Suppose G acts on a measure space (S, μ) so that the action map $S \times G \to S$ is measurable and μ is quasi-invariant, i.e., $\mu(A) = 0$ if and only if $\mu(Ag) = 0$. The action is called ergodic if $A \subset S$ is measurable and G-invariant implies $\mu(A) = 0$ or $\mu(S \setminus A) = 0$.

Definitions and Notation

Now that we have stated the goal of the paper, let us immediately make a detour. We will state definitions and relevant theorems (without proof) in compact form with ample references so that a reader can catch up if necessary. The advanced reader can skip this section and move straight to the next topic without issue.

Measure Spaces

A *measurable space* is a pair $(X, \mathscr{B})$ where X is a set and $\mathscr{B}$ is a σ-algebra of subsets of X. Elements of $\mathscr{B}$ are called *measurable sets.* A function of measurable spaces $f : X \to Y$ is called *measurable* if $f^{-1}(A)$ is a measurable set in X for all measurable sets A of Y.

A *measure* on a measurable space $(X, \mathscr{B})$ is a map $\mu : \mathscr{B} \to [0, \infty]$ such that - $\mu(\emptyset) = 0$, and - $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for every countable collection $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathscr{B}$ (countable additivity).

The Borel σ-algebra of a topological space X is the σ-algebra $\mathscr{B}$ generated by the open subsets of X , and the members of $\mathscr{B}$ are called Borel sets.

A measure μ is called *finite* if the whole space has finite measure $\mu(X) < \infty$, and *σ-finite* if X is the countable union of sets with finite measure, meaning, there exist sets $\{A_i\}_{i \in \mathbb{N}}$ such that $\cup_{i=1}^{\infty} A_i = X$ and $\mu(A_i) < \infty$ for all i.

Groups

We are interested in Lie groups. Primarily for its nature as a topological group. A *Lie group* is a group that is also a manifold. A *locally compact* group is locally compact as a topological space. We require groups to be locally compact, so that the Haar measure exists, which is, up to scaling, the unique measure on Borel sets which satisfies the following: For all $g \in G$ $\mu(gS) = \mu(S)$, μ is finite on compact sets and is inner and outer regular. Unless otherwise specified, we talk about these types of groups.

A *lattice* is a discrete subgroup Γ of a locally compact group G such that there exists a finite measure on the quotient space G/Γ.

Group Actions

By an *action* of the group G on a set X we mean a map $\alpha : G \times X \to X$ such that, writing the first argument as a subscript, $\alpha_s(\alpha_t(x)) = \alpha_{st}(x)$ and $\alpha_e(x) = x$ for all $x \in X$ and $s, t \in G$. Most of the time we will not give this map a name and write the image of a pair (s, x) written as sx. For sets $A \subset X$ and $K \subset G$ and an $s \in G$ we write

$$sA = \{sx : x \in A\}, \quad Kx = \{sx : s \in K\}, \quad KA = \{sx : x \in A \text{ and } s \in K\}.$$

The *G-orbit* of a point $x \in X$ is the set Gx.

Representations

A *representation* is a group-homomorphism from a group into the general linear group of a vector space, $\pi : G \to GL(V)$. We consistently use lowercase Greek letters to refer to representations. Most often π. The *dimension* of a representation is the dimension of the vector space that is being represented onto.

A *unitary operator* on a Hilbert space $\mathscr{H}$ is a bounded linear operator U, such that $U^*U = UU^* = \mathrm{Id}_{\mathscr{H}}$. A *unitary representation* is a representation into the unitary group of a vector space $\pi : G \to \mathcal{U}(V) \subset GL(V)$, where the unitary group consists of all unitary operators on $\mathscr{H}$.

For a representation π onto a (complex) Hilbert space $\mathscr{H}$, $\pi : G \to GL(\mathscr{H})$ and two vectors $v, w \in \mathscr{H}$, a *matrix coefficient* is a map $f(g) : G \to \mathbb{C}$ defined by

$$f(g) = \langle \pi(g)v, w \rangle$$

In the case of a finite dimensional Hilbert space and, for a given choice of basis, and two basis vectors e_i, e_j, the inner product $\langle e_i\pi(g), e_j \rangle$ works out to be the coefficient of the matrix associates to $\pi(g)$.

“direct difference” notation

Zimmer, and we, use the symbol “$\ominus$” to denote “subtraction” of linear subspaces of Hilbert spaces. If $A \subset B$ are linear subspaces of a Hilbert space, $B \ominus A = \{x \in B : (x, y) = 0 \text{ for all } y \in A\}$.

The specifically we will use it on $L^2(\mathscr{H}) \ominus \mathbb{C}$, to denote the square integrable functions on $\mathscr{H}$"minus" the subspace of constant functions.

Ergodicity

We have successfully made our way back to ergodicity. We will try to illuminate the definition a bit by examples and non-examples.

To reiterate

Definition 2
Ergodicity For a group G, a measurable separable space S, and a G-invariant measure μ. An action is called ergodic if all G-invariant subsets $A \subset S$ are either null or conull. Which means

$$\forall g \in G : \; gA = A \quad \Rightarrow \quad \mu(A) = 0 \text{ or } \mu(S \setminus A) = 0$$

Let us try to build some intuition for what this means. Notice that orbits are, by definition G-invariant, so one way to constructively build invariant sets is to consider orbits of actions. Inversely as well, any invariant set can be considered a union of orbits of all its points. Recall from basic group theory that orbits partition the space, so saying that these must be either null or conull means there is no straightforward “divide and conquer” strategy for understanding ergodic actions. In this regard ergodicity resembles a sort of “irreducibility”-property. To put it in Zimmer’s words “Ergodic theory is (in large part) the study of complicated orbit structure in the presence of a measure.”

Note, that the adjective “ergodic” sometimes applied to either the action, the measure or the space. What that means is that, for two out of three given, the third completes the definition. All three are necessary to be ergodic but when, for example, we have a group action on a space, we call a measure ergodic if together with the others they are ergodic.

Example Let $\mathbb{T}$ be the circle group of $\{z \; \mathbb{C} \mid |z| = 1\}$ and $A : \mathbb{T} \to \mathbb{T}$ multiplication by $e^{i\alpha}$ with $\alpha/2\pi$ irrational. This induces an action $\mathbb{Z} \curvearrowright \mathbb{T} \to \mathbb{T}$ by $n \cdot z \mapsto e^{in\alpha} z$. As a measure we take the arc-length measure, which is preserved under the action of A.

This is an example of an ergodic action.

To prove this, suppose $S \subset \mathbb{T}$ is A-invariant. We take $\chi_S(z) = 1$ for $z \in S$ and 0 for $z \notin S$, the characteristic function of S and take the L^2-Fourier expansion $\sum a_n z^n$. Then, by invariance, $\chi_S(z) = \chi_S(e^{i\alpha}z) = \sum a_n e^{in\alpha} z^n$. Therefore $a_n e^{in\alpha} = a_n$. By assumption $\alpha/2\pi \notin \mathbb{Q}$, so $a_n = 0$ for all $n \neq 0$. This implies χ_S is constant, meaning either constant 0 or constant 1, which implies ergodicity.

definition; explanation of definition; Examples; why the prerequisites come in, like quasi-invariance; clarify edge cases of properly ergodic.

The Direct Integral and Unitary Representations

Now that we've laid out the prerequisites, we can turn to what we'll actually need in terms of this specific subject. We have to take a detour into unitary representations and define the direct integral to make statements about certain subgroups, in particular $\mathbb{R}^n$. It turns out, we can transform questions about ergodicity into questions about representations. Thereby opening up the problems to more tractable linear algebra and matrix groups.

The question about ergodicity, that hangs in the background of the theorem is: "what happens at the boundary?". Boundary means we are interested in the limit behavior of an ergodic action, which explains why our theorem makes an assertion about matrix coefficients at infinity.

The way there will lead us through the direct integral, unitary representations and in particular the representation of $\mathbb{R}^n$. To jump ahead of ourselves, we'll later look at the upper diagonal group and its subgroup $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, which is isomorphic to $\mathbb{R}$ and whose representation we'll care about.

The Direct Integral

In simple terms, the direct integral is a way to patch together locally defined functions into a function on the whole domain. Let us first consider the simple case where we have global functions on a measure space M, that takes values in some Hilbert space $\mathscr{H}$, $f : M \to \mathscr{H}$. The 'sensible' space to put these functions into is the space of square integrable functions on M, denoted $L^2(M, \mathscr{H})$. The word 'sensible' here is justified by being again a Hilbert space by integration $\langle f, g \rangle = \int_M \langle f(x), g(x) \rangle$.

The next step towards locality is to use two function, by defining $L^2(M_1 \sqcup M_2, \mathscr{H}_1 \oplus$

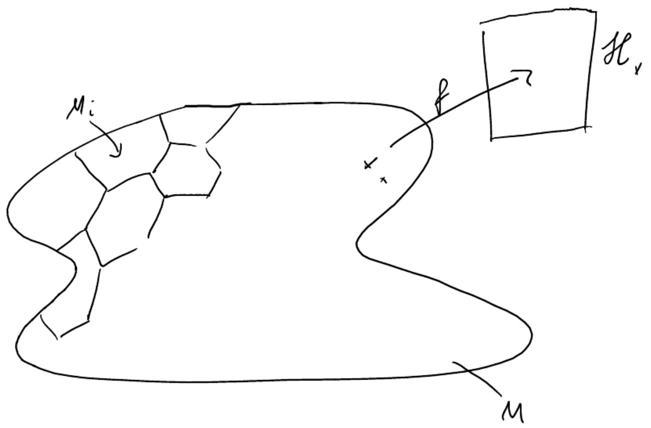

Figure 1: aa

$\mathscr{H}_2$), where every function is defined separately on each M_i, and taking values in $\mathscr{H}_i$.

Suppose we have a measure space M, and for each $x \in M$ a Hilbert space $\mathscr{H}_x$ such that $x \mapsto \mathscr{H}_x$ is piecewise constant, that is, we have a disjoint decomposition of M into $\cup_{i=1}^{\infty} M_i$ such that for $x, y \in M_i$, $\mathscr{H}_x = \mathscr{H}_y$. Interesting aside: the condition that the assignment $x \mapsto \mathscr{H}_x$ be piecewise constant is not necessary. We can allow the Hilbert spaces to be arbitrary, and in fact uncountably infinite. Short answer: magic; slightly less short answer: von Neumann. A *section* on M is an assignment $x \mapsto f(x)$, where $f(x) \in \mathscr{H}_x$. Since $\mathscr{H}_x$ is piecewise constant, the notion of measurability carries over in an obvious manner, namely that a measurable function on M is measurable on each M_i into the appropriate Hilbert space. Let $L^2(M, \{\mathscr{H}_x\})$ be the set of square integrable sections $\int \|f\|^2 < \infty$ where we identify two sections if they agree almost everywhere. This set is then also a Hilbert space with the inner product $\langle f|g\rangle = \int_M \langle f(x)|g(x)\rangle$.

Suppose now we have for each $x \in M$ a unitary representation π_x of a group G on $\mathscr{H}_x$. We say this is measurable when for $g \in G$, $\pi_x(g)$ is a measurable function on each $M_i \times G$.

This allows us to define the relevant representation we intermediately care about.

> **Remark (On the notation of the direct integral)** The above notation of $\pi_{\mu,\mathscr{H}}$ is generally fine, but putting an already hard to read typeface in a small font size into the subscript is hard to read. We have introduced it as is to conform with the notation in the literature, but in the next section we will encounter a number of operations that manipulate these subscripts. For that reason we'll write them also in square brackets like so:
>
> $$\pi[\mu, \mathscr{H}]$$
>
> meaning the same thing as the subscript notation.

Unitary Representations

irreducible unitary representations to understand the action(s) of $SL(n, \mathbb{R})$.

Representation of $\mathbb{R}^n$

Theorem 1 (Zimmer 2.3.3)

- For any unitary representation π of $\mathbb{R}^n$, there exist $\mu, \mathscr{H}_\lambda$, on $\hat{\mathbb{R}}^n$ such that $\pi \cong \pi_{\mu,\mathscr{H}_\lambda}$.

- $\pi_{\mu,\mathscr{H}_\lambda}$ and $\pi_{\mu',\mathscr{H}'_\lambda}$ are unitarily equivalent if and only if
  - $\mu \sim \mu'$, i.e., they are in the same measure class
  - and $dim\mathscr{H}_\lambda = dim\mathscr{H}'_\lambda$ a.e.

proof

Theorem 2 (Zimmer 2.3.4)
Let $\pi = \pi_{\mu,\mathscr{H}_\lambda}$, $A \in \mathrm{Aut}(\mathbb{R}^n)$, α the adjoint automorphism of $\hat{\mathbb{R}}^n$. Then

- $\alpha(\pi)$ is unitarily equivalent to $\pi[\alpha_* \mu]$

proof

Theorem 3 (Zimmer 2.3.5, from Mackey [3])
Suppose $\mathbb{R}^n \subset G$ is a normal subgroup and π is a unitary representation of G. Write $\pi|\mathbb{R}^n \cong \pi_{(\mu,\mathscr{H}_\lambda)}$ for some $(\mu, \mathscr{H}_\lambda)$ by 2.3.3. Then

- μ is quasi-invariant under the action of G on $\hat{\mathbb{R}}^n$.
- If $E \subset \mathbb{R}^n$ is measurable, let $\mathscr{H}_E = L^2(E, \mu, \{\mathscr{H}_\lambda\})$. Then $\pi(g)\mathscr{H}_E = \mathscr{H}_{g \cdot E}$
- If π is irreducible, then μ is ergodic and $dim\mathscr{H}_\lambda$ is constant on a μ-conull set.

proof

Theorem 4 (Zimmer 2.3.6)
Let πbe a unitary representation of $P = AN$.

- either $\pi|N$ has non-trivial invariant vectors or
- or for $g \in A$ and any vectors, v, w, the matrix coefficients $\langle \pi(g)v, w\rangle \to 0$ as $g \to \infty$.

proof We identify N with $\mathbb{R}$via the map $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto x$

All the irreducible unitary representations of $\mathbb{R}^n$ are one-dimensional.

It turns out that the group unitary representations on $\mathbb{R}^n$ are isomorphic to $\mathbb{R}^n$. So we define a map from $\mathbb{R}^n$ to $\mathcal{U}(\mathbb{C})$ and show that it's in fact bijective. Let $\theta.$ t be in $\mathbb{R}^n$ and let $\lambda_\theta(t) = e^{i\langle\theta|t\rangle}$. This is in fact a unitary automorphism on $\mathbb{C}$ by multiplication. To clarify, for every $\theta \in \mathbb{R}^n$ we have a representation given by

$$\begin{aligned} \lambda_\theta : \mathbb{R}^n &\to \mathcal{U}(\mathbb{C}) \\ t &\mapsto e^{i\langle\theta|t\rangle} \end{aligned}$$

We denote the group of representations by $\hat{\mathbb{R}}^n$. It is in fact a group under pointwise multiplication.

This definition is maybe a bit dense, so here is the assignment formatted in pseudo code. This might help some reader more familiar with programming than mathematics. The more mathematically inclined may ignore it. It is not relevant other than to further the understanding of the above definition. Note here that lambda denotes the programming term of a lambda function, an unfortunate notation collision.

$$\begin{aligned}
&\text{func}\ \ \pi_{\mu,\mathscr{H}_\lambda}(t : \mathbb{R}^n) \to \mathcal{U}(L^2(\hat{\mathbb{R}}^n))\ \{\\
&\qquad \text{return lambda}(f :\ L^2(\hat{\mathbb{R}}^n)) \to L^2(\hat{\mathbb{R}}^n)\ \{\\
&\qquad\qquad \text{return lambda}(\lambda :\ \hat{\mathbb{R}}^n) \to \mathscr{H}_\lambda\ \{\\
&\qquad\qquad\qquad \text{return}\ \lambda(t)f(\lambda)\\
&\qquad\qquad \}\\
&\qquad \}\\
&\}
\end{aligned}$$

The Connection between Ergodicity and Unitary Representations

approach: - char func - char func in L2(S) and non-trivial - if A invariant then char func invariant as a vector in L2(S) - due diligence: make sure measure works

To see why we care about unitary representations at all if we really want ergodicity, we neeedd to make the folllowing connection. We use the characteristic function of a set to connect the set to a vector in $L^2(S)$. The characteristic function of a subset $A \subset S$, is defined as $\chi_A(x) = 1$ for $x \in A$ and 0 otherwise.

This representation allows us to pass from talking about sets to talking about vectors, while retaining the properties we care about.

Theorem 5 ()
An action $G \curvearrowright S$, with **finite** invariant measure is ergodic on S if and only if the restriction of the above representation to in $L^2(S) \ominus \mathbb{C}$ has no invariant vectors.

Since S has finite measure, assume $\mu(S) = 1$.

proof "$\Leftarrow$": Proof by contrapositive: If $A \subset S$ is G-invariant with measure $0 < \mu(A) < \mu(S) = 1$ then χ_A is also G-invariant in $L^2(S)$ as well as the projection $\chi_A - \mu(A) \cdot 1$ in $L^2(S) \ominus \mathbb{C}$. Therefore there exists an invariant vector in $L^2(S) \ominus \mathbb{C}$. "$\Rightarrow$": ([2](Prop 2.7)) Suppose the action is ergodic and $f \in L^2(S) \ominus \mathbb{C}$ is G-invariant. We can find a measurable set $D \subset \mathbb{C}$ such that $0 < \mu(f^{-1}(D)) < 1$ and denote $\widetilde{A} = f^{-1}$. Now we verify ergodicity. For every $g \in G$ the symmetric difference $g\widetilde{A}\Delta\widetilde{A}$, for which all points are in the set $\{x \in X| \ |f(x) - sf(x)| > 0\}$, which has measure zero because $\|f - sf\|_2 = 0$. Therefore the action fails to be ergodic.

The adjective "finite" on the measure is necessary, because for a set A of infinite measure the statement is no longer true as χ_A will no longer be in L^2.

If $A \subset S$ is G-invariant then $\chi_A \in L^2(S)$ will also be G-invariant. For A neither null nor conull then χ_A, $f_A \neq 0$, where f_A is the projection of χ_A onto $L^2(S) \ominus \mathbb{C}$.

Proof for $SL(2, \mathbb{R})$

We start here because it is an easy example of the theorem and a general group G has many subgroups locally isomorphic to $SL(2, \mathbb{R})$. Later we extend the proof, first to $SL(n, \mathbb{R})$ and then to a general G.

To state our intentions: we first show that either the matrix coefficients vanish as we want, or there exist invariant vectors. Then we show that there are no invariant vectors, completing the statement.

We're going to use the following decomposition, which we take for granted The so called Iwasawa decomposition of $SL(2, \mathbb{R})$ into three matrices K, A, and N, defined as

$$K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \subset SL(2, \mathbb{R}) \mid \theta \in \mathbb{R} \right\} \tag{1}$$

$$A = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \subset SL(2, \mathbb{R}) \mid r > 0 \right\} \tag{2}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \subset SL(2, \mathbb{R}) \mid x \in \mathbb{R} \right\} \tag{3}$$

$$\tag{4}$$

Theorem for P

Lemma 6 (decomposition of $\boldsymbol{SL(2,\mathbb{R})}$and $\boldsymbol{P}$)

1. The upper triangular group P and $\bar{P}$ generate $SL(2,\mathbb{R})$.
2. The upper triangular group can be decomposed into the semidirect product:
$$P = AN = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$
3. N is normal in P

proof We look at the subgroup

$$P \subset SL(2,\mathbb{R}) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

of upper triangular matrices. Together with the lower diagonal matrices $\bar{P}$, they generate $SL(2,\mathbb{R})$. To see this, decompose as follows:

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & \beta x \\ \alpha x & \alpha\beta x + 1/x \end{pmatrix}$$

For any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,\mathbb{R})$ with matrix coefficient $a \neq 0$, we can solve for x, α, β. In the case of $a = 0$ we can use the following construction:

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\beta\gamma & \delta(1+\beta\gamma)+\beta \\ \alpha(1+\beta\gamma)+\gamma & \alpha\delta(1+\beta\gamma)+\alpha\beta+\gamma\delta+1 \end{pmatrix}$$

If $1 + \beta\gamma = 0$, the above product becomes $\begin{pmatrix} 0 & \beta \\ \gamma & 1+\alpha\beta+\gamma\delta \end{pmatrix}$ and we can make suitable choices for $\alpha, \beta, \gamma, \delta$ to construct A.

Note first, that N is normal in P. To see this, first calculate that the inverse of a matrix $\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}$ in P is $\begin{pmatrix} a^{-1} & -x \\ 0 & a \end{pmatrix}$. Next note that the result of conjugation with an element in P is again in N: $\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & -x \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & a^2x \\ 0 & 1 \end{pmatrix}$. This defines a group action $P \curvearrowright N \to N$ by multiplication with a^2.

Theorem for Cartan decomposition

Polar decomposition to Cartan

$T = US$ for some unitary U and a sym pos def S. S can be diagonalized into $U_0 D U_0^{-1}$ so we can write $T = UU_0DU_0^{-1} = U_1DU_2$ for $U_i \in SO(2,\mathbb{R})$. Then

$SL(2,\mathbb{R}) = KAK$ for $K = SO_2$ and A the diagonal group. This is the Cartan decomposition.

Lemma 7 If π is a unitary representation of a Group G (which is assumed to be second countable) and we can write $G = KAK$, with K compact, then it suffices to check that the matrix coefficients vanish on A as $g \to \infty$.

proof We take vectors v, w and write $g \in G$ as $g = k_1 a k_2$. Then the corresponding matrix coefficient can be written as $\langle \pi(g)v|w\rangle = \langle \pi(a)\pi(k_2)v|\pi(k_1)^{-1}w\rangle$.

We make a proof via contraposition. If there exists a matrix coefficient that fails to vanish as $g \to \infty$ we can find a sequence $g_n = k_{1,n} g_n k_{2,n} \to \infty$ as $n \to \infty$ with $|\langle \pi(g_n)v|w\rangle| \geq \varepsilon$ for some $\varepsilon > 0$.

Because G, and therefore K is second countable and compact, it is also sequentially compact. So we can suppose $k_{1,n} \to k$ and $k_{2,n}^{-1} \to k'$. Then, for n sufficiently large, $|\langle \pi(a_n)\pi(k)v|\pi(k')w\rangle| \geq \varepsilon/2$. This follows from the following estimation, where we ommit the representation π for legibility:

$$\begin{aligned}
&= |\ \langle a_n k_n v, k'_n w\rangle - \langle a_n k v, k' w\rangle\ | \\
&= |\ \langle a_n k_n v - a_n k v, k'_n w\rangle\ +\ \langle a_n k v, k'_n w - k' w\rangle\ | \\
&\leq |\ \langle a_n k_n v - a_n k v, k'_n w\rangle\ |\ +\ |\ \langle a_n k v, k'_n w - k' w\rangle\ | \quad \text{Triangle Inequality} \\
&\leq \|a_n k_n v - a_n k v\| \|k'_n v\|\ +\ \|a_n k v\| \|k'_n w - k' w\| \quad \text{Cauchy-Schwarz}
\end{aligned}$$

From here, we can pick an n large enough to assert the inequality.

But since K is compact and $g_n \to \infty$, we must have $a_n \to \infty$. This shows that the must be a matrix coefficient in $\pi|A$ that fails to vanish at infinity.

Proof for $SL(n,\mathbb{R})$

Theorem for $SL(2,\mathbb{R})$)

If π is a unitary representation of $G = SL(2,\mathbb{R})$ with no invariant vectors, then all matrix coefficients of π vanish at ∞.

We can now start on the statement for $SL(2,\mathbb{R})$. Thanks to the work we did in the preceeding chapter, the statement is actually not very difficult to prove. The theorem 4 and the preceeding lemma 7 does the bulk of the heavy lifting here.

proof By assumption, Ghas no invariant vectors. By theorem 4, There are two possible cases. Either N has non-zero invariant vectors, or the matrix coefficients vanish along A.

Should there be no non-zero invariant vectors, as we'll show, then the matrix coefficients vanish along A, and, by lemma 7, vanishing along A implies vanishing along G.

To see that there are no N-invariant vectors, we assume towards a contradiction that there are N-invariant vectors and show that these must be G-invariant as well, which contradicts our assumption.

Suppose there is a vector v that is N-invariant, meaning $\pi(n)v = v$ for all $n \in N$. As a shorthand, define the function $f(g) = \langle \pi(g)v, v\rangle$. This defines a continuous bi-N-invariantfunction on G.

This is because $f(gn) = \langle \pi(gn)v, v, \rangle = \langle \pi(g)\pi(n)v, v\rangle = \langle \pi(g)v, v\rangle = f(g)$, and $f(ng) = \langle \pi(n)\pi(g)v, v\rangle \xrightarrow{unitary} \langle \pi(g)v, \pi(n)^{-1}v\rangle = f(g)$.

Thus f lifts from a continuous bi-N-invariantfunction on G/N.

Gacts transitively on $\mathbb{R}^2 \setminus \{0\}$ by matrix multiplication, and, using the fact that N is exactly the stabilizer of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, we get an isomorphism $G/N \cong \mathbb{R}^2 \setminus \{0\}$ [1].

Calculating the orbits of this action we have $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+bx \\ b \end{smallmatrix}\right)$. So there exist two kinds of orbits: for $b \neq 0$. the orbit is the horizontal line at height b and for $b = 0$ every individual point $(a\ 0)$ on the x-axix. (See Figure 2). As f is N-invariant, f will be constant along these orbits. Because f is continuous, f will also be constant along the x-axis.

But we can also identify the x-axis with P/N by $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax \\ 0 \end{smallmatrix}\right)$. Therefore f is also constant on P. So it follows that v is P-invariant. And as we've seen in the intoduction, we can identify G/P with the real projective line and P has a dense orbit in G/P so f is constant on G and therefore v is actually G-invariant, contradicting our assumption.

In this section we'll prove the statement for $G = SL(n, \mathbb{R})$ and later show how the proof is extended to a general group G. We begin just as for $SL(2, \mathbb{R})$, by applying lemma 7. Thus it suffices to show that matrix coefficients vanish on $A \subset G$ to imply that they vanish on G.

[1] This is due to the fact that for a transitive action $G \curvearrowright X$ there is an isomorphism $G/Stab_G(x) \to X$ sending $g \cdot Stab_G(x) \mapsto gx$.

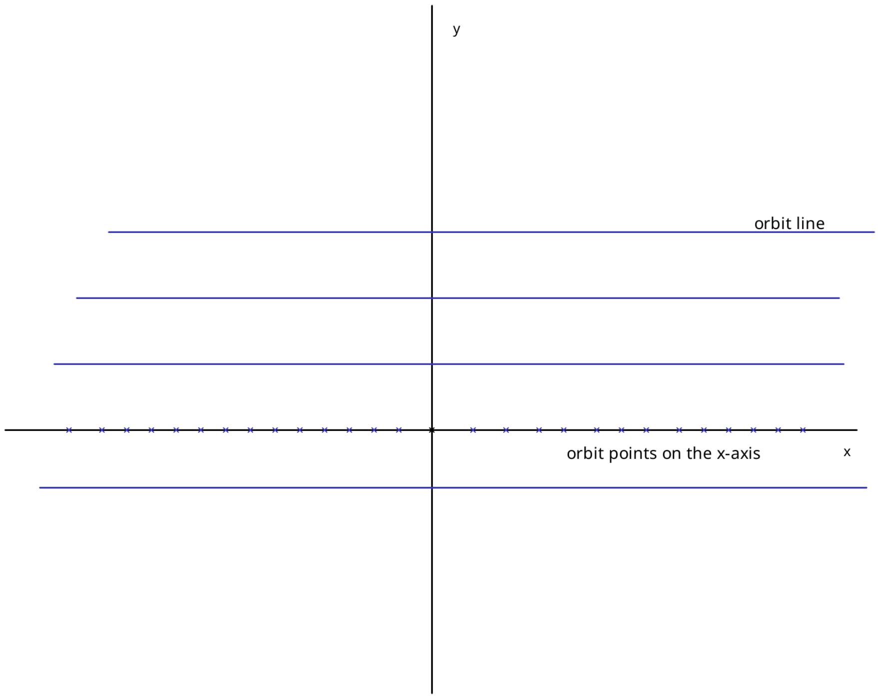

Figure 2: The orbits of N on G/N correspond either to the horizontal lines parallel to the x-axis or to the individual points on the x-axis.

$$\begin{pmatrix} 1 & b_{1,2} & \cdots & b_{1,n} \\ 0 & & & \\ \vdots & & \mathrm{Id}_{n-1} & \\ 0 & & & \end{pmatrix}$$

Note: in the case of $n = 2$, which reduces this to $SL(2, \mathbb{R})$and the above matrix to N from the previous proof.

Following our remark in the preface, we shall prove this in detail for G = SL(n, IR), and then indicate how the proof carries over to general G. Let A c SL(n, IR) be the group of diagonal matrices. We denote an element aEA by (at, ... , an), where these are to be interpreted as the diagonal elements of a matrix. We note Ila; = 1. Let B be the set of matrices (cii) with cu = 1, and cii = 0 fori =f. j and i ~ 2. We denote an element bEB by b = (1, b 2 , •.• , bn) where this is to be interpreted as the first row of the corresponding matrix. Then30 Ergodic theory and semisimple groups aBa- 1 = B for aEA, and hence H = AB is a subgroup of G, and B c H is normal. We observe B ~ IRn- 1 . As with SL(2, IR), by Lemma 2.4.1, it suffices to show that the matrix coefficients of n: IA vanish at oo. For SL(2, IR) we obtained this using knowledge of the representation of P. In our more general situation, we will examine the representation of H. (Note that H = P for n = 2.) Express n: IB ~ n:<~.x .) (by 2.3.3) via the above identification of B with IRn- 1 . Matrix multiplication shows that for aEA, bEB, aba- 1 = (1, a 1 ai 1 b2, ... , a 1 a;; 1 bn)EB. The adjoint action on !Rn- 1 will be given by the same expression, replacing b; by the dual variables h i = 2, ... , n. Therefore, if E, F c !Rn- 1 are compact subsets which are disjoint from the union of the hyperplanes).; = 0, i = 2, ... , n then for aEA outside a sufficiently large compact set, we have a · En F = 0. Therefore, arguing exactly as in the proof of Theorem 2.3.6, we deduce that if f.J. assigns measure 0 to the union of the hyperplanes A.; = 0, then all matrix coefficients vanish along A, and by our comments above, this suffices to prove the theorem. Therefore, it remains to show that f.J.({A.; = 0}) > 0 is impossible. If f.J.({J.; = 0}) > 0, then by definition of f.J.< 11 .x,J, the subgroup B; c B, B; = {bEBibi = 0 fori #j} leaves non-trivial vectors invariant (namely, the subspace .#p.;=o 1.) However B; c H; c G where H; ~ SL(2, IR) and is defined as follows H; = {(cik)ESL(n, IR)Icjj = 1 for j # 1, i, and for j # k and {1, i} # {j, k}, Cjk = 0}. From the vanishing of matrix coefficients for SL(2, IR), (2.4.2), the existence of a B;-invariant vector implies the existence of a H;-invariant vector (since B; is clearly non-compact). In particular, A;= H; n A has non-trivial invariant vectors. Let W= {vEYl'ln:(a)v = vforallaEA;}.Itsufficestoshowthat WisG-invariant. For then the representation n:w ofG on Whas kernel (n:w) ::::J A; which by simplicity of G implies that kernel(n:w) = G, so that G itself leaves all vectors in W fixed, contradicting our assumptions. (For the analogous argument in the semisimple case the fact that dim(kernel n:w) > 0 contradicts the assumption that no simple factor of G leaves vectors invariant.) We now turn to G-invariance of W. For k # j, let Bki c G be the one- dimensional subgroup defined by Bki = {(c,.)lc„ = 1, and for r #sand (r, s) # (k, j), c,. = 0}. We consider two possibilities. (i) k # i or 1 andj # i or 1. Then Bki commutes with A;, and hence Bki leaves W invariant. (ii) If { k, j} n { i, 1} # 0 then A; normalizes Bki · Hence A;Bki is a 2-dimensional subgroup and is isomorphic to P in such a way that A;+-+(diagonal matricesMoore's ergodicity theorem 31 in P), Bki- N. By Corollary 2.3.7, all A;-invariant vectors are also Bki invariant. Hence in this case, too, Bki leaves W invariant. Finally, we remark that since A; c A, A abelian, A also leaves W invariant. However, A and all Bki together generate G. Therefore G leaves W invariant, completing the proof.

Proof for a general G

In concluding this section, we indicate the modifications necessary in the above argument for a general semisimple G. Let A c G be a maximal IR-split torus. Then A c G' c G where G' is semisimple and split over IR, and A is the maximal IR-split torus of G'. Choose a maximal linearly independent set S of positive roots of G' relative to A such that for a, {3ES, a+ {3 is not a root. Then the direct sum of the root spaces is the Lie algebra of an abelian subgroup B c G', with dim B =dim A, and B is normalized by A. The representations of AB can be analyzed exactly as in the case of SL(n, IR), and since the relevant copies of s1(2, IR) are present, we deduce that either we are done, or some one-dimensional subgroup A 0 c A leaves a non-trivial vector fixed. (Actually to obtain this we may need to use the universal covering G of SL(2, IR) rather than SL(2, IR) itself. Namely, we need that for N c SL(2, IR) as in the proof of2.4.2, N c G the connected component of the lift of N to G (so that N ~ N), that N invariant vectors are G-invariant. However, this follows by elementary covering space arguments applied to the picture in the proof of 2.4.2. If G is algebraic, which will be our main concern, consideration of SL(2, IR) suffices.) The proof then proceeds as in the case of SL(n, IR); G is generated by elements that either commute with Ao or lie in a suitable copy of the group P.

Outro

The return of the initial example

circle back to fractional linear transforms. hyperbolas! 3 cases comp eucl and non-comp. if we want to go to infinity and don't want boring examples, hyperbolic geometry is necessary. fractional linear transforms. riemann sphere model?

Auxilliary Statements

Proposition 8 In a second countable topological space, compactness and sequential compactness are equivalent.

proof no proof

List of Theorems

List of Figures

References

[1] Roger E. Howe and Calvin C. Moore. "Asymptotic properties of unitary representations". In: *Journal of Functional Analysis* 32.1 (Apr. 1979), pp. 72–96. ISSN: 0022-1236. DOI: 10.1016/0022-1236(79)90078-8. URL: https://www.sciencedirect.com/science/article/pii/0022123679900788 (visited on 03/16/2024).

[2] David Kerr and Hanfeng Li. *Ergodic Theory*. Springer International Publishing, 2016. ISBN: 9783319498478. DOI: 10.1007/978-3-319-49847-8. URL: http://dx.doi.org/10.1007/978-3-319-49847-8.

[3] G. Mackey. “The theory of unitary group representations”. In: 1976. URL: https://www.semanticscholar.org/paper/The-theory-of-unitary-group-representations-Mackey/956fcae01ce6826f64b08badcd921493aad18440 (visited on 03/07/2024).

[4] George W. Mackey. “Ergodic theory and its significance for statistical mechanics and probability theory”. In: *Advances in Mathematics* 12.2 (Feb. 1974), pp. 178–268. ISSN: 0001-8708. DOI: 10.1016/S0001-8708(74)80003-4. URL: https://www.sciencedirect.com/science/article/pii/S0001870874800034 (visited on 03/18/2024).

[5] Toshitsune Miyake. *Modular Forms*. Springer Berlin Heidelberg, 1989. ISBN: 9783540295938. DOI: 10.1007/3-540-29593-3. URL: http://dx.doi.org/10.1007/3-540-29593-3.

[6] Calvin C. Moore. “Ergodicity of Flows on Homogeneous Spaces”. In: *American Journal of Mathematics* 88.1 (1966), pp. 154–178. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2373052 (visited on 02/27/2024).

[7] Robert J. Zimmer. *Ergodic Theory and Semisimple Groups*. Birkhäuser Boston, 1984. ISBN: 9781468494884. DOI: 10.1007/978-1-4684-9488-4. URL: http://dx.doi.org/10.1007/978-1-4684-9488-4.