Solutions to Problems on Decidability

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Exercise 1

Classify each language as:

- Decidable
- Recursively Enumerable (r.e.)
- co-Recursively Enumerable (co-r.e.)

1. $L_1 := \{M \mid M \text{ halts on itself}\}$
   This is a version of the **diagonal halting problem**, which is known to be:
   - **Not decidable** (by reduction from the halting problem),
   - **Not r.e.** (there's no TM that semi-decides it),
   - **Not co-r.e.** (its complement is not r.e. either).
2. $L_2 := \{(M, w) \mid M \text{ halts on } w\}$
   This is the classic **Halting Problem**, which is:
   - **Not decidable**,
   - **Recursively enumerable (r.e.)** (can simulate M on w and accept if it halts),
   - **Not co-r.e.**.
3. $L_3 := \{(M, w, k) \mid M \text{ halts on } w \text{ in at most } k \text{ steps}\}$
   This is **decidable**, because:
   - You can simulate M on w for at most k steps.
   - Halt and accept if M halts within k steps; otherwise, reject.

Exercise 2

1. **True.** The union of two decidable languages is also decidable. Run both deciders and accept if either accepts.

2. **True.** The class of decidable languages is closed under complement. If L is decidable, we can construct a TM that rejects when L accepts and vice versa.

3. **True.** If L is decidable, then L^* is also decidable. You can nondeterministically split the input string and test each part with the decider for L.

4. **True.** The union of two r.e. languages is also r.e. You can dovetail both TMs and accept if either one accepts.

5. **False.** The complement of an r.e. language is not necessarily r.e. Counterexample: the Halting Problem is r.e., but its complement is not.

6. **False.** L^* may not be r.e. even if L is. For example, let $L = \{w \mid M_w \text{ halts}\}$. Then L^* is not r.e. in general.