Homework 10 and 11

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Exercise 1: Convert to CNF

1. $\varphi_1 := \neg((a \wedge b) \vee (\neg c \wedge d))$

$$\begin{aligned}\varphi_1 &= \neg((a \wedge b) \vee (\neg c \wedge d)) \\ &= \neg(a \wedge b) \wedge \neg(\neg c \wedge d) \quad \text{(De Morgan)} \\ &= (\neg a \vee \neg b) \wedge (c \vee \neg d)\end{aligned}$$

CNF: $(\neg a \vee \neg b) \wedge (c \vee \neg d)$

2. $\varphi_2 := \neg((p \vee q) \to (r \wedge \neg s))$

$$\begin{aligned}\varphi_2 &= \neg(\neg(p \vee q) \vee (r \wedge \neg s)) \\ &= \neg((\neg p \wedge \neg q) \vee (r \wedge \neg s)) \quad \text{(De Morgan)} \\ &= \neg(\neg p \wedge \neg q) \wedge \neg(r \wedge \neg s) \\ &= (p \vee q) \wedge (\neg r \vee s)\end{aligned}$$

CNF: $(p \vee q) \wedge (\neg r \vee s)$

Exercise 2: Satisfiability

1. $\psi_1 := (a \vee \neg b) \wedge (\neg a \vee b) \wedge (\neg a \vee \neg b)$

Try: $a = 0$, $b = 0$

- $(a \vee \neg b) = 0 \vee 1 = 1$
- $(\neg a \vee b) = 1 \vee 0 = 1$
- $(\neg a \vee \neg b) = 1 \vee 1 = 1$

Satisfiable. Assignment: $a = 0$, $b = 0$

2. $\psi_2 := (\neg p \vee q) \wedge (\neg q \vee r) \wedge \neg(\neg p \vee r)$

$$\psi_2 = (\neg p \vee q) \wedge (\neg q \vee r) \wedge (p \wedge \neg r)$$

From p and $\neg r$, we must have $q = 1$ (from $\neg p \vee q$), but then $\neg q = 0$ implies $r = 1$ (from $\neg q \vee r$), which contradicts $\neg r$.
Unsatisfiable.

3. $\psi_3 := (x \vee y) \wedge (\neg x \vee y) \wedge (x \vee \neg y) \wedge (\neg x \vee \neg y)$

Test all 4 assignments:

- $x = 0, y = 0$: $(x \vee y) = 0$
- $x = 0, y = 1$: $(x \vee \neg y) = 0$
- $x = 1, y = 0$: $(\neg x \vee y) = 0$
- $x = 1, y = 1$: $(\neg x \vee \neg y) = 0$

Unsatisfiable.

Exercise 3: Encoding Sudoku in CNF

Let $x_{r,c,v}$ be true iff cell (r, c) has value v, where $r, c, v \in \{1, \ldots, 9\}$.

$$\varphi := C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5 \wedge C_6$$

C_1: Each cell has at least one value

$$C_1 := \bigwedge_{r=1}^{9} \bigwedge_{c=1}^{9} \left(\bigvee_{v=1}^{9} x_{r,c,v} \right)$$

C_2: Each cell has at most one value

$$C_2 := \bigwedge_{r=1}^{9} \bigwedge_{c=1}^{9} \bigwedge_{1 \le v_1 < v_2 \le 9} (\neg x_{r,c,v_1} \vee \neg x_{r,c,v_2})$$

C_3: Each row has all numbers

$$C_3 := \bigwedge_{r=1}^{9} \bigwedge_{v=1}^{9} \left(\bigvee_{c=1}^{9} x_{r,c,v} \right)$$

C_4: Each column has all numbers

$$C_4 := \bigwedge_{c=1}^{9} \bigwedge_{v=1}^{9} \left(\bigvee_{r=1}^{9} x_{r,c,v} \right)$$

$\mathbf{C_5}$: Each 3×3 block has all numbers

Let block rows $br = 0, 1, 2$ and block columns $bc = 0, 1, 2$:

$$C_5 := \bigwedge_{v=1}^{9} \bigwedge_{br=0}^{2} \bigwedge_{bc=0}^{2} \left(\bigvee_{r=3br+1}^{3br+3} \bigvee_{c=3bc+1}^{3bc+3} x_{r,c,v} \right)$$

$\mathbf{C_6}$: Respect the given clues

Let the given clues be $G = \{(r_i, c_i, v_i)\}$:

$$C_6 := \bigwedge_{(r_i,c_i,v_i)\in G} x_{r_i,c_i,v_i}$$