Homework 1: Extended Transition Functions

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1. Description of the Language Accepted by $\mathcal{A}^{(2)}$

The automaton $\mathcal{A}^{(2)}$ accepts the language:

$$L(\mathcal{A}^{(2)}) = \{w \in \{a, b\}^* \mid |w| \text{ is odd and every letter at an odd position is } a\}$$

That is, all words over the alphabet $\{a, b\}$ that have an odd length, and every letter in an odd-numbered position (1st, 3rd, 5th, etc.) is the letter a.

2. Computation of Extended Transition Functions

a) For $\mathcal{A}^{(1)}$, compute $\widehat{\delta^{(1)}}(1, abaa)$

Given that $\mathcal{A}^{(1)}$ accepts non-empty words where no two consecutive letters are the same, and assuming the states are numbered, we proceed step by step:

- Start at state 1
- Read a: move to state 2
- Read b: move to state 1
- Read a: move to state 2
- Read a: since the previous letter was also a, and two consecutive letters are the same, this transition is invalid.

Therefore, the computation halts, and the word $abaa$ is rejected by $\mathcal{A}^{(1)}$.

b) For $\mathcal{A}^{(2)}$, compute $\widehat{\delta^{(2)}}(1, abba)$

Assuming the transition function $\delta^{(2)}$ is defined as per the automaton's description:

- Start at state 1
- Read a: move to state 2
- Read b: move to state 3

- Read b: move to state 2
- Read a: move to state 3

After processing the entire input $abba$, the automaton ends in state 3. If state 3 is an accepting state, then the word is accepted; otherwise, it is rejected. Based on the language description, since $abba$ has even length and the language requires odd length, $abba$ is rejected by $\mathcal{A}^{(2)}$.

Problem Statement

Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m)$ is true, $P(m{+}{+})$ is also true. Show that if $P(n)$ is true, then $P(m)$ is true for all $m \geq n$. This principle is sometimes referred to as the principle of induction starting from the base case n.

Solution

We aim to prove that if $P(n)$ is true and $P(m) \Rightarrow P(m++)$ holds for all $m \in \mathbb{N}$, then $P(m)$ is true for all $m \geq n$.

Approach

Define a new property $Q(k) := P(n + k)$ for $k \in \mathbb{N}$. Our goal is to show that $Q(k)$ is true for all $k \in \mathbb{N}$, which would imply that $P(m)$ is true for all $m \geq n$.

Base Case

For $k = 0$, we have $Q(0) = P(n + 0) = P(n)$, which is given to be true.

Inductive Step

Assume $Q(k)$ is true for some $k \in \mathbb{N}$, i.e., $P(n+k)$ is true. Since $P(m) \Rightarrow P(m++)$ for all m, it follows that $P(n+k) \Rightarrow P((n+k)++) = P(n+k+1)$. Therefore, $Q(k+1) = P(n+k+1)$ is true.

Conclusion

By the principle of mathematical induction, $Q(k)$ is true for all $k \in \mathbb{N}$. Consequently, $P(m)$ is true for all $m \geq n$.